Normal Base Compactifications

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In 1964, Orrin Frink (see [2]) in his paper on 'Compactifications and semi-normal spaces' introduced the notion of a normal base Z to construct Hausdorff compactifications of completely regular T_1 spaces. For a normal base Z of a completely regular T_1 space, he constructed for his compactification the space ω(Z) of all Z - ultrafilers. He showed that the Alexandroff and Stone-Cech compactifications of a space X can be obtained as spaces ω(Z) . He then raised the question as to whether every Hausdorff compactification Y of a space X can be obtained in this way.

Recently Olav Njastad ([3]) gave an affirmative answer to this question for several important compactifications. His approach utilized a condition on the unique proximity associated with a Hausdorff compactification.

In another paper (see [1]) the authors gave a similar answer for a wider class of compactifications. Our approach utilized necessary and sufficient conditions on the normal base Z(X) associated with the space X .

In this paper we consider another approach to the solution of Frink's problem. We consider the normal base Z(Y) associated with the Hausdorff compactification Y of the space X . We give sufficient conditions on the base Z(Y) for the family Z(X) of intersections of X with members of Z(Y) to be a normal base,. We also give sufficient conditions for Y to be homeomorphic to co(Z(X)) .

A normal base is said to be regular if each of its members is a regular closed set^ that is a set which is the closure of its interior. It will follow from our theorems that every compactification which has a normal base that is also regular can be obtained as a Frink compactification. Thus Frink*s problem can be solved if it can be shown that every compactification has a normal base that is also regular.

We have not shown this in general but we can show it for particular cases like a closed cube - a product of closed intervals. Thus we have shown that the closed cube is a Frink compactification of each of its dense subspaces. In particular the unit disk is such a compactification. In his paper [3] Njastad was not sure if this was the case.

DEFINITIONS. A family Z of subsets of a space Y is a ring if it is closed under finite unions and finite intersections. It is disjunctive if for every closed set F in Y and for every point x not in F there is a Z in Z such that x is in Z and F D Z = *fi* . We say that Z is normal if disjoint members A , B of Z are contained in disjoint complements of members of Z .

We say that Z is space separating if whenever a member Z of Z is disjoint from a subspace X of Y there is a

Z^1 in Z that is disjoint from Z and contains S. We say X is 7-dense in Y if $cl_y(X \cap Z) = Z$ for all Z in Z.

A normal base for X is a base z for the closed subsets of X that is a normal disjunctive ring of sets.

THEOREM 1. Suppose that X is_a. subspace of Y, and that $2(Y)$ is a normal base on Y. Ij. $Z(Y)$ is space separating then the trace $z(X)$ of. zOO with X JL£ a. normal base on X.

Proof. It is easy to verify that $z(X)$ is a ring of sets. Let F_Y be a closed set in Y whose trace in X is the closed set F. If x in X is any point not in F then x is not in F_Y. The disjunctive property of $z(Y)$ gives a Z_Y in $z(Y)$ that is disjoint from F_Y and does not contain x. Hence $z(X)$ is disjunctive.

Let A_Y and B_Y be subsets of $z(Y)$ whose traces with X give the disjoint subsets A and B of $z(X)$. Since $A_Y \cap B_Y \cap X = fi$ and since $z(Y)$ is space separating there is a Z_Y in $z(Y)$ such that X is included in Z_Y and Z_Y is disjoint from $A_Y \cap B_Y$. Let $C_Y - Z_Y \cap A_Y$ and $D_Y = Z_Y \cap B_Y$. The sets C_Y and D_Y are disjoint members of $z(Y)$ and hence must be contained in disjoint complements E_Y' and F_Y' of members of $z(Y)$. The traces of E_Y' and F_Y' with X are complements of members of $z(X)$ which include A and B respectively. Hence $z(X)$ is normal and this completes the proof.

In trying to show that a normal base for a space Y is hereditary to subspaces X, it is the property of normality which produces the most difficulty. As with normal spaces, the property that a family of subsets is a normal family is not hereditary. It is interesting however, that if the subspace X is a member of the normal base for the space Y , then the normality of the family is inherited. This is given in the following corollary of our theorem.

