# Proof Theory

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**PROOF THEORY**

*Proof theory is a branch of mathematical logic that was founded by David Hilbert around 1920 to pursue* HILBERT's PROGRAM. *The problems that were addressed by the program had been formulated, in some sense, already at the turn of the century, for example, in Hubert's famous address to the First International Congress of Mathematicians in Paris. They were closely connected to the set theoretic foundations for analysis investigated by Cantor and Dedekind, in particular, to difficulties with the unrestricted notion of system or set; they were also related to the philosophical conflict with Kronecker on the very nature of mathematics. At that time, the central issue for Hilbert was the "consistency of sets" in Cantor's sense. Hilbert suggested that the existence of consistent sets, e.g., the set of real numbers, could be secured by proving the consistency of a suitable, characterizing axiom system, but indicated only vaguely how to give such proofs model theoretically. Four years later, Hilbert departed radically from these indications and proposed a novel way of attacking the consistency problem for theories. This approach required, first of all, a strict formalization of mathematics together with logic; then, the syntactic configurations of the joint formalism would be considered as mathematical objects; finally, mathematical arguments would be used to show that contradictory formulas cannot be derived by the logical rules.*

*This two-pronged approach of developing 'substantial parts of mathematics in formal theories (be they set theory, second order arithmetic, finite type theory, or still other theories) and of proving their consistency (or the consistency of significant sub-theories) was sharpened in lectures beginning in 1917/18 and then pursued systematically in the twenties by Hilbert and a group of collaborators including Paul Bernays, Wilhelm Ackermann, and Johan von Neumann. In particular, the formalizability of analysis in a second-order theory was verified by Hilbert already in those very early lectures; the record of that was published as Supplement IV of (Hilbert and Bernays 1939). So it was possible to focus on the second prong, namely to establish the consistency of "arithmetic" (second order number theory and set theory) by elementary mathematical, so-called finitist means. This part of the task proved to be much more recalcitrant than expected, and only limited results were obtained. That results were limited by necessity was explained in 1931 by* GÖDEL'S THEOREMS; *indeed, they refuted the attempt of establishing consistency on a finitist basis — as soon as it was* *realized that finitist considerations could be carried out in a small fragment of first-order arithmetic. This led to the formulation of a general reductive program; see sections 2 and 4.*

*Gentzen and Gödel made the first contributions to the general reductive program by establishing the consistency of classical first-order arithmetic, PA, relative to intuitionistic arithmetic, HA, as formalized by Heyting. Gentzen proved in 1936 the consistency of PA relative to a quantifier-free theory of arithmetic that included transfinite recursion up to the first epsilon number; in his 1941 lectures at Yale, Gödel proved the consistency of the same theory relative to a theory of computable functionals of finite type. These two "Ansätze" turned out to be most important for subsequent proof theoretic work. Currently it is known how to analyze, in Gentzen-style, strong subsystems of second order arithmetic and set theory. The first prong of proof theoretic investigations, the actual formal development of parts of mathematics, has also been pursued — with a surprising result: the bulk of classical analyis can be developed in theories that are conservative over (fragments of) first-order arithmetic.*



## 1 Metamathematics

Consistency is the crucial logical notion connected with Hilbert's investigations. In traditional, Aristotelian logic it was viewed as a semantic notion: two or more statements are consistent, if they are simultaneously true under some interpretation. In modern logic there is a syntactic definition that fits complex theories since Frege's *Begriffsschrift:* a set of statements is consistent with respect to a logical calculus, if no statement of the form (P&-iP) is derivable from the statements by rules of the calculus. If these definitions are equivalent for a logic we have a significant fact, as the equivalence amounts to the completeness of the logic's system of rules. The first such completeness theorem was obtained for sentential logic by Bernays in (1918) and, independently, by Emil Post in (1921); the completeness of predicate logic was proved by Kurt Gödel in (1930). The crucial step in such proofs shows that "syntactic consistency" implies "semantic consistency". The other direction is established quite directly, but involves the notion of truth. Here we have located one central issue that has motivated proof theoretic investigations, namely, to avoid the uncritical use of the broad concept of classical truth for infinite mathematical structures.

