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A Variational Finite Element Method for Stationary Nonlinear Fluid-Solid Interaction

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Abstract. We consider the problem of the interaction of a stationary viscous fluid with an elastic solid that undergoes large displacement. The fluid is modeled by the stationary incompressible Navier-Stokes equations in an Eulerian frame of reference, while a Lagrangian reference frame and large displacement-small strain theory is used for the solid. A variational formulation of the problem is developed that insures satisfaction of continuity of interface tractions and velocities. The variational formulation is approximated by a Galerkin finite element method, yielding a system of nonlinear algebraic equations in unknown fluid velocities and pressures and solid displacements. A Newton-like method is introduced for solution of the discrete system. The method employs a modified Jacobian that enables decomposition into separate fluid and solid subdomains. This domain decomposition avoids possible ill-conditioning of the Jacobian, as well as the need to compute and store geometric coupling terms between fluid and interface shape. The capability of the methodology is illustrated by solution of a problem of the flow-induced large displacement of an elastic infinite cylinder.

Keywords. Nonlinear fluid-solid interaction, nonlinear aeroelasticity, Navier-Stokes equations, finite elasticity, variational methods, Galerkin method, finite elements.

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1. Introduction

Despite its importance, the problem of modeling the nonlinear interaction of a viscous fluid with a solid undergoing large deformation has remained a challenging problem in mechanics. Its resolution is of significant practical importance to such disciplines as aerospace, marine, automotive, and wind engineering. Such problems may arise, for instance, in the large-amplitude vibration of flexible aerodynamic components such as high aspect ratio wings and blades, in wind-induced deformation of towers, antennas, and lightweight bridges, in hydrodynamic flows around offshore structures, and in the interaction of biofluids with elastic vessels. The greatest difficulty here lies when two-way coupling occurs between fluid and solid: viscous flow produces tractions that deform the solid, while deformation of the solid influences the flow field and thus fluid tractions. Solid deformation influences the flow both by altering the fluid domain as well as by creating solid tractions that must be in equilibrium with the fluid tractions.

Because of its critical importance in aerospace applications, the problem of such nonlinear fluid-structure interaction has received considerable attention within the aerospace literature, where it is known as aeroelasticity. Classical approaches based on linear theory are well established [4], [8]. Certain nonlinear aeroelasticity phenomena have been amenable to analytical and semi-analytical study, and significant understanding of the physics of these problems *has* been elucidated in recent years [9]. Recently, interest has increased in *computational aeroelasticity*, i.e. in developing methods for direct numerical approximation of the governing nonlinear partial differential equations of the fluid-solid system [15], [16], [17], [3], [18], [10]. This interest has been motivated by advances in computational fluid dynamics and computational structural mechanics, and in the rapid growth in computational power.

The methods that have emerged within the past several years in the computational aeroelasticity literature employ different numerical approximations in fluid and solid domains, typically finite difference or volume methods for the fluid and finite elements for the solid. Fluid and solid are thus coupled *after* discretization. Since coupling is achieved after numerical approximation, and approximations may not be consistent across the interface, continuity of interface tractions cannot be rigorously assured. In some approaches, fluid and solid are solved separately using existing numerical codes; thus coupling consists of a mechanism to transmit interface tractions between the two codes. Examples include [15], [16], [17], and [18]. Other approaches solve the coupled discrete equations simultaneously as a single set of nonlinear algebraic equations (in the context of steady problems) [10]. These methods have been criticized for resulting in possibly ill-conditioned Jacobian matrices of the coupled system, due to the disparity in solid and fluid behavior [18]. However, one ought to be able to apply various numerical linear algebraic devices to overcome this problem, as we shall do here. Nonlinear fluid-solid interaction problems have also been approximated by purely finite element methods, e.g. [19], [2], [1]. Coupling is again effected at the discrete level.

In this article, we address the stationary fluid-solid interaction problem. We target this problem because of its importance in optimal design, in which a design is optimized under steady-state conditions. In steady problems, if the solid deformation is so small that it does not change the fluid domain, the coupling is only one-way. In other words, we may solve

the fluid equations, assuming a rigid solid, to obtain fluid tractions on the interface, and then apply them to the solid to deform it. On the other hand, if the solid undergoes large deformation, coupling is two-way, even if the flow is steady; thus, we are not able to solve the fluid problem independently of the solid. This latter case is the subject of this article.

We are not aware of any variationally-coupled numerical solutions of viscous fluid-finite elasticity interaction problems, a fact that motivates the present work. We develop a variational formulation that couples fluid and solid at the continuous level, and thus assures continuity of interface tractions. We model the fluid by the stationary incompressible Navier-Stokes equations in an Eulerian frame of reference, while a Lagrangian reference frame and large displacement-small strain theory are used for the solid. Once coupled, we can systematically apply a numerical approximation—a Galerkin finite element method—to obtain a single set of nonlinear algebraic equations. We may then seek appropriate numerical methods for its solution, which may include linear algebraic devices such as domain decomposition to avoid ill-conditioning. The two main advantages of this method are that the variational formulation automatically insures continuity of interface tractions, and it can be systematically translated into a unified finite element method for the coupled problem.

The rest of this paper is organized as follows. In §2, we develop the variational form of the viscous flow-finite elasticity interaction problem. A finite element approximation is constructed in §3, while §4 introduces a modified Newton method for solution of the resulting discrete system. The method is illustrated in §5 through the solution of a problem of flow-induced deformation of an infinite, elastic cylinder. We conclude with some remarks in §6.

2. Variational formulation

In this section we develop a variational formulation of the fluid-solid interaction problem in the context of a stationary, viscous, incompressible, Newtonian fluid, described by the Navier-Stokes equations, interacting with an isotropic piecewise-homogeneous elastic solid in a Lagrangian frame of reference. We assume that the solid is capable of large displacements, but that strains are small—a reasonable assumption for problems arising in aerospace, civil, and mechanical engineering.

