ON SUBOBJECTS IN CATEGORIES

by

Oswald Wyler

Report 67-2

January, 1967

ON SUBOBJECTS IN CATEGORIES

by Oswald Wyler [1]

1. Introduction

Grothendieck [1] defined a subgadget (sous-truc) of an object A of a category $\mathscr{C}$ as an equivalence class of monomorphisms of $\mathscr{C}$ with codomain A. In operational categories [7], a subobject of an object A is basically a subset of the underlying set of A which, with operations induced by the operations of A, becomes an object of the category. In many operational categories (e.g. sets, groups, rings), subgadgets correspond bijectively to subobjects. In others (e.g. topological spaces), there are subgadgets which do not correspond to subobjects.

Various categorical remedies have been considered for this situation. Isbell [2] introduced bicategories. These were generalized by the author in [6]. Sonner [5] introduced canonical categories, with extremal monomorphisms as subobjects. Other methods have been suggested by Isbell [3] and others.

None of these methods seem to be satisfactory for all situations. Thus we propose in this note an axiomatic theory of categories with injections as a common generalization, with subobjects represented by injections. This also generalizes a situation encountered by the author in the study of operational

---

[1] Research partially supported by Research Grant DA-ARO-D-31-124-G680, Army Research Office (Durham).

categories [7]. We obtain the basic properties of categories with injections, we consider the question whether monomorphisms obtained by limit constructions are injections, and we discuss some related topics.

We shall use the language of Mitchell [4], with some modifications. For instance, we use "map" as a synonym of "morphism", and we often identify objects and their identity maps. We find it more convenient to write composition of maps "from left to right", and not "from right to left" as in [4].

2. Categories with injections

Let $\mathcal{C}$ be a category, with composition of maps written "from left to right". We shall denote by αu the domain or left identity of a map u of $\mathcal{C}$, and by βu the codomain or right identity of u, so that $(\alpha u)\, u = u\, (\beta u) = u$. For monomorphisms m and m' with the same codomain, we put $m' \leqslant m$ if $m' = x\, m$ for a map x of $\mathcal{C}$, and we call m and m' equivalent, in symbols $m' \simeq m$, if $m' \leqslant m$ and $m \leqslant m'$, i.e. if $m' = x\, m$ for an isomorphism x of $\mathcal{C}$. In particular, $m \leqslant \beta m$ for any monomorphism m, and $m \simeq \beta m$ if and only if m is an isomorphism.

In many categories (example: topological spaces) there are monomorphisms which one does not want to associate with subobjects of their codomains. This leads us to the following definition.

2.1. Definition. A category with injections is a pair $(\mathcal{C}, \mathcal{J})$ consisting of a category $\mathcal{C}$ and a subcategory $\mathcal{J}$ of $\mathcal{C}$, subject to the follow-

ing conditions.

J 1. Every map of $\mathcal{J}$ is a monomorphism of $\mathcal{C}$, and every isomorphism of $\mathcal{C}$ is in $\mathcal{J}$.

J 2. Whenever $u\,v$ is defined in $\mathcal{C}$, and $u\,v$ and v are in $\mathcal{J}$, then $u\,v$ is in $\mathcal{J}$.

J 3. For any map u of $\mathcal{C}$ there is a map j of $\mathcal{J}$ with the following properties. (i) $u = p\,j$ for a map p of $\mathcal{C}$. (ii) If also $u = u'\,j'$ with $j' \in \mathcal{J}$, then $j \leqslant j'$, i.e. $j = x\,j'$ for a map x of $\mathcal{C}$.

We call the maps of $\mathcal{J}$ injections of $(\mathcal{C},\mathcal{J})$, and we usually write $\mathcal{C}$ for $(\mathcal{C},\mathcal{J})$. The map j of J 3 is called a $\mathcal{J}$-image, or just an image, of the map u. Examples will be given in (2.8).

2.2. We assume from now on that a category with injections $(\mathcal{C},\mathcal{J})$ is given. If $j \in \mathcal{J}$ is an image of $u \in \mathcal{C}$, then we call the factorization $u = p\,j$ a decomposition of u, and the factor p a preimage of u.

In order to make our theory independent of the underlying axiomatic set theory, and only for this reason, we impose the following condition on $\mathcal{J}$.

J 0. There is a subclass $\mathcal{J}_0$ of $\mathcal{J}$ such that every map j of $\mathcal{J}$ is equivalent to exactly one map of $\mathcal{J}_0$, and all identity maps of $\mathcal{C}$ are in $\mathcal{J}_0$.