Cantor applied consistency in an informal way to sets. He distinguished, for example in a letter to Dedekind, a consistent from an inconsistent multiplicity. The latter is such "that the assumption that all of its elements 'are together' leads to a contradiction", whereas the elements of the former "can be thought of without contradiction as 'being together'". Cantor had conveyed these distinctions by letter to Hilbert already in 1897; see (Purkert and Ilgauds) for the text of the letters and (Sieg 1990) for the wider historical context. Hilbert pointed out, implicitly in (1900) and explicitly in (1905), that Cantor had not given a rigorous criterion for distinguishing between consistent and inconsistent multiplicities. In (1900) Hilbert suggested to remedy the problem for analysis and to give the proof of the "existence of the totality of real numbers or — in the terminology of G. Cantor — the proof of the fact that the system of real numbers is a consistent (complete) set" by establishing the consistency of an axiomatic characterization of the reals. Indeed, he claimed that the consistency could be established "by a suitable modification of familiar methods". Hilbert's own hints, partly in unpublished lecture notes, and remarks of Bernays make it plausible that he had a model theoretic proof in mind. This "problematic" can be traced back to considerations in Dedekind's essay (1888) as explicated carefully in his letter to Keferstein, where he asked of his notion *simply infinite system:* " ... does such a system *exist* at all in the realms of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions." Clearly, Dedekind tried to establish the consistency by exhibiting a suitable "logical", i.e., set theoretic model.

In 1904 Hilbert began to pursue a completely different strategy for giving consistency proofs. His new way of proceeding was still aimed at securing the existence of sets, but intended to exploit the formalizability of the theory at hand. Formalizations had to satisfy requirements stricter than those imposed on the structure of theories by the traditional (Euclidean) axiomatic-deductive method. The additional requirement was the regimentation of inferential steps in proofs: not only axioms had to be given in advance, but the rules representing steps in mathematical arguments had to be taken from a predetermined list. To avoid a regress in the definition of proof and to achieve intersubjectivity at an absolutely minimal level, the rules had to be "formal" or "mechanical" and had to depend only on the form of statements. Thus, to exclude any ambiguity, a precise and effectively described language was also needed to formalize particular theories; the indications in (Hilbert 1905) were very sketchy, however.

The general kind of requirements had been clear to Aristotle and were explicitly formulated by Leibniz; but only Frege presented in his *Begriffsschrift* -- in addition to an expressively rich language with relations and quantifiers -- an adequate logical calculus. Through the formalization of mathematical proofs Frege pursued a clear philosophical aim, namely, to recognize the "epistemological nature" of theorems. In the introduction to his *Grundgesetze der Arithmetik* Frege wrote: "By insisting that the chains of inference do not have any gaps we succeed in bringing to light every axiom, assumption, hypothesis or whatever else you want to call it on which a proof rests; in this way we obtain a basis for judging the epistemological nature of the theorem." An epistemological analysis of theorems was also aimed for by Hilbert in his work on the foundations of geometry; that was done quite differently, however, relying on a more traditional axiomatic presentation and pushing forward genuinely metamathematical investigations. For the emerging proof theoretic work the formal aspect Frege had emphasized would be exploited by Hilbert, but in a distinctively novel way.

Hilbert gave repeatedly courses on the foundations of mathematics in the period from 1904 to 1917; the spirit of his lectures is captured in his Zürich talk *Axiomatisches Denken*. The mathematical development and philosophical clarification of a new consistency program began to be given only in Hilbert's lectures on *Prinzipien der Mathematik* presented in the winter term of 1917/18. These lectures mark the beginning of the fruitful collaboration with Bernays, who supported Hilbert's preparation in essential ways and also wrote careful notes. The notes of these and later lectures are amazing documents, as one finds in them for the first time a detailed modern presentation of the syntax and informal semantics of predicate logic and of finite type theories. The 1917/18 notes served as the basis for Hilbert and Ackermann's book *Grundzüge der theoretischen Logik* published only in 1928; indeed, the notes contain with minor exceptions all the material presented in the book. In the lectures during the winter terms 1920 and 1921/22, proof theory **and the *finitist consistency program* emerged.**

## 2 Hilbert's Program

For the purpose of the consistency program, metamathematics was taken in Hubert's 1921/22 lectures to be included in, if not co-extensive with, the part of mathematics acceptable to constructivists like Kronecker and Brouwer. The point of consistency proofs was no longer to guarantee the existence of sets, but to establish the instrumental usefulness of classical mathematical theories T, say set theory, with respect to finitist mathematics. That focus rested on the observation that the statement formulating the consistency of T is equivalent to the reflection principle $/^{>}r(a,{}^{l}a^{f}) \Rightarrow a$; here, *Pr* is the finitist proof predicate for T, a a finitistically meaningful statement, and V its translation into the language of T. A finitist consistency proof for T would thus ensure that T is a reliable instrument for the proof of finitist statements. Other important metamathematical issues were the completeness and decidability of theories.

