ON FINITE GROUPS WHOSE p-SYLOW SUBGROUP IS A T.I. SET

by

Henry S. Leonard, Jr.

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ON FINITE GROUPS WHOSE p-SYLOVI SUBGROUP IS -A T, I# SET [X]

Henry S* Leonard$ Jr,

Throughout this note we let p be a fixed prime and let G be a finite group whose fixed p-Sylow subgroup P is a T. I. set (trivial intersection set)* · That is, the intersection of any* two distinct conjugates of P is <]>• Denote |p| by p^a # It is conjectured that if G has a faithful complex character X with J\'(l) $\leq p^{a/2} - 1$ then P <J G • This has been confirmed in certain cases [4 , page 287 and Lemma 4.2], [6 , Theorem 4*33• In fact under certain conditions it is sufficient to assume #(l) < $(p^a \sim l)/2$ [1 , Theorem 3]> [6 , Theorem 4«2], but in general the conclusion P< G does not hold under this weaker assumption because of the presence of Suzúki's simple groups.

Our purpose here is to use Brauer's theory of the correspondence between p-blocks of a subgroup of G and p-blocks of G [2], [3] together with a result of Gorenstein and Walter [5 % (46)] to obtain the theorem below which verifies the conjecture in the case that C(V) C N(P) , where V is the group of p-regular elements of C(P) e In particular for any counterexample of minimal order of the conjecture9 we would have C(P) £P Z(G)#

The notation is standard. If H is a subgroup of G then N(H), C(H), and Z(H) denote the normalizer, centralizer, and center of H • Denote Z(G) by Z , All characters are over the complex field#

Assuming P is a T. I# set, let B be a p-block of G of defect =/ 0 ,

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and let D be a defect group of B with $D \supseteq P$ and with $|D| = p^d$ • Then $N(D) \subseteq N(P)$, and the p-Sylow group of N(D) is normal in N(D), Furthermore by [2 , (80)] there is a block $\tilde{B}$ of N(D) which corresponds to B in the sense of [2 , (75)]. The defect group of B is the p-Sylow group of N(D) [2 , (9F)], and is contained in D [2 , (8D)]. We must have D = P • Thus every p-block of G has defect 0 or full defect a •

We know that

(1) $$P\ C(P) = P \times V$$

where V is a group of order prime to p. Then every p-block of P C(P) consists of the p^a irreducible characters A^ where ^ is a fixed irreducible character of V and A ranges over all the irreducible characters of P , We shall denote this block by b (^) ,

There is a one-to-one correspondence between the p-blocks of defect a of G and the classes {^f} of irreducible characters of V associated in N(P) [2 , (12A)]. Denote the block of G corresponding to {$} by B(£f) , Then, according to [3 , (2D)],

(2) $$b(\$r)^G = B(\mathfrak{F}) ,$$

in the sense defined there. Every p-block of N(P) is of defect a and must be of the form $b(\theta')^{N(P)}$ for some (3f « We denote this block by $\tilde{B}$(^T) . Then [3 , (2C)] implies

(3) $$\tilde{B}(\theta')^G = B(\mathfrak{F}) .$$

LEMMA 1* <u>An irreducible character</u> ty <u>of</u> N(P) <u>belongs to</u> $\tilde{B}(\mathfrak{F})$ <u>if and only if</u> ψ | V <u>has</u> J^ <u>as a constituent</u>.

Proof, Let XL be an algebraic number field of finite degree containing the- |N(P)|-th roots of unity* Let o be the ring of algebraic integers in jfi, and let p be a•prime ideal of ^o containing p *

If we apply (2) to N(P) , it follows- from [2 > (12,2)] that for ty e ^(c^) and v e V we have

$$\frac{|N(P)| \, \vartheta(1)}{|C(v) \cap N(P)|} \; \frac{\psi(v)}{\psi(1)} \equiv \sum \vartheta(w) \quad (\mathrm{mod}\ \mathfrak{p}) \; .$$

