DISCONJUGACY CRITERIA FOR LINEAR DIFFERENTIAL EQUATIONS

by

Zeev Nehari

Report 67-8

March, 1967

<u>Disconiugacy criteria for linear differential equations</u>

Zeev Nehari

1. A linear homogeneous n-th order differential equation is said to be disconjugate on an interval I if none of its solution have more than $n - 1$ zeros on I (where the zeros are counted with their multiplicities). If it is merely known that no solution has an infinite number of zeros on I, the equation is said to be non-oscillatory pn the interval. The question as to how the disconjugacy or non-disconjugacy of an equation of order larger than 2 is reflected in its coefficients is of obvious interest, and it has been studied by a number of authors [1-11]. While this work has resulted in some necessary conditions for the disconjugacy of certain classes of equations, no nontrivial sufficient conditions seem to be known for equations of order higher than 4 if I is an interval (a,∞) (the case of principal interest). The following theorem furnishes conditions of this type.

<u>Theorem</u> I. <u>If</u> R(x) <u>is positive and non-increasing on</u> $[0,\infty)$, <u>and</u>

$$\int^{\infty} R^{p}(x)\, x^{p\,n-1}\, dx < \infty, \tag{1}$$

<u>for any</u> $p\in[1,n]$, <u>then the equations</u>

$$y^{(n)} + R(x)y = 0 \tag{2}$$

<u>and</u>

$$y^{(n)} - R(x)y = 0 \tag{3}$$

<u>are non-oscillatory on</u> $[0,\infty)$. <u>Moreover, there exists a positive number</u> c <u>such that the equations are disconjugate in</u> (c,∞).

<u>For given</u> p, <u>and</u> $n \geq 3$ <u>(and also for</u> $n = 2$ <u>and equation</u> (2)), <u>condition</u> (1) <u>is sharp in the sense that</u> $x^{\wedge}$ <u>cannot be</u>

replaced by a lower power of x.

We note here that, for $n = 2,3,4$ the non-oscillation of equations (2) and (3) implies the existence of a positive c such that the equations are disconjugate in (c,∞). Whether or not this is also true for $n > 4$ is an open question.

Theorem I will be a consequence of the following stronger result.

Theorem II. *If* $p \geq 1$, *there exists a positive constant* A, *which depends on* n *and* p, *but not on* a *and* b, *such that*

$$\text{(4)} \qquad \int_a^b R^p(x)(x-a)^{p-1}\,dx \geq A \qquad (0 \leq a < b < \infty)$$

if either (2) *or* (3) *has a solution which has* n *zeros in* $[a,b]$.

It is easy to see that, except for the statement concerning sharpness, Theorem II implies Theorem I. If $p \leq n$ and condition (1) holds, the left-hand side of (4) can be made smaller than A by taking a large enough. According to Theorem II, no solution of (2) or (3) can then have more than $n - 1$ zeros in (a,∞).

2. If y is a function of class $C^n[0,\infty)$ which has a zero of order k $(1 \leq k \leq n - 1)$ at $x = a$ and a zero of order $n - k$ at $x = b$ $(b > a)$, we shall say that y satisfies the boundary conditions $U_k(y;a,b) = 0$. It is known [7,10] that, if (2) or (3) has a solution with n zeros in $[a,c]$, there exists a number b in $(a,c]$ and a solution y of the equation such that y satisfies the conditions $U_k(y;a,b) = 0$ for some k. It is thus sufficient to prove (4) for the interval $[a,b]$ corresponding to this solution y.

If $g_k(x,t)$ is the Green's function of the differential operator $Mu = u^{(n)}$ for the boundary conditions $U_{n-k}(u;a,b) = 0$, then

$$(5) \qquad y(t) = \int_a^b g(x,t)R(x)y(x)\,dx \equiv Ly.$$

This formula holds for both equations (2) and (3). The reason a negative sign does not appear in one of the two representations (5) is that, in the case of equation (2), $n - k$ is an odd number, while $n - k$ is even for equation (3) [7,10]; the two Green's functions are thus different.