The finite nature of solid displacements implies a geometric dependence of the flow field on the solid displacement. Consider a solid of finite extent surrounded by an infinite fluid, as depicted in Figure 1. Define *Clp* as the fluid domain, *Qs* as the *undeformed* solid domain, *T]r* as a boundary approximating the fluid far-field on which tractions are prescribed, *Fj?* as the portion of the far-field fluid boundary on which velocity is prescribed, T^{x}_{s} as the undeformed solid boundary on which tractions are prescribed, F| as the undeformed solid boundary on which displacements are prescribed, $F/_{o}$ as the undeformed interface between solid and fluid, and F/ as the deformed interface between solid and fluid. The fluid field quantities are the pressure p, the velocity vector v, the stress tensor <7/r, and the rate of strain tensor d. In the solid, the field quantities are the displacement vector u, the Piola-Kirchhoff stress tensor S, and the Green strain tensor E. We shall have occasion to refer to the solid Eulerian stress

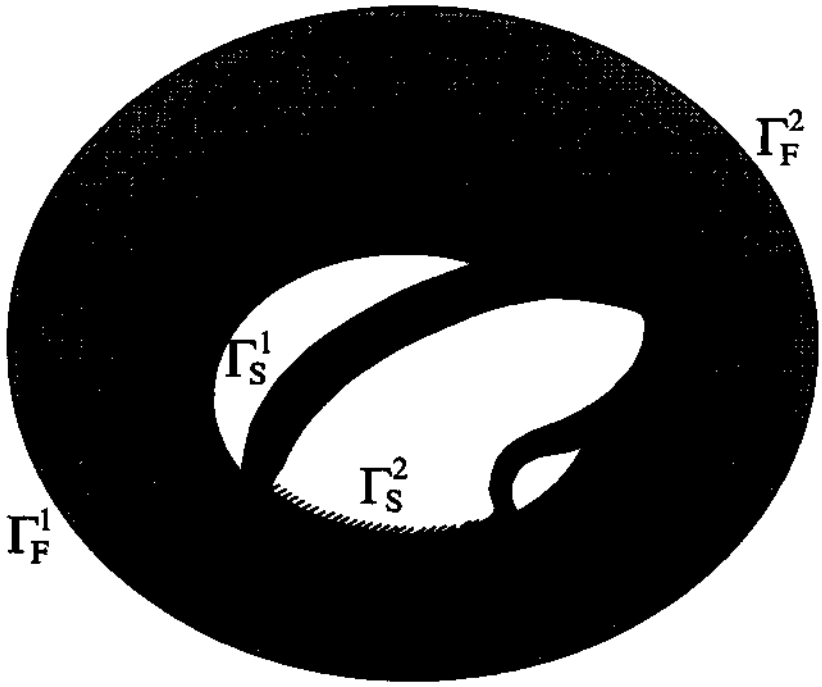

Figure 1: Problem of Fluid-Solid Interaction

tensor, which we denote <*Ts*- Material constants are the fluid viscosity \$x_F$ and density /), and the Lamé moduli of the solid, A and /is. We take ˆv as the prescribed far-field fluid velocity, ˆtp as the prescribed fluid traction, ûˆ as the prescribed solid displacement, and *ît$* as the prescribed solid traction. The fluid and solid body forces are denoted i_F and *is*- Define $T_F = V^l_F$ U F|r and $Ts = T^l_s$ U r|. We also define n as the unit outward normal to a deformed surface, no as the unit outward normal to an undeformed surface, $n/_F$ as the unit normal to the deformed fluid-solid interface, directed away from the fluid, and $n/_s$ as the unit normal to the deformed fluid-solid interface, directed away from the solid.

The conservation of momentum, conservation of mass, constitutive law, and strain rate-velocity equations of the fluid are:

$$p_F(v \cdot V)v - \&\backslash vcr_F = ¥F \qquad \text{in } fl_F \tag{1}$$

$$V \cdot v = 0 \qquad \text{in } t_F \tag{2}$$

$$<r_F = -pi + 2fi_F d \qquad \text{in } Q,_F \tag{3}$$

$$d = \frac{1}{2}(Vv + Vv^T) \qquad \text{in } n_F \tag{4}$$

The constitutive law, equilibrium equations, and strain-displacement relations of the solid are given by:

$$S = A\,tr(E)I + 2/i_5 E \qquad \text{in } Sl_s \tag{5}$$

$$div\,[(I + Vu)\,S] = f_s \qquad \text{in } il_s \tag{6}$$

$$E = \frac{1}{2}[Vu + Vu^T + VuVu^T] \qquad \text{in } Sl_s \tag{7}$$

At the interface, coupling between fluid and solid requires that tractions and velocity be continuous:

$$cr_s ni_s + <r_F ni_F = 0 \qquad \text{on } T/ \tag{8}$$

$$\mathbf{v} = li = 0 \qquad \text{on } T/ \tag{9}$$

Here, $\dot{u}$ is the solid velocity, which is zero according to the steady nature of the problem. Thus we have a no-slip condition on fluid velocity at the interface. Finally, the boundary conditions take the form

$$(T_F n_F = \hat{t}_F \qquad \text{on } T_p^l \tag{10}$$

$$[(I + Vu)\, S]\, no = \hat{t}_5 \qquad on T^l_s \tag{11}$$

$$\mathbf{v} = \hat{\mathbf{v}} \qquad \text{on } T^\wedge \tag{12}$$

$$u = \hat{u} \qquad \text{on } T| \tag{13}$$

Here we have used the symbols **div** and V to denote the spatial divergence and spatial gradient, respectively. See for example [14] for derivations of the governing equations of fluid and solid. Notice that the consequence of the small strain assumption is to allow the use of Hooke's law for the solid constitutive relation (5).

We now proceed to establish the variational form of the problem. Let us assume, for simplicity of presentation, that the fluid and solid do not experience body forces, and that the fluid and solid prescribed tractions are zero, i.e. f/r, fs, $\hat{t}p$, and $\hat{t}s$ are all zero. First, we substitute the strain rate-velocity relationship (4) into the fluid constitutive law (3), which is in turn substituted into the conservation of momentum equation (1). Then, multiplying the residual of the resulting equation by the test function w, integrating over the fluid domain, and applying Green's formula, we obtain the weak form of the conservation of momentum equation:

$$a(v, w) + 6(p, w) + c(v, v, w) = \int_{JTj} w \bullet <r_F n\, dTi(u) + \int_{JV_F} w \bullet <r_F n\, dT_F(u) \tag{14}$$

where

$$a(v, w) = \int_{Jsip} \frac{}{2} (Vv + Vv^{-}) : (Vw + Vw^{-})\, dQ,_F(u) \tag{15}$$

$$b(p, w) = - \int_{J\Omega_F} pV \bullet w\, dft_F(u) \tag{16}$$

$$c(v, v, w) = \int_{J\Omega_F} jow \bullet (v \bullet V)\, \mathbf{v}\, dQ,_F(u) \tag{17}$$

and where the symbol : denotes the scalar product of two tensors. Since we wish to consider problems in which the displacement of solid may be large enough to influence the flow, we indicate the dependence of the fluid domain on the solid displacement in the definition of the domains of integration of the the bilinear functional a(*, •) and &(•, •) and the trilinear functional c(-, •, •). The second term on the right side of (14) can be rewritten as:

$$\int_{Jr_F} w\, cr_F n\, dV_F(u) = \int_{Jr^*} w \bullet <r_F n\, dVUxi) + \int_{Jr^2_F} w\text{-}a_F n dT^2_F(u) \tag{18}$$

