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**K-GRAPH MACHINES: Generalizing
Turing's Machines and Arguments**

by

Wilfried Sieg and John Byrnes

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Abstract

The notion of *mechanical process* has played a crucial role in mathematical logic since the early thirties; it has become central in computer science, artificial intelligence, and cognitive psychology. But the discussion of Church's Thesis, which identifies the informal concept with a mathematically precise one, has hardly progressed beyond the pioneering work of Church, Gödel, Post, and Turing. Turing addressed directly the question: *What are the possible mechanical processes a human computer can carry out in calculating values of a number-theoretic function?* He claimed that all such processes can be simulated by machines, in modern terms, by deterministic Turing machines. Turing's considerations for this claim involved, first, a formulation of boundedness and locality conditions (for linear symbolic configurations and mechanical operations); second, a proof that computational processes (satisfying these conditions) can be carried out by Turing machines; third, the central thesis that all mechanical processes carried out by human computers must satisfy the conditions. In Turing's presentation these three aspects are intertwined and important steps in the proof are only hinted at. We introduce *K-graph machines* and use them to give a detailed mathematical explication of the first two aspects of Turing's considerations for general configurations, i.e. K-graphs. This generalization of machines and theorems provides, in our view, a significant strengthening of Turing's argument for his central thesis.

Introduction

I

Turing's analysis of effective calculability is a paradigm of a foundational study that (i) led from an informally understood concept to a mathematically precise notion, (ii) offered a detailed investigation of the new mathematical notion, and (iii) settled an important open question, namely the *Entscheidungsproblem*. The special character of Turing's analysis was recognized immediately by Church in his review of Turing's 1936 paper. The review was published in the first issue of the 1937 volume of the *Journal of Symbolic Logic*, and Church contrasted in it Turing's mathematical notion for effective calculability (via idealized machines) with his own (via A-definability) and Gödel's general recursiveness and asserted: "Of these, the first has the advantage of making the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately ___"

Gödel had noticed in his (1936) an "absoluteness" of the concept of computability, but found only Turing's analysis convincing; he claimed that Turing's work provides "a precise and unquestionably adequate definition of the general concept of formal system" (1964, p. 369). As a formal system is simply defined

to be a mechanical procedure for producing theorems, the adequacy of the definition rests on Turing's analysis of mechanical procedures. And with respect to the latter Gödel remarked (pp. 369-70): "Turing's work gives an analysis of the concept of 'mechanical procedure' (alias 'algorithm' or 'computation procedure' or 'finite combinatorial procedure'). This concept is *shown* [our emphasis] to be equivalent with that of a 'Turing machine'." Nowhere in Gödel's writings is there an indication of the nature of Turing's conceptual analysis or of a proof for the claim that the analyzed concept is equivalent with that of a Turing machine.

Gödel's schematic description of Turing's way of proceeding is correct: in section 9 of (Turing 1936) there is an analysis of effective calculability, and the analysis is intertwined with a sketch of an argument showing that mechanical procedures on linear configurations can be performed by very restricted machines, i.e., by deterministic Turing machines over a two-letter alphabet. Turing intended to give an analysis of mechanical processes on planar configurations; but such processes are not described, let alone proved to be reducible to computations on linear objects. This gap in Turing's considerations is the starting-point of our work. We formulate broad boundedness and locality conditions that emerge from Turing's conceptual analysis, give a precise mathematical description of planar and even more general computations, and present a detailed reductive argument. For the descriptive part we introduce *K-graph machines*; they are a far-reaching generalization of Post production systems and thus, via Post's description of Turing machines, also of Turing machines.

