Estimating Dirichlet's Integral and Electrical Conductance for Systems Which Are Not Self-Adjoint

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Abstract

In simple electrical flow problems the Dirichlet integral of the potential function gives the electrical conductance. Moreover the Dirichlet integral of an arbitrary function satisfying the boundary condition gives an upper bound for the conductance. This last property (Dirichlet's principle) does not hold if the boundary value problem is not self-adjoint. This paper develops new algorithms for estimating the conductance. The proof of these algorithms replaces the Dirichlet principle with the elliptic maximum principle. There is an analogous discrete problem for conductance of electrical networks of the non-reciprocal type. The conductance problem both for continuous bodies and discrete networks can be treated by a single postulational theory.

Estimating Dirichlet's Integral and Electrical Conductance For Systems Which Are Not Self-Adjoint*

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1. Introduction

Let the function $v(x,y,z)$ be harmonic in a region R and be prescribed on the boundary of R. Then the Dirichlet integral is the quadratic functional

$$\mathcal{D}(v) = \iiint_R [(\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial v}{\partial y})^2]\, dx\, dy\, dz.$$

a classical problem of numerical analysis is to estimate $\mathcal{D}$ from a knowledge of the boundary values of v. The interest of this problem in applied mathematics is that the value of $\mathcal{D}$ can be used to determine electrical conductance of a uniform conducting body in the shape R. (An equivalent problem concerns thermal conductance). In previous papers [1,2,3] we treated this problem by 'lumping' the body R into an electrical network with a finite number of junction points. By means of such lumped networks it was possible to obtain upper and lower bounds for $\mathcal{D}$.

This paper is also concerned with estimating quadratic functionals such as $\mathcal{D}$ but now it is not required that the boundary value problem be self-adjoint. The difficulty of the non self-adjoint case is that it cannot be assumed that $\mathcal{D}$ is a minimum as in self-adjoint problems. In other words, Dirichlet's principle cannot be used. However the maximum principle solutions of elliptic partial differential equations remains valid and proves

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effective in estimating $\mathcal{D}$. More precisely it is shown here that there are simple formulae giving upper and lower bounds for $\mathcal{D}$ in terms of a function v which satisfies the boundary condition but is arbitrary otherwise. These formulae serve to give an approximate evaluation of the electrical conductance between two parts of the conductor.

Lumped electrical networks which are not self-adjoint have been termed 'non-reciprocal.' In a previous paper [4] we analyzed a class of non-reciprocal networks termed positively connected networks. It is shown here that an analog of the maximum principle applies to such networks. It is then seen that the maximum principle can be employed to give estimation formulae for the joint electrical conductance between two terminals of the network.

It seems desirable to have an abstract theory which would lead to the above formulae both for continuous bodies and discrete networks. To this end a postulational treatment, is formulated. We term this postulational system an 'elliptic fluency.'

To motivate the analysis we first discuss a very common situation leading to a boundary value problem not in self-adjoint form. It is common because the earth's magnetic field is ubiquitous.

2. Dirichlet's principle is false on Earth

Let R be an isotropic and homogeneous conducting body and over a part A of the surface of the body let the electrical potential v be prescribed. Let C denote the rest of the surface of R and let this part be insulated. Then on C the boundary condition is $\partial v/\partial n = 0$ where n denotes the exterior normal. In the steady state the current density vector j satisfies

$$(1) \qquad j = -G \nabla v \qquad \text{(Ohm's law)}$$

where the scalar G is the specific conductivity and v is a harmonic function taking on the prescribed boundary values.

This is a self-adjoint boundary value problem with mixed Dirichlet and Neumann boundary conditions. According to Dirichlet's principle the solution can be obtained by minimizing the Dirichlet integral $\mathfrak{D}$ over the class of functions which satisfy the boundary condition on A. It is not necessary to require that the boundary condition on C be satisfied because this is a natural boundary condition.

