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Using D-separation to Calculate  
Zero Partial Correlations  
in Linear Models  
with Correlated Errors

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# Using D-separation to Calculate Zero Partial Correlations in Linear Models with Correlated Errors

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### Abstract

It has been shown in Spirtes(1995) that  $X$  and  $Y$  are d-separated given  $Z$  in a directed graph associated with a recursive or non-recursive linear model without correlated errors if and only if the model entails that  $p^{XZ}_Y = 0$ . This result cannot be directly applied to a linear model with correlated errors, however, because the standard graphical representation of a linear model with correlated errors is not a directed graph. The main result of this paper is to show how to associate a directed graph with a linear model  $L$  with correlated errors, and then use d-separation in the associated directed graph to determine whether  $L$  entails that a particular partial correlation is zero.

In a linear structural equation model (SEM) some partial correlations may be equal to zero for *all* values of the model's free parameters (for which the partial correlation is defined). (When we refer to "all values" of the free parameters, we assume that there are no constraints upon the model's parameters except for the coefficients and the correlations among the error variables that are fixed at zero.) In this case we will say that the SEM **linearly entails** that the partial correlation is zero. It has been shown in Spirtes(1995) that  $X$  and  $Y$  are d-separated given  $Z$  in a directed graph associated with a recursive or non-recursive linear model with uncorrelated errors if and only if the model linearly entails that  $p^{XYZ} = 0$ . This result cannot be directly applied to a linear model with correlated errors, however, because the standard graphical representation of a linear model with correlated errors is not a directed graph. The main result of this paper is to show how to associate a directed graph with a linear model  $L$  with correlated errors, and then use d-separation in the associated directed graph to determine whether  $L$  linearly entails that a particular partial

correlation is zero. The standard graph terminology in this paper, the standard terminology for linear structural equation models, and the relationship between the two terminologies are described in the Appendix.

If  $G$  is the graph of SEM  $L$  with correlated errors, let  $\text{Transform}(G)$  be the graph resulting from replacing a double headed arrow between correlated errors  $E_i$  and  $E_j$  with a new latent variable  $T^{\wedge}$  ( $i < j$ ) and edges from  $T^{\wedge}$  to  $X_i$  and  $X_j$ , and then removing the error terms from the graph. See Figure 1. A **trek** between  $X_i$  and  $X_j$  is an undirected path between  $X_i$  and  $X_j$  that contains no colliders. If there is a trek  $X_i \leftarrow TV_{\wedge} \rightarrow X_j$  in  $\text{Transform}(G)$ , we will say that  $X_i$  and  $X_j$  are **d-adjacent** in  $\text{Transform}(G)$ . A trek  $X_j \leftarrow TV_{\wedge} \rightarrow X_i$  is called a **correlated error trek** in  $\text{Transform}(G)$ . In  $\text{Transform}(G)$ , a **correlated error trek sequence** is a sequence of vertices  $\langle X_i, \dots, X_k \rangle$  such that no pair of vertices adjacent in the sequence are identical, and for each pair of vertices  $X_r$  and  $X_s$  adjacent in the sequence, there is a correlated error trek between  $X_r$  and  $X_s$ . For example in Figure 1, the sequence of vertices  $\langle X, A, B, C, D, Y \rangle$  is a correlated error trek sequence between  $X$  and  $Y$ .

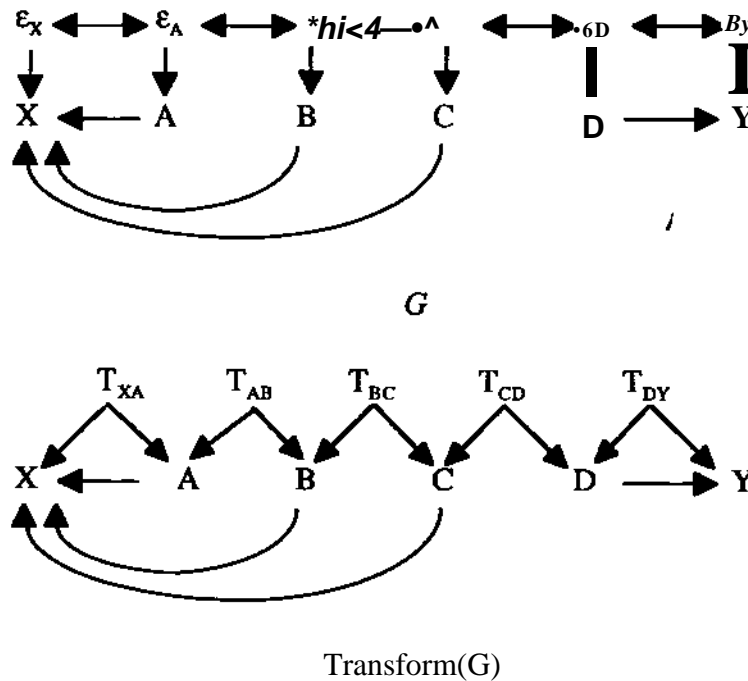


Figure 1

It might at first glance appear that for every parameterization of  $G$ , there is a parameterization of  $\text{Transform}(G)$  with the same covariance matrix. The following theorem shows that this is not the case.

**Theorem 1:** There exists a SEM  $L$  with measured variables  $\mathbf{X}$ , correlated errors, graph  $G$ , and correlation matrix  $\Sigma(\mathbf{X})$  such that no linear parameterization of  $\text{Transform}(G)$  has marginal correlation matrix  $\Sigma(\mathbf{X})$ .

**Proof.** Assume that  $L$  has no structural equations, but every pair of errors is correlated in  $L$ .  $G$  and  $\text{Transform}(G)$  are shown in Figure 2.



