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Title: Formal Systems, properties of

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Formal systems or theories must satisfy requirements that are sharper than those imposed on the structure of theories by the axiomatic-deductive method; that method can be traced back to Euclid's *Elements*. The crucial additional requirement is the regimentation of inferential steps in proofs: not only axioms have to be given in advance, but also the logical rules representing argumentative steps. To avoid a regress in the definition of proof and to achieve intersubjectivity on a minimal basis, the rules are to be "mechanical" and must take into account only the syntactic form of statements. Thus, to exclude any ambiguity, a precise symbolic language is needed and a logical calculus. Both the concept of a "formula" (i.e., statement in the symbolic language) and that of a "rule" (i.e., inference step in the logical calculus) have to be effective; by the CHURCH-TURING THESIS that means they have to be recursive.

Frege presented in his (1879) a symbolic language (with relations and quantifiers) together with an adequate logical calculus, thus providing the means for the completely formal representation of mathematical proofs. The Fregean frame was basic for the later development of mathematical logic; it influenced the work of Whitehead and Russell that culminated in *Principia Mathematics*. The next crucial step was taken most vigorously by

Hilbert; he built on Whitehead and Russell's work and used an appropriate frame for the development of parts of mathematics, but took it also as an object of mathematical investigation. The latter metamathematical perspective proved to be extremely important. Clearly, in a less rigorous way it goes back to the investigations concerning non-Euclidean geometry and Hilbert's own early work on independence questions in geometry in his (1899).

Hilbert's emphasis on the mathematical investigation of formal systems really marked the beginning of *mathematical* logic. In the lectures (1918), prepared in collaboration with Paul Bernays, he isolated the language of first order logic as the central language (together with an informal semantics) and developed a suitable logical calculus. Central questions were raised and partially answered; they concerned the completeness, consistency, and decidability of such systems and are still central in mathematical logic and other fields, where formal systems are being explored. Some important results will be presented paradigmatically; for a real impression of the richness and depth of the subject readers have to turn to (classical) textbooks or to up-to-date handbooks listed in the bibliography.

Completeness has been used in a number of different senses, from the *quasi-empirical completeness* of Zermelo Fraenkel set theory (being sufficient for the formal development of mathematics) to the *syntactic completeness* of formal theories (shown to be impossible by Gödel's First Theorem for theories containing a modicum of number theory). For logic the central concept is, however, *semantic completeness*: a calculus is (semantically)

complete, if it allows to prove all statements that are true in all interpretations (models) of the system. In sentential logic these statements are the tautologies; for that logic Hilbert and Bernays in (1918) and Post, independently in (1921), proved the completeness of appropriate calculi; for first order logic completeness was established by Gödel (1930). Completeness expresses obviously the adequacy of a calculus to capture all logical consequences and entails almost immediately the logic's compactness: if every finite subset of a system has a model, so does the system. Ironically, this immediate consequence of its adequacy is at the root of real inadequacies of first order logic: the existence of non-standard models for arithmetic and the inexpressibility of important concepts (like "finite", "well-order"). The relativity of "being countable" (leading to the so-called Skolem paradox) is a direct consequence of the proof of the completeness theorem.

Relative *consistency* proofs were obtained in geometry by semantic arguments: given a model of Euclidean geometry one can define a Euclidean model of, say, hyperbolic geometry; thus, if an inconsistency could be found in hyperbolic geometry it could also be found in Euclidean geometry. Hilbert formulated as the central goal of his program to establish by elementary, so-called finitist means the consistency of formal systems. This involved a direct examination of formal proofs; the strongest results before 1931 were obtained by Ackermann, von Neumann, and Herbrand: they established the consistency of number theory with a very restricted induction principle. A basic limitation had indeed been reached, as was made clear by Gödel's Second Theorem; see GÖDEL'S THEOREMS.

Modern proof theory, by using stronger than finitist, but still "constructive" means, has been able to prove the consistency of significant parts of analysis. In pursuing this generalized consistency program, important insights have been gained into structural properties of proofs in special calculi ("normal form" of proofs in sequent and natural deduction calculi).