If

$$(u,v) = \int_a^b u(x)v(x)\,dx$$

and we define the operator L^* by

$$(6) \qquad L^*y = R(t) \int_a^b g(t,x)y(x)\,dx,$$

we have $(u,Lv) = (L^*u,v)$, *i.e.,* L^* is the operator adjoint to L.

We now consider the integral equation

$$(7) \qquad w = \lambda L^*Lw$$

or, written explicitly,

$$(8) \qquad w(t) = \lambda \int_a^b K(x,t)w(x)\,dx$$

where $K(x,t)$ is the symmetric kernel

$$(9) \qquad K(x,t) = R(x)R(t) \int_a^b g(x,\xi)g(t,\xi)\,d\xi = R(x)R(t)G(x,t).$$

The kernel $K(x,t)$ is positive-definite, and the smallest eigenvalue λ_1 of (7) is given by

$$(10) \qquad \frac{1}{\lambda_1} = \sup(u,L^*Lu) = \sup(Lu,Lu),$$

where u ranges over all functions in $L^2(a,b)$ for which $(u,u) = 1$. If y is the (normalized) solution of (5), it follows from (10) that

$$\frac{1}{\lambda_1} \geq (Ly,Ly) = (y,y) = 1,$$

and thus

$$(11) \qquad \lambda_1 \leq 1.$$

We shall show that, under the assumptions made,

$$(12) \qquad \lambda^{2p} \int_a^b R^p(x)(x-a)^p\, dx \geq A, \qquad (p > -1),$$

where A depends on p and n only. In view of (11), this will prove (4).

If we set $w(x) = u(x)R(x)$,

the integral equation (8) takes the form

$$(13) \qquad u(t) = \lambda \int_a^b G(x,t)R^2(x)u(x)dx$$

where, according to (9), G(x,t) is the symmetric kernel

$$(14) \qquad G(x,t) = \int_a^b g(x,\xi)g(t,\xi)d\xi .$$

The integral equation (13) is equivalent to the differential equation

$$(15) \qquad u^{(2n)} - (-1)^n \lambda R^2(x)u = 0$$

with the boundary conditions

(16) u - u' — • • • — u — u — u —...— u — u, x— a

and

(17) u = u' =...= u ^ ^ " 1 ^ $u^{(n)} = u^{(n+1)} = \ldots = u^{(n+k-1)} = 0$, x = b.

This follows from the observation that, by the definition of g(x,t), the function

$$W(t) = \int_a^b g(x,t)w(x)dx$$

satisfies the boundary conditions $U_k(W;a,b) = 0$ and the relation

$$W^{(n)}(t) = w(t) .$$

Similarly, if $g_1(x,t)$ is the Green's function of Mv 2 v^n^ for the 'adjoint' boundary conditions $U_v(v;a,b) = 0$, the function

$$S(t) = \int_a g_1(x,t)s(x)dx$$

satisfies $S^{(n)}(t) = s(t)$ and the boundary conditions $U_{n-k}(S;a,b) = 0$.

It is well known (and easily confirmed with the help of Green's identity for the operator M) that $g(x,t) = (-1)^n g(t,x)$. In view of the definition (14) of $G(x,t)$ it follows therefore that the function

$$T(t) := \int_a^b G(x,t)R^2(x)u(x)dx$$

satisfies the boundary conditions (16)-(17) and the identity

$$T^{(2n)}(t) = (-1)^n R^2(t)u(t).$$

Since, by (13), $u(t) = \lambda T(t)$, this establishes the equivalence of the integral equation (13) and the differential system (15)-(16)-(17).

3. By classical results, the lowest eigenvalue λ of this system may also be defined by

$$\frac{1}{\lambda} = \sup \int_a^b R^2(x)y^2(x)dx, \tag{18}$$

where the functions y satisfy the boundary conditions (16)-(17), are normalized by

$$\int_a^b [y^{(n)}]^2 dx = 1, \tag{19}$$

and possess continuous derivatives of the order $\max[2n - k - 1, n + k - 1]$. Hence, the number λ defined by (18) is subject to the inequality (11).

