Common Reasoning about Admissibility

by

Cristina Bicchieri and Oliver Schulte

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Cristina Bicchieri
Department of Philosophy
Carnegie Mellon University
E-mail: cb36@andrew.cmu.edu

Oliver Schulte
Department of Philosophy
Carnegie Mellon University
E-mail: schulte+@cmu.edu

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Abstract

We analyze common reasoning about admissibility in the strategic and extensive form of a game. We define a notion of sequential proper admissibility in the extensive form, and show that, in finite extensive games with perfect recall, the strategies that are consistent with common reasoning about sequential proper admissibility in the extensive form are exactly those that are consistent with common reasoning about admissibility in the strategic form representation of the game. Thus in such games the solution given by common reasoning about •*iitti«qfcliifcy does not depend on how the strategic situation is represented. We further explore the links between iterated admissibility and backward and forward induction.

1 Introduction

A weU known problem with non-cooperative game theory is that Nash equilibria are seldom relevant for predicting how the players will play. The equilibria of a game do not represent all the possible outcomes. Rather, they represent the set of self-enforcing agreements: had the players known their respective choices before playing the game, then they must have constituted an equilibrium. Some game theorists have argued that predictabffity must involve what Binmore (1987/88) has called an "eductive" procedure. When asking how the players[1] deductive processes might unfold, one must usually specify some basic principles of rationality, and then examine what choices are consistent with common knowledge of the specified principles. The advantage of this approach is that it is possible to refine our predictions about how players might choose without assuming that they will coordinate on a particular equilibrium. Principles such as iterated strict dominance or rationalkabOity (Pearce 1984), (Bernheim 1984) are examples of how it is possible to restrict the set of predictions

yields exactly the backward induction solution. And in finite games of imperfect information, common reasoning about arimissihility yields typical forward induction solutions. Thus backward and forward induction seem to follow from one principle, namely that players' choices should be ≈w***tmnt* with common reasoning about admissibility. This result may seem questionable, as it is also commonly held that backward and forward induction principles are mutually inconsistent. That is, if we take backward and forward induction principles to be restrictions imposed on equilibria, then they lead to contradictory conclusions about how to play. We show that the problem with the fncamplfs one finds in the literature is that no constraints are set on players' forward induction "signals". We define a credible forward induction signal in an extensive game as a signal consistent with common reasoning about sequential admissibility. Thus the examples in the literature which purport to show the conflict between backward and forward induction principles involve forward induction signals that are not credible.

2 Extensive Form Games

We introduce the bask notions for describing games in extensive form. Note that our formaliiation is limited to finite games, and that we restrict players to only play pure strategies. A finite extensive form game for players $N = 1,2,\ldots,n$ is given by a *game tree T* with finitely many nodes V, root r, payoff functions *+ which assigns a payoff to each player i at each terminal node in T, and information sets *1%* for each player i. For each node *x* in T, *I(x)* is the information set containing s. A pure *strategy s%* for player t in a game tree *T* assigns a unique action, called a *move*, to each information set *h* of player t in T. We denote the set of *Vs* pure strategics in *T* by &(T) (in what follows, the term •strategy* always refers to pure strategies.) A *strategy profile* in *T* is a vector (*i,*2, ••',*») consisting of one strategy for each player t. We denote the set of pure strategy profiles in *T* by S(T); Le. $S(T) = \times_{i \in N} \&(r)$. We use V to denote a generic strategy profile. It is useful to denote a vector of length n — 1 consisting of strategy choices by player t*s opponents by *s-%*. We write *S-i(T)* for the set of strategy profiles of *Ve* opponents, i.e. $S_{-i}(T) = \times \quad S_j(T)$.

Given a strategy profile *, we use *s[i* to denote the strategy of player t in J, and *s[-t{* to denote the strategy profile of t's opponents in *s*.