1 Turing's Analysis¹

In 1936, the very year in which Turing's paper appeared, Post published a computation model strikingly similar to Turing's. Our brief discussion of Post's model is not to emphasize this well-known similarity, but rather to bring out the strikingly dissimilar methodological attitudes underlying Post's and Turing's work. Post has a worker operate in a symbol space consisting of "a two way infinite sequence of spaces or boxes ...".² The boxes admit two conditions: they can be unmarked or marked by a single sign, say a vertical stroke. The worker operates in just one box at a time and can perform a number of *primitive acts*: make a vertical stroke [*V*], erase a vertical stroke [*23*], move to the box immediately to the right [*M_r*] or to the left [*M_l*] (of the box he is in), and determine whether the box he is in is marked or not [*D*]. In carrying out a "combinatory process" the worker begins in a special box and then follows directions from a finite, numbered sequence of instructions. The *t*-th direction,

¹This section is based on (Sieg 1994) which was completed in June 1991; for details of the reconstruction of Turing's analysis and also for the broader systematic and historical context of our investigations we refer the reader to that paper.

²Post remarks that the infinite sequence of boxes is ordinally similar to the series of integers and can be replaced by a potentially infinite one, expanding the finite sequence as necessary.

i between 1 and n , is in one of the following forms: (i) carry out act V , E , M_r , or M_i and then follow direction $\#$, (ii) carry out act D and then, depending on whether the answer is positive or negative, follow direction $j|$ or $j|'$. (Post has a special stop instruction, but that can be replaced by the convention to halt, when the number of the next direction is greater than n .)

Are there intrinsic reasons for choosing this formulation as an explication of effective calculability, except for its simplicity and Post's expectation that it will turn out to be equivalent to recursiveness? An answer to this question is not clear from Post's paper, at the end of which he wrote:

The writer expects the present formulation to turn out to be equivalent to recursiveness in the sense of the Gödel-Church development. Its purpose, however, is not only to present a system of a certain logical potency but also, in its restricted field, of psychological fidelity. In the latter sense wider and wider formulations are contemplated. On the other hand, our aim will be to show that all such are logically reducible to formulation 1. We offer this conclusion at the present moment as a *working hypothesis*. And to our mind such is Church's identification of effective calculability with recursiveness.

Investigating wider and wider formulations and *reducing* them to *Formulation 1* would change for Post this "hypothesis not so much to a definition or to an axiom but to a *natural law*". It is methodologically remarkable that Turing proceeded in *exactly* the opposite way when trying to justify that all computable numbers are machine computable or, in our way of speaking, that all effectively calculable functions are Turing computable: He did not extend a narrow notion reducibly and, in this way, obtain quasi-empirical support, but rather analyzed the intended broad concept and reduced it to a narrow one, once and for all. The intended concept was mechanical calculability by a human being, and in the reductive argument Turing exploited crucially limitations of the computing agent.

Turing's *On computable numbers* opens with a description of what is ostensibly its subject, namely, "real numbers whose expressions as a decimal are calculable by finite means". Turing is quick to point out that the problem of explicating "calculable by finite means" is the same when considering, e.g., computable functions of an integral variable. Thus it suffices to address the question: "What does it mean for a real number to be calculable by finite means?" But Turing develops first the theory of his machines.³ A Turing machine consists of a finite, but potentially infinite tape; the tape is divided into squares, and each square may carry a symbol from a finite alphabet, say, the two-letter alphabet consisting of 0 and 1. The machine is able to scan one square at a time and perform, depending on the content of the observed square and its own internal

³Note that the presentation of Turing machines we give is not Turing's, but rather the one that evolved from Post's formulation in (1947).

state, one of four operations: print 0, print 1, or shift attention to one of the two immediately adjacent squares. The operation of the machine is given by a finite list of commands in the form of quadruples $q_i s_k c_l q_m$ that express: if the machine is in internal state q_i and finds symbol s_k on the square it is scanning, then it is to carry out operation c_l and change its state to q_m . The deterministic character of the machine operation is guaranteed by the requirement that a program must not contain two different quadruples with the same first two components.