The formalizability of mathematics was obviously crucial for this proof theoretic approach, and the programmatic goal was seen as a way to circumvent some philosophical issues, e.g., concerning the nature of infinite sets in the case of set theory. For these reasons Hilbert's philosophical position is (still) frequently equated with formalism in the sense of Frege's articles *Über die Grundlagen der Geometrie* and of Brouwer's inaugural address *Intuitionism and Formalism.* Such a view is not completely unsupported by some of Hilbert's polemical remarks during the twenties, but on balance his philosophical views developed into a sophisticated instrumentalism, if that label is taken in Ernest Nagel's judicious sense; see (Nagel 1961). Hilbert's is an instrumentalism emphasizing the contentual motivation of mathematical theories; that is perhaps most clearly expressed in the first chapter of (Hilbert and Bernays 1934). A sustained philosophical analysis of proof theoretic research in the context of broader issues in the philosophy of mathematics was provided by Bernays; his penetrating essays stretch over five decades and have been collected in his (1976). (Feferman 1988) and

(Sieg 1988) give complementary accounts; for altogether contrasting approaches, see (Detlefsen 1986) and (Simpson 1988).

The mathematical work is most remarkable for what it started, as it constitutes the beginnings of modern mathematical logic. Even before Gödel and Gentzen's work it was rich in accomplishments: consider, as examples, the completeness proof of Bernays for sentential logic; the partial solutions to the decision problem for predicate logic (i.e., Hilbert's *Entscheidungsproblem)* obtained by Behmann, Bernays, Schönfinkel, and Herbrand; the consistency proofs given by Ackermann, von Neumann, and Herbrand. Taking for granted the broader conceptual clarifications and the focus on first order logic, Herbrand's "theorème fondamentale" is perhaps the most significant result on purely logical grounds, but also because of its applicability in consistency proofs. Gödel pointed to its essence when he gave in (1933o) the following formulation of HERBRAND's THEOREM: "If we take a theory which is constructive in the sense that each existence assertion made in the axioms is covered by a construction, and if we add to this theory the non-constructive notion of existence and all the logical rules concerning it, e.g., the law of excluded middle, we shall never get into any contradiction."

The results obtained in the twenties, including Herbrand's Theorem, were disappointing when measured against the hopes and ambitions in the Hilbert school: Ackermann, von Neumann, and Herbrand had established essentially the consistency of first order arithmetic with a very restricted principle of induction (for quantifier-free formulas). Actual limits on finitist considerations for consistency proofs had been reached; that became clear in 1931 through GÖDEL'S THEOREMS and the realization that finitist proofs could be formalized in a weak fragment of number theory. Initially, Gödel did not share the view on the limits of finitist reasoning, as is clear from the final remarks in his (1931) and contemporaneous correspondence with von Neumann and Herbrand. But by December 1933, when he lectured in Cambridge, he had changed his position: having isolated a "system A", essentially a version of primitive recursive arithmetic (PRA), he made the following claim: "Now all the intuitionistic [i.e., finitist] proofs complying with the requirements of the system A which have ever been constructed can easily be expressed in the system of classical analysis and even in the system of classical arithmetic, and there are reasons for believing that this will hold for any proof which one will ever be able to construct."

**The solvability of the Entscheidungsproblem had been made implausible by Gödel's results, but the actual proof of unsolvability had to wait until 1936 for a conceptual clarification of "mechanical procedure" or "algorithm". Such a clarification was achieved mainly through the work of Church and Turing, see COMPUTABILITY THEORY and CHURCHs THESIS. A precise notion of mechanical procedure was also needed to prove the incompleteness theorems for *general* "formal" theories satisfying basic representability and derivability conditions; after all, Gödel had established limits only for (formalizations of) particular theories, like the system of *Principia Mathematica* and then current axiomatic set theories. In his attempt to characterize a proper extension of the class of primitive recursive functions, Gödel introduced in his Princeton lectures of 1934 the general recursive functions through an equational calculus. The informal concept underlying Gödel's and also Church's approach, calculability of functions in a "formal" calculus, was carefully analyzed by Hilbert and Bernays in Supplement II of their book (1939); they formulated recursiveness conditions for general deductive formalisms and showed that the number theoretic functions whose values can be calculated in formalisms satisfying these conditions *{reckonable functions)* are exactly the general recursive ones.**