In the games we consider, the root is the only member of its information set (Le. *I(r)* = {r}), so that a strategy profile J in T determines a unique maximal path < r,«i,«j,...,«» > from the root r to a terminal node x_m; we refer to this path as the *play sequence* resulting from *, and denote it by pJay(j). When a strategy profile *s* in *T* is played, each player receives as payoff the payoff from the terminal node reached in the play sequence resulting from *. With some abuse of notation, we use *m* to denote both a function from strategy profiles to payoffs for player t, as well as a function from terminal nodes to a payoff for player t, and define strategy, whereas in the second case also weakly dominated strategies are eliminated.

A player who is reasoning, say, with the help of admissibility would not go very far in «dimin»tain{r plays of the game inconsistent with it, unless he assumes that the other players are also applying the same principle. In the game of Figure 1, for example, player 1 could not eliminate a *priori* any play of the game unless he assumed player 2 never plays a dominated strategy. [1] In general, even assuming that other players are rational might not be enough to rule out possibilities about how a given game might be played. Players must reason about other players' reasoning, and such mutual reasoning must be common knowledge. Unless otherwise specified, we shall assume that players have common knowledge of the structure of the game and of rationality, and examine how common reasoning about rationality unfolds.

3.1 Strict Dominance and Subgame Perfection

This section explores in detail the implications of common reasoning about weak admissibility, the requirement that players should avoid strictly dominated actions. We show that in finite games of perfect information, common reasoning about weak admissibility gives exactly the same results as Zermelo's backward induction algorithm, which in finite games of perfect information corresponds to Selten's notion of *subgame perfection* [3]. We th*n show by examples that the tight connection between common reasoning about weak admissibility and subgame perfection breaks down in games of imperfect information.

We define a strategy to be sequentially weakly admissible in a game tree T if it is weakly admissible at each information set in T. A strategy **i* for player t is not weakly admissible at a given information set /« if the strategy is strictly dominated at I*. This means that there is some other strategy J£ that yields *i* a better outcome than *Si* at every node *x* in *Ii*. For example, in the game of Figure 1, playing right (CR') at 2's information set is strictly dominated by playing left fL[1]).

The formal definition of sequential weak admissibility is the following.

DEFINITION 2 *Strict Dominance and Weak Admissibility in Extensive Form Games*

- *LetT be a finite game tree for N = 1,2,..., n players.*
- *We define the payoff to player i from strategy *i and strategy profile *-• atx, written Ui(si,s-i,x),* to 6e «>(*, «.t, s) = m(*|T*,*-i|T*).
- *A strategy s% is* strictly dominated *by another strategy s[at an information set Ii belonging to tin T just in ease for all strategy profiles s-i in T_f and for ally in I_{it}* **(*,«_»; y) < *i(*J, *_i, y).

[1]Here and elsewhere, the payoff at a terminal node is given as & pair (x,y), where x is the payoff for player 1 and y is the payoff for player 2.
[3]cf. (Osborne and Rubinstein, 1994, Ch. 6).

- *A strategy a is* weakly Admissible at an information set *U inT just in ease Si is not strictly dominated at* I».
- *A strategy a is* sequentially *weakly admissible inT if and only if *i is weakly admissible at each information set U inT 1hat belongs to player i.*

Our procedure for capturing common reasoning about sequential weak admissibility in *T* is the following. First, eliminate at each information set in T all moves that are inconsistent with weak admissibility, Le. strictly dominated chokes. The result is a restricted game tree 7".

Repeat the pruning procedure with *T* to obtain another restricted game tree, and continue until no moves in the resulting game tree are strictly dominated. Note that the recursive pruning procedure does not start at the final information sets. Our procedure allows players to consider the game tree as a whole and start eliminating branches anywhere in the tree by applying weak admissibility. To illustrate the procedure, look at the game of figure 1. *R* is eliminated at 2>s information set in the first iteration, and then c is eliminated for player 1 because, after *R* is eliminated, either a or 6 yield player 1 a payoff of 1 for sure, while *c* yields 0. The pruning procedure is formally defined as follows. Fbr a given game tree T, let *Weak-Adi(T)* = {* € &(T) : * is sequentially weakly admissible in T}, and let *Weak - Ad(T)* = x_{ieN}*Weak - Adi(T)* .