A map j of $\mathcal{J}_0$ will be called a subobject of its codomain βj. Every object of $\mathcal{C}$ is a subobject of itself.

2.3. Images are defined up to equivalence. Thus every map u of $\mathcal{C}$ has exactly one image in $\mathcal{J}_0$ which we denote by $\operatorname{im} u$. More generally, we put

$$u[j] = \operatorname{im}(j\,u)$$

if $j\,u$ is defined in $\mathcal{C}$ and $j \in \mathcal{J}$. We note the following properties.

(2.3.1) $\operatorname{im} u = u[\alpha u]$ *for any* $u \in \mathcal{C}$, *and* $\operatorname{im} u \simeq u$ *if* $u \in \mathcal{J}$.

(2.3.2) *If* $u[j]$ *and* $u[j']$ *are defined and* $j' \leq j$, *then* $u[j'] \leq u[j]$.

This follows immediately from the definitions.

2.4. Injections form one half of a bicategory [2]; we replace the other half by a definition. We call $p \in \mathcal{C}$ a *projection* of $(\mathcal{C},\mathcal{J})$ if p is a preimage (2.2) of some map u of $\mathcal{C}$. We shall denote by $\mathcal{P}$ the class of all projections of $(\mathcal{C},\mathcal{J})$. This class has the following properties.

(2.4.1) *A map* v *of* $\mathcal{C}$ *is in* $\mathcal{P}$ *if and only if* $\operatorname{im} v = \beta v$.

(2.4.2) *If* $u\,v$ *is defined in* $\mathcal{C}$ *and* $u\,v \in \mathcal{P}$, *then* $v \in \mathcal{P}$.

(2.4.3) $\mathcal{J} \cap \mathcal{P}$ *is the class of all isomorphisms of* $\mathcal{C}$.

(2.4.4) *If* $u = p\,j$ *in* $\mathcal{C}$, *with* $p \in \mathcal{P}$ *and* $j \in \mathcal{J}$, *then* j *is an image, and* p *a preimage, of* u.

Proof. If $\operatorname{im} v = \beta v$, then $v \in \mathcal{P}$. Conversely, let j be an image of $v\,j$, and let $v = v'\,j'$, with $j' \in \mathcal{J}$. Then $v\,j = v'\,j'\,j$, and hence $j \leq j'\,j$. It follows that j' is isomorphic, and thus $\operatorname{im} v = \beta v$.

If $v = v'\,j$ with $j \in \mathcal{J}$, then $\operatorname{im}(u\,v) \leq j$. Thus $\operatorname{im}(u\,v) \leq \operatorname{im} v$. If $\operatorname{im}(u\,v) = \beta v$, it follows that $\operatorname{im} v = \beta v$. This proves (2.4.2).

A map u of $\mathcal{J}$ is in $\mathcal{P}$ if and only if $\beta u = \operatorname{im} u \simeq u$, and hence if and only if u is isomorphic. This proves (2.4.3).

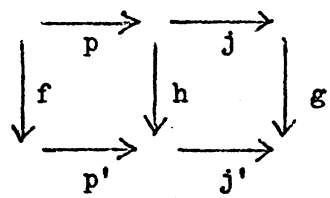
If $\operatorname{im} u = j_1$ in (2.4.4), then $j_1 \leqslant j$. If $j_1 = j' j$ and $u = p_1 j_1$, then $j' \in \mathcal{J}$ by J 2, and $p = p_1 j'$. But then $j' \geqslant \operatorname{im} p = \beta p$, and j' is isomorphic. Thus $j \simeq j_1$, and j is an image of u.

2.5. Proposition. The following statements are logically equivalent.

(i) $v[u[j]] = (u\,v)[j]$ whenever $u\,v$ and $u[j]$ are defined in $\mathcal{C}$.

(ii) $\operatorname{im}(u\,v) = v[\operatorname{im} u]$ whenever $u\,v$ is defined in $\mathcal{C}$.

(iii) If $u\,g = f\,u'$ in $\mathcal{C}$, and if $u = p\,j$ and $u' = p'\,j'$ are decompositions, then there always is a map h in $\mathcal{C}$ such that the diagram



is commutative.

(iv) Projections form a subcategory of $\mathcal{C}$.

Proof. (i) $\Longrightarrow$ (ii) by putting $j = \alpha u$ in (i).