**The impact of the incompleteness theorems on Hilbert's Program was profound and yet limited. On the one hand, as remarked above, they pointed out definite limits of finitist considerations; on the other hand, they left open the possibility of modifying the program to a *general reductive program* that was no longer aiming for "absolute" finitist consistency proofs, but rather for consistency proofs relative to "appropriate" constructive theories. Before Gödel's results were known, Bernays had given in his (1930) a detailed and searching analysis of the philosophical aims of Hilbert's proof theory; in the *Postscriptum* to that paper published in (1976), Bernays expressed clearly what was lost due to the incompleteness theorems: the sharp distinction of what is intuitive and what is non-intuitive, a distinction that was basic for the proposed philosophical treatment of the problem of the infinite. Thus, it is the particular "solution" to a philosophical problem that was shown to be impossible.**

**Work in proof theory continued with the explicit goal of achieving relative consistency proofs. Such work is in a venerable mathematical tradition, as the many examples of significant results**

show, e.g., the consistency of non-Euclidean relative to Euclidean geometry, that of Euclidean geometry relative to analysis; the consistency of set theory with the axiom of choice relative to set theory (without the axiom of choice), that of set theory with the negation of the axiom of choice relative to set theory. The mathematical significance of relative consistency proofs is often brought out by sharpening them to conservative extension results. Such results may ensure, for example, that the theories have the same class of provably total functions, see section 4. However, the initial motivation for such arguments is most frequently philosophical: one wants to guarantee the coherence of the original theory on an epistemologically distinguished basis. In this spirit one has to see the specific results that have been obtained in the pursuit of the general reductive program.

## 3 Mathematical work and logical tools

The development of the general reductive program is characterized by modifications of the two-pronged approach of Hilbert's original program, namely: (i) weakening the theories in which parts of mathematics are formalized, and (ii) strengthening the theories in which the metamathematical considerations are carried out and, consequently, relative to which constructive consistency proofs can be given. The first modification reaches back to Weyl's book *Das Kontinuum* and culminated in the seventies, when it was realized that the classical results of mathematical analysis can be obtained in conservative extensions of first order arithmetic. The second modification started with work of Gödel and Gentzen in 1933, when they established independently the consistency of classical arithmetic relative to intuitionistic arithmetic; it led in the seventies and eighties to consistency proofs of subsystems of second order arithmetic or, synonymously, subsystems of analysis relative to intuitionistic theories of constructive ordinals. Obviously, only a sketch of some main results can be attempted here.

Second order arithmetic has been used for some time as a framework for the formal development of classical mathematical analysis; i.e., the theory of the continuum set-theoretically described by Dedekind and Cantor. Because of this mathematical adequacy, second order arithmetic and a variety of subsystems have been thoroughly investigated. The main set theoretic principles for these systems are the comprehension axiom

CA $(\exists X)(\forall y)(y \varepsilon X \Leftrightarrow S(y))$

and the axiom of choice in the form

AC $(\forall x)(\exists Y)S(x,Y) \Rightarrow (\exists Z)(\forall x)S(x,(Z)_x)$ ,

where S is in each case an arbitrary formula of the language and thus may contain set quantifiers; $y\varepsilon(Z)_x$ is defined as $<y,x> \varepsilon Z$. These principles are impredicative, as the sets X and Z whose existence is postulated are in general characterized by reference to all sets of natural numbers. The induction principle is formulated either as a schema or as a second-order axiom

IND $(\forall X)[0\varepsilon X \,\&\, (\forall y)(y\varepsilon X \Rightarrow y'\varepsilon X) \Rightarrow (\forall x)x\varepsilon X]$ .

Theories are denoted by the name of their set existence principle enclosed in parentheses; thus (CA) names full analysis. If $\restriction$ follows its name, a theory uses the second-order axiom IND to formalize induction. Two general results are of interest, as they show that second order arithmetic has a certain robustness: (CA) is proof-theoretically equivalent to Zermelo-Fränkel set theory without the power set axiom; (AC) is conservative over (CA) for $\Pi^1_4$-formulas and properly stronger, as there is a $\Pi^1_2$-instance of AC not provable in (CA). In the presence of full CA, the theories with the induction schema, respectively IND are equivalent, but they can be of strikingly different strength, when the set existence principles are restricted; for example, $(\Pi^0_\infty\text{-CA})\restriction$ is conservative over PA, whereas $(\Pi^0_\infty\text{-CA})$ proves the consistency of PA.