DEFINITION 3 *Common Reasoning about Sequential Weak Admissibility*

- *Let T be a finite game tree for N =* 1,2,... ,n *players.*
- *Hie strategies in T* consistent with common reasoning about sequential weak admissibility *are denoted by CRWA(T), and are defined as follows:*

  1. *WA°(T) = S(T).*
  t. JPA^{i+1}(T) = *Weak - Ad(T\WA'(T)).*
  3. *se CRWA(T)* <=* Vj: *s\[T\WA*(T)]* 6 *WA*+\T).*

If T is a finite game tree, the set of strategies for player *i*$_t$*£k(T)* is finite, and our procedure will go through only finitely many iterations. lb be precise, let max = *J^ien* 1*1" *» ^{thcntac} *P^ocodme* will terminate after max iterations, Le, for all;_> max, WA^T) = *WA>+\T).*

We introduce the concept of Nash equilibrium and one of its refinements, subgame perfection, for generic finite games in extensive form. A strategy *s%* in a game tree *T* is a *best reply* to a strategy profile *s-i* of f*s opponents if there is no strategy *J for player *i* such that «<(«£, *_<) > «<(««, «-i). A strategy profile a is a *Nash equilibrium* if each strategy *s[t* in *s is* a best reply against *s[-i*. A strategy profile *s* is a *subgame perfect equilibrium* if for each subgame T_m of T, (*|T«) is a Nash equilibrium of T_K. We say that a strategy *• in *T* is *consistent with subgame perfection* if there is a subgame perfect strategy profile *s* of which «« is a component strategy, Le. «; = *s[%*. We denote the set of player t's strategies in T that are consistent with subgame perfection by *SPEi(T),*

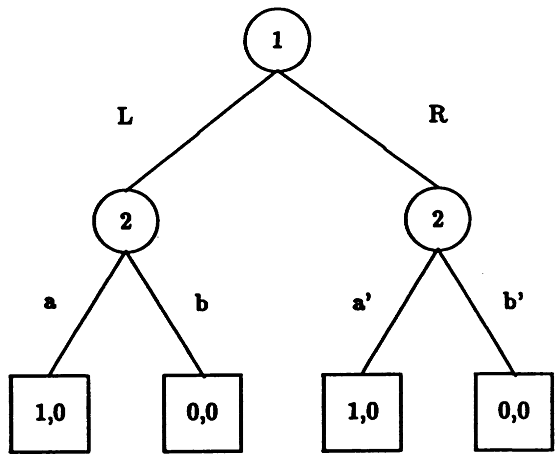

Figure 2: A game of perfect information.

4 Sequential Weak Dominance and Forward Induction

4.1 Weak Dominance

Informally, a strategy *s%* is weakly dominated by another strategy *s* at an information I* in a game tree T if aj never yields less to t at *1%* than 5* does, and sometimes yields more. For example, in the game of Figure 3, a b weakly 4mninitH at 2's information set. And in the game of Figure 4, △rvwing 6 |s weakly dominated for 2 because a yields player 2 the payoff 2 for sure, while *b* may yield only 0 if player 1 plays Jfe. As in the case of weak admissibility, we call a strategy * sequentially admissible just in case *Si* is admissible at each information set belonging to player *i*.

DEFINITION 4 *Weak Dominance and AdmitsibiUiy in Extensive Form Games*

- *LetT be a finite game tree for* $N = 1, 2, \ldots, n$ *players.*
- *A strategy n is* weakly dominated *by another strategy s\ at an information set U belonging to i in T just in case*
  1. *for all strategy profiles s-i in* T, *and for ally in /**, tti(*i,«-», y) $\leq$ $u_i(s'_i, s_{-i}, y)$, *and*
  2. *for some strategy profile s-i and some node y in* I*, $u_i(s_i, s_i, y) <$ $u_i(s'_i, s_{-i}, y)$.
- *A strategy s% is* admissible at an information *set U in T just in ease Si is not weakly dominated at /».*
- *A strategy Si is* sequentially admissible *in T if and only if Si is admissible at each information set Ii inT that belongs to* *i*.