Hilbert's Entscheidungsproblem, the decision problem for first order logic, was one issue that required a precise characterization of "effective methods"; see CHURCH-TURING THESIS. Though partial positive answers were found during the 1920s, Church and Turing proved in 1936 that the general problem is undecidable. The result and the techniques involved in its proof (not to mention the very mathematical notions) inspired the investigation of the recursion theoretic complexity of sets that led at first to the classification of the arithmetical, hyper-arithmetical, and analytical hierarchies, and later to that of the computational complexity classes.

Some general questions and results were described for particular systems; as a matter of fact, questions and results that led to three branches of modern logic: model theory, proof theory, and computability theory. However, to re-emphasize, from an abstract recursion theoretic point of view any system of "syntactic configurations" whose "formulas" and "proofs" are effectively decidable (by a Turing machine) is a formal system. In a footnote to his 1931 paper added in 1963, Gödel made this point most strongly: "In my opinion the term 'formal system'¹ or 'formalism'¹ should never be used for anything but this notion. In a lecture at Princeton [[in 1946]] I suggested certain transfinite generalizations of

formalisms; but these are something radically different from formal systems in the proper sense of the term, whose characteristic property is that reasoning in them, in principle, can be completely replaced by mechanical devices."

Bibliography:

Barwise, Jon (ed.)

1977 *Handbook of mathematical logic*; North-Holland Publishing Company.

Börger, Egon, Erich Grädel, and Yuri Gurevich

1997 *The Classical Decision Problem*; Springer Verlag.

Church, Alonzo

1936 A note on the Entscheidungsproblem; *J. Symbolic Logic* 1, 40-41; correction, *ibid.*, 101-2.

Frege, Gottlob

1879 *Begriffsschrift*, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens; Louis Nebert Verlag.

1893 *Grundgesetze der Arithmetik*, Begriffsschriftlich abgeleitet. volume I; Pohle.

Gabbay, Dov e.a. (eds.)

199 M *Handbook of mmmm*;

Gentzen, Gerhard

1934/5 Untersuchungen über das logische Schließen I, II; *Math. Zeitschrift* 39, 176-210, and 405-431; translated in (1969).

1936 Die Widerspruchsfreiheit der reinen Zahlentheorie; *Mathematische Annalen* 112, 493-565; translated in (1969).

1969 *The collected papers of Gerhard Gentzen*, M.E. Szabo (ed.), North-Holland Publishing Company.

Gödel, Kurt

1930 Die Vollständigkeit der Axiome des logischen Funktionenkalküls; *Monatshefte für Mathematik und Physik* 37, 349-360; translated in *Collected Works I*.

1931 Über formal unentscheidbare Sätze Principia Mathematica und verwandter Systeme I; *Monatshefte für Mathematik und Physik* 38, 173-198; translated in *Collected Works I*.

1986 *Collected Works I*; Oxford University Press.

Hilbert, David

1899 *Grundlagen der Geometrie*; Teubner.

1918 *Die Prinzipien der Mathematik*; lectures given during the winter term 1917-18, written by Paul Bernays; typewritten manuscript, 246 pages, Mathematical Institute, University of Göttingen.

Hilbert, David and Wilhelm Ackermann
1928 *Grundzüge der theoretischen Logik*, Springer Verlag.

Kleene, Stephen C.
1952 *Introduction to metamathematics*; Wolters-Noordhoff Publishing.

Post, Emil
1921 Introduction to a general theory of elementary propositions; *Amer. J. Math.* 43,
163-185; reprinted in (van Heijenoort 1967).

Prawitz, Dag
1966 *Natural deduction: A proof-theoretical study*; Almqvist & Wiskell.

Rogers, Hartley Jr.
1967 *Theory of recursive functions and effective computability*, McGraw Hill.

Shoenfield, Joseph R.
1967 *Mathematical Logic*; Addison-Wesley.

van Dalen, Dirk
1989 *Logic and Structure*, 2nd edition; Springer Verlag.