In the diagram of (iii), we always have $\operatorname{im}(f\,u') \leqslant j'$. If (ii) is valid, then $\operatorname{im}(f\,u') = \operatorname{im}(u\,g) = g[j] = \operatorname{im}(j\,g)$, and it follows that $j\,g = h\,j'$ for a map h. But then $p\,h\,j' = p\,j\,g = f\,p'\,j'$, and hence $p\,h = f\,p'$. Thus (ii) $\Longrightarrow$ (iii).

Consider now the diagram of (iii) with $u = p$ in $\mathcal{P}$, $j = \beta p$, $f = \alpha p$, and $u' = p\,g$. If (iii) is valid, then $g = h\,j'$ for a map h, and it follows that $\operatorname{im} g \leqslant \operatorname{im}(p\,g)$. If $g \in \mathcal{P}$, so that $\operatorname{im} g = \beta g$, this implies that $\operatorname{im}(p\,g) = \beta g$, and hence $p\,g \in \mathcal{P}$. Thus (iii) $\Longrightarrow$ (iv).

Finally, if $u[j] = j_1$ and $v[j_1] = j_2$, with decompositions $j\,u = p_1\,j_1$ and $j_1\,v = p_2\,j_2$, then $j\,u\,v = p_1\,p_2\,j_2$ is a decomposition, and $(u\,v)[j] = j_2 = v[u[j]]$, if $p_1\,p_2 \in \mathcal{P}$. Thus (iv) $\Longrightarrow$ (i).

2.6. Proposition. The following two statements are logically equivalent.

(i) Every map u of $\mathcal{C}$ has a factorization $u = e\,j$ with e epimorphic in $\mathcal{C}$ and $j \in \mathcal{J}$.

(ii) Every projection of $\mathcal{C}$ is epimorphic in $\mathcal{C}$.

Proof. If $p = e\,j$ with $p \in \mathcal{P}$, $j \in \mathcal{J}$, and e epimorphic, then $\beta p = \operatorname{im} p \leqslant j$, and j is isomorphic. But then p is epimorphic, and thus (i) $\Longrightarrow$ (ii). The converse is trivial.

2.7. Remarks. Definition (2.1) is easily dualized. We call the dual of a category with injections a category with projections. A decomposition $u = p\,j$ in a category with projections defines a coimage p and a postimage j of u, and postimages are injections of the category.

We have used J 2 exactly once, in the proof of (2.4.4). It is easily seen that, conversely, J 2 follows from J 1, J 3, and (2.4.4).

If $(\mathcal{C},\mathcal{J},\mathcal{P})$ is a bicategory in the sense of Isbell [2], then $(\mathcal{C},\mathcal{J})$ is a category with injections and $\mathcal{P}$ the class of its projections. Conversely, if $(\mathcal{C},\mathcal{J})$ is a category with injections and $\mathcal{P}$ the class of its projections, then the following three statements are easily seen to be logically equivalent.

(i) $\mathcal{P}$ is a subcategory of $\mathcal{C}$ and consists of epimorphisms of $\mathcal{C}$.

(ii) $(\mathcal{C},\mathcal{P})$ is a category with projections.

(iii) $(\mathcal{C},\mathcal{J},\mathcal{P})$ is an Isbell bicategory.

Let now $\mathcal{C}$ be a category, $\mathcal{M}$ the class of its monomorphisms, and $\mathcal{E}$ the class of its epimorphisms. $\mathcal{M}$ always satisfies conditions J 1 and J 2 of (2.1), and $\mathcal{M}$-images in our sense are images in the sense of Grothendieck [1], and of [4]. $\mathcal{M}$ satisfies J 3 if and only if $\mathcal{C}$ has images, in the sense of [4], I.10.

Let us call $\mathcal{C}$ factored if every map u of $\mathcal{C}$ has a factorization $u = e\,m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. If $\mathcal{C}$ is factored, and if $u = e\,m$, then m is an image of u in the sense of [1] and [4] if and only if e is a coimage of u in the sense of Sonner [5]. In other words, the images and coimages of [5] are post-images and preimages in our terminology. It follows that a canonical category, in the sense of [5], is the same as a factored category $\mathcal{C}$ such that $(\mathcal{C},\mathcal{M})$ is a category with injections, and $(\mathcal{C},\mathcal{E})$ a category with projections.

If $\mathcal{C}$ is factored and $(\mathcal{C},\mathcal{M})$ a category with injections, then the projections of $(\mathcal{C},\mathcal{M})$ are the extremal epimorphisms ([3], [5]) of $\mathcal{C}$.

2.8. Examples. In a pointed category, the class of normal monomorphisms, as defined e.g. in [6], satisfies J 1 and J 2, and in many cases also J 3, but this class is in general not a subcategory.