In section 9 Turing argues that the operations of his machines “include all those which are used in the computation of a number”. But he does not try to establish the claim directly; he rather attempts to answer what he views as “the real question at issue”: “What are the possible processes which can be carried out [by a computer⁴] in computing a number?” Turing imagines a computer writing symbols on paper that is divided into squares “like a child’s arithmetic book”. As the two-dimensional character of this computing space is taken—*without any argument*—not to be essential, Turing considers the one-dimensional tape divided into squares as the basic computing space and formulates one important restriction. The restriction is motivated by limits of the human sensory apparatus to distinguish *at one glance* between symbolic configurations of sufficient complexity and states that only finitely many distinct symbols can be written on a square. Turing suggests as a reason that “If we were to allow an infinity of symbols, then there would be symbols differing to an arbitrarily small extent”, and we would not be able to distinguish at one glance between them. A second and clearly related way of arguing this point uses a finite number of symbols and strings of such symbols. E.g., Arabic numerals like 9979 or 9989 are seen by us at one glance to be different; however, it is not possible for us to determine immediately that 9889995496789998769 is different from 98899954967899998769. This second avenue suggests that a computer can operate directly only on a finite number of (linear) configurations.

Now we turn to the question: What determines the steps of the computer, and what kind of elementary operations can he carry out? The behavior is *uniquely* determined at any moment by two factors: (i) the symbolic configuration he observes and (ii) his internal state. This uniqueness requirement may be called the **determinacy condition (D)**; it guarantees that computations are deterministic. Internal states, or as Turing also says “states of mind”, are introduced to have the computer’s behavior depend possibly on earlier observations and, thus, to reflect his experience. Since Turing wanted to isolate operations of the computer that are “so elementary that it is not easy to imagine them further divided”, it is crucial that symbolic configurations that help fix the conditions for a computer’s actions are immediately recognizable. We are thus led to postulate that a computer has to satisfy two **boundedness conditions**:

⁴Following Gandy, we distinguish between a *computer* (a human carrying out a mechanical computation) and a *computer* (a mechanical device employed for computational purposes); cf. (Gandy 1988), p. 81, in particular fn. 24.

(B.1) *there is a fixed bound for the number of symbolic configurations a computer can immediately recognize;*

(B.2)⁵ *there is a fixed bound for the number of internal states that need be taken into account.*

For a given computer there are consequently only boundedly many different combinations of symbolic configurations and internal states. Since his behavior is, according to (D), uniquely determined by such combinations and associated operations, the computer can carry out at most finitely many different operations. These operations are restricted by the following **locality conditions**:

(L.1) *only elements of observed symbolic configurations can be changed;*

(L.2) *the distribution of observed squares can be changed, but each of the new observed squares must be within a bounded distance of an immediately previously observed square.*

Turing emphasized that "the new observed squares must be immediately recognisable by the [computer]"; that means the observed configurations arising from changes according to (L.2) must be among the finitely many ones of (B.1). Clearly, the same must hold for the symbolic configurations resulting from changes according to (L.1). Since some steps may involve a change of internal state, Turing concluded that the most general single operation is a change *either* of symbolic configuration and, possibly, internal state *or* of observed square and, possibly, internal state. With this restrictive analysis of the steps a computer can take, the proposition that his computations can be carried out by a Turing machine is established rather easily.⁶ Thus we have:

Theorem 1 (Turing's Theorem for calculable functions) *Any number theoretic function F that can be calculated by a computer satisfying the determinacy condition (D) and the conditions (B) and (L) can be computed by a Turing machine.*

As the Turing computable functions are recursive, F is recursive. This argument for F 's recursiveness does not appeal to any form of Church's Thesis; rather, such an appeal is replaced by the assumption that the calculation of F is done by a computer satisfying the conditions (D), (B), and (L). If that assumption is to be discharged a substantive thesis is needed. We call this thesis—that a mechanical computer must satisfy the conditions (D) and (B), and that the

⁵Gödel objected in (1972) to this condition for a notion of human calculability that might properly extend mechanical calculability; for a computer it seems quite unobjectionable.

⁶Turing constructed machines that mimic the work of computers on linear configurations directly and observed: "The machines just described do not differ very essentially from computing machines as defined in § 2, and corresponding to any machine of this type a computing machine can be constructed to compute the same sequence, that is to say the sequence computed by the computer [in our terminology: compute[^]." Cf. section 2 below for this reductive claim.

elementary operations he can carry out must be restricted as conditions (L) require—**Turing's Central Thesis**.