Now suppose the body is subjected to a uniform magnetic field H in the z-direction. Then as Hall found, Ohm's law changes to

$$(2) \qquad \begin{aligned} j_x &= -G \frac{\partial v}{\partial x} - \zeta \frac{\partial v}{\partial y} \\ j_y &= \zeta \frac{\partial v}{\partial x} - G \frac{\partial v}{\partial y} \\ j_z &= \qquad\qquad -G \frac{\partial v}{\partial z} \,. \end{aligned}$$

Here ζ is a constant proportional to the magnetic field strength. By the conservation of electricity the divergence of the current vanishes. Thus $\nabla \cdot j = 0$ and because of (2) $\nabla^2 v = 0$. Thus v is again harmonic inside R. The boundary condition on C, the insulated part of the boundary, is that the normal component of j vanishes, $j_n = 0$. This is not the same as $\partial v/\partial n = 0$ unless $\zeta = 0$. This boundary condition for the Hall effect is not of the standard self-adjoint form. It may be shown that the problem adjoint to this one corresponds to replacing ζ by $-\zeta$. This is equivalent to reversing the direction of the magnetic field.

The power supplied to a unit volume is $-j \cdot \nabla v = G(\nabla v)^2$. Thus the total power input is $Q = G\mathcal{D}(v)$, and hence proportional to the Dirichlet integral $\mathcal{D}(v)$. However $\mathcal{D}(v)$ is not a minimum value or a stationary value relative to the class of functions which satisfy the boundary conditions. This fact was known to Maxwell [5]. However the power input Q is still an important physical quantity not only because of its electrical significance but also because it is transformed into heat. Consequently it is desirable to have formulae which estimate $\mathcal{D}(v)$.

3. <u>The concept of an elliptic fluency</u>

This paper treats estimation formulae which pertain to a large class of physical systems. In these systems there is a steady flow of a "fluid" resulting from a distribution of "potential." The system just described concerning the Hall effect is an example. To bring out the salient features and to unify the proofs, an abstract postulational treatment is now developed for such systems.

An <u>elliptic fluency</u> S is defined by the following conditions:

A. Let R be a region of space

B. There is a Borel measure μ defined on R.

C. There is a class of functions P termed "potentials" which are continuous and bounded on R.

D. With each potential v there is associated an absolutely integrable function w termed a "current source."

E. If a non-constant potential v has a global supremum (infimum) $M > 0$ ($M < 0$) then there is a point t_0 such that

$M = v(t_o)$ and in any neighborhood of t_o there are points t where $w(t)$ is positive (negative).

F. There exists a related system S^* satisfying A, B, C, and D such that

$$\int_R vW \, d\mu = \int_R Vw \, d\mu$$

where $v \epsilon P$ and $V \epsilon P^*$, the class of potentials for S^*.

System S^* is termed the adjoint of S. Postulate E is termed the maximum principle and postulate F is termed the reciprocity principle. The current I entering R through the measurable set A is defined to be

$$I = \int_A w \, d\mu.$$

The power input Q is defined to be

$$Q = \int_R vw \, d\mu.$$

Theorem 1. Suppose that a potential function v of an elliptic fluency satisfies the relation $E \geq v \geq 0$ on a measurable set A for some constant E. Suppose that the corresponding current source function w vanishes on the set K complementary to A. Then the power input

$Q = \int_R v\, w \, d\mu$ satisfies the following inequalities

$$\int_A VW d\mu + E \int_K W^- d\mu \leq Q \leq \int_A VW d\mu + E \int_K W^+ d\mu$$

Here W is the current source in the adjoint system corresponding to a potential V such that $V = v$ on A but otherwise V is arbitrary.

Proof. Here

$W^+ = \max\ (W,0)$ is the positive part of W.

$W^- = \min\ (W,0)$ is the negative part of W.