Figure 2

Suppose that the marginal correlation matrix  $\Sigma(\mathbf{X})$  is the following:

$$\Sigma(\mathbf{X}) = \begin{pmatrix} 1.0 & 0.99 & 0.99 \\ 0.99 & 1.0 & 0.99 \\ 0.99 & 0.99 & 1.0 \end{pmatrix}$$

Every parameterization of  $\text{Transform}(G)$  is of the following form:

$$\begin{aligned} X_1 &= a_{11}T_1 + a_{13}T_3 + b_{11}\epsilon'_1 \\ X_2 &= a_{21}T_1 + a_{22}T_2 + b_{22}\epsilon'_2 \\ X_3 &= a_{32}T_2 + a_{33}T_3 + b_{33}\epsilon'_3 \end{aligned} \tag{1}$$

Suppose first that the variance of each of the variables is equal to 1. It follows then that

$$\begin{aligned}
\text{var}(X_1) = 1 &= a_{11}^2 + a_{13}^2 + b_{11}^2 \\
\text{var}(X_2) = 1 &= a_{21}^2 + a_{22}^2 + b_{22}^2 \\
\text{var}(X_3) = 1 &= a_{32}^2 + a_{33}^2 + b_{33}^2
\end{aligned} \tag{2}$$

$$\begin{aligned}
\text{corr}(X_1, X_2) &= 0.99 = a_{11}a_{21} \\
\text{corr}(X_2, X_3) &= 0.99 = a_{22}a_{33} \\
\text{corr}(X_1, X_3) &= 0.99 = a_{13}a_{33}
\end{aligned} \tag{3}$$

From (1), the absolute values of each of the coefficients is less than one. From (2), it follows that  $a_{11}$ ,  $a_{21}$ ,  $a_{22}$ ,  $a_{32}$ ,  $a_{13}$ , and  $a_{33}$  all have absolute values greater than 0.99. Hence  $\text{var}(X_1)$  is greater than 1, which is a contradiction. It follows that there are no solutions to (2) and (3).

Suppose now that we do not fix the variances of the exogenous variables at one. We will show that if the corresponding set of equations has a solution, then so do (2) and (3), which is a contradiction.

$$\begin{aligned}
\text{var}(X_1) = 1 &= a_{11}^2 \text{var}(T_1) + a_{13}^2 \text{var}(r_3) + b_{11}^2 \text{var}(\epsilon_1) \\
\text{var}(X_2) = 1 &= a_{21}^2 \text{var}(T_1) + a_{22}^2 \text{var}(T_2) + b_{22}^2 \text{var}(\epsilon_2) \\
\text{var}(X_3) = 1 &= a_{32}^2 \text{var}(T_2) + a_{33}^2 \text{var}(T_3) + b_{33}^2 \text{var}(\epsilon_3)
\end{aligned} \tag{2'}$$

$$\begin{aligned}
\text{corr}(X_1, X_2) &= 0.99 = a_{11}a_{21} \text{var}(T_1) \\
\text{corr}(X_2, X_3) &= 0.99 = a_{22}a_{33} \text{var}(T_2) \\
\text{corr}(X_1, X_3) &= 0.99 = a_{13}a_{33} \text{var}(T_3)
\end{aligned} \tag{3'}$$

Suppose now that 2' and 3' have a solution. Then set

$$\begin{aligned}
a_{11} &= a'_{11} \sqrt{\text{var}(T_1)} & a_{21} &= a'_{21} \sqrt{\text{var}(T_1)} & a_{22} &= a'_{22} \sqrt{\text{var}(T_2)} \\
a_{32} &= a'_{32} \sqrt{\text{var}(T_2)} & a_{13} &= a'_{13} \sqrt{\text{var}(r_3)} & a_{33} &= a'_{33} \sqrt{\text{var}(T_3)} \\
b_{11} &= b'_{11} \sqrt{\text{var}(\epsilon_1)} & b_{22} &= b'_{22} \sqrt{\text{var}(\epsilon_2)} & b_{33} &= b'_{33} \sqrt{\text{var}(\epsilon_3)}
\end{aligned}$$

These now form a solution to (2) and (3), which is a contradiction.  $\therefore$

**Lemma 1:** If  $E$  is a positive definite matrix, then there exists a positive definite matrix  $E' = E - \delta I$ , where  $\delta$  is a real positive number.

**Proof.** Suppose that  $E$  is a positive definite matrix. It follows then that for all solutions of  $\det(E - \lambda L) = 0$ ,  $\lambda$  is positive. Let the smallest solution of  $\det(E - \lambda L) = 0$  be  $\lambda_{\min}$ . Let  $\delta$  be

less than  $\lambda$  and greater than 0. Let  $Z' = Z - 5I$ . We will now show that all of the solutions of  $\det(Z' - VI) = 0$  are positive.  $Z' - VI = Z - 8I - VI = Z - (V + 5)I$ . If we set  $V = X - 5$ , then for each solution of  $\det(Z - XI) = 0$ , there is a solution of  $\det(Z - (V + 8)I) = 0$ . Since  $V = X - 8$ , and 8 is less than  $X_{\min}$ , the smallest solution of  $\det(Z' - VI) = 0$  is greater than 0.  $\square$

A linear transformation of a set of random variables is **lower triangular** if and only if there is an ordering of the variables such that the matrix representing the transformation is zero for all entries  $a_{ij}$ , when  $j > i$ .

**Lemma 2:** If  $X_1, \dots, X_n$  have a joint normal distribution  $N(0, Z)$ , where  $Z$  is positive definite, then there is a set of  $n$  mutually independent standard normal variables  $T_1, \dots, T_n$ , such that  $X_1, \dots, X_n$  are a lower triangular linear transformation of  $T_1, \dots, T_n$  and for each  $i$ , the coefficient of  $T_i$  in the equation for  $X_i$  is not equal to zero.

**Proof.** For every positive definite correlation matrix  $Z$ , a complete directed graph can be given a linear parameterization that represents  $Z$  (Spirtes et al. 1993). The reduced form of a complete directed graph is a lower triangular transformation of independent error variables that is non-zero on the diagonal, because  $Z$  is positive definite.  $\square$

**Theorem 2:** If  $G$  is the graph of SEM  $L$  with measured variables  $X$ , normally distributed correlated errors, and marginal correlation matrix  $Z(X)$ ,  $\{X, Y\} \cup Z \subseteq X$ , and  $X$  is d-separated from  $Y$  given  $Z$  in  $\text{Transform}(G)$ , then  $p_{XZ}^Y = 0$  in  $Z(X)$ .