The injections of an operational category $\mathcal{C}$ (see [7] for the definitions and notations used in this paragraph) form a subcategory $\mathcal{J}$ of $\mathcal{C}$ which satisfies J 1 and J 2. If $\mathcal{C}$ is an Ω-category and all range functors R_ω, $\omega \in \Omega$, preserve intersections, then $\mathcal{C}$ also satisfies J 3. If these functors preserve inverse images as well as intersections, then $\mathcal{C}$ has inverse images (see below), and projections form a subcategory of $\mathcal{C}$.

The categories of sets and of groups, with subsets and subgroups respectively as subobjects, are categories with injections, with all monomorphisms as injections, and all epimorphisms as projections. The categories of topological spaces and of Hausdorff spaces can be considered as categories with injections in at least three ways. We may regard just the closed subspaces, or all subspaces, of a space A as subobjects of A, or we may regard all monomorphisms as injections.

In all examples of the preceding paragraph, inverse images exist, and $\mathcal{P}$ is a subcategory. For topological spaces with closed subspaces as subobjects, all epimorphisms are projections, but not all projections epimorphic. For Hausdorff spaces, with all subspaces as subobjects, all projections are epimorphic, but not all epimorphisms projections.

3. Limits in categories with injections

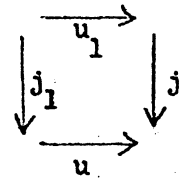
Let $(\mathcal{C},\mathcal{J})$ be a category with injections. We consider the following question. If a limit construction preserves, or induces, monomorphisms in any category, does it preserve, or induce, injections in $\mathcal{C}$?

3.1. Proposition. *Let $(j_\lambda)_{\lambda\in I}$ be a family of injections with a common codomain. If $\bigcap j_\lambda$ is defined in $\mathcal{C}$, then $\bigcap j_\lambda$ is an injection.*

Proof. Let $m = \bigcap j_\lambda$. Then $m = x_\lambda j_\lambda$, for a map x_λ, for every $\lambda\in I$. But then $\operatorname{im} m \leq j_\lambda$ for all λ, and hence $\operatorname{im} m \leq m$. Now $m \leq \operatorname{im} m$ in any case. Thus $m \simeq \operatorname{im} m$, and $m\in\mathcal{J}$.

3.2. Inverse images in $\mathcal{C}$ are defined by pullback diagrams

(3.2.1)



with j, and hence j_1, monomorphic. We say that pullbacks in $\mathcal{C}$ preserve injections if $j_1 \in \mathcal{J}$ in every pullback diagram (3.2.1) with $j \in \mathcal{J}$. We say that $(\mathcal{C},\mathcal{J})$ has inverse images if for any maps u of $\mathcal{C}$ and j of $\mathcal{J}$, with the same codomain, there is a pullback diagram (3.2.1) in $\mathcal{C}$, with $j_1 \in \mathcal{J}$.

A pullback diagram (3.2.1) is determined by u and j up to an isomorphism in the upper lefthand corner. Thus if $j_1 \in \mathcal{J}$, we can determine $j_1 \in \mathcal{J}_o$ in exactly one way. We put

$$j_1 = u^{\leftarrow}[j]$$

if $j \in \mathcal{J}$ and $j_1 \in \mathcal{J}_o$ in a pullback diagram (3.2.1).

3.3. The following statements are valid whenever all their terms are defined.

(3.3.1) $u^{\leftarrow}[\beta u] = \alpha u$.

(3.3.2) $(u\,v)^{\leftarrow}[j] = u^{\leftarrow}[v^{\leftarrow}[j]]$.

(3.3.3) $u^{\leftarrow}[\bigcap j_\lambda] = \bigcap u^{\leftarrow}[j_\lambda]$.

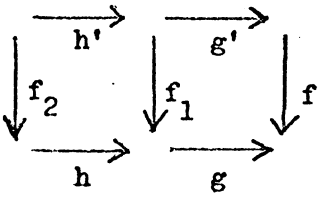
(3.3.4) If $j' \leq j$, then $u^{\leftarrow}[j'] \leq u^{\leftarrow}[j]$.

(3.3.5) $u[j_1] \leq j$ if and only if $j_1 \leq u^{\leftarrow}[j]$.

(3.3.6) $u[u^{\leftarrow}[j]] \leq j$ and $u^{\leftarrow}[u[u^{\leftarrow}[j]]] = u^{\leftarrow}[j]$.

(3.3.7) $j_1 \leq u^{\leftarrow}[u[j_1]]$ and $u[u^{\leftarrow}[u[j_1]]] = u[j_1]$.