Author: Wilfried Sieg

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Church proposed at a meeting of the American Mathematical Society in April 1935, "that the notion of an effectively calculable function of positive integers should be identified with that of a recursive function ...". This proposal of identifying an informal notion, *effectively calculable function*, with a mathematically precise one, *recursive function*, has been called Church's Thesis since Kleene used that name in his (1952). Turing independently made in 1936 a related proposal, Turing's Thesis, suggesting the identification of effectively calculable functions with functions whose values can be computed by a particular idealized computing device, a *Turing machine*. As the two mathematical notions are provably equivalent, the theses are "equivalent", and are jointly referred to as the Church-Turing Thesis.

The reflective, partly philosophical and partly mathematical, work around and in support of the thesis concerns one of *the* fundamental notions of mathematical logic. Its proper understanding is crucial for making informed and reasoned judgments on the significance of limitative results - like Gödel's Theorems or Church's Theorem. The work is equally crucial for computer science, artificial intelligence, and cognitive psychology as it provides also for these subjects a basic theoretical notion. For example, the

thesis is the cornerstone for Newell's delimitation of the class of physical symbol systems, i.e. universal machines with a particular architecture. Newell views this delimitation in his (1980) "as the most fundamental contribution of artificial intelligence and computer science to the joint enterprise of cognitive science". In a turn that had *almost* been taken by Turing in (1948) and (1950), Newell points to the basic role physical symbol systems have in the study of the human mind: "... the hypothesis is that humans are instances of physical symbol systems, and, by virtue of this, mind enters into the physical universe. ... this hypothesis sets the terms on which we search for a scientific theory of mind." The restrictive "almost" in Turing's case is easily motivated: he viewed the precise mathematical notion as a crucial ingredient for the investigation of the mind, but did not subscribe to a "mechanist" theory of mind. It is precisely for an understanding of such, sometimes controversial, claims that (the background for) Church's and Turing's work has to be presented carefully.

The informal notion of an effectively calculable function, effective procedure, or algorithm had been used in 19th century mathematics and logic, when indicating that a class of problems is solvable in a "mechanical fashion", by following fixed elementary rules. Hilbert suggested in 1904 to take formally presented theories as objects of mathematical study, and metamathematics has been pursued vigorously and systematically since the 1920s. In its pursuit concrete issues arose that required for their resolution a precise characterization of the class of effective procedures.

Hilbert's *Entscheidungsproblem*, the decision problem for first order logic, was one such issue. It was solved negatively - relative to the precise notion of recursiveness, respectively Turing machine computability; though obtained independently by Church and Turing, this result is usually called Church's Theorem. A second significant issue was the formulation of Gödel's Incompleteness Theorems as applying to *all* formal theories (satisfying certain representability and derivability conditions), see GÖDEL'S THEOREMS. Gödel had established the theorems in his ground-breaking 1931 paper for specific formal systems like type theory of *Principia Mathematica* or Zermelo-Fraenkel set theory. The general formulation required a convincing characterization of "formality"; see FORMAL SYSTEMS.

According to Kleene and Rosser, Church proposed in late 1933 the identification of effective calculability with λ -definability. That proposal was not published at the time, but in 1934 Church mentioned it in conversation to Gödel who judged it to be "thoroughly unsatisfactory". In his Princeton Lectures of the same year Gödel later defined the concept of a (general) recursive function using an equational calculus, but he was not convinced that all effectively calculable functions would fall under it. The proof of the equivalence between λ -definability and recursiveness (found by Church and Kleene in early 1935) led to Church's first published formulation of the thesis as quoted above. The thesis was reiterated in Church's 1936 paper. Turing introduced also in 1936 his notion of computability by machines. Post's 1936 paper contains a model of computation that is strikingly similar to Turing's, but he

did not provide any analysis in support of the generality of his model. On the contrary, he suggested considering the identification of effective calculability with his concept as a working-hypothesis that should be verified by investigating ever wider formulations and reducing them to his basic formulation. The classical papers of Gödel, Church, Turing, Post, and Kleene are all reprinted in (Davis 1965), and good historical accounts can be found in (Davis 1982), (Gandy 1988) and (Sieg 1997).