The potential v has a supremum M by postulate C. First suppose that $M > E$ then v is not constant. If follows from the maximum principle that $v(t_o) = M$ at a certain point t_o and that $w > 0$ at points t_i in any neighborhood of t_o. The points t_i are in A because $w = 0$ in K and so $v(t_i) \leq E$. By the continuity of v it results that $v(t_o) \leq E < M$. This contradiction shows that $v \leq E$ in R. A similar argument shows that $v \geq 0$ in R.

From the assumed conditions of the theorem it is seen that

$$Q = \int_R vwd\mu = \int_A vwd\mu = \int_A Vwd\mu = \int_R Vwd\mu \ .$$

Now from the reciprocity principle

$$Q = \int_R Vwd\mu = \int_R vWd\mu \ .$$

Thus by the bounds of $v(t)$ in K we have

$$\int_A vWd\mu + E\int_K W^- d\mu \leq Q \leq \int_A vWd\mu + E\int_K W^+ d\mu$$

and this completes the proof.

Corollary 1. <u>Let</u> A <u>and</u> B <u>be measurable sets of</u> R. <u>Suppose that the potential</u> v <u>has the constant value</u> $E > 0$ <u>on</u> A <u>and that the potential</u> v <u>has the value zero on</u> B. <u>Suppose that the current source</u> w <u>vanishes on the set</u> K <u>which is the complement of</u> $A \cup B$. <u>Then the total current entering</u> A <u>is</u> $I = \int_A wd\mu$ <u>and it satisfies the inequalities</u>

$$\int_A Wd\mu + \int_K W^- d\mu \leq I \leq \int_A Wd\mu + \int_K W^+ d\mu .$$

Here W is the current source in the adjoint system corresponding to a potential V which is arbitrary except that $V = E$ on A and $V = 0$ on B .

Proof. This follows directly from Theorem 1 by noting that $Q = EI$.

A set where the potential is constant is called a terminal. Thus A and B in the Corollary are terminals. The conductance γ between terminals A and B may be defined as

$$\gamma = \frac{I}{E}$$

Thus the inequalities of Corollary 1 give an estimation of the conductance.

Corollary 2. If a boundary value problem and its adjoint are both solvable then the power input is the same in both systems.

Proof. In Theorem 1 we may take $V = v$ on A and $W = 0$ in K. This means that $W^+ = 0$ in K and $W^- = 0$ in K and the inequality of Theorem 1 becomes the equality $Q = \int_A VWd\mu$. However $\int_A VWd\mu = \int_R VWd\mu$ so the proof is complete.

4. Non-reciprocal conductors

It is now to be shown that a body R which is a conductor of electricity is an example of an elliptic fluency S. It is not required that the body be isotropic or homogeneous. It is required that Ohm's law holds in the following form at interior points of R

(1) $$j = -G \nabla v$$

Here G is the conductance tensor, v is the electric potential in volts, and j is the current vector in amperes per cm^2. Thus the <u>interior current source</u> in amperes per cm^3 is

(2) $$w = - \nabla\cdot(G \nabla v)$$

The <u>boundary current source</u> in amperes per cm^2 is

(3) $$w = n\cdot(G \nabla v)$$

where n is the exterior normal to R.

In components, relation (1) is written

$$j_a = - \sum_{b=1}^{3} G_{ab}(t)\frac{\partial v(t)}{\partial t_b} \qquad a = 1,2,3 \;.$$

Here at each point t the matrix G_{ab} is assumed to be positive definite but not necessarily symmetric. The adjoint tensor has a matrix

(4) $$G^*_{ab}(t) = G_{ba}(t) \,.$$

The adjoint system S^* has the current vector J and the potential V related by

(5) $$J = - G^* \nabla V$$

and the current source W is defined in terms of V by formulae analogous to (2) and (3).

The power supplied to a unit volume is $-j\cdot\nabla v = (G \nabla v)\cdot\nabla v$ so the total power is

(6) $$Q = \iiint_R (G \nabla v)\cdot\nabla v \; dx\, dy\, dz \;.$$

The problem of concern is to estimate Q when there are no interior sources of current. This will be done by appealing to Theorem 1 proved for an elliptic fluency.