In the historical and systematic context in which Turing found himself, he asked exactly the right question: What are the processes a computer can carry out in calculating a number? The general problematic *required* an analysis of the idealized capabilities of a computer, and exactly this feature makes the analysis epistemologically significant. The separation of conceptual analysis (leading to the axiomatic conditions) and rigorous proof (establishing Turing's Theorem) is essential for clarifying on what the correctness of his central thesis rests; namely, on recognizing that the axiomatic conditions are true for computers who proceed mechanically. We have to remember that clearly when engaging in methodological discussions concerning artificial intelligence and cognitive science. Even Gödel got it wrong, when he claimed that Turing's argument in the 1936 paper was intended to show that "mental processes cannot go beyond mechanical procedures".

2 Post Productions & Puzzles

Gödel's misunderstanding of the intended scope of the analysis may be due to Turing's provocative, but only figurative attribution of "states of mind" to machines; it is surprising nevertheless, as Turing argues at length for the eliminability of states of mind in section 9 (HI) of his paper. He describes there a modified computer and avoids the introduction of "state of mind", considering instead "a more physical and definite counterpart of it". The computer is now allowed to work in a desultory manner, possibly doing only one step of the computation at a sitting: "It is always possible for the [computer] to break off from his work, to go away and forget all about it, and later to come back and go on with it." But on breaking off the computer must leave a "note of instruction" that informs him on how to proceed when returning to his job; such notes are the "counterparts" of states of mind. Turing incorporates notes into "state formulas" (in the language of first order logic) that describe states of a machine mimicking the computer and formulates appropriate rules that transform a given state into the next one.

Post used in (1947) a most elegant way of describing Turing machines purely symbolically via his production systems (on the way to solving, negatively, the word-problem for semi-groups).⁷ The configurations of a Turing machine are given by *instantaneous descriptions* of the form $aqiSk/3$, where a and $/?$ are possibly empty strings of symbols in the machine's alphabet; more precisely, an id contains exactly one state symbol, and to its right there must be at least one symbol. Such id 's express that the current tape content is $as*/?$, the machine is in state qi , and it scans (a square with symbol) $s*$. Quadruples $qiSkCiQm$ of

⁷Post's way of looking at Turing machines underlies also the presentation in (Davis 1958); for a more detailed discussion the reader is referred to that classical text.

the program are represented by rules; for example, if the operation Q is *print 0*, the corresponding rule is:

$$ctqiS_kp \Rightarrow aq_m0/3.$$

That can be done, obviously, for all the different operations; one just has to append 0 or 5n to a (*ft*) in case $c/$ is the operation *move to the left (right)* and a ($/?$) is the empty string—reflecting the expansion of the only potentially infinite tape by a blank square. This formulation can be generalized so that machines operate directly on finite strings of symbols; operations can be indicated as follows:

$$ayqiS/3 \Rightarrow a-y^*q_m\delta^*0.$$

If in internal state qi a *string machine* recognizes the string jS (i.e., takes in the sequence at one glance), it replaces that string by $y^*\delta^*$ and changes its internal state to g_m . Calling ordinary Turing machines *letter machines*, Turing's claim reported in note 6 can be formulated as a **Reduction Lemma**: Any computation of a string machine can be carried out by a letter machine.

The rule systems describing string machines are semi-Thue systems and, as the latter, not deterministic, if their programs are just sequences of production rules. The usual non-determinism certainly can be excluded by requiring that, if the antecedents of two rules coincide, so must the consequents. But that requirement does not remove every possibility of two rules being applicable simultaneously: consider a machine whose program includes in addition to the above rule also the rule

$$\alpha\gamma^{\#}q_l\delta^{\#}\beta \Rightarrow \alpha\gamma^{\perp}q_n\delta^{\perp}\beta,$$

where δ^* is an initial segment of rf , and $\delta^{\#}$ is an end segment of 7 ; then both rules would be applicable to $^{\wedge}qiS$. This kind of non-determinism can be excluded in a variety of ways, for example, by ordering the rules and always using the first applicable rule; this approach was taken by Markov in his 1954 *Theory of Algorithms*.