We now make use of the fact that any non-negative non-increasing function on $[a,b]$ can be approximated by finite sums of the form

$$\alpha_1 r_1(x) + \dots + \alpha_m r_m(x), \quad \alpha_\nu > 0, \quad \nu = 1,\dots,m, \tag{20}$$

where $r_\nu(x)$ is the characteristic function of the interval $[a,x_\nu]$ and $a < x_1 < x_2 < \dots < x_m \leq b$. We apply this, in particular, to the non-negative, non-increasing function $R^{\frac{1}{p}}(x)$ $(p \geq 1)$. If $R^{\frac{1}{p}}$ is of the form (20) we have, by Minkowski's inequality,

$$(21)\quad \Big[\int_a^b R^{2p} y^2\,dx\Big]^{\frac{1}{2p}} = \Big[\int_a^b \Big(\sum_{\nu=1}^m \alpha_\nu r_\nu\Big)^{2p} y^2\,dx\Big]^{\frac{1}{2p}}$$

$$\le \sum_{\nu=1}^m \alpha_\nu \Big[\int_a^b r_\nu^{2p} y^2\,dx\Big]^{\frac{1}{2p}} \le \sum_{\nu=1}^m \alpha_\nu \lambda_\nu^{-\frac{1}{2p}},$$

where

$$(22)\quad \frac{1}{\lambda_\nu} = \sup_y \int_a^b r_\nu^{2p} y^2\,dx = \sup_y \int_a^{x_\nu} y^2\,dx .$$

Since (21) holds for all the functions y admissible in (18), we thus have

$$\frac{1}{\lambda^{\frac{1}{2p}}} \le \sum_{\nu=1}^m \frac{\alpha_\nu}{\lambda_\nu^{\frac{1}{2p}}} .$$

Hence, if $\sigma(x)$ is a non-decreasing function in $[a,b]$, it follows that

$$\frac{1}{\lambda^{\frac{1}{2p}}} \le \sum_{\nu=1}^m \frac{\alpha_\nu \int_a^b r_\nu\,d\sigma(x)}{\lambda_\nu^{\frac{1}{2p}} \int_a^b r_\nu\,d\sigma(x)},$$

whence, in view of

$$\sum_{\nu=1}^m \alpha_\nu \int_a^b r_\nu\,d\sigma(x) = \int_a^b R^p\,d\sigma(x)$$

and

$$\int_a^b r_\nu\,d\sigma(x) = \int_a^{x_\nu} d\sigma(x) = \sigma(x_\nu) - \sigma(a),$$

$$\lambda^{\frac{1}{2p}} \int_a^b R^p\,d\sigma(x) \ge \inf_\nu \lambda_\nu^{\frac{1}{2p}}\,[\sigma(x_\nu) - \sigma(a)],$$

If we set $\sigma(x) = (x-a)^p$, we thus obtain the inequality

$$\lambda^{\frac{1}{2p}} \int_a^b R^p (x-a)^{p-1}\,dx \ge \frac{1}{p}\inf_\nu \lambda_\nu^{\frac{1}{2p}} (x_\nu - a)^p .$$

Accordingly, (12) will be proved if we can show that there exists a positive constant B such that

(23) $$\bar{A}_\nu (x_\nu - a)^{2n} \geq B,$$

where $\bar{A}_\nu$ is defined in (22) (and the admissibility conditions for the functions y are the same as in the definition of A in (18)).