To simplify the proof that such conductors give rise to an elliptic fluency, rather restrictive assumptions are now imposed.

It is then possible to appeal to well known existence theorems:

A'. Let R be a compact region of three-space having interior R_o and boundary ∂R. Then R_o is to be connected and the boundary ∂R is to be a smooth surface with bounded curvature.

B'. The measure μ on R is defined as

$$\int_R f d\mu = \int_{R_o} f d\tau + \int_{\partial R} f\, d\sigma$$

there τ is the ordinary volume measure in R_o and σ is the ordinary surface measure on ∂R.

C'. The potential function class P is taken to be the class of C^2 functions in R.

D'. Given a potential v the current source function w is defined as

$$w = -\nabla\cdot(G\ \nabla v) \quad \text{in} \quad R_o$$

$$w = n\cdot(G\ \nabla v) \quad \text{in} \quad \partial R$$

Here G is termed the conductance tensor:

(a) G is continuously differentiable.

(b) G is positive definite.

It is not assumed that G is necessarily symmetric.

E'. Suppose that v is a non-constant potential. Let H be the set where v takes on its global maximum M. Then let t_o be a point on the boundary of H and first consider the case that t_o is in the interior R_o.

Suppose that $w \leq 0$ in a neighborhood of t_o. In other words $\nabla\cdot(G\ \nabla v) \geq 0$ in the neighborhood. Clearly $\nabla\cdot G v$ is an elliptic operator and so it follows by the well known maximum principle for elliptic operators that v can not have a maximum at t_o unless v is constant in a neighborhood of t_o. However this contradicts the assumption that t_o was a boundary point of H. Thus it must be so that $w(t) < 0$ at some points t in the neighborhood of t_o.

Next consider the case that t_o is in H and also on the boundary ∂R. Since v has a maximum at a boundary point t_o it follows that the tangential derivatives vanish. We are now making use of the assumption that ∂R has bounded curvature. Thus ∇v is in the normal direction and at t_o

$$w = n \cdot (G \, \nabla v) = g \frac{\partial v}{\partial n} \quad .$$

Since G is positive definite it follows that g is positive. It is desired to show that $w > 0$ at some points in a neighborhood N of t_o. The neighborhood N includes both points of R_o and ∂R. Suppose that

$$\nabla (G \, \nabla v) \geq 0 \quad \text{in} \quad N \cap R_o$$

Moreover v is not constant in N so the extension of the maximum principle due to E . Hopf states that $\partial v/\partial n > 0$ at the boundary maximum point t_o. At this point $w = g\partial v/\partial n > 0$. This verifies that t_o is a global maximizing point satisfying the statement of postulate E. The verification of the statement of E concerning a global minimizing point is treated analogously.

F'. The adjoint elliptic fluency is defined with the same class of regular potentials, i.e., $P^* = P$. The conductance tensor G^* is the transpose of G. If v and V are regular potentials then clearly the following form of Green's theorem is valid.

$$\int_{R_o} (G\ \nabla v)\cdot \nabla V\ d\tau = \int_{R_o} Vwd\tau + \int_{\partial R} Vwd\sigma = \int_{R} vwd\mu$$

But clearly $(G\ \nabla v)\cdot\nabla V = (G^*\ \nabla V)\cdot \nabla v$ and so

$$\int_R vWd\mu = \int_R Vwd\mu \ .$$

This completes the verification of the postulates.

We are now able to apply Theorem 1 to the case of non-reciprocal conductors. In particular we have the following special result.