We omit the simple proofs. (3.3.2) is a special case of the following result ([4], Prop. I.7.2). If the righthand square of the commutative diagram



is a pullback, then the outer rectangle is a pullback if and only if the lefthand square is a pullback. We shall use this result in (4.5) and (4.6).

3.4. Proposition. If $\mathcal{P}$ is a subcategory of $\mathcal{E}$, then pullbacks in $\mathcal{E}$ preserve injections. Conversely, if $(\mathcal{E},\mathcal{J})$ has inverse images, then $\mathcal{P}$ is a subcategory of $\mathcal{E}$.

Proof. Let $j \in \mathcal{J}$ in a pullback diagram (3.2.1), and let $\operatorname{im} j_1 = j'$. Then $j_1 \leqslant j'$. If $\mathcal{P}$ is a subcategory, then

$$u[j'] = \operatorname{im}(j_1\, u) = \operatorname{im}(u_1\, j) \leqslant j$$

by (2.5), and thus $j'\, u = u'\, j$ for a map u'. Thus $j' = x\, j_1$, $u' = x\, u_1$, for a map x. But then $j_1 \simeq j'$, and $j_1 \in \mathcal{J}$ since $j' \in \mathcal{J}$.

Conversely, let $(\mathcal{E},\mathcal{J})$ have inverse images. If $v\, u = p\, j$ in $\mathcal{E}$, with p, u, v in $\mathcal{P}$ and j in $\mathcal{J}$, construct a pullback diagram (3.2.1), with $j_1 \in \mathcal{J}$. Then $v = x\, j_1$, $p = x\, u_1$, for a map x, and $\beta v = \operatorname{im} v \leqslant j_1$. But then j_1 is isomorphic, and $u = j_1^{-1}\, u_1\, j$. Now $\beta u = \operatorname{im} u \leqslant j$, and j is isomorphic. Thus $v\, u = p\, j$ is in $\mathcal{P}$, and $\mathcal{P}$ is a subcategory.

3.5. Proposition. Let $(j_\lambda : A'_\lambda \longrightarrow A_\lambda)_{\lambda \in I}$ be a family of injections for which a product $\times j_\lambda : \times A'_\lambda \longrightarrow \times A_\lambda$ is defined in $\mathcal{C}$. If $\mathcal{P}$ is a subcategory of $\mathcal{C}$, then $\times j_\lambda$ is in $\mathcal{J}$.

Proof. Let $p_\lambda : \times A_\lambda \longrightarrow A_\lambda$ and $p'_\lambda : \times A'_\lambda \longrightarrow A'_\lambda$ be the projections, so that $(\times j_\lambda)\, p_\lambda = p'_\lambda\, j_\lambda$ for all $\lambda \in I$, and let $\times j_\lambda = q\, j$ be a decomposition. By (2.5.iii) there are maps r_λ, one for each $\lambda \in I$, such that $p' = q\, r_\lambda$, and $r_\lambda\, j_\lambda = j\, p_\lambda$, for all λ. There is a map r such that $r_\lambda = r\, p'_\lambda$ for all λ, and then $q\, r\, p'_\lambda = q\, r_\lambda = p'_\lambda$ for all λ, and hence $q\, r = \alpha q$. On the other hand, $r\, q\, j\, p_\lambda = r\, q\, r_\lambda\, j_\lambda = r\, p'_\lambda\, j_\lambda = r_\lambda\, j_\lambda = j\, p_\lambda$ for all λ, so that $r\, q\, j = j$. But then $r\, q = \beta q$, and q is isomorphic. Thus $\times j_\lambda = q\, j$ is in $\mathcal{J}$.

3.6. Proposition. If all projections of $\mathcal{C}$ are epimorphisms, then all equalizers in $\mathcal{C}$ are injections. Conversely, if $\mathcal{C}$ has equalizers, and all equalizers are injections, then all projections are epimorphic.

Proof. If m is an equalizer of f and g in $\mathcal{C}$, then m is monomorphic. If $m = p\, j$ is a decomposition, then $m \leq j$, and $p\, j\, f = p\, j\, g$. If p is epimorphic, then $j\, f = j\, g$ follows. But then $j \leq m$, so that $j \simeq m$, and $m \in \mathcal{J}$.