Hilbert and Bernays used ramified type theory with the axiom of reducibility in (1917/18) to develop analysis; the presentation in Supplement IV of their (1939) is based on this early work, but employs full second order arithmetic as the formal framework. They encouraged developments with restricted means already in (1920), where they wrote w.r.t. Brouwer and Weyl: "The positive and fruitful part of the investigations into the foundations of mathematics carried out by these two researchers fits into the mold of the axiomatic method and is exactly in the spirit of this method. For one investigates here, how a part of analysis can be delimited by a certain narrower system of assumptions." (p. 34) Subsystems of analysis are now mainly defined by restricting S in the set existence schemata to particular classes of formulas. The set theoretic demands can be reduced dramatically: Hilbert and Bernays's 1939 presentation, for example, can be given quite readily in (Ill-CA)l\ Strictly mathematical work continued to accompany work on consistency proofs for subsystems; it had the aim of establishing the mathematical significance of subsystems and made use of work in the constructivist tradition.

By the mid-seventies, through final efforts of Feferman, Friedman, and Takeuti it was clear that classical analysis could be carried out in conservative extensions of number theory, for example, (n£-CA)l\ In this context Friedman suggested to pursue a strategy, familiar from investigations of the axiom of choice in set theory, namely, to establish the equivalence of certain set existence principles with mathematical theorems. This theme is played with surprising variations in Friedman's and Simpson's work on subsystems and gave rise to the enterprise of *Reverse Mathematics.* To mention just two, by now "classical", examples: CA for arithmetic formulas is equivalent to the theorem that every bounded sequence of reals has a least upper bound and to König's lemma. — Friedman introduced a second order theory WKLo that extends primitive recursive arithmetic conservatively for $\Pi^0_2$-formulas; the theory is weak, but still provides a very good basis for developing parts of analysis and algebra. Simpson considers in (1988) the development of mathematics in such a conservative extension o^ PRA not only as a "reductionist program", but equates it with Hilbert's Program. One should recall, however, that Hilbert did not propose to redo all of mathematics in (a conservative extension of) PRA, but rather to justify — via finitist consistency proofs — the use of strong classical theories sufficient for the direct formalization of mathematical practice.

Hilbert's central idea for the metamathematical treatment of the consistency problem found its expression in the e-calculus and the associated substitution method. In (Hilbert and Bernays 1939) is a presentation of what was achieved in its terms; even Herbrand's (difficult) work was recast in terms of the e-calculus. This tradition was kept up by Tait (1965) and more recently by Mints (1994). However, other logical tools turned out to be more useful for proof theoretic investigations: Gentzen's sequent calculi and Gödel's so-called Dialectica Interpretation. As to the latter, Gödel used computable functionals of finite type to obtain a reduction of intuitionistic arithmetic; joining this reduction with the consistency of PA relative to HA, a consistency proof relative to the system of these functionals was obtained. Influenced by the considerations in (Hilbert 1926), Gödel presented this work already in a lecture at Yale (1941), but published it only in (1958). Spector gave in (1962) a consistency proof for full classical analysis using bar recursive functionals of finite type; this proof prompted a searching analysis in the *Stanford Report on the foundations of analysis*, Stanford, 1963.

Here the focus is on sequent calculi, as they have found the most extensive use and widest applicability. In the form given to them by (Tait 1968), they allow the proof of finite sets of formulas built up from literals (atomic formulas or negations of such), conjunction, disjunction, universal and existential quantification, and — depending on the theory — infinitary conjunction and disjunction. Thus, for just first order logic the basic logical symbols are A, V, 3, V, and the rules of the calculi include the following ones, where P is used as a syntactic variable ranging over finite sets of formulas and F, $\varphi$ stands for the union of r and the singleton $\varphi$:

LA: r, 9,-19, cp atomic

$\wedge$: $$\frac{\Gamma,\varphi_0 \qquad \Gamma,\varphi_1}{\Gamma,\varphi_0\wedge\varphi_1}$$

$\vee_i$: $$\frac{\Gamma,\varphi_i}{r, \varphi_0\vee\varphi_1}, \qquad i = 0,1$$

C: $$\frac{\Gamma,\varphi \qquad \Gamma,\neg\varphi}{r}$$

V: DS? a * P(D

r,(Vx)(px

3: r,g>t

r, (3x)<px

ae P(F) means that the parameter a occurs in one of the formulas in T. The crucial claim established by Gentzen was his Hauptsatz or cut-elimination theorem: every derivation in the logical system using the cut-rule C_ can be transformed into a cut-free or normal derivation.