We define a procedure to capture common reasoning about sequential admissibility analogous to common reasoning about sequential weak admissibility. To illustrate the procedure, consider figure 4. Common reasoning about admissibility rules out *b* as a choice for player 2 because b is weakly dominated. Then given that only *a* remain at 2[9]s decision node, *Ri* (strictly) dominates *In* for player 1. So the only play consistent with common reasoning about sequential admissibility is for player 1 to play *Ri* and end the game. Note however that common reasoning about sequential weak admissibility, Le. the standard backwards induction procedure, is consistent with both *Ri* and the play sequence Iri,b,L3. So even in games of perfect information, common reasoning about sequential admissibility may lead to stronger results than common reasoning about sequential weak admissibility.

For a given game tree T, let *Seq - Adi(T)* = {* € *S(T)* : *Si* is sequentially admissible in T}, and let *Seq - Ad(T)* - x_{ieN}*Seq - Aa\(T)*.

DEFINITION 5 *Common Reasoning about Sequential Admissibility*

- *Let T be a finite game tree with players* $N = 1, 2, \ldots n$.

- *The strategies in T* **consistent with common reasoning about sequential admissibility** *are denoted by CRs*q(T), and are defined as follows:*

  1. *Seq*(T) = S(T).*

  3. *s eCRs^CT) <=•* Vi : *sWTlSe^iT)) €* $\boldsymbol{Seq^{j+1}(T)}$.

We have seen that common reasoning about sequential admissibility can lead to stronger results than common reasoning about sequential weak admissibility; we next show that the former never leads to weaker results than the latter. The key is to observe that if a strategy * is strictly dominated in a game tree T, «« will be strictly dominated in a restriction of T. The next lemma asserts the contrapositive of this observation: If a strategy * is admissible in a restriction of T, ***Si*** **is not strictly dominated in T.**

LEMMA 2 ***IfT is a restriction ofF and «« is sequentially admissible in T, then there is an extension s'{ of a to T such that*** **J£** ***is sequentially weakly admissible in T*.***

This means that our procedure ***Seq*** **yields, at each stage** ***j,*** **a result that is at least as strong as that of common reasoning about weak admissibility, the procedure** ***WA.*** **Hence we have the following proposition.**

PROPOSITION 3 ***Let T be a finite game tree. If a play sequence is consistent with common reasoning about sequential admissibility in T, then that play sequence is consistent with common reasoning about sequential weak admissibility. That is, {play{s) : s € CRs^T)} C {ptay(s) : s e CRWA(T)}.***

4.2 Forward Induction

It is commonly held that iterated weak dominance (Le., iterated sequential admissibility) captures some of the features of backward and forward induction. Fudenberg and Tiroie (1993, p.461) thus state that: "Iterated weak dominance incorporates backward induction in games of perfect information: The suboptimal choices at the last information sets are weakly dominated; once these are removed, all subgame-tmperfect choices at the next-to4ast information sets are removed at the next round of iteration; and so on. Iterated weak dominance also captures part of the forward induction notions implicit in stability, as a stable component contains a stable component of the game obtained by deleting a weakly dominated strategy".

Indeed, we have previously shown that, in finite game of perfect information, common reasoning about weak aHmingihiiHy yields exactly the backward induction solution. In this section we show how, in finite games of imperfect information, common reasoning about admissibility yields typical forward induction solutions. Thus backward, and forward induction seem to follow from one principle, namely that players' choices should