**Proof.** First we will construct a latent variable model of  $e_1, \dots, e_n$ . Then we will use this model to form the latent variable model  $L'$  with graph  $G'$  that has marginal correlation matrix  $Z(X)$  but no correlated errors, and in which  $X$  is d-separated from  $Y$  given  $Z$  in  $G'$ . It follows that  $p_{XZ}^Y = 0$  in  $Z(X)$ .

Order the variables so that  $X$  is first,  $Y$  is second, followed by each variable with a descendant in  $Z$ , followed by any remaining variables that have  $X$  or  $Y$  as descendants, followed by the rest of the variables. Given this ordering, we will now refer to the variables as  $X_1, \dots, X_n$ , where for all  $i$ ,  $X_i$  is the  $i^{\text{th}}$  variable in the ordering. Suppose for the graph in Figure 1 we are interested in whether  $p_{XZ}^Y = 0$  (i.e.  $Z = 0$ ). One renaming of the variables for the graph in Figure 1 that is compatible with the ordering rules given above is shown in Figure 3.

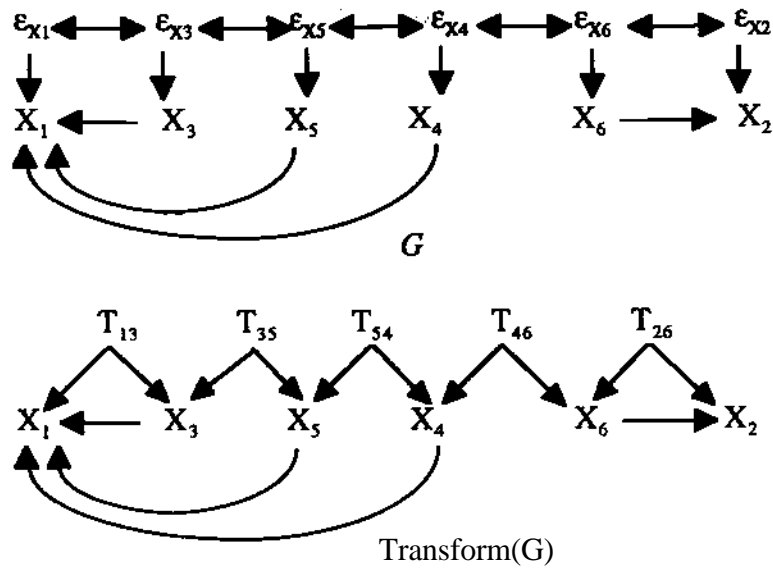


Figure 3

Suppose that the correlation matrix among the error terms of \$L\$ is \$E\$. We will show that there is a latent variable model of \$Z\$ of the form

$$\epsilon_i = \sum_{j < i} a_{ij} T_j + \epsilon''_i$$

where each of the \$T\_l\$ and \$\epsilon''\_l\$ are uncorrelated.

By hypothesis, \$\Sigma\$ is a positive definite matrix. By Lemma 1, there is a set of variables \$e'\_1, \dots, e'\_n\$ with positive definite matrix \$\Sigma' = \Sigma - \Sigma\_1\$. As a first step to constructing a latent variable model of \$e\$, we will construct a latent variable model of \$e\$ represented by a directed graph \$H\$. Note that \$H\$ does not contain any of the \$X\$ variables or \$e\$ variables.

By Lemma 2, there is a set of variables \$T\_1, \dots, T\_n\$ such that \$e'\_1, \dots, e'\_n\$ with correlation matrix \$\Sigma'(s)\$ are a lower triangular linear transformation of \$T\_1, \dots, T\_n\$ and for each \$i\$, the coefficient of \$T\_i\$ in the equation for \$e'\_i\$ is not equal to zero. That is

$$\epsilon'_i = \sum_{j < i} a_{ij} T_j$$

where \$a\_{ij} \neq 0\$. The transformation can be represented by a directed graph \$H\$ in which for each \$i\$, there are edges from \$T\_j\$ to \$e'\_i, j < i\$.



From the construction of  $//$ , there are no edges from  $T_j$  to  $e_i$  unless  $j = 1$ . Hence, for every  $j \neq 1$ , in  $H$  every every trek between  $e_i$  and  $e_j$  contains  $T_1$ . It follows that there is at most one trek between  $e_i$  and  $e_j$ . The edge from  $T_1$  to  $e_i$  is not zero. Hence if  $e_i$  and  $e_j$  are not correlated in  $L$  (i.e.  $X_i$  and  $X_j$  are not d-adjacent in  $\text{Transform}(G)$ ) then the edge from  $T_1$  to  $e_j$  is zero. In the example from Figure 3,  $a_{11} = a_{14} = a_{15} = a_{16} = 0$ .

Applying this strategy to each of the  $T_i$  variables in turn, we can now show that for each  $i$  and  $r > i$ , if there is no trek between  $e_r$  and  $e_i$  containing a variable  $T_j$  where  $j < i$ , and  $X_r$  is not d-adjacent to  $X_i$  in  $\text{Transform}(G)$  (i.e.  $E_i$  and  $E_r$  are uncorrelated in  $L$ ), then the  $T_i \rightarrow e_r$  edge can be removed from the graph (i.e.  $a_{ir}$  can be set to zero.) Suppose on the contrary that there is no trek between  $e_r$  and  $e_i$  containing a variable  $T_j$  where  $j < i$ , and  $X_r$  is not d-adjacent to  $X_i$  in  $\text{Transform}(G)$  (i.e.  $e_i$  and  $E_r$  are uncorrelated in  $L$ ), but the  $T_i \rightarrow e_r$  edge is not removed from the graph by this procedure (i.e.  $a_{ir}$  is not set to zero.) By the construction of  $//$ , if  $k > i$ , then there is no edge from  $T_k$  to  $e_i$ . It follows that if in  $H$  there is no trek between  $e_r$  and  $e_i$  containing a variable  $T_j$  where  $j < i$ , then every trek between  $E_i$  and any other variable contains the edge from  $T_i$  to  $e_i$  which is not equal to zero. The  $T_i \rightarrow e_r$  edge exists by hypothesis, so there is exactly one trek between  $E_i$  and  $E_r$  in  $//$ . Hence  $E_i$  and  $E_r$  are correlated in every parameterization of  $//$ . (Note that this could not be claimed if there were more than one trek between  $E_i$  and  $E_r$  since in that case the treks might cancel each other.) Since the covariances between distinct  $E$  variables are equal to the correlations between the corresponding  $X$  variables, it follows that  $E_i$  and  $E_r$  are correlated in  $L'$ , and hence d-adjacent in  $\text{Transform}(G)$ . This is a contradiction. The end result of this process of edge removal for the graph in Figure 3 is shown in Figure 4.