Here w ranges over the elements of V which are conjugate to v in N(P) . Hence'

$$\frac{|N(P)| \, \vartheta(1)}{|C(v) \cap N(P)|} \; \frac{\psi(v)}{\psi(1)} \equiv \frac{|N(P)|}{|C(v) \cap N(P)|} \; \frac{1}{q} \sum_{\{\vartheta\}} \vartheta_i(v) \quad (\mathrm{mod}\ \mathfrak{p}) \; ,$$

where ϑ_i ranges over the associates of ϑ in N(P) and q is the number of these associates. But, since V < N(P)>

$$\text{(4)} \qquad \psi|V = \frac{\psi(1)}{q' \, \vartheta'(1)} \sum_{\{\vartheta'\}} \vartheta'_j$$

for some class $\{\vartheta'\}$ where q' is the number of members of this class* These last two relations yield a congruence relating the values of ϑ and its associates to those of ϑ' and its associates. However, the irreducible characters of V are linearly independent (mod $\bar{\mathfrak{p}}$) [2 , (3C)]. Therefore ϑ and ϑ' are associates in N(P) , and the lemma follows from (4)*

Let D denote the set of p-singular elements of G whose p-factor is in the fixed p-Sylow subgroup P_0 Let B be a p-block of G , and let $\chi_i \in B$ • Then

$$\text{(5)} \qquad \chi_i \,|\, N(P) = \sum_j a_{ij} \; \psi_j$$

where the ψ_j are the irreducible characters of N(P) and the a_{ij} are integers. Then according to [5 > (46)]

(6) $$/CiH>- L' \ a_{ij} \psi_j | D$$

where we have summed only those terms for which ψ_j e B and B = B > for some block $\tilde{B}$ of N(P) «

LEMMA 2f If -9f . • e -B(& > and*\(l) < p^a then every constituent of $P(_i$ | V is an associate in K(P) of ^ « In particular, if £f == 1 then the kernel of Of_i contains V •

Proofs For $p(._i$ vie have an equation of the form (5) • It follows from (3) and (6) that

$$* \sim \sum_{\psi_j \notin \tilde{B}(\mathfrak{F})} a_{ij} \psi_j$$

vanishes on P - {l} • Hence ty | P must be a multiple of the character of the regular representation of P , so p^a | ^ (l) • Since ^C ^ is not of defect 0 > χ_i(l) < p^a . Hence ty is identically zero, and (5) and (6) have.the same terms* The lemma now follows from Lemma 1 *

In particular, B(ly) is the principal block (containing the principal character 1Q of G) ,

PROPOSITION, ~~Suppose the~~ p~~-Sylow subgrou~~p P ±1 £ T. I. ~~set~~. .~~If~~ G ~~has an irreducible character~~ 5C ^iicjh ttiat -X i V ^s ~~reducible and~~ Of (l) < $(p^a + 1)^{1/2}$ ~~then~~ G has a ^12f212i £Hk££2HP ^^ ^ ~~containing~~ V •

Proqfo It follows from Lemma 2 that $\rho(\bar{y}($ has a non-principal constituent in B(l) , and that this constituent has V in its kernel, REMARK* If G has a non-principal character /*(such that XI^v is irreducible then without use of the lemmas we see easily that G has a normal subgroup M ^ G containing either P or V.

THEOREM . Suppose the p-S^rlow subgroup P of G is a T. I$_0$ set and. thajb C\V) S K(P) , If G has a. l^ithful character % all of whose constituents have degress < (p + l) ~ tiien P^ G ,

Proof* Suppose the statement is false and that G is a counterexample of minimal order# If for every constituent pTo of %, X$_o$ I^{PV} is irreducible then Z(P) £ Z(G) and P<3 G ; which is not the case. Hence for some constituent %$_Q$ of X > %$_o$ I^{pv} is reducible* Then X$_o$ X$_o$ has a constituent p(. ^ 1 such that lp$_V$ £ PC -, PV • By Lemma 2 ^ V £ K , the kernel of X$_x$ • Either K N(P) = G or P <3 K N(P) • In the first case, %, J}(P) is irreducible and then P C^K<QG * By the minimality of G> P 4 K < G , which is not the case.

Thus P<3 KN(P) ^ Then KH P = 1 since P^ G . Hence K P = KX P , so V j£ K<£ V , and V<G$_e$ Then P A V C(V) <J G so P <3 G. This Is a contradiction and the proof Is complete*

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CARNEGIE INSTITUTE OF TECHNOLOGY