(Church 1936) presented one central reason for the proposed identification, namely that other plausible explications of the informal notion lead to mathematical concepts weaker than or equivalent to recursiveness. Two paradigmatic explications, calculability of a function *via algorithms* and *in a logic*, were considered by Church. In either case, the steps taken in determining function values have to be effective; if the effectiveness of steps is taken to mean recursiveness, then the function can be *proved* to be recursive. This requirement on steps in Church's argument corresponds to one of the "recursiveness conditions" formulated by Hilbert and Bernays (1939). That condition is used in their characterization of functions that are evaluated according to rules in a deductive formalism: it requires the proof predicate for a deductive formalism to be primitive recursive. Hilbert and Bernays show that all such "reckonable" functions are recursive and can actually be evaluated in a very restricted number theoretic formalism. Thus, in any formalism that (satisfies the recursiveness conditions and) contains this minimal number theoretic system, one

can compute exactly the recursive functions: recursiveness or computability consequently has, as Gödel emphasized, an absoluteness property not shared by other metamathematical notions like provability or definability; the latter notions depend on the formalism considered.

All such indirect and ultimately unsatisfactory considerations were bypassed by Turing. He focused directly on the fact that *human mechanical* calculability on symbolic configurations was the intended notion. Analyzing the processes that underly such calculations (by a *computer*), Turing was led to certain boundedness and locality conditions. To start with, he demanded the immediate recognizability of symbolic configurations so that basic computation steps need not be further subdivided. This demand and the evident limitation of the computer's sensory apparatus motivate the conditions. Turing also required that the computer proceed deterministically. The above conditions, somewhat hidden in Turing's 1936 paper, are formulated now following (Sieg 1994); first the *boundedness conditions*:

(B.1) there is a fixed bound for the number of symbolic configurations a computer can immediately recognize;

(B.2) there is a fixed bound for the number of a computer's internal states that need to be taken into account.

Since the behavior of the computer is uniquely determined by the finitely many combinations of symbolic configurations and internal states, he can carry out only finitely many different operations. These operations are restricted by the *locality conditions*:

- (L.1) only elements of observed configurations can be changed;
- (L.2) the computer can shift his attention from one symbolic configuration to another only if the second is within a bounded distance from the first.

Thus, on closer inspection, Turing's Thesis is seen as the result of a two part analysis. The first part yields the above conditions and Turing's *central thesis*, that any mechanical procedure can be carried out by a computer satisfying these conditions. The second part argues that any number theoretic function calculable by such a computer is computable by a Turing machine. Both Church and Gödel found Turing's analysis convincing; indeed, Church wrote in (1937) that Turing's notion makes "the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately". From a strictly mathematical point, the analysis leaves out important steps, and the claim that is actually established is the more modest one that Turing machines operating on strings can be simulated by machines operating on single letters; a way of generalizing Turing's argument is presented in (Sieg and Byrnes 1996).

Two final remarks are in order. First, all the arguments for the thesis take for granted that the effective procedures are being carried out by human beings. Gandy, by contrast, analyzed in his 1980 paper *machine* computability; that notion involves crucially parallelism. Gandy's mathematical model computes nevertheless only recursive functions. Second, the effective procedures are taken, in addition, to be mechanical, not general cognitive ones - as claimed by Webb and many others. Also Gödel was wrong when

asserting In a brief note from 1972 that Turing intended to show in his 1936 paper that "mental procedures cannot go beyond mechanical procedures". Turing, quite explicitly, had no such intentions; even after having been engaged in the issues surrounding machine intelligence, he emphasized in his 1953 paper that the precise concepts (recursiveness, Turing computability) are to capture the mechanical processes that can be carried out by human beings.