We shall require that the test function w satisfy the homogeneous essential boundary condition v = 0 on F|r, implying that that the second term on the right of (18) is zero. Furthermore, since i_F is zero, and in light of (10), the first term on the right side of (18) vanishes. Therefore,

$$\int_{\Gamma_F} \mathbf{w} \cdot \langle r_F n \, dT_F(u) = 0 \tag{19}$$

Next, we write the conservation of mass equation in weak form by multiplying (2) by the test function q and integrating over the domain of the fluid:

$$\int_{\Omega_F} q \nabla \cdot \mathbf{v} \, d\Omega_F(\mathbf{u}) = -b(q, \mathbf{v}) = 0 \tag{20}$$

Again, note the dependence of the fluid domain, and thus weak form, on the solid displacement.

The weak form of the solid equilibrium equation is established by first substituting the expression for the Green's strain tensor (7) into the constitutive law (5), and then substituting the resulting expression for the Piola-Kirchhoff stress into the equilibrium equation- Multiplying the residual of the resulting equation by the test function r, integrating over the domain, and applying Green's formula, we obtain the weak form of the displacement form of the equilibrium equations:

$$\int \nabla \mathbf{r} : [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, dQ_s =$$

$$\int \mathbf{r} \cdot [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, \mathbf{n}_0 \, dT_{Io} + \int \mathbf{r} \cdot [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, \mathbf{n}_0 \, dT_s \tag{21}$$

where the relationship between Piola-Kirchhoff stress and displacement is given by:

$$\mathbf{S}(\mathbf{u}) = \frac{1}{2} \lambda \, \mathrm{tr} \, [\nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u} \nabla \mathbf{u}^T] \, \mathbf{I} + \mu_s \, [\nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u} \nabla \mathbf{u}^T] \tag{22}$$

Notice that, since we are in a Lagrangian frame of reference, the unit normal is with respect to the *undeformed* geometry, and the interface is between fluid and undeformed solid, denoted Γ_o. The solid boundary T_s consists of the portion on which displacements are specified, Γ_1, and the portion on which tractions are specified, Γ_2. Thus, the second term on the right of (21) can be rewritten as:

$$\int \mathbf{r} \cdot [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, \mathbf{n}_0 \, dr_s =$$

$$\int_{\Gamma_s^1} \mathbf{r} \cdot [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, \mathbf{n}_0 \, dT_s^1 + \int_{\Gamma_s^2} \mathbf{r} \cdot [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, \mathbf{n}_0 \, dT_s^2 \tag{23}$$

Since i_s is zero, and in light of (11), the first term on the right of this equation is zero. Furthermore, we shall require that the test function r satisfy the homogeneous essential boundary condition u = 0 on Γ_1. Therefore, the second term vanishes over Γ_1. Thus,

$$\int_{\Gamma_s^2} \mathbf{r} \cdot [(\mathbf{I} + \nabla \mathbf{u}) \, \mathbf{S}(\mathbf{u})] \, \mathbf{n}_0 \, dT_s = 0 \tag{24}$$

The first term on the right of (21) can be transformed to the deformed geometry by noting that (I + Vu)S = T, the Lagrangian stress tensor. The surface traction expressed in terms of the Lagrangian stress at a point s' on the undeformed interface is identical to the surface traction expressed in terms of the Eulerian stress at its image s" on the deformed interface. Thus, we may write:

$$T(s')n_o(s') = <r_5(s'')n(s'') \tag{25}$$

Therefore, provided

$$r(s') = w(s'') \tag{26}$$

i.e. the restriction of the test functions r to the undeformed interface is equal to the restriction of w to the deformed interface, the first term on the right side of (21) can be rewritten as

$$\boldsymbol{J}_{-I_0} \mathbf{r} \bullet [(\mathbf{I} + \mathbf{Vu})\, \mathbf{S}(\mathbf{u})]\, \mathbf{n}_0\, dT_{Io} = \boldsymbol{j}_{-I} \mathbf{w} \bullet <\mathbf{r}_s \mathbf{n}\, dT_t \tag{27}$$

To simplify the left side of (21), we separate S into S^L, a tensor that depends linearly on displacement, and S", one whose dependence is nonlinear, in fact quadratic:

$$\mathbf{S}^L = \lambda\, \mathrm{tr}\,(\nabla \mathbf{u})\, \mathbf{I} + \mu_S \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \tag{28}$$

$$S^* = \wedge A\ \mathrm{tr}\ (VuVu^T)\ I + (i_s\ (VuVu^T) \tag{29}$$

so that (22) can be rewritten as $S = S^L + S\wedge$. Thus, the domain integral on the left side of (21) can be rewritten as the sum of terms that depend linearly, quadratically, and cubically on the derivatives of u:

$$/\ \ Vr : [(I + Vu)\ S(u)]\ dSl_s = <^*(u, r) + e(u, u, r) + /(u, u, u, r) \tag{30}$$

where

$$d(\mathbf{u}, \mathbf{r}) = \int_{\Omega_S} \nabla \mathbf{r} : \mathbf{S}^L(\mathbf{u})\, d\Omega_S \tag{31}$$

$$e(u,u,r) = \int_{Jn_s} Vr : [s''(u) + Vu\ S^L(u)]_L\ <Kl_s \tag{32}$$

and

$$/(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{r}) = \boldsymbol{J}_{\Omega_S} \mathbf{Vr} : [\mathbf{Vu}\ \mathbf{S}^N(\mathbf{u})]\ dSi_s \tag{33}$$

The condition of continuity of interface tractions can now be imposed. Adding equations (14) and (21), and making use of (19), (24), and (27), gives:

$$o(v, w) + b(p, w) + c(v, v, w) + rf(u, r) + e(u, u, r) + /(u, u, u, r) =$$

$$/\ \ w\ \{<r_F \bullet n_{lF} + as \bullet n_{Is})\ dTr\{n) \tag{34}$$

The right side of this equation is just zero, in view of the continuity of traction condition (8).