However, as we emphasized already, Turing had intended to analyze genuine planar computations, not just string machines or letter machines operating in the plane.⁸ To formulate and prove a Reduction Lemma for planar computations, one has to specify the finite symbolic configurations that can be operated on and the mechanical operations that can be performed. Turing recognized the significance of Post's presentation for achieving mathematical results, but also for the conceptual analysis of calculability: as to the former, Turing extended in his (1950) Post's and Markov's result concerning the unsolvability of

⁸Such machines are also discussed in Kleene's *Introduction to Metamathematics*, pp. 376-381, in an informed and insightful defense of Turing's Thesis. However, in Kleene's way of extending configurations and operations, much stronger normalizing conditions are in place; e.g., when considering machines corresponding to our string machines the strings must be of the same length.

the word-problem for semi-groups to semi-groups with cancellation; as to the latter, we look at Turing's semi-popular and most informative presentation of *Solvable and Unsolvable Problems* (1953).

Turing starts out with a description of puzzles: square piece puzzles, puzzles involving the separation of rigid bodies or the transformation of knots; i.e., puzzles in two and three dimensions. "Linear" puzzles are described as Post systems and called *substitution puzzles*. They are viewed by Turing as a "normal" or "standard" form of describing puzzles; indeed, a form of the Church-Turing thesis is formulated as follows:

Given any puzzle we can find a corresponding *substitution puzzle* which is equivalent to it in the sense that given a solution of the one we can easily find a solution of the other. If the original puzzle is concerned with rows of pieces of a finite number of different kinds, then the substitutions may be applied as an alternative set of rules to the pieces of the original puzzle. A transformation can be carried out by the rules of the original puzzle if and only if it can be carried out by the substitutions... (1953, p.15)

Turing admits, with some understatement, that this formulation is "somewhat lacking in definiteness" and claims that it will remain so; he characterizes its status as lying between a theorem and a definition: "In so far as we know *a priori* what is a puzzle and what is not, the statement is a theorem. In so far as we do not know what puzzles are, the statement is a definition which tells us something about what they are." Of course, Turing continues, one could define puzzle by a phrase beginning with 'a set of definite rules', or one could reduce its definition to that of 'computable function' or 'systematic procedure'. A definition of any of these notions would provide one for puzzles.

Even before we had seen Turing's marvelous 1953 paper, our attempts of describing mechanical procedures on general symbolic configurations had made use of the puzzle-metaphor. The informal idea had three distinct components: a computer was to operate on finite *connected* configurations; such configurations were to contain a unique *distinguished element* (corresponding to the scanned square); the operations were to *substitute neighborhoods* (of a bounded number of different forms) of the distinguished element by appropriate other neighborhoods resulting in a new configuration, and such substitutions were to be given by *generalized production rules*. Naturally, the question was how to transform this into appropriate mathematical concepts; referring to Turing's statement above, we were unwittingly trying to remove (as far as possible) the *lack of definiteness* in the description of general puzzles. But in contrast to Turing, we wanted to analyze deterministic procedures and follow more closely his own analysis given in 1936. For this purpose we introduced *K-graph machines*. These machines were inspired, in part, by Kolmogorov and Uspensky's 1958 analysis of algorithms.