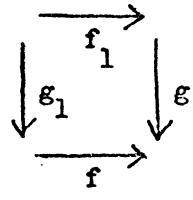
Conversely, let $p\, f = p\, g$, with $p \in \mathcal{P}$. If f and g have an equalizer j in $\mathcal{C}$, then $p = p'\, j$ for a map p'. If $j \in \mathcal{J}$, it follows that $\beta p = \mathrm{im}\, p \leq j$. But then j is isomorphic, and $f = g$.

3.7. We shall denote by p_1^{AB} and p_2^{AB}, or just by p_1 and p_2, the projections of a product $A \times B$ in $\mathcal{C}$. If $f : C \longrightarrow A$ and $g : C \longrightarrow B$ are maps of $\mathcal{C}$ with the same domain C, and if a product $A \times B$ is defined in $\mathcal{C}$, then we denote by $\{f,g\} : C \longrightarrow A \times B$ the map of $\mathcal{C}$ characterized by

$$\{f,g\} p_1^{AB} = f \ , \quad \{f,g\} p_2^{AB} = g \ .$$

If $\mathcal{C}$ has finite products, then any pullback diagram

(3.7.1)



defines a monomorphic map $\{f_1,f_2\}$.

3.8. Proposition. If $\mathcal{C}$ has finite products, and if all projections of $\mathcal{C}$ are epimorphic in $\mathcal{C}$, then every map $\{f_1,g_1\}$ obtained from a pullback diagram (3.7.1) is an injection. Conversely, if $\mathcal{C}$ has finite products and pullbacks, and if all maps $\{f_1,g_1\}$ obtained from pullback diagrams (3.7.1) are injections, then all projections of $\mathcal{C}$ are epimorphic in $\mathcal{C}$.

Proof. If (3.7.1) is given, and if $\{f_1,g_1\} = q\, j$ is a decomposition, then $q\, j\, p_1\, g = q\, j\, p_2\, f$. If q is epimorphic, $j\, p_1\, g = j\, p_2\, f$ follows. But then $j\, p_1 = x\, f_1$, $j\, p_2 = x\, g_1$, for a map x of $\mathcal{C}$, and $j = x\{f_1,f_2\} = x\, q\, j$ follows. But then $x\, q = \beta q$, and as q is epimorphic, it follows that q is isomorphic, so that $\{f_1,g_1\} = q\, j$ is in $\mathcal{J}$.

Conversely, let p be a projection with codomain $\beta p = A$, and assume that

all possible pullback diagrams (3.7.1) exist, with $\{f_1,g_1\}$ in $\mathcal{J}$. Then $\{1_A,1_A\}$ is in $\mathcal{J}$, since the square with four sides 1_A is a pullback.

If $p\,f = p\,g$, construct a pullback diagram (3.7.1) for this f and g. Then $p = x\,f_1 = x\,g_1$ for some map x of $\mathcal{C}$, and hence

$$x\,\{f_1,g_1\} = \{p,p\} = p\,\{1_A,1_A\}\ .$$

As $\{1_A,1_A\}\in\mathcal{J}$, this is a decomposition, and thus $\{1_A,1_A\}\leq\{f_1,g_1\}$. This means that $1_A = y\,f_1 = y\,g_1$ for a map y of $\mathcal{C}$. But then

$$f = y\,g_1\,f = y\,f_1\,g = g\ ,$$

and p is epimorphic.

4. Complements

We consider some functors, extremal monomorphisms and epimorphisms, coretractions, and pullbacks preserving projections.

4.1. Let $\mathrm{Map}\,\mathcal{C}$ be the category with maps of $\mathcal{C}$ as objects and commutative squares in $\mathcal{C}$ as maps, with composition defined by juxtaposition of squares. If $\mathcal{C}$ is a category with injections, and $\mathcal{P}$ a subcategory of $\mathcal{C}$, then the diagram of (2.5.iii), with j and j' in $\mathcal{J}_0$, defines an image functor and a preimage functor, both from $\mathrm{Map}\,\mathcal{C}$ to $\mathrm{Map}\,\mathcal{C}$.

4.2. We call a category with injections $(\mathcal{C},\mathcal{J})$ locally small if, for every object A of $\mathcal{C}$, the maps in $\mathcal{J}_0$ with codomain A, i.e. the subobjects

of A, form a set. This is an ordered set; we denote it by $A\,P$. For a map $u : A \longrightarrow B$ of $\mathcal{C}$, we define an order preserving mapping $u\,P : A\,P \longrightarrow B\,P$ by putting

$$j\,(u\,P) = u[j]$$

for $j \in A\,P$. If $(\mathcal{C},\mathcal{J})$ is locally small and $\mathcal{P}$ a subcategory of $\mathcal{C}$, this defines a covariant direct image functor P, from $\mathcal{C}$ to the category of ordered sets.