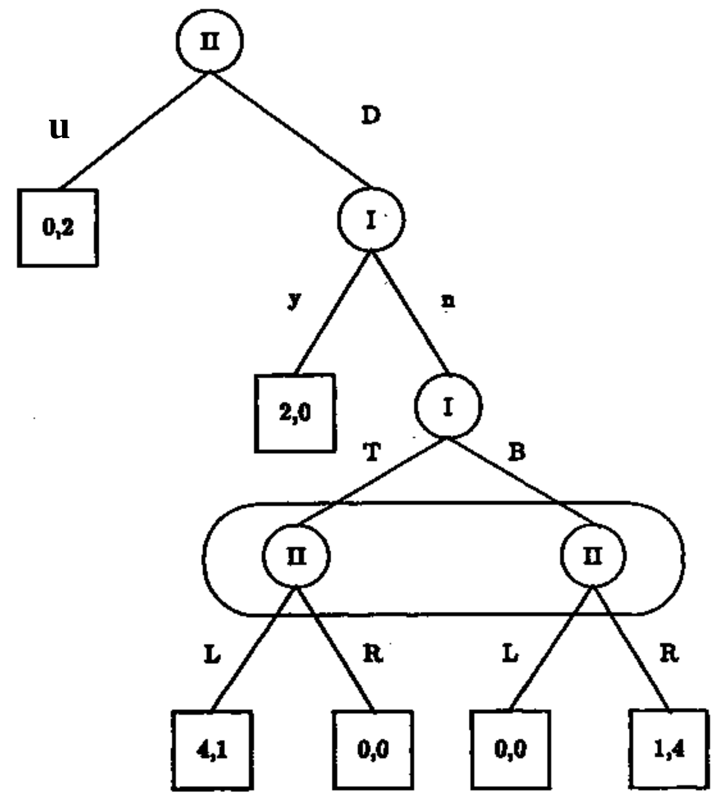

Figure 5: Backward vs. Fbrward Induction Prindples

DEFINITION 6 *Let T be a game tree with information set /»: Let T\I% denote the restriction ofT to nodes in U and successors of nodes in* J».

- *A strategy s% is* **consistent with forward induction** *at U if s% is sequentially admissible at* /«.
- *A move matan information set U is a* **forward induction signal** *for SI at a lower information set* IJ *(written* < /»: m, ij : 5_t* >*), where* $s_i \in S_i^* \iff$

  1. $s_i(I_i) = m$;
  t. *Ii is reachable from 1% with* *+;
  3. **i is consistent with forward induction at* /«.
- *A forward induction signal* < *1%* : m,/J : S? > *is* **credible** *if some strategy s% in S1 is consistent with common reasoning about sequential admusibiUty in* T, *Le. Si* € *CRs**(T)<.

Let us illustrate these concepts in the game of figure 5. According to our definitions, the only strategy that chooses n at Fs first information set and is consistent with forward induction is nT. So < i£ : *n^lf* : *{nT}* > *is* a forward induction signal, where /} denotes Fs first information set and *If* denotes Fs second information set. However, < / | : n , i | : *{nT}* > is not a credible signal. For *nT* is inconsistent with common reasoning about sequential admissibility, since such reasoning rules out *L* at IFs second information set. Similarly for player II, < Jjj : *DJ*}± : *{DR}* > is a forward induction signal. But it is not a credible signal, since *DR* is inconsistent with common reasoning about sequential admissibility. Hence neither forward induction signal is credible, as "sending" either signal is inconsistent with common reasoning about sequential admissibility as denned by *CRseq-*

In terms of reasoning about admissibility, the difference between Kohlberg's and our analysis is this. Kohlberg applies admissibility once to argue that *D* b a forward induction signal for *R* and n is a forward induction signal for T. But if we assume that admissibility is common knowledge among the players, then neither *D* nor n are credible signals. Indeed, common knowledge b not even needed to get to this conclusion: it b sufficient to apply admissibility twice to get the same result.