Theorem 2. Suppose that v is a potential function such that

$$\nabla\cdot(G\ \nabla v) = 0$$

in the interior of R. On a part A of the boundary of R suppose

$$0 \le v \le E$$

for some constant E. On the part of the boundary C complementary to A suppose

$$n\cdot(G\ \nabla v) = 0$$

where n is the normal. Then the power input Q satisfies the inequalities:

$$Q \le \int_A V(n\cdot\nabla V)\,d\sigma + E\int_C [n\cdot(G\nabla V)]^+ d\sigma + E\int_R [\nabla\cdot(G^*\nabla v)]^+ d\tau .$$

$$Q \ge \int_A V(n\cdot\nabla V)\,d\sigma + E\int_C [n\cdot(G\nabla V)]^- d\sigma + E\int_R [\nabla\ (G^*\nabla V)]^- d\tau .$$

Here V is any C^2 function such that $V = v$ on A.

Theorem 2 can be used to estimate the conductance γ between a pair of terminal surfaces A and B. The statement is analogous to Corollary 1.

It should be recognized that the restrictions enforced on the functions v and the region R are far more stringent than is necessary or desirable. The relaxation of these restrictions leads to various technical questions concerning mixed boundary value problems. There does not seem to be a satisfactory reference for such questions.

If the conducticity tensor G is symmetric then it is well known that γ satisfies the inequality

$$\gamma \leq \int_R (G\ \nabla v) \cdot \nabla v\ d\tau \tag{7}$$

where v is arbitrary except that $v = 1$ on A and $v = 0$ on B. The right hand side of this inequality is essentially the Dirichlet integral and (7) is a formulation of Dirichlet's minimum principle. It is not difficult to show that (7) is incorrect if G is not a symmetric tensor.

The elliptic maximum principle also applies to equations of the form

$$-\nabla^2 v + \lambda v = w$$

where λ is a positive constant. The term λv is interpreted as a 'leakage current to ground' in reference [3]. It is seen that this generalization satisfies the postulates for an elliptic fluency.

5. <u>The discrete elliptic fluency</u>.

The postulates for an elliptic fluency when R is a finite set are given the following interpretation.

A discrete elliptic fluency S is defined by the following conditions:

A_o. Let R be a set of $n+1$ distinct points designated as $(0,1,2,\ldots,n)$.

B_o. Each point is given unit measure.

C_o. There is a class of functions P termed potentials. A function v is a set of $n+1$ numbers $(v_o, v_1, \ldots, v_n)$.

D_o. Each potential v has an associated 'current source' function a set of $n+1$ numbers $(w_o, w_1, \ldots, w_n)$.

E_o. If a non-constant potential v has a global maximum (minimum) value $M > 0$ $(M < 0)$ then there is a point i where $v_i = M$ and $w_i > 0$ $(w_i < 0)$.

F_o. There exists another elliptic fluency S^* satisfying A, B, C, and D. Let P^* be the class of potentials for S^* then

$$\sum_{i=0}^{n} v_i W_i = \sum_{i=0}^{n} V_i w_i$$

where $v \epsilon P$ and $V \epsilon P^*$.

Postulate E_o is termed the maximum principle and postulate F_o is termed the reciprocity principle. The current I entering R through the set A is defined to be

$$I = \Sigma_A W_i$$

The power Q is defined to be

$$Q = \sum_{i=o}^{n} v_i W_i .$$

The points of R are called terminals. The discrete elliptic fluency will now be related to electric networks.

6. Non-reciprocal networks

Here we are concerned with a black box S with electric terminals on the outside. It is desired to specify conditions on S so that it is an elliptic fluency. First consider the case when there are only two terminals on the box and suppose Ohm's law holds. Then

$$(1)\qquad \begin{aligned} w_1 &= gv_1 - gv_2 \\ w_2 &= -gv_1 + gv_2 \,. \end{aligned}$$

where w_1 is the current (in amperes) entering terminal 1 and w_2 is the current in amperes entering terminal 2. The electric potentials (in volts) at terminals 1 and 2 are v_1 and v_2. The power input to the black box is

$$(2)\qquad Q = w_1 v_1 + w_2 v_2 = g(v_1 - v_2)^2 .$$

According to Ohm's law g is a positive constant so power is non-negative.

