ON COMPLETIONS OF UNIFORM LIMIT SPACES

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Weil and Bourbaki fl] introduced the category of uniform spaces, to be denoted by V; in this note, and constructed a coreflective ' completion functor from tr to the subcategoiy U of separated complete uniform spaces. Cook and Fischer J3J introduced the category of uniform limit spaces which we shall denote by JJ • They pointed out that Jf can be regarded as a coreflective full subcategory of JJ • Subsequently, the author f?J constructed a coreflective completion functor from the subcategory of separated uniform limit spaces to the smaller subcategory JJ of separated complete uniform limit spaces* Cochran f2] studied this completion further, supplying the proofs not given in fs]# He also raised the following question* A uniform space (E, j) has a completion (E f J[ ) in JJ f and also the gourbaki completion (E°, £) in JJ^{pC} • How are these two completions related? We answer this question in the present note by showing that £ is the finest uniform structure of E coarser than the uniform limit structure jj^c

We introduce first the notations and definitions which we shall use# If J? and £ are filters on a set B , we put £ ^ G, if £ is finer than £ , i.e# if £ C £ . With this order relation, filters on E form a complete lattice • If JP and £ are filters on E , then P uC consists of all sets A u B ,

• Footnotes are given at the end of this report, after the references.

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A €Z » B££ ' and L n<i consists of all sets A n B , A £F B <££ •

We shall use the following notations for filters on EX E • If £ and £ are filters on E t we denote by ,FX G. the filter on E V E generated by all sets A X B , A £ JP , B£r J3" . For subsets U and V of E X E , we denote by \J~ the set of all pairs (y,x) with (x,y)e U , and by U ° V the set of all pairs (x,z) with (x,y) (£ U and (y,z)6LV for some y. For filters 4> and V on EX E , we denote by &~ 1 the filter consisting of all sets lT 1 f U S <p f and by <po^ the filter[4] generated by all sets U o V , V £ (£) f V ^: y* • We refer to f2j and f3j for the laws satisfied by these operations*

Cook and Fischer f3] have defined a uniform limit structure on a set E as a set £ of filters on E X E with the following three propertie$«

UL 1. If <p&£ and ty £ <$) f then ^ 6 J .

UL 2. Let A be the filter of all subsets of E x E containing the diagonal A of J. If <£>, V^ are in J , then (#>v_/A) ® (<4^^A) is in J .

DL 3. If Cb e I , then cf?"1^ £ .

A uniform limit space is a pair (Efij) consisting of a set E and a uniform limit structure J on E . The filters of £ are called uniform filters of (EfJj) •

A uniformly continuous mapping f : (E,J[) —> (Ef,£f) of uniform limit spaces is a mapping f : E —> Bf such that the filter (f xf)((£>) is in Jf for every filter (p in £ • Uniform limit spaces and their uniformly continuous mappings are the objects and morphisms of a category which we denote by JO and call the category of uniform limit spaces.

Since (d> KJK) ^ (4^^A) * A u ^ u f u f ^ 0 ^) , it follows from

the axioms UL that *(P^J^* and *(p&y* are in *£* whenever *(p_f ^* are in J ,
* .5

and that *A € £* • Thus £ is a dual filter of filters on E X B •

If £ is a principal dual filter, i.e. £ consists of all filters *($>* such that *(P £ <t*_o ,* for a generator *(p* of j; , then *(p_o* is the filter of entourages of a uniform structure of E . All uniform structures are obtained in this way; see f?]. This means that we may (and do) regard the category of uniform spaces as a full subcategory of JJ *%* we denote this subcategory by JJ .

For any uniform limit space (E_fi j) there is, as shown in feL a finest uniform structure £^P on E such that £ ^ J^P (i.e. *£ C* J^P) • If *(fy* is the filter of entourages of (E_>: J^P) , then *(£> £ (£>* for every filter *(p* of £ , but as Cochran f2] has pointed out, *<$)* is in general not the supremum of all filters in J[• The correspondence from (B,J[) to (E_t J^P) defines a coreflective functor from JJ to JJ^P , as shown in f3j•

We shall need the following additional definitions. Let (E_f< j) be a uniform limit space. A filter £ on B is called a Cauchy filter of (E,£) if jF is not the null filter[4] of B , and JPXjP (5£ • Two Cauchy filters £ and £ on B are called equivalent if P *K* £ 6 J • This is easily seen to be an equivalence relation. We denote by E the set of all equivalence classes of Cauchy filters *of* (B,£) _f and by q(p) the equivalence class of a Cauchy filter £ .

For *x€* E _f we denote by x• the filter of all sets A *CZ* E with x£.A . This is a Cauchy filter. We say that a filter F on B converges to *x* £ E if JPX *xCjl* • This defines the induced limit space[6] of (E,J[) ; see f3J. Every convergent filter on E is a Cauchy filter. The uniform limit space (E_fi j) is

called complete if, conversely, every Cauchy filter converges to some point of E .
We call (E,J[) separated if •xX ẏ ^ J, for points x, y of E_f only if x » y •
We denote by IJ^c the subcategory of separated complete uniform limit spaces and their uniformly continuous mappings. As shown in [2J, JJ^c is basically the same as the category of separated (i.e. Hausdorff) limit spaces• We denote by JJ^{pc} the intersection of the subcategories tJ^p and J2^c of JJ •

For any uniform limit space $(E_{f<}j)$ we construct a separated complete uniform limit space (E ,J[ ) as follows. E is the set of equivalence classes of Cauchy filters of (E,J[) , We define j : E —> $E^°$ by putting j x = q(x) for x £ B. . For a Cauchy filter J? of (£,j) and jf ~<p(F), we put

$$y\ l \cdot« (5(1) \wedge f)\ x\ (j(F) \cup \xi)\ .$$

With these notations, $£^c$ is the dual filter of filters on $E^c \times E^c$ generated by all filters of the form

$$(j \times j)(\Phi) \cup \dot{\Delta}_c \cup \gamma \underline{F}_1 \cup \ \dots\ \cup \gamma \underline{F}_r\ ,$$

where (þ £ £ , A is the diagonal of $E^c \times E^c$, and £, , ... , F are Cauchy filters of $(E_{f}j)$. As pointed out in $J5]_f$ the following theorem is valid.