We are in a position now to state the unified variational form of the viscous flow-finite elasticity interaction problem:

Find v 6 HJ (ft), *p e* L^2 (ft), and u € H£ (ft) such that:

a(v, w) + *b(p,* w) + c(v, v, w) + d(u, r) + e(u, u, r) + /(u, u, u, r) = 0

for all w G Hj (ft_F) and **r** G Hj (ft_5) (35)

6(g,v) = 0 for all *q* € L^2 (ft_F)

where the functionals a(-,-), &(•,•)> c(-,-,-), d(-,-)> e(*>*>')> an(^ /("?'?'»') are defined by expressions (15), (16), (17), (31), (32), and (33), respectively. Here, Hj(ft_F) is the Sobolev subspace of all functions having one square integrable derivative over ft_F and that vanish on *Tjr* and outside of ft_F, L^2 (ft_F) is the space of functions that are square integrable over ft_F and that vanish outside of ft_F, and HQ (fts) is the Sobolev subspace of all functions having one square integrable derivative over fts and that vanish on F| and outside of fts. The essential boundary conditions v = v̂ and u = û must be enforced on F^ and F|, respectively, as must the no-slip condition **v** = **0** on F/.

3. Finite element approximation

Let us define the finite element approximations v^, p^, and u^:

Vfc = i>,-(x)v,- (36)

$$p_h = \sum_{j=1}^{n^p} \chi_j(\mathbf{x}) p_j \tag{37}$$

$$\mathbf{u}_h = \sum_{A;=1}^{n^u} \psi_k(\mathbf{x}) u_k \tag{38}$$

where *V{*, Pj, and *Uk* are approximations of velocity, pressure, and displacement at nodes *i. j,* and fc, respectively. The basis function families <*f*>^ *Xji* an(i ^*k* define finite element spaces *Vh, Vht* and *Uh* for velocity, pressure, and displacement, respectively:

V_h= span {<£!,..., </v} (39)

V_h= span{xi,...,Xnp} (40)

U_h = span{^1,...,'0nu} (41)

Let *Vh* C H^(ft_F), V_h C $L^2(n_F)$, and U_h C Hj(fts), i.e. the finite element spaces V_h, V_h, and *Uh* are subspaces of the infinite dimensional spaces in (35). In order to satisfy the condition (26), we require that fluid velocity and solid displacement shape functions be identical, when restricted to the interface between solid and fluid. An example of this is given by combining quadratic triangles in the solid with the Taylor-Hood element in the fluid. The Taylor-Hood element employs a quadratic approximation of velocity in conjunction with a linear approximation of pressure; thus, solid displacement and fluid velocity shape functions are identical on the interface.

Applying the Galerkin method to the problem (35) yields the discrete problem:

Find v_h eVh, Ph ZPh, and u^ G U_h such that

$$a(\mathbf{v}_h,\ vr_h)+b(p_h,\ w^*)+c(v^*,\ v_h,\ w_h)+d(u_h,r_h)+e(u_h,\ u_h,\ r_h)+f(u_h,\ u_h,\ u_h,\ r_h) = 0$$

$$\text{for all } Wc\ e\ V_h \text{ and } r^\wedge\ G\ S_h \qquad (42)$$

and

$$K^\wedge{}^V 0 = 0 \quad \text{for} \quad \text{all } qeV_h$$

The discrete problem (42) is a system of nonlinear algebraic equations. To show the explicit form of these equations, let us first distinguish between nodes lying in the interior and those on the interface. Let

$$\begin{aligned} n^v &= n_F^v + n_I^v \\ n^p &= n_F^p + n_I^p \\ n_s &= n_S^s + n_I^s \end{aligned} \qquad \textbf{(43)}$$

where the subscript F indicates the number of nodal unknowns belonging strictly to the fluid domain, S the number of nodal unknowns belonging strictly to the solid domain, and / the number of unknowns belonging to the interface. So, for example, the n^v nodal velocities are composed of $n^v{}_F$ fluid domain velocities as well as n\ interface nodal velocities. Notice that the satisfaction of condition (26) implies that the number of interface nodal velocities and displacements are equal. Let us call this number n/:

$$m \equiv n^v{}_F = n^u{}_s \qquad (44)$$

Let the fluid nodal velocities be ordered such that velocities of nodes 1,...,nj lie on the interface and the ones of n/ + 1,..., n^v lie in the fluid domain as well as that portion of the fluid boundary on which tractions are prescribed. Similarly, the solid displacement nodes are ordered such that displacements of nodes 1,..., nj lie on the interface and the ones of ni + 1,..., n^u lie in the solid domain as well as that portion of the solid boundary on which tractions are prescribed.

We are able now to elucidate the structure of the discrete problem (42). In the fluid, we have the $n^v{}_F$ discrete conservation of momentum equations

$$\sum_{i=1}^{n^v} a(\phi_i, \phi_\ell)v_i + \sum_{j=1}^{n^p} b(\chi_j, 4>e)Pi + \sum_{z\backslash r=1}^{n^u} E\ ^c(\&> <^*V, <t>t)Wr = 0 \qquad t = n, + 1,\ldots, n^v \qquad (45)$$

and the $n^p{}_F$ discrete conservation of mass equations

$$\overset{n^v}{J2}\ \&(Xm, 4>i>i = 0 \qquad m = n_I^p + 1,\ldots, n^p \qquad (46)$$

In the solid, the n£ discrete equilibrium equations are given by

$$\sum_{k=l}^{n^u} d(\psi_k, \psi_n)u_k + \sum_{k,s=1}^{n^u} e(\psi_k, \psi_s, \psi_n)u_k u_s + \sum_{k,s,t=1}^{n^u} f(\psi_k, \psi_s, \psi_t, \psi_n)u_k u_s u_t = 0$$

$$n = n/ + 1, \ldots, n^{u} \tag{47}$$

Finally, on the interface, we have the nj discrete traction continuity equations:

$$\sum_{i=1}^{n^v} a(\phi_i, \phi_y) v_i + \sum_{j=1}^{nP} b(\chi_h <f>y) Pj + \sum_{t\backslash r=1}^{n^v} {}^{C}(<\wedge> tr, <f>y) ViV_r + \sum_{k=l}^{n^u} d(\psi_k, \psi_y) u_k +$$

$$\sum_{k,s=1}^{n^u} e(\psi_k, \psi_s, \psi_y) u_k u_s + \sum_{k,s,t=1}^{n^u} f(\psi_k, \psi_s, \psi_t, \psi_y) u_k u_s u_t = 0 \qquad y = 1, \ldots, n/ \tag{48}$$

and the $n^P j$ discrete conservation of mass equations

$$\sum_{i=1}^{n^v} b(\chi_z, \phi_i) v_i = 0 \qquad z = 1, \ldots, n^p{}_I \tag{49}$$