The value of the right-hand side of (22) cannot decrease if we enlarge the class of admissible functions y by dropping the boundary condition (17), and we may thus conclude that

(24) $$\frac{1}{\bar{\lambda}_\nu} \leq \bar{A}_\nu ,$$

where $\bar{\lambda}_\nu$ is the lowest eigenvalue of the differential equation

$$y^{(2n)} - (-1)^n \lambda y = 0$$

with the boundary condition (16) at $x = a$ and the 'free' boundary condition

$$y^{(n)} = y^{(n+1)} = \cdots = y^{(2n-1)} = 0$$

at $x = x_\nu$. From the way the value of

$$\frac{1}{\bar{\lambda}_\nu} = \sup \frac{\int_a^{x_\nu} y^2\, dx}{\int_a^{x_\nu} [y^{(n)}]^2 dx}$$

changes under the coordinate transformation $x - a \to x_\nu (x - a)$, it is evident that the expression $\bar{A}_\nu (x_\nu - a)^{2n}$ is independent of x_ν. If its value is denoted by B, (24) is seen to imply (23).

4. This completes the proof of Theorem II. As shown above, the main assertion of Theorem I is a direct consequence of Theorem II. All that remains to be shown is that equations (2) and (3) can have oscillatory solutions if the coefficient $R(x)$ satisfies the condition -_ ,

(25) $$\int^{\infty} R^p(x) x^{pn-1-\epsilon}\, dx < \infty , \qquad \epsilon > 0.$$

That this is indeed the case is shown by the Euler equation

(26) $$y^{(n)} + \wedge y = 0,$$
$$x$$

which has the solutions x , where V is a solution of the algebraic equation .

(27) $$(V -. 1)(V - 2) \cdots (y - n + 1) + <* = 0.$$

If n is even, this equation evidently has precisely two real solutions if ^ is chosen sufficiently small, and it has no real solution if *x£* is taken large enough. Hence, (27) has complex solutions for sufficiently large positive ^, and it has complex solutions if n > 2 and ^ is a negative number of large enough modulus. For odd n, (27) has precisely one ral solution if |c^» | is sufficiently large and ^ is either negative or positive; the remaining roots of the equation are complex. A complex root of (27) corresponds to an oscillating solution of (26). Since, for equation (26),

$$R^{P}(x)x^{P}\sim " = x^{1-6} ,$$

the existence of oscillating solutions is thus seen to be compatible with condition (25).

This argument fails if n = 2 and the equation is of the form (3). However, in this case the equation is trivially non-oscillatory, and there is nothing to prove.

References

1. G. B. Anan'eva and V. I. Balaganskii, 'Oscillation of the solutions of certain differential equations of higher order,' Usp. Mat. Nauk 14 (1959) 135-140.

2. J. H. Barrett, 'Disconjugacy of a self-adjoint differential equation of the fourth order,' Pacific J. Math, 11 (1961) 25-37.

3. P. R. Beesack, 'On the Green's function of an n-point boundary value problem,' Pacific J. Math, 12 (1962), 801-812.

4. D. B. Hinton, 'Disconjugate properties of a system of differential equations,' J. Diff. Equations 2 (1966) 420-

5. V. A. Kondrat'ev, 'Oscillatory properties of solutions of the equation $y^{(n)} + p(x)y = 0$', Trudy Moskov. Mat. Obšč, 10 (1961), 419-436.

6. W. Leighton and Z. Nehari, 'On the oscillation of solutions of self-adjoint linear differential equations of the fourth order', Trans. Am. Math. Soc, 89 (1958) 325-377.

7. A. Yu. Levin, 'Some questions concerning the oscillation of solutions of linear differential equations,' Dokl. Akad. Nauk. SSSR, 148 (1963) 512-515.

8. A. Yu. Levin, 'On the distribution of zeros of the solutions of linear differential equations,' Dokl. Akad. Nauk. SSSR, 156 (1964), 1281-1284.

9. Z. Nehari, 'Non-oscillation criteria for n-th order linear differential equations', Inke Math. J., 32 (1965) 607-616.

10. Z. Nehari, 'Disconjugate linear differential operators', Trans. Am. Math. Soc.

11. T. L. Sherman, 'Properties of solutions of n-th order linear differential equations', Pacific J. Math. 15 (1965) 1045-1060.