5 Common Reasoning about Admissibility in the Extensive and Strategic Forms

A game *G* in *strategic form* is a triple (AT, <S»€*9 **»€*)» where *N is* the number of players and, for each player t € *N, Si is* the set of pure strategies available to t, and u» is player t*s utility function. Given a strategy profile *s* = (*i,..., 0»),ffti(*) denotes the payoff to player *i* when players follow the strategies («!,...,«»). Consider the set of strategy profiles *S* = *Si* x S2 x ... x S«, and two strategies *,«< 6 *Si* of player t. Player t's strategy *Si* is *weakly dominated* by her strategy «< given *S* just in case:

1. $PSeq^0(T) = S(T)$.
2. $PSeq^{j+1}(T) = Seq - PA(T|PSeq^j(T))$.
3. $s \in CR_{PSeq}(T) \iff \forall j : s|[T|PSeq^j(T)] \in PSeq^{j+1}(T)$.

The two notions of sequential admissibility are equivalent in terms of their predictions about how the game will be played. That is, exactly the same play sequences are consistent with both restrictions.

LEMMA 4 *Let T be a finite game tree. Then the play sequences consistent with sequential admissibility are exactly those consistent with sequential proper admissibility. That is,* $\{play(s) : s$ *is sequentially admissible in* $T\} = \{play(s) : s$ *is sequentially properly admissible in* $T\}$.

From this fact it follows immediately that common reasoning about sequential admissibility yields the same predictions as common reasoning about proper sequential admissibility.

PROPOSITION 5 *Let T be a finite game tree. Then the play sequences consistent with common reasoning about sequential admissibility are exactly those consistent with common reasoning about sequential proper admissibility. That is,* $\{play(s) : s \in CR_{Seq}(T)\} = \{play(s) : s \in CR_{PSeq}(T)\}$.

However, it is not always the case that a strategy that is admissible in the strategic form of a game is properly admissible in an extensive form of the game. For example, in the game of figure 6, the strategy L is properly weakly dominated for player 2 at her information set: at node y, R yields a higher payoff than L, and starting at node x, both choices yield the same. On the other hand, node y cannot be reached when 2 plays L, so that L is admissible in the strategic form of the game, yielding 2's maximal payoff of 1. The game in figure 6 has the strange feature that if 2 plays R at x to arrive at y, she has 'forgotten' this fact and cannot distinguish between x and y. Indeed, this is a game without perfect recall. Perfect recall is defined as follows.

DEFINITION 9 (KUHN) *Let T be a finite game tree. Then T is an extensive game with* perfect recall *if and only if for each information set I_i belonging to player i, and each strategy s_i in T, all nodes in I_i are consistent with s_i if any node in I_i is.*

We note that if T is a game with perfect recall, then all restrictions of T satisfy perfect recall. The next proposition shows that in extensive form games with perfect recall, the notion of proper weak dominance coincides exactly with admissibility in the strategic form.

PROPOSITION 6 *Let T be a finite game tree with perfect recall. Then a strategy s_i for player i is admissible in the strategic form $S(T)$ if and only if s_i is sequentially properly admissible in T.*

Consider a game G in strategic form. We define an *order-free* iterative procedure for eliminating weakly dominated strategies. If S is a set of strategy profiles, let $Admiss_i(S)$ be the set of all strategies s_i for player i that are consistent with S and admissible given S, and let $Admiss(S) = \times_{i \in N} Admiss_i(S)$.

	L	M	R
a	1.3	3,2	1.2
b	2,2	2,0	0,0
c	2,1	1,2	0,0

Figure 7: Order-Free Elimination of Weakly Dominated Strategies

DEFINITION 10 ***Common Reasoning About Admissibility in the Strategic Form***

- **Let the strategic form of a finite game *G* be given by (AT, &€*,<*•€*)> and let S = Si x £2 x ... x S» be the set of strategy profiles in G.**
- **The strategies in *S consistent with common reasoning about admissibility* are denoted by *CR$_A$d{S)*, and are defined as follows.**

1. ***AdP(S)* = 5.**
2. **$Ad^{j+1}(S) = Admiss(Ad^{j}(S))$.**
3. **$CR_{Ad}(S) = \bigcap_{j=0}^{\infty} Ad^{j}(S)$.**

The procedure goes through at most £)$_{i\,e\,N}$ \S% - 1 | iterations; that is, for all i> £ * * \Si - 1\$_9$A*W = Ad?+\S) .