Let us define vectors of unknown nodal quantities: let vp G 5f?$^{n\wedge}$ represent the fluid nodal velocities, p^ € 3?$^{n\wedge}$ the fluid pressures, V/ £ 3ft$^{n/}$ the interface velocities, p/ G 3fJ$^{n/}$ the interface pressures, *Us* € 3ft^{n}£ the solid displacements, and U/ G 3£^{nj} the interface displacements. We can rewrite the discrete equations (45)-(49) symbolically as

$$\begin{aligned} \mathbf{h}_F^v(\mathbf{v}_F, \mathbf{p}_F, \mathbf{v}_I, \mathbf{p}_I, \mathbf{u}_I) &= 0 \\ \mathbf{h}_F^p(\mathbf{v}_F, \mathbf{v}_I, \mathbf{u}_I) &= 0 \\ \mathbf{h}_S^u(\mathbf{u}_S, \mathbf{u}_I) &= 0 \\ \mathbf{h}_I(\mathrm{v}_F, \mathrm{p}_F, \mathrm{V}/, \mathrm{p}/, \mathrm{U}_5, \mathrm{U}_7) &= 0 \\ \mathrm{h}^p(\mathrm{v}_F, \mathrm{v}/, \mathrm{u}/) &= 0 \end{aligned} \tag{50}$$

where h£ G 5R$^{n\wedge}$ represents conservation of momentum in the fluid, h^p G 9?$^{n/r}$ conservation of mass in the fluid, h£ G 5R$^{n\wedge}$ equilibrium in the solid, h/ G 3?$^{n/}$ continuity of interface tractions, and h^p G 3ft^{n} conservation of mass on the interface. It appears that we have $n^v + n^p + n^u$ — ni equations in $n^v + n^p + n^u$ unknowns. However, the continuity of interface velocity condition (9) implies that V/ = 0, and we are thus left with an equal number of equations and unknowns upon enforcing this condition in (50).

Note that, in addition to h^, the fluid and interface residuals h£, h^p_F, h^p, and h/ depend on the interface displacements u/. This is implied in the domain of integration of the functional a(-, •) (15), 6(«, •) (16), and c(-, •, •) (17), i.e. in the dependence of the flow on the interface geometry.

4. Solution of the discrete system

We discuss in this section a Newton-like method for solving the system of nonlinear algebraic equations (50). Our discussion will be kept brief; a more extensive discussion of this and other solution methods for finite element approximations of viscous flow-finite elasticity interaction will be presented in the future [12].

Let us first begin by rewriting (50) as

$$
\begin{aligned}
h_F(x_F, x_J) &= 0 \\
h_5(x_5, x/) &= 0 \\
h/(x_F, x_S, x) &= 0
\end{aligned}
\tag{51}
$$

where

$$
h_F = \begin{Bmatrix} h_M \\ h_F^p \\ h_I^p \end{Bmatrix} \qquad x_F = \begin{Bmatrix} f^{VF} \\ p_F \\ I_{P/} \end{Bmatrix} \qquad x_s = u/ \qquad x_7 = u_5 \tag{52}
$$

Note that the fluid equations $hp = 0$ include the equations for conservation of mass on the interface, $h!j = 0$, and the fluid variables x_F include the interface pressures p/. Accordingly, the interface variables consist only of the interface displacements. The reason for this choice of partitioning will become apparent.

Newton's method for the nonlinear system $h(x) = 0$ consists of iterating on solution of the linear system

$$
J(x^*)(x^{*+1} - x^*) = -h(x^*) \tag{53}
$$

until convergence, given an initial iterate x°. Here, J is the Jacobian of h with respect to x. A Newton step for the discrete system (51) takes the form:

$$
\begin{bmatrix} J_{FF}^{A} & J^{r} & J_{FI}^{jk} \\ n & i^{k} & i^{k} \\ j_{P/F}^{A} & J_{IS}^{TA} & J_{II_F}^{TA} + J_{II_S}^{TA} \end{bmatrix} \begin{Bmatrix} \Delta x_F \\ \Delta x_S \\ \Delta x_I \end{Bmatrix} = - \begin{Bmatrix} h_F^k \\ h_S^k \\ h_I^k \end{Bmatrix} \tag{54}
$$

where

$$
Ax = x^{*+1} - x^{fc} \tag{55}
$$

Here, the superscript k indicates evaluation of the residual h and the Jacobian J at the point x^{fc}, and the interface-interface coupling matrix J// includes contributions from both solid and fluid:

$$
J// = J//_F + 3ii_s \tag{56}
$$

The Newton iteration (54) entails two difficulties. First, the Jacobian matrix, because of the disparity between fluid and solid behavior, can be very ill-conditioned. Second, the coupling terms between fluid and interface variables in general render the matrices *3pi* and $J//_F$ dense. The density of these matrices is a consequence of the dependence of the domains of integration of a(-, •), &(•, •), and c(-, •, •) on the interface displacements. In the case of JF/, all fluid nodal velocities and pressures may be coupled to all interface nodal displacements, since a change in any interface displacement potentially moves the fluid mesh everywhere. The matrix $J//_F$ derives its density from the fact that the interface traction continuity equation (48) includes contributions from the first layer of fluid elements, which change with a movement in the interface. Thus, the potential exists for coupling between all interface variables. $J//_s$ contains nonzeroes contributed by the solid terms in (48), i.e. the terms involving d(-, •), e(«, •, •), and /(•, •, •, •). These are just the standard solid stiffness matrix coupling terms, so the coupling is local in nature.