If $(\mathcal{C},\mathcal{J})$ is locally small and has inverse images, then for any map $u : A \longrightarrow B$ of $\mathcal{C}$, we define a mapping $u\,P^* : B\,P \longrightarrow A\,P$ by putting

$$j\,(u\,P^*) = u^{\leftarrow}[j] ,$$

for $j \in B\,P$. This defines a contravariant inverse image functor P^*, from $\mathcal{C}$ to the category of ordered sets.

4.3. Using the terminology of (2.7), we have the following result.

Proposition. If a factored category $\mathcal{C}$ has images and inverse images, then the extremal epimorphisms of $\mathcal{C}$ form a subcategory of $\mathcal{C}$.

Proof. If $\mathcal{C}$ is factored and has images, then $(\mathcal{C},\mathcal{M})$ is a category with injections, with extremal epimorphisms as projections. If $\mathcal{C}$ has inverse images, these projections form a subcategory of $\mathcal{C}$, by (3.4).

We note the dual result, in a somewhat weaker form.

Proposition. If a factored category $\mathcal{C}$ has coimages and pushouts, then the

extremal monomorphisms of $\mathcal{C}$ form a subcategory of $\mathcal{C}$.

4.4. We recall that a map u of $\mathcal{C}$ is called a coretraction of $\mathcal{C}$ if u has a right inverse, i.e. if $u\,v = \alpha u$ for some map v of $\mathcal{C}$.

Proposition. If all projections of $\mathcal{C}$ are epimorphic in $\mathcal{C}$, then all coretractions of $\mathcal{C}$ are injections. Conversely, if $\mathcal{C}$ has finite products and all coretractions of $\mathcal{C}$ are injections, then all projections of $\mathcal{C}$ are epimorphic.

Proof. Let $u\,v = \alpha u$, and let $u = p\,j$ be a decomposition, so that $p\,(j\,v) = \alpha p$. If p is epimorphic, it follows that $(j\,v)\,p = \beta p$, so that p is isomorphic, and $u \in \mathcal{J}$.

Conversely, let $p\,u = p\,v$ for $p \in \mathcal{P}$ and u, v from A to B in $\mathcal{C}$. Using the notations of (3.7), we put $f = \{1_A, u\}$ and $g = \{1_A, v\}$. Then $p\,f = p\,g$ and $f\,p_1 = g\,p_1 = 1_A$. If coretractions are injections, then $p\,f$ and $p\,g$ are decompositions of the same map. But then $g = x\,f$, $p = p\,x^{-1}$, for an isomorphism x of $\mathcal{C}$. It follows that $1_A = x\,1_A$ and $v = x\,u$. But then $x = 1_A$ and $v = u$, so that p is epimorphic.

4.5. We say that pullbacks in a category with injections $\mathcal{C}$ preserve projections if $f \in \mathcal{P} \Longrightarrow f_1 \in \mathcal{P}$ for every pullback diagram (3.7.1) in $\mathcal{C}$.

Proposition. If $(\mathcal{C}, \mathcal{J})$ has inverse images, then the following two statements are logically equivalent.

(i) Whenever a pullback diagram (3.7.1) is given in $\mathcal{C}$, and $f[j]$ is defined, then $g^{\leftarrow}[f[j]] = f_1[g_1^{\leftarrow}[j]]$.

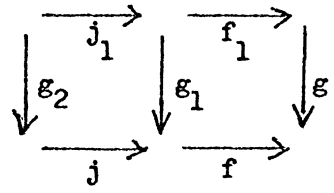
(ii) Pullbacks in $\mathcal{C}$ preserve projections.

Proof. If $f \in \mathcal{P}$ in a pullback diagram (3.7.1), and if (i) holds, then

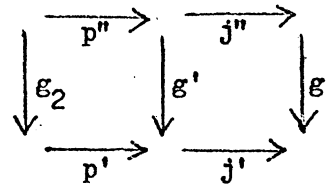
$$\text{im } f_1 = f_1[g_1^{\leftarrow}[\alpha f]] = g^{\leftarrow}[f[\alpha f]] = g^{\leftarrow}[\beta f] = \beta f_1 ,$$

and $f_1 \in \mathcal{P}$ by (2.4.1). Thus (i) $\Longrightarrow$ (ii).