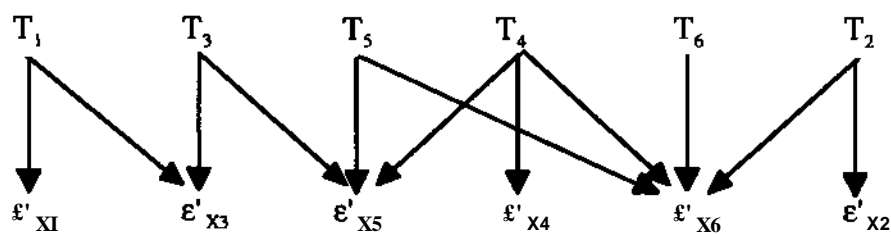


Figure 4:  $H$  after extra edges are removed

From the latent variable model without correlated errors of the  $E$  variables, we can now form a latent variable model without correlated errors of the  $X$  variables. For each  $i$ , let  $E_i$  be a normally distributed variable with variance  $\sigma^2$  that is independent of all of the  $T_j$ , and all of the other  $E$  variables. It follows then that

$$\varepsilon_i = \sum_{j \leq i} a_{ij} T_j + \varepsilon'_i$$

because the addition of the  $\varepsilon'$  term does not change any of the correlations, and adds 8 to the variance of  $eV$

From the latent variable model without correlated errors of the  $e$  variables, we can now form a latent variable model without correlated errors of  $Z(X)$ . If we use the above equation to replace each  $e_i$  in the SEM  $L$ , we form a SEM  $L'$  which has no correlated errors, but has the same marginal covariance matrix as  $L$ . If the equations in  $L$  are:

$$X_i = \sum_{j \neq i} b_{ij} X_j + \varepsilon_i$$

then the equations in  $L'$  are:

$$X_i = \sum_{j \neq i} b_{ij} X_j + \sum_{j < i} a_{ij} T_j + \varepsilon'_i$$

$L'$  has a graph  $G'$  obtained from  $G$  and  $H$  in the following way: Remove each error variable from  $G$  (because all of the error terms are uncorrelated in  $L'$ ), add each of the  $T_j$  variables to  $G$ , and add an edge from  $T_j$  to  $X_j$  if in  $H$  there is an edge from  $T_j$  to  $e_j$ . Note that the ancestor relations among the  $X$  variables in  $G'$  is the same as the ancestor relations among the  $X$  variables in  $\text{Transform}(G)$ . Given the graph  $G$  from Figure 3 and the graph  $H$  from Figure 4, the end result is shown in Figure 5. As in  $\text{Transform}(G)$ , we will call a trek  $X_j \leftarrow T_m \rightarrow X^i$  that contains a  $T$  variable a **correlated error trek** in  $G'$

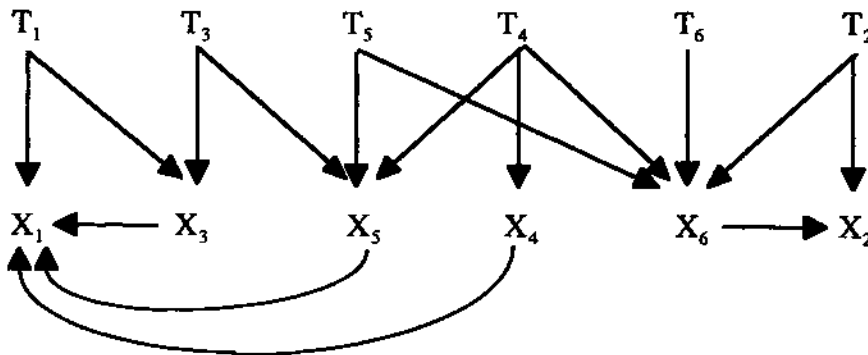


Figure 5:  $G'$

We will now show that if there is a correlated error trek between  $X_i$  and  $X_j$  in  $G'$  that contains a variable  $T_r$ , then in  $\text{Transform}(G)$  there is a correlated error trek sequence between  $X_i$  and  $X_j$ , such that every variable in the correlated error trek sequence, with the possible exception of the endpoints, has index (i.e. subscript) less than or equal to  $r$  (henceforth referred to as the correlated error trek sequence in  $\text{Transform}(G)$  corresponding to the correlated error trek between  $X_i$  and  $X_j$  in  $G'$ .) The proof is by induction on  $r$ . Suppose first that  $r = 1$ . If there is a correlated error trek between  $X_i$  and  $X_j$  in  $G'$  that contains  $T_1$  then there are correlated error treks between  $X_i$  and  $X_1$  in  $\text{Transform}(G)$ , and between  $X_j$  and  $X_1$ . The concatenation of these two correlated error treks forms a correlated error trek sequence in which (trivially) every variable in the sequence except for the endpoints has an index less than or equal to 1. The induction hypothesis is that for all  $r \leq n$ , if there is a correlated error trek between  $X_i$  and  $X_j$  in  $G'$  that contains  $T_r$ , then in  $\text{Transform}(G)$  there is a correlated error trek sequence between  $X_i$  and  $X_j$ , such that every variable in the sequence, with the possible exception of the endpoints has an index less than  $r$ . Suppose now that in  $G'$  there is a correlated error trek between  $X_i$  and  $X_j$  such that the trek contains  $T_{n+1}$ , where  $i, j \geq n+1$ . Since the edge between  $T_{n+1}$  and  $X_i$  exists in  $G'$ , it follows from the method of construction of  $G'$  that either there is a correlated error trek between  $X_i$  and  $X_{n+1}$  in  $G'$  that contains some  $T_r$ ,  $r < n+1$ , or  $X_{n+1}$  and  $X_i$  are  $d$ -adjacent in  $\text{Transform}(G)$ . In the former case, by the induction hypothesis there is a correlated error trek sequence between  $X_i$  and  $X_{n+1}$  that, except for the endpoints, contains only vertices whose indices are less than or equal to  $n+1$ . In the latter case,  $\langle X_i, X_{n+1} \rangle$  is a correlated error trek sequence between  $X_i$  and  $X_{n+1}$ . Similarly, there is a correlated error trek sequence between  $X_{n+1}$  and  $X_j$  that, except for the endpoints, contains only vertices whose indices are less than or equal to  $n+1$ . These two correlated error trek sequences can be concatenated to form a correlated error trek sequence between  $X_i$  and  $X_j$  that, except for the endpoints, contains only vertices whose indices are less than or equal to  $n+1$ . For the  $G'$  shown in Figure 5, there is a correlated error trek between  $X_5$  and  $X_6$ , and a corresponding correlated error trek sequence  $\langle X_5, X_4, X_6 \rangle$  in the graph  $\text{Transform}(G)$  in Figure 3.