For example, consider the game in figure 7. In the first iteration, player 1 will eliminate c, which is weakly dominated by 6, and player 2 will eliminate A, which is dominated by *L* and *M*. Since admissibility is common knowledge, both players know that the reduced matrix only contains the strategies a, 6 and L, Af. Common reasoning about admissibilhy means that both players will apply admissibility to the new matrix (and know that they both do it), and since now *L* dominates Af, both will know that *M* is being eliminated. Finally, common reasoning about admissibility will leave 6, *L* as the unique outcome of the game.

(=»): Let *s* be a strategy profile consistent with common reasoning about sequential weak admissibility (Le. *s e* $CRWA(T_X)$). Suppose that it is player f's turn at x. For each player j,*[j]|7V k consistent with subgame perfection in each proper subgame T, of T«, by the inductive hypothesis and the fact that *s\j* is consistent with common reasoning about sequential weak admissibility in T«. So the implication *(=>)* is established if we show that *s[i* is consistent with subgame perfection in T_x. Let y be the successor of * that is reached when t plays **[%[* at x. Let max(y) be the w*«™«**i thftr t ^n achieve given common reasoning about sequential weak admissibility when he follows «[t] (Le. max(y) = max{ui(/[i], *_», *x)* : «-»is consistent with $CRWA(T_X)\})$. Fbr each j / that is a successor of x, let mind/) be $\wedge^e$ minimum that t can achieve given common reasoning about sequential weak admiwahflity when he follows *[t] in Ty#. Then we have (•) that max(y) $\geq$ mind/) fbr each successor y' of x. Fbr otherwise player t can ensure himself a higher payoff than *s[t* can possibly yield, by moving to some successor *yf* of x and continuing with «[tj. That is, the strategy jj which moves to y' at x and follows *s[i* below *y'* strictly dominates *s[t* in $T_X\backslash CRWA(T_X)$. But since *T* and hence T* is finite, this contradicts the assumption that *s[l* is consistent with $CRWA(T_X)$. NOW by inductive hypothesis, $CRWA(T^\wedge)$ = *SPE(T^)* for each successor y' of x. So there is a subgame perfect equilibrium *Sm**** in T_y which yields t the payoff max(y) in T_y and in which player *%* follows «[t] (Le. «[t] = «m*x[t]). Again by inductive hypothesis, for each successor node]/ of x there is a subgame perfect equilibrium $s\backslash_{oia}$ in *T^t* which gives player t the payoff min(y') and in which player t follows *s[{* in *Tj*. Now we define a subgame perfect equilibrium *s** in T_x in which player *i* follows *s*[*i*]:

1. $s^*[i](\{x\}) = s[i](\{x\})$,
2. in T_y,0* follows s_{max},
3. in Ty/,4* follows ^J^, where j/ is a successor of x other than y. By our observation (*), there is no profitable 1-deviation from *s** for player i at x, and hence by lemma 0, ** is a subgame perfect equilibrium in T_x.

(<=) Let *s* be consistent with subgame perfection in T*. Let *%* be the player moving at x. Consider any strategy «[/] in *s*, where *j* ^ i. Since y is not moving at x, *s[j]* is consistent with common reasoning about sequential weak admissibility in T« if and only if *s\j\fT₉* is consistent with common reasoning about sequential weak admissibility in each subgame T_9 of T_s. Since *s* is consistent with subgame perfection in T_c, there is a subgame perfect equilibrium *8** in T« in which *j* follows *s\j*. Since *3** is subgame perfect, $s^*\backslash T_t$ is subgame perfect in T,. Hence «[/]|T_y = **b1|T_y is consistent with subgame perfection in T_y. By inductive hypothesis, this entails that «[j]|T_y **&* consistent with common reasoning about sequential weak admissibility in T_9. Since this is true for any subgame T_9 of T«, *s\j* is consistent with common reasoning about sequential weak admissibility in T_x. Next, consider *[t], the strategy followed by the player who

is a restriction of $T\backslash WA^{j+l}(T)$. This completes the proof by induction. D