Bibliography:

Church, Alonzo

- 1935 An unsolvable problem of elementary number theory (abstract); *Bulletin of the Amer. Math. Soc.* 41, 332-3.
- 1936 An unsolvable problem of elementary number theory; *Amer. J. Math.* 58, 345-63.
- 1937 Review of (Turing 1936); *J. Symbolic Logic* 2, 42-3.

Davis, Martin

- 1965 (ed.) *Undecidable: basic papers on undecidable propositions, unsolvable problems and computable functions*; Raven Press.
- 1982 Why Gödel didn't have Church's Thesis; *Information and control* 54, 3-24.

Gandy, Robin

- 1988 The confluence of ideas in 1936; in: *The Universal Turing Machine*, (Rolf Herken, ed.), Oxford University Press, 55-111.
- 1980 Church's Thesis and principles for mechanisms; in: *The Kleene Symposium*, Barwise, Keisler, and Kunen (eds.), North-Holland Publishing Company, 123-148.

Gödel, Kurt

- 1931 Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I; translated in: *Collected Works I*, 144-195.
- 1934 On undecidable propositions of formal mathematical systems (Princeton Lectures); in: *Collected Works I*, 346-69.
- 1972 A philosophical error in Turing's work; in: *Collected Works II*, 306.
- 1986 *Collected Works I*; Oxford University Press.
- 1990 *Collected Works II*; Oxford University Press.
- 1995 *Collected Works III*, Oxford University Press.

Hilbert, David

- 1904 Über die Grundlagen der Logik und der Arithmetik; translated in: (van Heijenoort 1967), 129-138.

Hilbert, David and Wilhelm Ackermann

- 1928 *Grundzüge der theoretischen Logik*; Springer Verlag.

Hilbert, David and Paul Bernays

1939 *Grundlagen der Mathematik*, vol. 2; Springer Verlag.

Kleene, Stephen

1952 *Introduction to Metamathematics*; Wolters-Noordhoff Publishing.

1981 Origins of recursive function theory; *Annals Hist. Computing* 3, 52-66.

Mundici, Daniele and Wilfried Sieg

1995 Paper machines; *Philosophia Mathematica* 3, 5-30.

Newell, Alan

1980 Physical Symbol Systems; *Cognitive Science* 4, 135-183.

Post, Emil

1936 Finite combinatory processes. Formulation 1; *J. Symbolic Logic* 1, 103-5.

1947 Recursive unsolvability of a problem of Thue; *J. Symbolic Logic* 12, 1-11.

Rosser, Barkley

1984 Highlights of the history of the lambda-calculus; *Annals Hist. Computing* 6, 337-49.

Sieg, Wilfried

1994 Mechanical procedures and mathematical experience; in: *Mathematics and Mind*, (A. George, ed.), Oxford University Press, 71-117.

1997 Step by recursive step: Church's analysis of effective calculability; *Bulletin of Symbolic Logic* 3, 154-180.

Sieg, Wilfried and John Byrnes

1996 K-graph machines: generalizing Turing's machines and arguments; in: *Gödel '96*, (P. Hájek, ed.), *Lecture Notes in Logic* 6, Springer Verlag, 98-119.

1997 Gödel, Turing, and K-graph machines; to appear in: *Logic in Florence*, A. Cantini, E. Casari, P. Minari (eds.), Kluwer Academic Publisher.

Soare, Robert

1996 Computability and recursion; *Bulletin of Symbolic Logic* 2, 284-321.

Turing, Alan

1936 On computable numbers, with an application to the Entscheidungsproblem; reprinted in (Davis 1965).

1948 Intelligent Machinery;

1950 Computing machinery and intelligence; *Mind* 59, 4-30.

1953 Solvable and unsolvable problems; *Science News* 31, 7-23.

van Heijenoort, Jean

1967 *From Frege to Gödel*, Harvard University Press.

Webb, Judson C.

1980 *Mechanism, Mentalism, and Metamathematics*; Reidel Publishing Company.