We will now show that if  $X_1$  and  $X_2$  are  $d$ -connected given  $Z$  in  $G'$ , then  $X_1$  and  $X_2$  are  $d$ -connected given  $Z$  in  $\text{Transform}(G)$  using Lemma 3.3.1+ (Richardson 1994, which is an extension to the cyclic case of Lemma 3.3.1 in Spirtes et al. 1993). Lemma 3.3.1+ states that there is a path in a directed graph  $G$  that  $d$ -connects  $X$  and  $Y$  given  $Z$  if and only if there is a sequence of vertices  $Q$  and a set  $P$  of paths in  $G$  between pairs of adjacent vertices in  $Q$  that have the following properties: (i) For each occurrence of a pair of adjacent variables  $X_i$  and  $X_j$  in  $Q$ ,  $i \neq j$ , and there is a unique path in  $P$  that  $d$ -connects  $X_i$  and  $X_j$

given  $Z \vee X^{\wedge} X_j$ ; (ii) if  $\langle X_i^{\wedge} C_j^{\wedge} C_k \rangle$  is a subsequence of  $Q$ , the corresponding path between  $X_i$  and  $X_j$  in  $P$  is into  $X_i$ , and **the corresponding path** between  $X_j$  and  $X_k$ , in  $P$  is into  $X_j$  (in which case we say that the occurrence of  $X_j$  is a **collider** in  $Q$ ) then  $X_j$  has a descendant in  $Z$ ; and (iii) if there is an occurrence of  $X_j$  that is a non-collider in  $Q$ , then  $X_j$  is not in  $Z$ . Note that we do not require that a vertex occur only once in  $Q$ . Hence one occurrence of a vertex in  $Q$  may be a collider, and another occurrence of the same vertex in  $Q$  may be a non-collider.

Suppose now that there is an undirected path  $U$  that  $d$ -connects  $X_1$  and  $X_2$  given  $Z$  in  $G$ . Intuitively, in  $G'$  we would like to form  $Q$  and  $P$  by breaking  $U$  into pieces, such that each correlated error trek occurs as a separate piece. More formally, form a sequence  $Q$  of vertices and an associated sequence  $P$  of paths in  $G'$  with the following properties: (i) every vertex in  $Q$  is in  $X$  and occurs on  $U$ ; (ii) no vertex occurs in  $Q$  more than once; (iii) if  $A$  occurs before  $B$  in  $Q$ , then  $A$  occurs before  $B$  on  $U$ ; (iv) if the subpath of  $U$  between  $A$  and  $B$  is a correlated error trek, then  $A$  and  $B$  both occur in that order in  $Q$ . The path in  $P$  associated with a pair  $A$  and  $B$  of adjacent vertices in  $Q$  is the subpath of  $U$  between  $A$  and  $B$ . In the example in Figure 5, in  $G^1$  the  $d$ -connecting path between  $X_1$  and  $X_2$  given  $Z = 0$  is  $X_1 \leftarrow X_5 \leftarrow T_4 \rightarrow X_6 \rightarrow X_2$ ,  $Q = \langle X_1, X_5, X_6, X_2 \rangle$ , and  $P = \langle X_1 \leftarrow X_5, X_5 \leftarrow T_4 \rightarrow X_6, X_6 \rightarrow X_2 \rangle$ . In this example, there are no colliders in  $Q$ .

Because  $U$  is a path that  $d$ -connects  $X_1$  and  $X_2$  given  $Z$  in  $G'$ , it is easy to see that the paths in  $P$  have the following properties in  $G'$ : (i) Each path in  $Q$   $d$ -connects its endpoints  $X_i$  and  $X_j$  given  $Z \vee X^{\wedge}$ ; (ii) if there is an occurrence of  $X_i$  in  $Q$  that is a collider then  $X_i$  has a descendant in  $Z$ ; and (iii) if there is an occurrence of  $X_i$  in  $Q$  that is a non-collider, then  $X_i$  is not in  $Z$ .

We will now show how to construct a sequence of vertices  $Q'$  and a set  $P'$  of paths in  $\text{Transform}(G)$  between pairs of adjacent vertices in  $Q'$  that have the following properties: (i) For each occurrence of a pair of adjacent variables  $X^{\wedge}$  and  $X_j$  in  $Q'$  there is a unique path in  $P'$  that  $d$ -connects  $X^{\wedge}$  and  $X_j$  given  $Z \vee X^{\wedge} X_j$ ; (ii) if there is an occurrence of  $X_j$  in  $Q'$  that is a collider, then  $X^{\wedge}$  has a descendant in  $Z$ ; and (iii) if there is an occurrence of  $X_i$  in  $Q'$  that is a non-collider, then  $X_i$  is not in  $Z$ . It will follow from Lemma 3.3.1+ that  $X$  and  $Y$  are  $d$ -connected given  $Z$  in  $\text{Transform}(G)$ .