Next suppose that the box has $n + 1$ terminals and take as a generalization of Ohm's law

$$(3)\qquad w_i = - \sum_{j=0}^{n} g_{ij} v_j \qquad\qquad i = 0, 1, 2, \ldots, n.$$

These equations define the steady flow of current into the black box. It is assumed that the g_{ij} are real constants. Here the variable v_i is the potential of the ith terminal and w_i is the current entering the ith terminal from outside the box.

There are two physical laws which any time invariant system must surely satisfy:

I. <u>The net current entering the system vanishes</u>.

II. <u>The level of potential has no significance</u>.

If we suppose that current can enter only at terminals then I demands that $\Sigma_0^n w_i = 0$. Since the potentials can be given arbitrary values it follows that

$$(4) \qquad \sum_{i=0}^{n} g_{ij} = 0 \qquad j = 0, 1, \ldots, n \ .$$

According to II, increasing all v_j by a constant amount cannot change w_i. Thus

$$(5) \qquad \sum_{j=0}^{n} g_{ij} = 0 \qquad i = 0, 1, \ldots, n \ .$$

Relation (5) may be used to eliminate the diagonal elements of the matrix g_{ij}. Thus

$$(6) \qquad w_i = \sum_{j=0}^{n} g_{ij}(v_i - v_j) \qquad i = 0, 1, \ldots, n$$

is seen to be equivalent to (3).

If the matrix g_{ij} is symmetric it is possible to interpret (6) as a statement of Kirchhoff's first law for a network of conductors. Then g_{ij} is interpreted as the conductance of the branch of the network connecting terminals i and j.

The principle of reciprocity concerns an adjoint system with matrix $g^*_{ij} = g_{ji}$. Obviously the adjoint system also satisfies (4) and (5). If V_i and W_i are the potential and current for the ith terminal of the adjoint system then

$$(3^*) \qquad W_i = -\sum_{0}^{n} g^*_{ij} V_j \qquad j = 0, 1, \ldots n \ .$$

Thus (3) and (3*) give

$$\Sigma_0^n v_i W_i = \Sigma_0^n V_i w_i \ .$$

This is the principle of reciprocity.

The power input to the system S is

(7) $$Q = \Sigma_o^n w_i v_i = - \Sigma_o^n \Sigma_o^n g_{ij} v_i v_j .$$

It is an easy consequence of relations (4) and (5) that the power input may also be written as

(8) $$Q = (1/2) \Sigma_o^n \Sigma_o^n g_{ij} (v_i - v_j)^2 .$$

It is then apparent from (8) that $Q \geq 0$ if

(9) $$g_{ij} + g_{ji} \geq 0 \quad \text{for} \quad i \neq j .$$

Thus (9) is a sufficient condition that Q be a semi-definite quadratic form.

An interesting special case is a four-terminal box such that

$$w_o = 0 + 0 + v_2 - v_3$$

$$w_1 = 0 + 0 - v_2 + v_3$$

$$w_2 = -v_o + v_1 + 0 + 0$$

$$w_3 = v_o - v_1 + 0 + 0 .$$

This special sytem was termed a gyrator by Tellegen [6]. Clearly the power input to a gyrator always vanishes. The gyrator has application to vibration problems [7] and to the network synthesis problem [8].

In a previous paper [4] the writer introduced the concept of a positively connected network. This may be defined as follows:

(i) $$\Sigma_{i=0}^n g_{ij} = 0 \qquad j = 0, 1, \ldots, n$$

(ii) $\Sigma_{j=0}^{n} g_{ij} = 0$ $\quad i = 0, 1, \ldots, n$

(iii) $g_{ij} \geq 0$ $\quad i \neq j$

(iv) For each pair of integers i and f there is a sequence of distinct integers $i, a, b, \ldots, e, f$ such that $g_{ia} g_{ab} \ldots g_{ef} \neq 0$.