Inspecting the rules, one notices that the premises of all rules (except for C) contain only subformulas of formulas in the conclusion. Consequently, a normal derivation of a sequent F contains only subformulas of elements in F. This is the crucial subformula property of normal derivations, providing a bound on the complexity of formulas that can occur in a proof of F. In (Gentzen 1934) this metamathematical fact is established and used to obtain in a most perspicuous way the most far-reaching consistency result that had been obtained, namely Herbrand's.

For full first order number theory PA this treatment was extended by Gentzen (1936) to a partial cut-elimination argument whose termination was established by quantifier-free transfinite induction up to the first epsilon number, TI(£o). In (1943), Gentzen showed that this induction schema for every ordinal a less than £Q can be established in PA; this is the first "ordinal analysis" of a formal theory. In the fifties Lorenzen and, much more extensively, Schiitte used infinitary extensions of Gentzen's finitary systems; in particular for the treatment of PA they used the so-called co-rule that allows one to infer F,(Vx) <|>x from the premises F,<|>n for each natural number n. Though derivations are now infinite, PA-derivations can be embedded into finitistically described ones and can be transformed effectively into cut-free derivations; the natural ordinal length of these derivations is bounded by £o. Schiitte extended these methods to treat systems of ramified analysis $RA_a$ (of order a) and obtained, in particular, ordinal bounds on the length of normal derivations in terms of the Veblen-hierarchy of ordinal functions. This work was used in 1963 by Feferman and Schiitte independently to characterize the ordinal FQ of predicative analysis, i.e., the first ordinal a such that TI(a) cannot be proved in RAp for p less than a.

## 4 Reductive Results

The mathematical work just described was to a large extent inspired by the attempt to establish the significance of relative consistency proofs and/or to focus on manageable subsystems that might be a proper next target for such proofs. Most of the reductive results mentioned below can be established by the metamathematical tools, the sequent calculi, though some of the original proofs used different techniques.

The Gödel-Gentzen result for number theory can be extended to ramified analysis and also to obtain conservativeness for II2-formulas. This provides a relative consistency proof, as ramified systems with intuitionistic logic are certainly acceptable constructively (as long as the ordinals along which the systems are iterated are acceptable). In the early sixties, partly through the study of predicativity, significant subsystems with S restricted to small classes of analytic formulas were isolated, among them (X\ -AC) and (A}-CA). Kreisel pointed out that (ZJ-AC) 2 (A†-CA). Friedman showed that (Z|-AC) is conservative over (A}-CA) for II^-formulas; that the inclusion is proper was established later by Steel. Indeed, (ZJ-AC) is conservative for n^-formulas over (nd-CA)$_{<£0}$, a theory based on the transfinite iteration of the jump-operator and equivalent to ramified analysis of level less than eo- This result allowed the determination of the proof-theoretic ordinal of the systems, but it showed also — and that was quite unexpected -- that these prima facie impredicative theories were predicative in the sense of Feferman and Schiitte. (Feferman 1964) and (Kreisel 1968) provide excellent summaries.

The above theorem for (X}-AC) turned out to be a special case of a general result: ($£^*_{n+1}$-AC) is conservative over (Fft-CA)^ for formulas in $F_n$, where $F_0$ is II2, $F_t$ is *Tl*, and $F_n$ is TI4 for n>l; this was established in (Friedman 1970). As just explained, the case n=0 was of special interest for the study of predicativity; the case n=l is also to be placed in the context of foundational investigations. The most immediate context is provided by Feferman's (1970) in which the systems (rfj-CA)^ were related to the classical theory for (less than $e_0$ times iterated) i.d. classes, i.e., classes defined by generalized inductive definitions. Well-known examples are the classes O of constructive ordinals and W of recursive well-founded trees. Together, the papers reduced the subsystems (£2-AC) and (A2-CA) to the classical theory of the tree classes $W_v$ with index v less than $e_0$. Feferman's (1977) gives a detailed survey of mathematical and proof theoretic work, including these last results.