Theorem 1• For any uniform limit space $(E_f j)$, (E^c, J^c) is a separated complete uniform limit space, and the mapping j : $(E_f«j)$ —;> $(E^c_f J[^c)$ is uniformly continuous. Moreover, whenever f : $(E_{fg}j)$—>$(E^f, J[^f)$ is uniformly continuous and $(E'jj;^1)$ separated and complete, then there is a unique uniformly continuous mapping f^c J $(E°_f J^c)$ —> $(E^f_f«P)$ such that f s f^c j •

In other words, Theorem 1 states that the correspondence from (B,jj) to (E˜fJJ˜) defines a coreflective functor from JJ to the subcategory tJ * We refer to *[2]* for the proof. In f5],.only separated uniform limit spaces (B,£) were considered, but the extension to the general case is trivial* The only change is this* If (Bfjj) is separated, and only in this case, j : (E,ç[) •–:> (E^c ,£^c) induces an isomorphism of (BfJj) and a dense subspace of (B^c ftJ^c) .

For a uniform space (E,J[) , with filter C|) of entourages, we have of course the completion (E^c,J^c) , and the finest uniform structure J[^cp of E^c * which is coarser than £^c. We also have the itouxbaki completion j : (E,£) —™> *(®°fJ^b.*). • constructed as follows* The set E^c and the mapping j s E – > E^c are the same as for (E^C f(J^C) • For an entourage U <= CJD, let U^b be the set of all pairs *(ffV)* £-E°X E^C such that there are Cauchy filters £ and £ of (E,J[) with I « q(F) f /W = q(G) , and U f F X G • The filter of entourages 0^b of (E!^C,J^b) is generated by the sets U^b , *V <£: 0.*

~~Lemma~~* JJ ^ ^ ^ jJ •

~~Proof~~* £^cp ° ^ £ is clear* For j : E –^ E^c there is, by Theorem 1, a uniformly continuous mapping j^c *(&»«^c[^c) –"^(B^c fiJ^b) such that j » j^c j #
For a Cauchy filter £ of (Ejj;)^1 f the filter jCjP) converges to q(£) in (^EC»2°) ^and ia (E^C,J^b) .. and thus j^C(q(P)) = q(P) . Thus j^C is the identity mapping, and J^c s* b ^ •$J^b, follows* Since £ is principal, also J^cp ^ J[^b •

Theorem j?» If (E,J[) is a uniform space, then the uniform structure jJ^b of E^c is the finest principal structure J[^Cp coarser than the uniform limit structure J[^c • JT SI is the supremum of all filters Ψ £±[^c and (p^b the filter of entourages of J^b , then (P^b =* SI o J7 ofl #

Proof, If 0^P is the filter of entourages of J^{Cp} , then J l £ < £ P ^ 0 b f and (j) P ^ P - (j) P . Thus SI ° SI oSI £ 0^P , and 0^b ^ il'o JZ p/1 implies that $(f)^h$ * SI °Sl o i l - 0^V , and hence J^{Cp} « J^b #

Let now V £ i T , Since (j X j)(<£>) ^ iT., and ^P ^ i l for every Cauchy filter P of (B,J) , there is an entourage U<=^ such that (j X j)(u) C V , and for every Cauchy filter F of (E,j) and J « q(£) f there is a set A (JjP such that A^1 X A^1 (Z V for A^1 « J(A) L ^ { ^ } . Now consider (^<rj) £ U^b • There are Cauchy filters £ and £ of (B,j) and sets A (f* P , B £ £ such that ξ = q(F) , η = q(£) , and A X B d U • Replacing A and B by smaller sets of P and £ if necessary, we can assume that A^1 X A^1 C V and B'XB'C V for A^1 * j(A) ^ { y and B^1 » J(B) ^ {^}. Thus if x£ A and. y£B , then (ξ, j(x)) ∈ V , (jW,j(y))6V , and (j(y),«)fi-V • But then (j~,^) is in V o V o V , and thus U^b CL V o V o V . As the sets V o v o V , V<E.XI, generate S\oSX oSl 9 we have proved ([) b>^jQLoj7- o^i f and hence the Theorem,

A simple example shows that the equation SL o SL oSl » 0^b cannot be improved in general* If (E,j) is the set of rational numbers with the usual uniform structure, then (B^c ,J^b) is the set of real numbers with the usual uniform structure* In this case, SX is strictly finer than S1&S2- \ and jTLoSh is strictly finer than the filter $(p^b$ of entourages of the real numbers, We leave the details of this to the reader.

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Footnotes

1. In the terminology of Mitchell [4].

2. Called uniform convergence spaces in [2] and [3].

3. Many authors prefer the dual notation, ^ for "finer". We shall consistently use ^ for "finer", regardless of inclusion relations, since this leads to a very manageable formalism.

4. We modify the definition of a filter on E by allowing the empty set to be an element of a filter. This adds the null filter on E, consisting of all subsets of E, to the collection of all filters on E.

5. We prefer this to the term "ideal" used by many authors.

6. Called induced convergence space in [2] and [3].