K-graph machines operate, not surprisingly, on *K-graphs*. These are finite connected graphs whose vertices are labeled by symbols and contain a uniquely labeled *central* vertex. They satisfy also the *principle of unique location*, i.e., every path of labels (starting with the label of the central vertex) determines a unique vertex. K-graph machines substitute distinguished K-subgraphs by other K-graphs; their programs are finite lists of generalized production rules specifying such substitutions. As these substitutions are local, we say that the machines satisfy the *principle of local action*. The subtle difficulties surrounding the principle of local action, even for the case of string machines, are discussed in the next section.⁹

Turing machines, when presented by production systems as above, are easily seen to be K-graph machines. Conversely, the theorem in section 4 shows that computations of K-graph machines can be carried out by Turing machines. Given this mathematical analysis, Turing's central thesis is turned into the thesis that K-graph machines, clearly satisfying the boundedness and locality conditions, subsume directly the work of computers. Our main theorem thus reduces mechanical processes carried out by computers to Turing machine computations. — We want to emphasize very forcefully that our generalization of Turing's analysis is a direct extension of the latter, both technically and conceptually. This is in striking contrast to other such generalizations, e.g., those of Friedman and Shepherdson, see (Shepherdson 1988); Gandy's penetrating analysis of *machine computability* is discussed briefly in the Concluding Remarks.

3 K-Graph Machines

To state and prove the main theorem we have to review some general concepts from graph theory and introduce some notions especially appropriate for our goals. As labeled graphs are going to be considered, we let \mathcal{U} be a (potentially infinite) universe of vertices, \mathcal{L} a finite set of labels, or *alphabet*, and lb a labeling function from \mathcal{U} to \mathcal{L} . \mathcal{L} -labeled finite graphs G are defined with reference to \mathcal{U} , \mathcal{L} , and lb ; thus, they can be given as ordered pairs $\langle V, E \rangle$, where V is a finite subset of \mathcal{U} and $E \subseteq \{\{u, v\} \mid u, v \in V\}$; a pair $\{u, v\} \in E$ is called an *edge* and may be denoted by uv . u and v are said to be *adjacent*. For a vertex $v \in V$ and an edge $uv \in E$, we write also $v \in G$ and $uv \in G$. The sets of vertices and edges of G are also denoted by V_G and E_G . Given $G = \langle V, E \rangle$ and $G' = \langle V', E' \rangle$, $G \cup G' = \langle V \cup V', E \cup E' \rangle$. We write $G' \subseteq G$ and say that G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E \upharpoonright V'$. $G \setminus G' = \langle V \setminus V', E \upharpoonright (V \setminus V') \rangle$. Here the symbol ' \upharpoonright ' indicates the restriction of a relation to a subset of its domain.

⁹Thus, our K-graph machines are deterministic graph rewriting systems; there is a considerable literature in computer science that discusses such systems, see for example the survey article (Courcelle 1990). The category theoretic way of presenting rewrite systems is for our purposes, however, not suitable: the substitution operations have to be graphically concrete and direct, not indirectly obtainable through pushout diagrams. Tim Herron (1995) used the category theoretic framework to characterize K-graph machines.

A *path* in G from u_1 to u_n is a sequence U_1, U_2, \dots, u_n of distinct vertices of G such that for every pair of consecutive vertices u_i and u_{i+1} the edge $U_i U_{i+1}$ is in G . A vertex v belongs to the path if v is an element of the sequence; an edge uv belongs to the path in case u and v are consecutive vertices in the sequence. The length of a path is defined as the number of edges belonging to the path; $\text{len}(u, v)$ is the length of a shortest path from u to v , if any path from u to v exists. A *component* of a graph G is a maximal subgraph G' of G such that for any two vertices u and v in G' , there is a path in G' from u to v ; if $G = G'$ then G is called *connected*.

The remaining definitions are tailored to our purposes and allow us to solve succinctly the central issue of representing symbolic configurations in a most general way—as labeled graphs. The *structure* of such configurations is fixed by the underlying, unlabeled vertices. As the specific nature of the vertices is irrelevant, we call a label-preserving bijection n from W to W' a *permutation* and use such bijections to specify isomorphisms. Clearly, two labeled graphs G and G' are *isomorphic* just in case there is a label and edge preserving bijection between G and G' ; we write $G \simeq G'$. Given a graph G , a permutation n picks out a unique graph G^* isomorphic to G , defined by $VG^* = \{K(U) \mid U \in VG\}$ and $EG^* = \{\{7r(tx), 7r(v)\} \mid \{u, v\} \in EG\}$. If in addition n is the identity over V^* for some graph G' , we write $G \simeq_K G'$. Finally, to fix the analogue of the scanned square, or rather the handle for the puzzle pieces, a distinguished label $* \in C$ is considered. *C*-labeled graphs* are those \wedge -labeled graphs that contain exactly one vertex v with $lb(v) = *$; this vertex is then called the graph's *central vertex* and is referred to simply as $*$ when the context is clear.