We will create  $Q'$  by several modifications of  $Q$ . Step (1) in creating  $Q'$  is to replace each subsequence  $KX^{\wedge}C^{\wedge}$  of  $Q$  such that  $X_r$  and  $X_s$  are on a correlated error trek in  $Q$ , with the corresponding correlated error trek sequence  $\langle X_r, \dots, X_s \rangle$  in  $\text{Transform}(G)$ . Note that each

occurrence of  $X_k$  between  $\langle X_r, \dots, X_s \rangle$  is a collider in  $Q$ . In the example, after the first step  $Q' = \langle X_1, X_5, X_4, X_6, X_2 \rangle$  and  $P^f = \langle X_1 \leftarrow X^{\wedge} X_5 \leftarrow T_{45} \rightarrow X_4 \rightarrow X_4 \leftarrow T^{\wedge} \rightarrow X_6, X_6 \rightarrow X_2 \rangle$ , i.e. we replaced the subsequence  $\langle X_4, X_6 \rangle$  in  $Q$  by  $\langle X_5, X_4, X_6 \rangle$ .

Recall that the ancestor relations among the  $X$  variables (which includes the variables in  $Z$ ) in  $G'$  is the same as the ancestor relations among the  $X$  variables in  $\text{Transform}(G)$ . After stage (1) in creating  $Q'$ , if  $X_k$  is not an ancestor of  $Z$  in  $\text{Transform}(G)$  (or in  $G'$ ), but has an occurrence in  $Q'$  that is a collider, it follows that  $X_k$  was added to  $Q'$  by replacing a subsequence  $\langle X_r, \dots, X_s \rangle$  of  $Q$  by a corresponding correlated error trek sequence  $\langle X_r, \dots, X_s \rangle$  in  $\text{Transform}(G)$ . Hence any such  $X_k$  lies between some pair of vertices  $X_r$  and  $X_s$  that are adjacent in  $Q$ . Because every vertex in  $\langle X_r, \dots, X_s \rangle$  in  $Q'$  (except for  $X_r$  and  $X_s$ ) has an index less than  $r$  and  $s$ , and  $X_k$  is not an ancestor of  $Z$  in  $G \setminus$  it follows from the ordering of the variables that we chose, that  $X_r$  and  $X_s$  are not ancestors of  $Z$  in  $G \setminus$ . Because  $X_r$  and  $X_s$  are on  $U$  but not ancestors of  $Z$  in  $G'$ , there is a subpath of  $U$  that is a directed path from  $X_r$  to  $X_l$  and a subpath of  $U$  that is a directed path  $X_s$  to  $X_2$ , or vice versa. In either case, in  $G'$ ,  $X_r$  is an ancestor of  $X_l$  and  $X_s$  is an ancestor of  $X_2$ , or  $X_r$  is an ancestor of  $X_2$  and  $X_s$  is an ancestor of  $X_p$ . Because in  $G \setminus$   $X_r$  is an ancestor of  $X_x$  and  $X_s$  an ancestor of  $X_2$  or vice-versa, and  $k < r$  and  $s$ , it follows from the ordering of the variables that  $X_k$  is also an ancestor of  $X_l$  or  $X_2$  in  $G \setminus$ . Hence  $X_k$  is an ancestor of  $X_l$  or  $X_2$  in  $\text{Transform}(G)$ . In the example, in  $\text{Transform}(G)$   $X_4$  is not an ancestor of the empty set but is an ancestor of  $X_p$  and it is between two vertices  $X_5$  and  $X_6$  which also are not ancestors of the empty set but are ancestors of  $X_j$  or  $X_2$ .

If there is some vertex  $X^{\wedge}$  in  $Q'$  that is not an ancestor of  $Z$ , but occurs in  $Q'$  as a collider, suppose without loss of generality that there is a vertex that is an ancestor of  $X_l$  but not of  $Z$ , that occurs as a collider in  $Q \setminus$ . Let  $X_a$  be the last occurrence of a collider in  $Q'$  that is an ancestor of  $X_l$  but not of  $Z$ , if there is one, otherwise let  $X_a = X_l$ . Step (2) in forming  $*Q'$  and  $P'$  is to replace the subsequence  $\langle X_p, \dots, X_a \rangle$  by  $\langle X_p X_a \rangle$  if  $X_a * X_p$  and replacing the corresponding paths in  $P'$  by a directed path from  $X_a$  to  $X_l$  if  $X_a * X_x$ . (Such a directed path exists if  $X_a * X_x$  because  $X_a$  is an ancestor of  $X_p$ ) This removes all occurrences of vertices between  $X_l$  and  $X_a$  that are not ancestors of  $Z$ , but are colliders in  $Q \setminus$ . In the example,  $X_a = X_4$ , and after step 2,  $Q' = \langle X_p X_4, X_6, X_2 \rangle$  and  $P' = \langle X_l \leftarrow X_4, X_4 \leftarrow T_{46} \rightarrow X_6, X_6 \rightarrow X_2 \rangle$ .

By definition, every vertex that occurs as a collider between  $X_a$  and  $X_2$  in  $Q'$  is an ancestor of  $Z$  or of  $X_2$ . Let  $X_b$  be the first vertex after  $X_a$  in  $Q'$  that is an ancestor of  $X_2$  but not of  $Z$ , if there is one, otherwise let  $X_b = X_2$ . Step (3) in forming  $Q'$  and  $P'$  is to replace the

subsequence  $\langle X_b, \dots, X_2 \rangle$  by  $\langle X_t, X_2 \rangle$  if  $X_b * X_2$ , and replacing the corresponding paths in  $P'$  by a directed path from  $X_b$  to  $X_2$  if  $X_b * X_2$ . This removes all occurrences of colliders between  $X_b$  and  $X_2$  that are not ancestors of  $Z$ . Note that all occurrences of colliders that are left are between  $X_a$  and  $X_t$ , and every occurrence of a collider between  $X_a$  and  $X_b$  is an ancestor of  $Z$  by construction. In the example,  $X_b = X_2$ , and after step (3),  $Q'$  and  $P'$  are unchanged.