COROLLARY 1. JjE X JLs ja subspace of ji space Y and if Z(Y) jLs <a normal base for Y which contains X then the trace Z(X) ojf z(Y) with X jLj3 a. normal base for X .

COROLLARY 2 . Suppose that X jLs. a. dense subspace of Y and that z(Y) is a normal base for Y . IjE X *is^* 7(Y) -dense in Y then the trace z (X) JLB ja normal base on X .

Proof. If X is dense in Y then X z(Y)-dense i^{n} Y implies vacuously that z(Y) is space separating. Thus our theorem holds.

COROLLARY 3# If X is a compact closed subset of a normal (not necessarily T_1) space Y and if z(Y) is a normal base on Y , then the trace z(X) is a normal base on X .

Proof. If X satisfies the conditions of the hypothesis, then z(Y) is space-separating with respect to the subspace X . The conclusion follows from Theorem 1:

If Y is a closed real interval and X is any dense subspace of Y , each having the usual topologies, Frink has found a normal base for Y such that X is z(Y)-dense in Y . Let L and R be the set of rationals and irrationals, respectively, in Y .

Then L and R are disjoint dense subsets of Y . A base for the closed sets of Y is the family, z > of finite unions of closed intervals [1 , r] where 1 and r are in L and R respectively. The intervals [1 , r] are regular closed intervals and it is easy to see that z is a normal base for Y which is also regular. Since each interval contains an open set and X is a dense subspace of Y , it follows that X is z-dense in Y .

The following lemma gives an equivalent condition for a subspace. X to be z (Y) ~ dense in Y .

LEMMA. JLL X JLS^ _a subspace of Y and if . z i§. JL normal base for Y , then the following statements are equivalent:

(1) X JLS z-dense in Y .

(2) For each Z jm z and for each open set V jLn Y 3 if V meets Z then V fl z meets X .

Proof. Suppose that condition (1) holds. Let V be any open subset of Y that meets a Z in z an<^ le^ p be any point in V fl Z . The point p is not in the closed set Y - V and so by the disjunctive property there is a Z< in z containing p and disjoint from Y - V . Then V n z Pi x is non-empty since p is in Z^1 n Z , $cl_Y(Z^1$ fl Z n X) = Z! n z. > and Z! is included in V . Thus (1) implies (2) .

Finally, it is clear that cl_Y(X n Z) is included in Z . If p is any point of Z and V is any neighborhood of it, then V D Z is non-empty. Assuming statement (2) it follows that p is in cl_Y(X n Z) and that cl_Y(X n Z) = Z . This completes the proof.

In [1] the authors showed the following theorem. We state it here so that we may refer to it in the proof of our final theorem.

THEOREM 2 . *Let* Y *be a, Hausdorff compactification of* X. *Then* Y *Js homeomorphic to a space* to(Z(X)) *if and only if* X *has a. normal base* Z(X) *that satisfies*:

(a) cl_Y (A n B) = cl_y A fl cl_y B *for aJA* A, B *in* ZX.

(b) *For each* y *jm* Y *arid each neighborhood* V *ojE* y *there is a.* Z *jln* Z(X) *such that* y *jLsi in* cl_Y Z c V.

THEOREM 3. *Let* Y *be a. Hausdorff compactification of ci space* X *and let* Z(Y) *bjB ci normal base for* X. *Then* Y *is homeomorphic to ja space* cc(Z(X)) *where* Z(X) *JLS the trace of* Z(Y) *in* X *if* X *is* Z(Y) *-dense rn* Y.

Proof. First we note that by Corollary 2 if X is Z(Y)-dense then Z(X) is indeed a normal base. We show that conditions (a) and (b) of Theorem 2 hold for Z(X) .

Let E_Y and F_Y be members of Z(Y) whose traces in X are the members E and F respectively of Z(X) . Then since X is Z(Y)-dense in Y we have that $cl_y(E_y$ D F_y fl X) = E_y OF_Y = $cl_y(E_y$ n X) n $cl_y(F_y$ n X) and condition (a) follows.