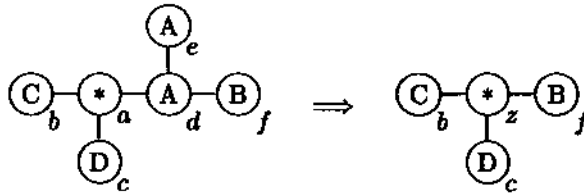
For an \wedge -labeled graph G , we let G^* be the (unique) component of G containing $*$. A sequence a of labels associated with a path from the central vertex $*$ to some vertex v is called a *label-sequence* for v ; the set of such sequences is denoted by $\text{Lbs}(v)$. If labeled graphs have the property that, for any vertex v , a label sequence from $\text{Lbs}(v)$ labels a path to v and not to any other vertex, then the labeling provides a coordinate system. Notice, however, that each vertex may have a number of different "coordinates". This leads to the following definition:

Definition 2 A finite connected \wedge -labeled graph K is a *Kolmogorov-graph*, or *K-graph*, over C if $(\forall a)(\forall u, v; \in K)[a \in \text{Lbs}(u) \wedge \text{Lbs}(v) \Rightarrow u = v]$.

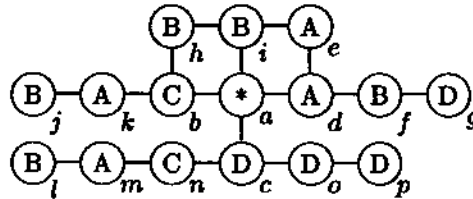
We refer to the above property of graphs as the *principle of unique location*; it guarantees that isomorphisms between K -graphs are uniquely determined. This principle and its relation to condition (a) in Kolmogorov and Uspensky's work is discussed in remark 1 below. — K -graphs constitute the class of finite symbolic configurations on which our machines operate, and we describe now what elementary operations are allowed on such configurations. The operations take the form of generalized production rules and are directly motivated as puzzle-piece substitutions.

Definition 3 A graph-rewrite rule, or simply rule, R is an ordered pair (A, C) , where (i)'s antecedent) A and (it's consequent) C are K-graphs. For a given R we let AR be A and CR be C . A sequence H of rules such that for every $Q, R \in H$ $[AQ \sim \Delta R \Rightarrow AQ \cup CQ \sim \Delta RUCR]$ is called a *program*.

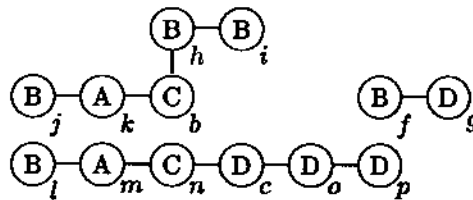
The application of a rule R to a K-graph K substitutes CR for Aj^* in K ; that requires, certainly, that AR is (isomorphic to) a subgraph of K and that CR can be "inserted into" $K \setminus AR$. The crucial work is done by the vertices which occur in both AR and CR . AS an easy example, consider rule R :



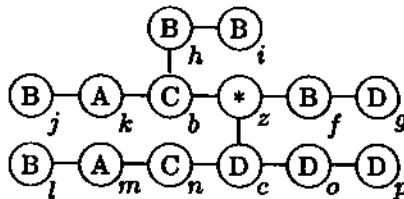
and K-graph K :



To apply R to K , we first remove AR from K , except for those vertices which appear also in CR ; this is given by $K \setminus (AR \setminus CR)$:



Inserting CR into $K \setminus (AR \setminus CR)$ leads to the graph $(K \setminus (AR \setminus CR))UCR$, denoted by $R[K]$:



The above case is special in two respects. First, $AR \subseteq K$. In general, we would like to apply i , if AR is isomorphic to a subgraph of K , and, clearly, for every $K' \simeq K$, $R[K']$ should be isomorphic to $R[K]$. This can be most easily accomplished by using a permutation n such that $AR \subseteq K$. Because of the key role played by the identity between certain vertices in AR and in CR , we of course also have to apply n to CR and make a similar replacement.