We will now show that every path between a pair of variables  $X_u$  and  $X_v$  in  $P'$  d-connects  $X_u$  and  $X_v$  given  $Z \setminus \{X_u, X_v\}$ . If the path between  $X_u$  and  $X_v$  is also in  $P$ , then it d-connects  $X_u$  and  $X_v$  given  $Z \setminus \{X_u, X_v\}$  because every path in  $P$  has this property. If the path between  $X_u$  and  $X_v$  is not in  $P$ , but was added in step (1) of the formation of  $P'$  then the path between  $X_u$  and  $X_v$  is a correlated error trek, which d-connects  $X_u$  and  $X_v$  given  $Z \setminus \{X_u, X_v\}$  because no  $T$  variable is in  $Z$ . If the path between  $X_u$  and  $X_v$  is not in  $P$ , but was added in step (2) of the formation of  $P'$  then  $X_u = X_p$ ,  $X_v = X_a$ , and the path between  $X_u$  and  $X_v$  is a directed path from  $X_a$  to  $X_p$  that does not contain any member of  $Z$ . Hence the path d-connects  $X_u$  and  $X_v$  given  $Z$ . Similarly, if the path between  $X_u$  and  $X_v$  is not in  $P$ , but was added in step (3) of the formation of  $P'$ , then  $X_u = X_b$ ,  $X_v = X_2$ , and the path between  $X_u$  and  $X_v$  is a directed path from  $X_b$  to  $X_2$  that does not contain any member of  $Z$ . Hence the path d-connects  $X_u$  and  $X_v$  given  $Z$ .

We will now show that every vertex that occurs as a collider in  $Q'$  has a descendant in  $Z$ , and every vertex that occurs as a non-collider in  $Q'$  is not in  $Z$ . Every vertex that occurs as a collider in  $Q'$  is an ancestor of  $Z$ , because steps (2) and (3) in the formation of  $Q'$  removed all occurrences of colliders that were not ancestors of  $Z$ . Every vertex that occurs as a non-collider in  $Q$  and as a non-collider in  $Q'$  is not in  $Z$ , because every vertex that occurs as a non-collider in  $Q$  is not in  $Z$ . The only vertices that may occur as non-colliders in  $Q'$  but not in  $Q$  are  $X_a$  and  $X_b$ .  $X_a$  is not in  $Z$ , because either it is equal to  $X_t$  or  $X_2$ , neither of which is in  $Z$ , or it is not an ancestor of  $Z$  by construction. Similarly,  $X_b$  is not in  $Z$ .

Hence  $Q'$  is a sequence of paths that satisfy properties (i), (ii), and (iii). It follows from Lemma 3.3.1+ that  $X_t$  and  $X_2$  are d-connected given  $Z$  in  $\text{Transform}(G)$ .

By contraposition, since  $X_t$  and  $X_2$  are d-separated in  $\text{Transform}(G)$ , they are d-separated given  $Z$  in  $G'$ . Because  $G'$  is the directed graph of a latent variable model  $L'$  with correlation matrix that has marginal  $\Sigma(X)$ , no correlated errors, and  $X_t$  and  $X_2$  are

d-separated given  $\mathbf{Z}$  in  $G'$ , it follows from Theorem 3 (Spirtes 1995) that  $\rho_{XY,Z} = 0$  in  $\Sigma(\mathbf{X})$ .  $\therefore$

**Theorem 3:** If  $G$  is the graph of SEM  $L$  with normally distributed correlated errors and marginal correlation matrix  $\Sigma(\mathbf{X})$ , and  $X$  is d-connected to  $Y$  given  $\mathbf{Z}$  in  $\text{Transform}(G)$  then  $L$  does not linearly entail that  $\rho_{XY,Z} = 0$ .

Proof. If  $X$  is not d-separated from  $Y$  given  $\mathbf{Z}$  in  $\text{Transform}(G)$ , then by Theorem 3 (Spirtes 1995) there is a parameterization of  $\text{Transform}(G)$  with correlation matrix  $\Sigma(\mathbf{X})$  such that  $\rho_{XY,Z} \neq 0$ . By the convention adopted for new latent variables names in  $\text{Transform}(G)$ , no new latent variable was called  $T_{ji}$  where  $j > i$ . For the sake of notational convenience, we will also use the name  $T_{ji}$  to refer to  $T_{ij}$ . In that parameterization,

$$X_i = \sum_{j < i} b_{ij} X_j + \sum_{j \neq i} a_{ij} T_{ij} + \varepsilon'_i$$

Now define

$$\varepsilon_i = \sum_{j \neq i} a_{ij} T_{ij} + \varepsilon'_i$$

It follows then that

$$X_i = \sum_{j < i} b_{ij} X_j + \varepsilon_i$$

which is a parameterization of  $L$  in which  $\rho_{XY,Z} \neq 0$ .  $\therefore$

Because the covariance matrix of the non-error variables in a linear SEM does not depend upon whether the error terms are normally distributed, but depends only upon the linear coefficients and the covariance matrix among the errors, Theorem 2 and Theorem 3 can obviously be extended to the case where the error terms are not normally distributed.

## Appendix

Sets of variables and defined terms are in boldface. A **directed graph** is an ordered pair of a finite set of vertices  $V$ , and a set of directed edges  $E$ . A directed edge from  $A$  to  $B$  is an ordered pair of distinct vertices  $\langle A, B \rangle$  in  $V$  in which  $A$  is the **tail** of the edge and  $B$  is the **head**; the edge is **out of**  $A$  and **into**  $B$ , and  $A$  is **parent of**  $B$  and  $B$  is a **child** of  $A$ . A sequence of edges  $\langle E_1, \dots, E_n \rangle$  in  $G$  is an **undirected path** if and only if there exists a sequence of vertices  $\langle V_1, \dots, V_{n+1} \rangle$  such that for  $1 \leq i < n$  either  $\langle V_i, V_{i+1} \rangle = E_i$  or  $\langle V_{i+1}, V_i \rangle = E_i$ . A path  $U$  is **acyclic** if no vertex occurring on an edge in the path occurs more than once. A sequence of edges  $\langle E_1, \dots, E_n \rangle$  in  $G$  is a **directed path** if and only if there exists a sequence of vertices  $\langle V_1, \dots, V_{n+1} \rangle$  such that for  $1 \leq i < n$   $\langle V_i, V_{i+1} \rangle = E_i$ . If there is an acyclic directed path from  $A$  to  $B$  or  $B = A$  then  $A$  is an **ancestor of**  $B$ , and  $B$  is a **descendant** of  $A$ . A directed graph is **acyclic** if and only if it contains no directed cyclic paths.<sup>1</sup>