1990 Introductory Note to (Gödel 1972); in: *Collected Works II*, 292-304.

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Kurt Gödel was undoubtedly one of the most influential logicians of the 20th century. A number of absolutely central facts were established by him, among them the completeness of first order logic and the relative consistency of the axiom of choice and of the generalized continuum hypothesis. However, the theorems which have been most significant (for the general discussion concerning the foundations of mathematics) are his two *Incompleteness Theorems* published in 1931; they are also referred to simply as Gödel's Theorems or *the Gödel Theorems*.

The early part of the 20th century saw a dramatic development of logic in the context of deep problems in the foundations of mathematics. This development provided for the first time the basic means to reflect mathematical practice in formal theories; see FORMAL SYSTEMS. One fundamental question was: Is there a formal theory such that mathematical truth is co-extensive with provability in that theory? Russell's type theory **P** of *Principia Mathematica* and axiomatic set theory as formulated by Zermelo seemed to make a positive answer plausible. A second question emerged from the research program that had been initiated by Hilbert around 1920 (with roots going back to the turn of the century): Is the consistency of mathematics in its formalized presentation provable by restricted mathematical, so-called finitist

means? -- The Incompleteness Theorems gave negative answers to both questions for the particular theories mentioned. To be more precise, a negative answer to the second question is given only, if finitist mathematics is considered to be formalizable in these theories; that was not clear to Gödel in 1931, only in his (1933) did he assert it with great force.

The First Incompleteness Theorem states (making use of an improvement due to Rosser):

If P is consistent, then there is a sentence σ in the language of P , such that neither σ nor its negation $\neg\sigma$ is provable in P .

σ is thus *independent* of P . As σ is a number theoretic statement it is either true or false for the natural numbers; in either case, we have a statement that is true and not provable in P . This incompleteness of P cannot be remedied by adding the true statement to P as an axiom: for the theory so expanded, the same incompleteness phenomenon arises. -- Gödel's Second Theorem claims the unprovability of a (meta-) mathematically meaningful statement:

If P is consistent, then cons, the statement in the language of P that expresses the consistency of P , is not provable in P .

Some, for example Church, raised the question, whether the proofs in some way depended on special features of P . In his Princeton lectures of 1934 Gödel tried to present matters in a more general way; he succeeded in addressing Church's concerns, but continued to strive for even greater generality in the formulation of the theorems. To understand in what direction, we review the very basic ideas underlying the proofs.

Crucial are the *effective presentation* of P's syntax and its *(internal) representation*. Gödel uses a presentation by primitive recursive functions, i.e., the basic syntactic objects (strings of letters of P's alphabet and strings of such strings) are "coded" as natural numbers, and the subsets corresponding to formulas and proofs are given by primitive recursive characteristic functions. Representability conditions are established for all syntactic notions R, i.e., really for all primitive recursive sets (and relations): if $R(m)$ holds then P proves $r(m)$, and if not $R(m)$ holds then P proves $\neg r(m)$, where r is a formula in the language of P and m the numeral for the natural number m . Thus, the metamathematical talk about the theory can be represented within it. Then the self-referential statement a (in the language of P) is constructed expressing of itself that it is not provable in P. An argument analogous to that showing the liar sentence not to be true establishes that a is not provable in P, thus we have part of the First Theorem. The Second Theorem is obtained, very roughly speaking, by formalizing the proof of the First Theorem concerning a , but additional derivability conditions are needed: this yields a proof in P of $(\text{cons} \rightarrow a)$. Now, clearly, cons cannot be provable in P, otherwise a were provable, contradicting the part of the First Theorem we just established. The proof of the Second Theorem was given in detail only by Hilbert and Bernays (1939). A gem of an informal presentation of this material is (Gödel 1931 A); for a good introduction to the mathematical details see (Smorynski 1977).