The exact sparsity pattern depends on the moving mesh scheme employed, but in general, the storage requirements and arithmetic complexity associated with *Jpi* and $J//_F$ can be quite severe. Therefore, we consider an *approximate* Newton's method obtained by ignoring the fluid-interface coupling matrix *Jpi* and the contribution of the fluid to the interface-interface coupling matrix, $J//_F$. The resulting Jacobian in (54) becomes block-lower triangular. Thus, the fluid variables can be found by solving the linear system

$$j£_F Ax_F = -h£ \tag{57}$$

for Axjr. The change in the displacements (both interior and interface) can then be found by solving:

$$[\ J^*_{I_s}\ \ 3i_{I_s}\ ]\ \{\ Ax,\ \ )\ -\ -\ \backslash\ h\} + J\$_F Ax_F\ \ J \tag{58}$$

This method avoids the ill-conditioning associated with the coupled problem by employing a "domain decomposition" into separate fluid and solid subdomains. Large storage requirements associated with geometric coupling matrices are avoided by ignoring these terms while constructing the Jacobian. However, since the residual in (54) is calculated correctly, we are guaranteed that, if the method converges, it must converge to the correct solution. This can be seen from (53): the only way that Ax can be zero is for h to be zero, provided only that J is nonsingular, regardless of whether or not it represents the true Jacobian. The price we pay for this approximate Jacobian is that we must give up the Newton guarantee of local quadratic convergence.

We now establish that the modified Jacobian is indeed nonsingular. First, the fluid step (57) can be seen to be just a Newton step for the Navier-Stokes equations, with a rigid boundary given by the current deformed interface, and a no-slip boundary condition imposed on the interface. Thus, the linear system (57) has a unique solution (provided of course that we are away from singular bifurcation or turning points). Second, the solid step (58) can also be regarded as a Newton step for the solid equilibrium equations. The term $J\$_F AXF$ is the incremental "loading" that the linearized fluid induces on the solid interface. So this linear step too must have a unique solution (provided again we are away from buckling points). Thus, the solution of (54) is unique, and the approximate Jacobian is nonsingular.

We stress that this decoupling is a numerical device designed to remedy the twin problems of ill-conditioning and storage and arithmetic complexity. The residual equations h = 0 still contain the correct variational coupling between fluid and solid, and the solution reflects that—only the convergence rate to the solution is affected.

5. Example: flow-induced deformation of an infinite elastic cylinder

We have built a code that implements the finite element approximation of §3 in two-dimensions, and solves the resulting nonlinear algebraic system using the method of Newton form described in §4. We employ a simple continuation strategy to help globalize the solution. Our code discretizes both solid and fluid with triangular elements, and uses quadratic

shape functions for velocity and displacement, and linear shape functions to approximate pressure. The Taylor-Hood element pair is known to satisfy the Ladyzhenskaya-Babuska-

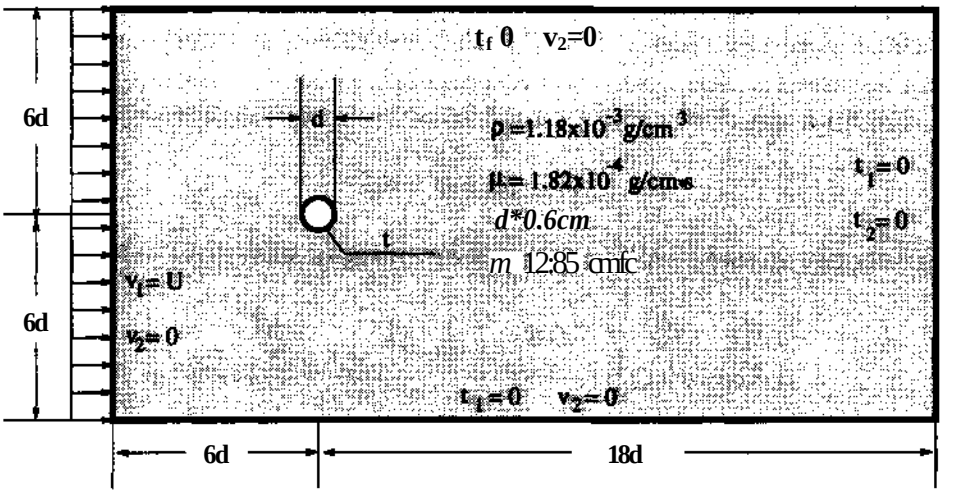

Figure 2: Stationary flow passing an elastic circular cylinder

Brezzi stability condition, e.g. [5], and the complimentary choice of quadratic triangles for the solid insures the satisfaction of the interface compatibility condition (26). The Taylor-Hood element produces L_2 norm errors of order h^3 for velocity and h^2 for pressure [13], while quadratic triangles for elasticity problems produce $L2$ norm errors of order h^3 for displacements [6] (provided in both cases the solution is sufficiently smooth). The linear solves (57) and (58) are performed using the unsymmetric multifrontal sparse LU factorization code UMFPACK[7].

As the solid deformations change from one iteration to the next, the movement of the fluid mesh is computed by the so-called elastic analogy, a common technique in the aeroelasticity literature. The fluid domain is treated as an elastic solid, and the move in the location of the interface is expressed through imposed displacements. Solution of this elastic analogy yields the change in location of fluid nodes. This technique was apparently first used in [1].

In order to illustrate our methodology, we next present results of a physical problem solved by our code. The problem is viscous flow about an infinite elastic cylinder, and is depicted in Figure 2. The problem thus is two-dimensional. The cylinder is of diameter d and thickness i, and is composed of material having elastic modulus E and Poisson ratio v. The fluid is characterized by density p and viscosity /x. The computational domain extends a distance of 6e/ upstream of the cylinder, 18d downstream, and *Yld* above and below its center. Flow is from left to right with a free-stream horizontal velocity of U and no vertical velocity. Boundary conditions downstream of the cylinder are that the flowfield is traction-free. On the top and bottom boundaries are imposed zero vertical flow and zero horizontal traction. Both vertical and horizontal components of displacement are fixed at nodes along the horizontal axis of symmetry at the downstream end of the cylinder. Nodes lying on the upstream end of the horizontal axis of symmetry are prevented from moving vertically but are free to translate horizontally. The inner boundary of the cylinder is traction-free. Two cases are solved: one for which the cylinder is nearly rigid, and one for which the cylinder

is quite flexible. Values of />, /z, d, and U used in the computation are given in the