Vertex  $X$  is a **collider** on an acyclic undirected path  $U$  in directed graph  $G$  if and only if there are two edges on  $U$  that are directed into  $X$ . Three disjoint sets  $X$ ,  $Y$ , and  $Z$ ,  $X$  and  $Y$  are **d-separated** given  $Z$  in  $G$  if and only if there is no acyclic undirected path  $U$  from a member of  $X$  to a member of  $Y$  such that every non-collider on  $U$  is not in  $Z$ , and every collider on  $U$  has a descendant in  $Z$ . For three disjoint sets  $X$ ,  $Y$ , and  $Z$ ,  $X$  and  $Y$  are **d-connected** given  $Z$  in  $G$  if and only if  $X$  and  $Y$  are not **d-separated** given  $Z$ .

The variables in a linear structural equation model (SEM) can be divided into two sets, the "error variables" or "error terms," and the substantive variables. Corresponding to each substantive variable  $X_i$  is a linear equation with  $X_j$  on the left hand side of the equation, and the direct causes of  $X_i$  plus the error term  $E_i$  on the right hand side of the equation. Since we have no interest in first moments, without loss of generality each variable can be expressed as a deviation from its mean.

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<sup>1</sup>An undirected path is often defined as a sequence of vertices rather than a sequence of edges. The two definitions are essentially equivalent for acyclic directed graphs, because a pair of vertices can be identified with a unique edge in the graph. However, a cyclic graph may contain more than one edge between a pair of vertices. In that case it is no longer possible to identify a pair of vertices with a unique edge.



Consider, for example, two SEMs  $S_x$  and  $S_2$  over  $X = \{X_1, X_2, X_3\}$ , where in both SEMs  $X_1$  is a direct cause of  $X_2$  and  $X_2$  is a direct cause of  $X_3$ . The structural equations<sup>2</sup> in Figure 6 are common to both  $S_1$  and  $S_2$ .

$$X_2 = \beta_1 X_1 + \epsilon_2$$

$$X_3 = \beta_2 X_2 + \epsilon_3$$

**Figure 6: Structural Equations for SEMs  $S_1$  and  $S_2$**

where  $\beta_1$  and  $\beta_2$  are free parameters ranging over real values, and  $\epsilon_2$  and  $\epsilon_3$  are error terms. In addition suppose that  $\epsilon_2$  and  $\epsilon_3$  are distributed as multivariate normal. In  $S_x$  we will assume that the correlation between each pair of distinct error terms is fixed at zero. The free parameters of  $S_1$  are  $\theta = \langle \beta, P \rangle$ , where  $\beta$  is the set of linear coefficients  $\{\beta_1, \beta_2\}$  and  $P$  is the set of variances of the error terms. We will use  $\Sigma$  to denote the covariance matrix parameterized by the vector  $\theta$  for model  $S_p$  and occasionally leave out the model subscript if the context makes it clear which model is being referred to. If all the pairs of error terms in a SEM  $S$  are uncorrelated, we say  $S$  is a SEM with **uncorrelated errors**.

$S_2$  contains the same structural equations as  $S_1$ , but in  $S_2$  we will allow the errors between  $X_2$  and  $X_3$  to be correlated, i.e., we make the correlation between the errors of  $X_2$  and  $X_3$  a free parameter, instead of fixing it at zero, as in  $S_x$ . In  $S_2$  the free parameters are  $\theta' = \langle \beta', P' \rangle$ , where  $\beta'$  is the set of linear coefficients  $\{\beta_1', \beta_2'\}$  and  $P'$  is the set of variances of the error terms and the correlation between  $\epsilon_2$  and  $\epsilon_3$ . If the correlations between any of the error terms in a SEM are not fixed at zero, we will call it a SEM with **correlated errors**.<sup>3</sup>

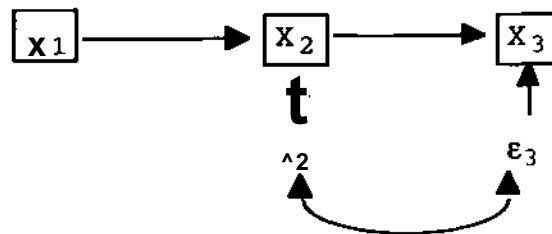
It is possible to associate with each SEM with uncorrelated errors a directed graph that represents the causal structure of the model and the form of the linear equations. For example, the directed graph associated with the substantive variables in  $S_x$  is  $X_1 \rightarrow X_2 \rightarrow X_3$ , because  $X_1$  is the only substantive variable that occurs on the right hand side of the equation for  $X_2$ , and  $X_2$  is the only substantive variable that appears on the right hand side of the equation for  $X_3$ . We generally do not include error terms in the directed graph

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<sup>2</sup> We realize that it is slightly unconventional to write the trivial equation for the exogenous variable  $X_1$  in terms of its error, but this saves to give the error terms a unified and special status as providing all the external sources of variation for the system

<sup>3</sup> We do not consider SEMs with other sorts of constraints on the parameters, e.g., equality constraints.

associated with a SEM unless the errors are correlated. We enclose measured variables in boxes, latent variables in circles, and leave error variables unenclosed.



**Figure 7. SEM  $S_2$  with correlated errors**

The typical path diagram that would be given for  $S_2$  is shown in Figure 7. This is not strictly a directed graph because of the double-headed arrow between error terms  $\epsilon_2$  and  $\epsilon_3$ , which indicates that  $\epsilon_2$  and  $\epsilon_3$  are correlated. It is generally accepted that correlation is to be explained by some form of causal connection. Accordingly if  $\epsilon_2$  and  $\epsilon_3$  are correlated we will assume that either  $\epsilon_2$  causes  $\epsilon_3$ ,  $\epsilon_3$  causes  $\epsilon_2$ , some latent variable causes both  $\epsilon_2$  and  $\epsilon_3$ , or some combination of these. In other words, double-headed arrows are an ambiguous representation of a causal connection.

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