Kreisel had introduced intuitionistic theories of iterated inductive definitions in the *Stanford Report;* these theories were viewed as codifying constructive principles that might be used in consistency proofs for subsystems of analysis. Feferman and

Friedman's work described above established connections between subsystems of analysis and classical theories of inductive definitions, making a crucial step towards answering the major problem posed in (Kreisel 1968): reduce (£$^1_2$-AC) to a constructive theory of inductive definitions; that would provide, as Kreisel put it then, "a solution to Hilbert's problem for the subsystem of analysis ... (E$^1_2$-AC)". The classical theories for i.d. classes are reducible to intuitionistic theories for accessible i.d. classes. This allowed, in particular, a satisfactory solution of Kreisel's open problem, as (E$^1_2$-AC) was shown to be reducible to (ID)<eo(0), the intuitionistic theory of constructive number classes with index less than e$_0$. These and many further results were obtained in (Buchholz e.a. 1981), in particular, the ordinal analysis of the subsystems at hand. This work was influenced by earlier considerations of Howard, Tait, and Takeuti.

Subsystems of set theory, in particular of admissible set theory, were used by Jäger and Pohlers in the early eighties to provide a unifying approach to the investigations and, so it was hoped, an avenue for analyzing even stronger systems than those corresponding to the subsystems mentioned above; cf. (Jäger 1986). This has indeed been a succesful strategy and reveals, in the work of Rathjen and others, a deep connection between large cardinals and the constructive ordinals needed for the proof theoretic investigation of such systems. Rathjen succeeded in analyzing (ΤΙ^-CA) ; it seems that the techniques developed for this case/ might allow the treatment of full analysis. (Rathjen 1995) gives an informative account of these proof theoretic investigations. They can no longer be motivated by the concern of "securing" mathematical practice: the systems that are investigated are much stronger than needed for practice; the constructive ordinals used in the metamathematical theory are obtained in analogy to "large cardinals" in set theory.

The systems WKL$_0$, (r£-CA)l\ and (n}-CA)r have been recognized as significant for the formalization of mathematical practice and are reducible to theories based on principles that are acceptable from constructive positions; after all, they are conservative for n^-formulas over primitive recursive arithmetic (PRA), intuitionistic number theory (HA), and the intuitionistic theory for the finite constructive number classes (ID)<Q)(O). This provides a coherent perspective bringing out the complementary character of mathematical and metamathematical work that

ultimately aims for relating significant parts of mathematical practice to distinctive foundational positions. But there is no obvious answer to the question "What is the mathematical significance of those subsystems, when taken as vehicles for the formal axiomatic study of ordinary mathematics?"; similarly there is no obvious answer to the question "What is the philosophical significance of the corresponding systems PRA, HA, and $(ID)_{<\omega}(O)$, when taken as formal expressions of foundational positions?". There is ample room for reflection on these questions; the work reported here and in the literature provides rich and crucial data; cf. (Sieg 1990). For the philosophical reflection on the foundations of mathematics the investigations of subsystems of set theory provide additional, significant material: what are constructions that lead to "accessible domains", how is it that we recognize their associated laws?

## 5 Outlook

"Internal" mathematical and philosophical challenges of work in proof theory were sketched at the end of the last section. However, the foundational goals of proof theoretic investigations have been complemented over the last few decades by other important directions.

First, Kreisel initiated in the fifties work that was to exploit the gap between provability in particular formal theories and truth. That led, on the one hand, to "global" characterizations of the provably total functions of theories and to related independence results. On the other hand, by attending "locally" to mathematical details of proofs and by using proof theoretic techniques, it led to explicit computational information of mathematical significance. This seems to have come to fruition through recent work of Luckhardt and Kohlenbach.

Second, methods and results of mathematical logic, but in particular of proof theory, are playing an increasing role in computer science. (Clearly, there has been significant and stimulating influence also in the other direction.) Various (type) systems, e.g., Martin-Löf's, Girard's F, Feferman's systems of explicit mathematics, have been used for the presentation of proofs and computations, but also for describing transformations on them.

Third, there is a direct connection to the general topic of theorem proving; investigations here, when focusing on automated proof search, might reflect back into proof theory by providing data **for a *structural theory of (mathematical) proofs.* Such a structural** proof theory would go beyond the representation of proofs in formal theories and articulate search heuristics expressing "leading mathematical ideas" for particular parts of mathematics; Saunders MacLane suggested already in his Göttingen dissertation of 1934 such an extension of proof theoretic investigations.

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