Gödel viewed in (1934) the primitive recursiveness of the syntactic notions as "a precise condition which in practice suffices

as a substitute for the unprecise requirement ... that the class of axioms and relation of immediate consequence be constructive", i.e., have an effectively calculable characteristic function. What was needed, in principle, was a precise concept capturing the informal notion of an effectively calculable function. Only that would allow a perfectly general characterization of *formal* theories. Such a notion emerged from the investigations of Church and Turing; see CHURCH-TURING THESIS. Finally it was possible to state and prove the Incompleteness Theorems for *all* formal theories satisfying representability (for all recursive relations) and derivability conditions. In the above statement of the theorems, the premise "P is consistent" can now be replaced by "P is any consistent formal theory satisfying the representability conditions", respectively "P is any consistent formal theory satisfying the representability and derivability conditions". It is this generality of his results, Gödel emphasized again and again; for example, in (1964): "In consequence of later advances, in particular of the fact that, due to A.M. Turing's work, a precise and unquestionably adequate definition of the general concept of formal system can now be given, the existence of undecidable arithmetical propositions and the non-demonstrability of the consistency of a system in the same system can now be proved rigorously for every consistent formal system containing a certain amount of finitary number theory."

Gödel analyzed the broader significance of his theorems for the philosophy of mathematics (and mind) most carefully in (1951). The first section is devoted to a discussion of the Incompleteness Theorems, in particular of the Second Theorem, and argues for a

"mathematically established fact" which seems to be of "great philosophical interest" to Gödel: either the humanly evident axioms of mathematics cannot be comprised by a finite rule (given by a Turing machine), or they can be and allow the successive development of all of demonstrable mathematics; but in the latter case there will be absolutely undecidable problems. That's indeed all that can be strictly inferred, counter to Lucas, Penrose, e.a.. Gödel thought that the first disjunct held and emphasized repeatedly, for example in (1964), that his results do not establish "any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics". Indeed, in the Gibbs lecture Gödel turned the first disjunct into the following dramatic and vague statement: "the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine".

Bibliography:

Gödel, Kurt

- 1931 Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I; translated in: *Collected Works I*, 144-195.
- 1931A Über unentscheidbare Sätze; translated in: *Collected Works III*, 30-35.
- 1933 The present situation in the foundations of mathematics; in: *Collected Works III*, 45-53.
- 1934 On undecidable propositions of formal mathematical systems (Princeton Lectures); in: *Collected Works I*, 346-69.
- 1951 Some basic theorems on the foundations of mathematics and their implications (Gibbs Lecture); in: *Collected Works III*, 304-323.
- 1964 Postscriptum to (Gödel 1934); in: *Collected Works I*, 369-71.
- 1986 *Collected Works I*; Oxford University Press.
- 1990 *Collected Works II*; Oxford University Press.
- 1995 *Collected Works III*; Oxford University Press.

Dawson, John W.

- 1997 *Logical Dilemmas - The life and work of Kurt Gödel*, A.K. Peters.

Hilbert, David and Paul Bernays

1939 *Grundlagen der Mathematik II*; Springer Verlag.

Lucas, J.R.

1961 Minds, machines and Gödel; *Philosophy* 36, 112-27.

Penrose, Roger

1989 *The emperor's new mind*, Oxford University Press.

Rosser, Barkley

1936 Extensions of some theorems of Gödel and Church; *J. Symbolic Logic* 2, 129-37.

Russell, Bertrand

1908 Mathematical logic as based on the theory of types; *Amer. J. Math.*, 222-62.

Smorynski, Craig

1977 The incompleteness theorem; in: *Handbook of Mathematical Logic* (J. Barwise, ed.); North-Holland Publishing Company, 821-65.

Smullyan, Raymond M.

1992 *Gödel's Incompleteness Theorems*. Oxford University Press.

Whitehead, Alfred North and Bertrand Russell

1910 *Principia Mathematica* vol. 1, Cambridge University Press.

1912 *Principia Mathematica* vol. 2.

1913 *Principia Mathematica* vol. 3.

Zermelo, Ernst

1908 Untersuchungen über die Grundlagen der Mengenlehre; *Math. Annalen* 65, 261-81; translated in: *From Frege to Gödel* (J. van Heijenoort, ed.), 1967, Harvard University Press, 199-215,