If p is in Y and G is any open set containing it then there is Z_Y in Z(Y) such that p is in Z_Y and Z_Y is disjoint from Y - G. Since Z_Y is non-empty and X is Z(Y) -dense then p is in Z_Y = $cl_Y(Z_Y$ n X) = cl_Y Z which is included in G. Thus condition (b) holds. This completes the proof.

If a subspace X is dense in a space Y then every non-empty regular closed set in Y will meet X. Thus our Theorem 3 shows that every Hausdorff compactification Y which has a normal base that is also regular is a compactification in the sense of Frink of each of its dense subspaces. In particular, our example of a closed real interval, as discussed previously, has this property.

Corollary. If the compact Hausdorff space Y has a regular normal base then Y is a Frink type compactification of each of its dense subspaces,

A space is <u>zero-dimensional</u> if it has a base for the closed sets of closed and open (clopen) sets. In a compact Hausdorff space this base is normal and thus it is a regular normal base.

THEOREM 4. <u>Every zero dimensional Hausdorff compactification is a Frink type compactification of each of its dense subspaces</u>.

Every cube Q, that is a product of closed real intervals, has a normal base of regular closed sets. This normal base is obtained in a manner similar to that for the closed interval. For a and b in Q we define $a \le b$ to mean $a_j \le b_j$ for all j in the indexing set J where $a = (a_j)_{j \in J}$ and $b = (b_j)_{j \in J}$. A <u>closed interval</u> $[a, b]$ in Q is the set of all x in Q such that $a \le x \le b$. A base for the closed sets in Q is the collection of all finite unions of closed intervals (see [4]). Let L and R be two disjoint dense subsets of Q. For a finite subset F of the indexing set J, let B(F) be the set of all x in Q such that $\ell_j \le x_j \le r_j$ for j in F where $\ell = (\ell_j)$ and $r = (r_j)$ are members of L and R respectively. It is easy to see that B(F) is the closure of the open set which consists of all x such that $\ell_j < x_j < r_j$ for j in F. The collection of sets which are finite unions of sets of the form $B(F)$ also form a base for the closed sets in Q. In fact this family forms a normal base that is also regular. Thus, as stated in the following theorem, we have answered Frink's question for the cube.

THEOREM 5. *The compact Hausdorff space Q which is the arbitrary product of closed real intervals is a Frink type compactification of each of its dense subspaces.*

Proof. Since Q has a normal base Z which is also regular, each member of the base must meet any dense subspace X. This implies that X is Z-dense in Q and our Theorem 3 applies.

Gillman and Jerison (see [5], page 95) state that a space X is pseudocompact if and only if every non-emtpy zero set in βX (the Stone-Cech compactification of X) meets X. It follows then that if Y is any Hausdorff compactification of a pseudocompact space X then every non-empty zero set in Y meets X. The collection of zero sets of a completely regular space X forms a normal base.

THEOREM 6. *Every compact Hausdorff space Y is a Frink type compactification of each of its dense pseudocompact (and therefore countable compact and sequentially compact) subspaces* X.

Proof. The theorem will follow from Theorem 3 if we show that the pseudocompact space X is $Z(Y)$ dense in Y where $Z(Y)$ is the normal base of zero sets of Y. Let Z be a non-empty zero set in Y. Clearly $c_Y(Z \cap X) \subset Z$. Let G be an open set in Y such that G contains a point p of Z. Since the complement of G is a closed set in Y, there is a member Z'

in Z(Y) such that p is in Z' and Z[1] is contained in G. The zero set Z' 0 z is non-emtpy and hence must meet the pseudocompact space X. Thus the arbitrary open set G meets X 0 z and c_y(Z n X) = Z.

In a compact Hausdorff space, zero-dimension is equivalent to totally disconnected (the component of each point of the space consists of the- point alone), and strongly zero-dimensional (every neighborhood of a closed set contains an open-closed neighborhood of the set).

A metric space (X,d) is an ultrametric space if the metric d satisfies the condition

$$d(x,y) \leq \sup(d(x,z),d(y,z)\}$$

for all x,y,zeX. In an ultrametric space the sets S(x,e) = {yeX: d(x,y) < e] are closed and open. Hence every ultrametric compactification is a Frink compactification of each of its dense subspaces.

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