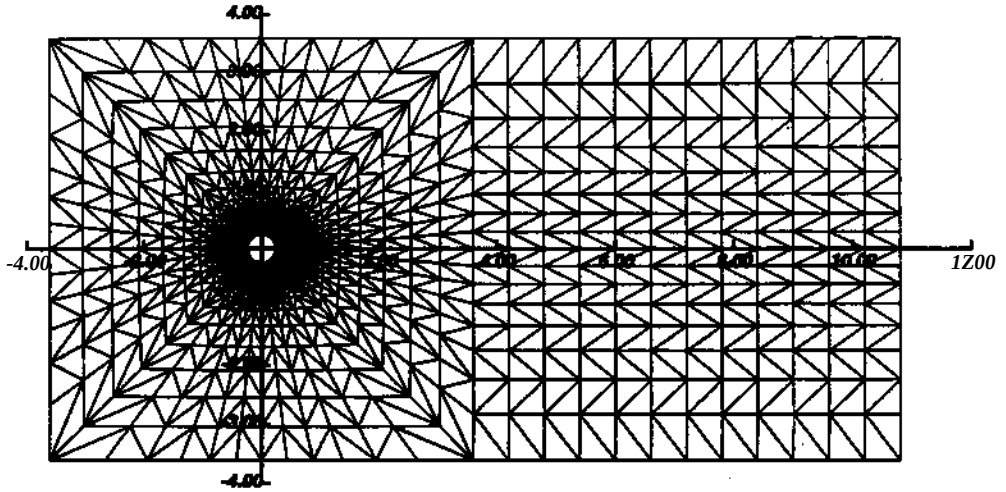

Figure 3: Finite element mesh

Table 1: Mesh data

item	**number**
solid nodes	**416**
fluid nodes	**3552**
interface nodes	**104**
boundary nodes	**152**
total nodes	4224
solid elements	208
fluid elements	1840
total elements	2048
velocity unknowns	7166
pressure unknowns	984
displacement unknowns	824
interface unknowns	205
total unknowns	9179

figure. These correspond to a Reynolds number of about 50, which is kept low to prevent the formation of the Kármán vortex street. A Poisson ratio of 0.3 is taken for the solid. The undeformed mesh is shown in Figure 3. Although it cannot be seen in the figure, two quadratic elements are used through the thickness of the cylinder.

Data describing the mesh are given in Table 1. A convergence criterion of $\| h \| \leq 10^{-6}$ is used to terminate the Newton iteration at each continuation step.

Figure 4 shows streamlines corresponding to the converged flowfield in the vicinity of the cylinder for the case of the nearly rigid cylinder (E'= 10000, t = 0.06).

Figure 5 shows a close-up of the velocity field for this case. The resulting displacement of the cylinder is negligible and does not affect the flow field. Two standing eddies of moderate size, symmetric about the horizontal axis, are observed behind the cylinder, as expected for

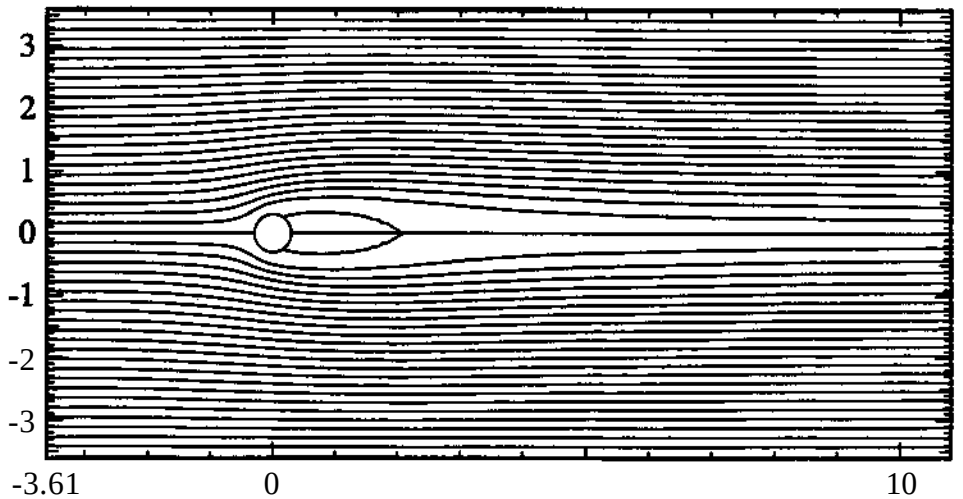

Figure 4: Streamlines about a nearly-rigid cylinder.

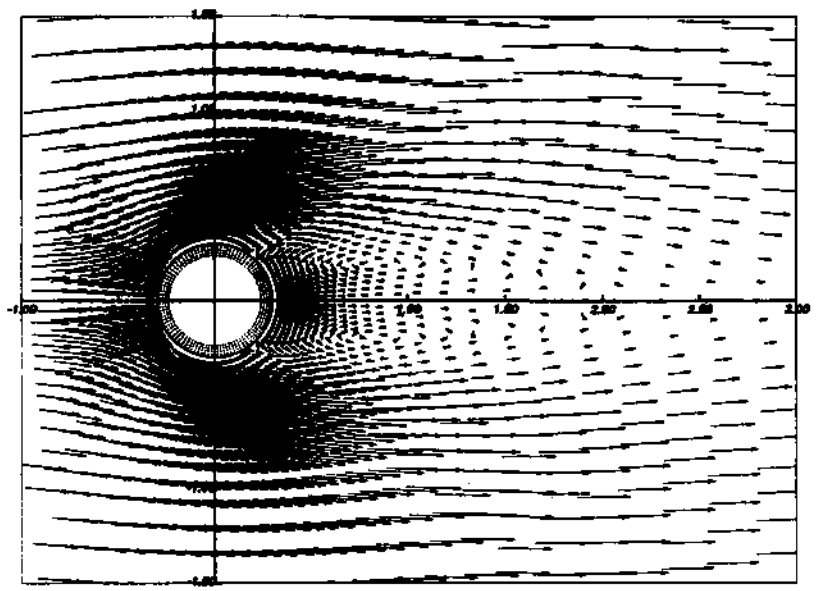

Figure 5: Close-up view of the velocity field corresponding to the nearly-rigid cylinder,

flow around a rigid cylinder in this regime.

Figure 6 shows the resulting displacement and a portion of the flow field when the cylinder is more flexible ($E = 1000, t = 0.02$) and thus undergoes large displacement. A close-up view of the velocity field for this case is depicted in Figure 7.

The undeformed shape of the cylinder is shown in addition to the deformed shape. The flow field depicted corresponds to the converged solution, i.e. to the deformed shape. The standing eddies extend more than twice as far downstream as in the rigid case, due to the

much more bluff shape assumed by the deformed cylinder.