LEMMA 4 *Let T be a finite game tree. Then the play sequences consistent with sequential admissibility are exactly those consistent with sequential proper admissibility. That is, {play(s) : s is sequentially admissible in T} = {play(s) : s is sequentially properly admissible in T}.*

Proof. (D) Let *s* be a sequentially properly admissible strategy profile in T, and let *x* be any node reached in *play(s)* such that *I(x)* belongs to player t. Then *s[i* is admissible at *I(x)* since *I(x)* is consistent with *s[i*. Now we may modify *s* to obtain a strategy profile «*, in which each player t follows «[t] at any information set containing a node in *piay(s)^* and follows an admissible strategy at every other information set. Then 5* is sequentially admissible, and *play(s*) = play(s)*.

(C) This is iinra'vti?** because all sequentially admissible strategies are sequentially properly admissibicD

PROPOSITION 5 *Let T be a finite game tree. Then the play sequences consistent with common reasoning about sequential admissibility are exactly those consistent with common reasoning about sequential proper admissibility. That is, {play(s) : s e CRs^T)} = {play(s) : s* € C7*PS«,(T)}.

Proof. We prove by induction on *j* that for each $j \geq 0, T|Seq^j(T) =$ $T|PSe$ '.

Base Case, $j = 0$. The claim is immediate since $Se4p(T) = PSeq^*(T) =$ sen.

Inductive Case: Assume that $T\backslash Se^\wedge(T)$ = T|P5e^(T), and consider $j+1$. The claim follows immediately from lemma 4.D

PROPOSITION 6 *Let T be a finite game tree with perfect recall. Then a strategy s% for player i is admissible in S(T) if and only if s% is sequentially properly admissible in T.*

Proof. Suppose that a strategy *s%* in *S(T)* for player t is weakly dominated in *S(T)*. Then there is a strategy a$'_i$ consistent with *S(T)* such that

1. for all strategy profiles *-« consistent with S(T),Ut(*i, «-i) ≤ ««(*', s_i), and
2. for some strategy profile «1{ consistent with 5(T), «»(*», *!.<) < u$_i$.

Let *x* be the first node that appears along both the plays of *s%* against *sti* and *Si* against «I{ at which «» deviates from «£, so that *x* 6 *range(play(si,* s$'_{-i}$))∩ *rangeiplayistbS^i))* and «i(/»(x)) / ^(/i(x)). Then *x* is consistent with *Si* and *s'i* in T. Let y be any node at *U(x)* consistent with *Si* and *s'^* and let 5_» be any strategy profile of t's opponents. Then u»(«i,«-», y) ≤ *Ui(s'i,* 5-t,y); for otherwise, by perfect recall, let «!.{ be a strategy profile of t's opponents such that both pfay(«i,«!_») and piay(«i, ^li) reach y, and such that *!Li|T$_y$ = *_»|T$_t$. Then t4»(«t,«!.») > t*»(«i, *-i)> contrary to that s_i is in $PSeq^j(T)$, and the second condition may be restated to say that s_i is admissible in $S(T|Ad^j(S(T)))$. By proposition 6, the second condition then implies that s_i is sequentially properly admissible in $T|Ad^j(S(T)) = T|PSeq^j(T)$. Hence s_i is in $PSeq_i^{j+1}(T)$. This shows that $PSeq^{j+1}(T) = Ad^{j+1}(S(T))$, and completes the proof by induction.□

PROPOSITION 8 *For all finite games G with pure strategy profiles* $S, CR_{Ad}(S) \neq \emptyset$.

Proof. The admissible elements in S_i^j survive at each iteration j, for each player i, and there always is a admissible element in each S_i^j since each S_i^j is finite. Hence $S^j \neq \emptyset$ for any j, and so $S^{\sum_{i \in N} |S_i - 1|} = CR_{Ad}(S) \neq \emptyset$.□

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