A positively connected network is an elliptic fluency. To prove this we note by (ii)(iii)(iv) that $g_{ii} < 0$. Thus equation (3) may be written in the form

$$v_i = \sum_{j \neq i} \left|\frac{g_{ij}}{g_{ii}}\right| v_j + \left|\frac{w_i}{g_{ii}}\right| \tag{10}$$

By virtue of (ii) it is seen that the summation here is a weighted average. Thus suppose that M is the maximum value of v_j then we have

$$v_i \leq M + w_i / |g_{ii}| .$$

Thus the maximum cannot occur where $w_i < 0$. Suppose $v_i = M$ at terminal i and that $w_f > 0$ at terminal f. Then if $w_i = 0$ we have

$$v_i = \sum_{j \neq i} \frac{g_{ij}}{|g_{ii}|} v_j$$

and it follows, according to (iv), that $v_a = M$ also. If $w_a = 0$ the same argument shows that $v_b = M$. Repeating this process shows that either there is a point d on the sequence $i, a, b, \ldots, e, f$ where $v_d = M$ and $w_d > 0$ or else $v_f = M$. But f has been selected so that $w_f > 0$. This proves the maximum principle when there is a terminal where $w > 0$. Otherwise $w_i \equiv 0$ and if follows from (10) that v_i is constant. It is now seen that a positively connected network satisfies all the requirements for an elliptic fluency.

Theorem 3. Let γ be the joint conductance between terminals 0 and 1 of a positively connected network. Then

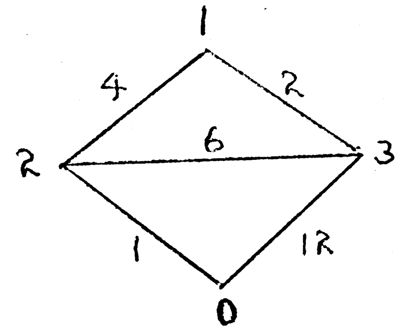
$$w_1 + \sum_2^n w_i^- \le \gamma \le w_1 + \sum_2^n w_i^+ .$$

Here w_i is the current flow resulting from a potential which is arbitrary except that $v_o = 0$ and $v_1 = 1$.

Proof: This follows from Corollary 1 and Corollary 2.

7. A numerical example of the estimation algorithms

It is instructive to apply Theorem 3 to a particular network such as the Wheatstone bridge network shown in this figure.



In this example $g_{01} = 0$, $g_{02} = 1$, $g_{12} = 4$, $g_{13} = 2$, $g_{03} = 12$, and $g_{23} = 6$. The network is taken to be reciprocal, $g_{ij} = g_{ji}$. It is easy to solve the network equations and find that the joint conductance between terminals 0 and 1 is $\gamma = 3.5$. Now employ Theorem 3 with $v_0 = 0$, $v_1 = 1$, $v_2 = .6$, $v_3 = .3$. Then by the formula (3) $w_1 = 3$, $w_2 = .8$ and $w_3 = .4$. This gives

$$3 \le \gamma \le 3 + .8 + .4 = 4.2 \quad .$$

Next try $v_2 = .5$, $v_3 = .2$ and so $w_1 = 3.6$, $w_2 = .3$, $w_3 = -1$. This gives $2.6 = 3.6 - 1 \le \gamma \le 3.6 + .3 = 3.9$

Since this network is reciprocal, the Dirichlet integral furnishes an upper bound. Thus

$$\gamma \le w_1 + \sum_2^n v_i w_i \quad .$$

In the first assignment of potentials this given

$$\gamma \leq 3 + .48 + .12 = 3.60 \ .$$

In the second assignment

$$\gamma \leq 3.6 + .15 - .2 = 3.55 \ .$$

This indicates that the Dirichlet principle given a better upper bound for the joint conductance of reciprocal networks.

For each pair of terminals of a network there is a joint conductance γ . These joint conductances are related to each other and to other properties of the network by simple equations and inequalities. Such relations are treated in references [1] and [4].

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