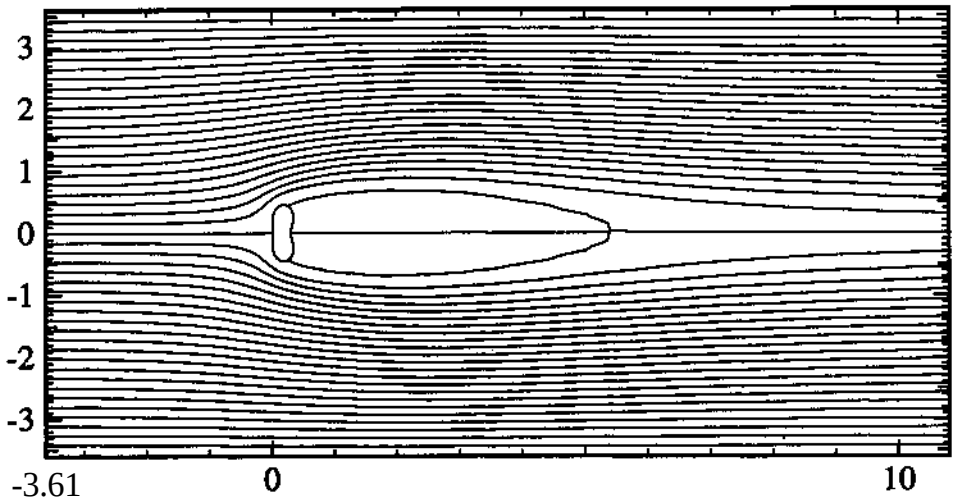

Figure 6: Streamlines about an elastic cylinder.

In the latter case of the elastic cylinder, convergence is obtained in a total of 38 iterations by a simple continuation scheme, first increasing the Reynolds number to the desired value, then decreasing the solid stiffness. The iteration history for the case of the elastic cylinder

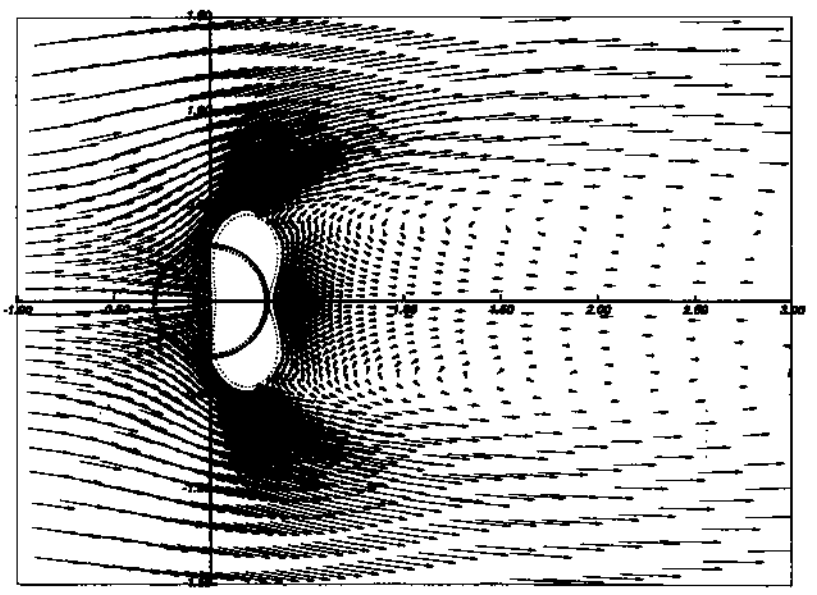

Figure 7: Close-up view of the velocity field corresponding to the elastic cylinder.

is shown in Figure 8. The abscissa represents cumulative Newton (linear) steps. The ordinate represents the value of the residual for the given values of Reynolds number and solid stiffness. As the figure shows, the appropriate nonlinear parameter is advanced when the residual falls below the convergence tolerance, always using the converged field quantities corresponding to the previous parameter to initiate the approximate Newton method. The dotted line indicates increasing Reynolds number, while the solid line represents decreasing

solid stiffness. Although the convergence scheme is conservative, it is quite effective, and we have been able to solve a number of flow-induced large displacement problems using it.

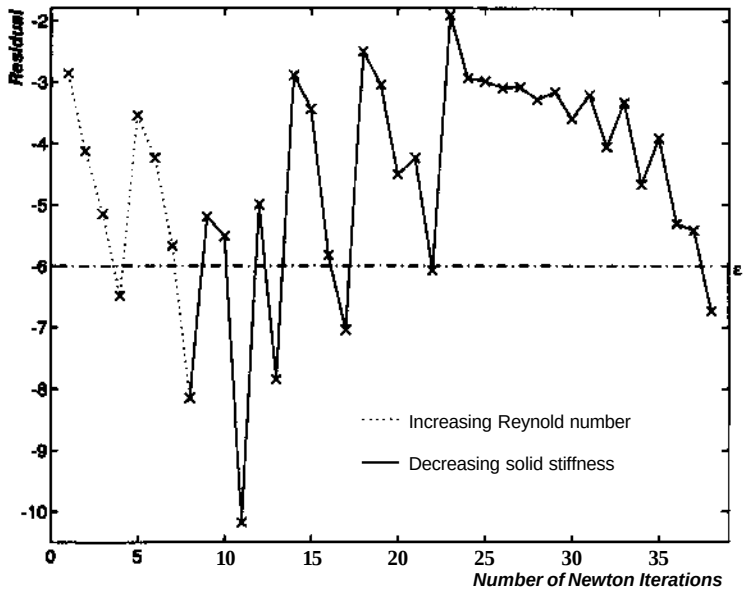

Figure 8: Iteration history showing continuation steps and cumulative Newton iterations.

6. Concluding remarks

We have developed a methodology for numerical approximation of the interaction of a stationary viscous fluid with a elastic solid that undergoes large displacement. The fluid is modeled with respect to an Eulerian frame of reference by the stationary incompressible Navier-Stokes equations, while a Lagrangian reference frame and large displacement-small strain theory is used for the solid. A variational formulation of the problem is developed that insures satisfaction of continuity of interface tractions and velocities. The variational formulation is approximated by a Galerkin finite element method, yielding a system of nonlinear algebraic equations in unknown fluid velocities and pressures and solid displacements. A Newton-like method is introduced for solution of the discrete system. The method employs a modified Jacobian that enables decomposition into separate fluid and solid subdomains. This domain decomposition avoids possible ill-conditioning of the Jacobian, as well as the need to compute and store geometric coupling terms between fluid and interface shape. The method is illustrated by solution of a problem of the flow-induced large deformation of an elastic cylinder.

Work is underway to extend the methodology to unsteady problems. In another thrust, we have developed methods for sensitivity analysis of the models described here [11]. Together, these methods should prove useful for solving problems in multidisciplinary design optimization.

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