Conversely, consider diagrams



and



in which all squares are pullbacks. If a pullback diagram (3.7.1) is given and $f[j]$ is defined, and if $j\,f = p'\,j'$ is a decomposition, then the three squares not involving p' and p'' are defined. Since $g_2\,p'\,j' = j_1\,f_1\,g$, there is a map p'' such that $g_2\,p' = p''\,g'$ and $j_1\,f_1 = p''\,j''$. Thus the fourth square is defined, and a pullback by the result of [4] quoted above in (3.3). If $p'' \in \mathcal{P}$, then $j_1\,f_1 = p''\,j''$ is a decomposition, and hence

$$f_1[g_1^{\leftarrow}[j]] = f_1[j_1] = j'' = g^{\leftarrow}[j'] = g^{\leftarrow}[f[j]]$$

if we assume, as we may, that $j'' \in \mathcal{J}_0$. Thus (ii) $\Longrightarrow$ (i).

4.6. We say that inverse images in $(\mathcal{C},\mathcal{J})$ preserve projections if $f \in \mathcal{P}$ $\Longrightarrow$ $f_1 \in \mathcal{P}$ in every pullback diagram (3.7.1) with g and g_1 in $\mathcal{J}$. In the following result, the intersections exist as inverse images and are in $\mathcal{J}$ by (3.1). Thus we may, and do, assume that they are in $\mathcal{J}_0$.

Proposition. If $(\mathcal{E},\mathcal{J})$ has inverse images, then the following four statements are logically equivalent.

(i) $f[j \cap f^{\leftarrow}[g]] = f[j] \cap g$ whenever $f[j]$ and $f^{\leftarrow}[g]$ are defined.

(ii) $f[f^{\leftarrow}[j]] = j \cap \operatorname{im} f$ whenever $f^{\leftarrow}[j]$ is defined.

(iii) $p[p^{\leftarrow}[j]] \simeq j$ whenever $p \in \mathcal{P}$ and $p^{\leftarrow}[j]$ is defined.

(iv) Inverse images in $(\mathcal{E},\mathcal{J})$ preserve projections.

Proof. Replacing j by $\propto f$ and g by j in (i), we have (i) $\Longrightarrow$ (ii).

Replacing f by p in (ii), with $\operatorname{im} p = \beta p$, we have (ii) $\Longrightarrow$ (iii).

In a pullback diagram (3.7.1), with g and g_1 in $\mathcal{J}$ and f in $\mathcal{P}$, we have $f^{\leftarrow}[g] \simeq g_1$, and $\operatorname{im}(f_1 g) = f[g_1] = f[f^{\leftarrow}[g]] \simeq g$ if (iii) holds. But then g is an image of $f_1 g$, and $f_1 \in \mathcal{P}$. Thus (ii) $\Longrightarrow$ (iii).

Now consider the two diagrams of the proof of (4.5), with $g \in \mathcal{J}$ and $p' \in \mathcal{P}$. If $p'' \in \mathcal{P}$, then $g_2 \, j \, f = p''(j'' g)$ is a decomposition, and

$$f[j \cap f^{\leftarrow}[g]] = f[j \cap g_1] = f[g_2 \, j] = j'' g = g \cap j' = g \cap f[j]$$

if we assume, as we may, that $j'' g$ is in $\mathcal{J}_0$. Thus (iv) $\Longrightarrow$ (i).

4.7. Remarks. In a bicategory, (4.4) and J 2 may be strengthened to the dual of (2.4.2): If u v is defined in $\mathcal{E}$ and in $\mathcal{J}$, then $u \in \mathcal{J}$. See [2].

The category of Hausdorff spaces, with closed subspaces as subobjects, is a bicategory in which inverse images do not preserve projections. This is easily verified. The author does not have at present an example of a category with injections in which projections are preserved by inverse images, but not by arbitrary pullbacks.

References

[1] Grothendieck, A., Sur quelques points d'algèbre homologique. Tôhoku Math. J. 9 (1957), 119 - 221.

[2] Isbell, J. R., Some remarks concerning categories and subspaces. Canad. J. Math. 9 (1957), 563 - 577.

[3] ----------, Subobjects, adequacy, completeness, and categories of algebras. Rozprawy Mat. 36 (1964).

[4] Mitchell, B., Theory of Categories. New York and London, 1965.

[5] Sonner, J., Canonical categories. Proceedings of the Conference on Categorical Algebra - La Jolla 1965. Berlin, Heidelberg, New York, 1966.

[6] Wyler, O., Weakly exact categories. Archiv der Math. 17 (1966), 1 - 19.

[7] ---------, Operational categories. Proceedings of the Conference on Categorical Algebra - La Jolla 1965. Berlin, Heidelberg, New York, 1966.