The second special property of our example is this: none of the "new" vertices of CR (i.e., vertices which did not occur in AR), occur in $K \setminus (AR \setminus CR)$. In general, however, $K \setminus A^*$ may contain vertices which appear also in CJ ; these vertices in $C\mathcal{E}$ must be replaced as well. This second replacement, say, to C'' , has to satisfy two properties: (i) none of the vertices in $V_C \setminus V_{C''}$ may be replaced; (ii) all vertices which occur in C but not in A^* should not occur anywhere in $K \setminus A$. (i) may be stated as $C \sim_{A^*} C''$; (ii) is given by $V_C \setminus V_{C''} \subseteq V_A \setminus V_{A^*}$.

Definition 4 Given a K -graph K and a rule JR , R is said to be *applicable* to K if for some permutation n , $A \subseteq K$. In this case, $R[K] = \{K \setminus (A \setminus C'')\} \cup C''$ for some $C'' \sim_{A^*} C$ such that $V_C \setminus V_{C''} \subseteq V_A \setminus V_{A^*}$.

If we restrict ourselves to programs, we rule out the kind of non-determinism that Turing considered for his machines, but it may still be that a number of different rules can be applied to a given state K . As suggested for the case of string machines in section 2, we avoid this difficulty by ordering the rules linearly and always using the first (in that ordering) applicable rule. Finally, having defined the configurations on which our machines operate and the steps they can take, we define the machines themselves.

Definition 5 Let \mathcal{E} be a finite alphabet with a distinguished element $*$ and let $M = (S, T)$, where S is the set of all K -graphs over C and T is a partial function from S to S . M is a *K-graph Machine over C* if and only if there is a program $II = (i_0, \dots, R_n)$ such that:

For every $5 \in S$, if there is an $R \in II$ that is applicable to 5 , then $T(5) = (i_{R+1}[S])^*$, where $i = m \setminus n \setminus j \setminus R_j \in II$ is applicable to 5 ; otherwise T is undefined for 5 .

The elements of S are the *states* of M , and T is the machine's *transition function*. We say that T satisfies the *principle of local action*. — We can give an obviously equivalent definition of K -graph machines that brings out the special principles more directly. Let $M = (S, T)$, where S is a set of \mathcal{E}^* -labeled graphs and T is a partial function from S to S ; M is a *K-graph Machine over C* if and only if (i) S is the largest set of \mathcal{E}^* -labeled graphs that satisfy the principle of unique location, and (ii) T satisfies the principle of local action.

Remarks

1. The principle of unique location. In the brief discussion preceding the definition of K -graph machines, we mentioned two sources of non-determinism

present already for letter and string machines (and the standard way of circumventing them). However, there is one additional source, when graphs, even labeled ones with a central vertex, are considered as the configurations on which production rules operate: if the antecedent of a rule can be embedded into a given graph, it usually can be done in a variety of ways; the result of the rule application will in general depend on the chosen embedding. It is precisely this kind of indeterminacy that is excluded by the principle of unique location, as it guarantees that there is a unique embedding, if a K-graph can be embedded into another K-graph at all. The principle is also related to a second conceptual issue, namely, immediate recognizability. For any K-graph machine M and any K-graph whatsoever, we can decide in constant time, whether any rule of M 's program is applicable. Mathematically, the principle is exploited for the reduction in section 4. It guarantees that, for any \mathbb{F}^* -labeled graph, paths starting at the central vertex are uniquely characterized by the sequence of symbols labeling their vertices. Thus, using the lexicographical ordering on strings of labels from \mathbb{F} , we can choose for each vertex a unique address which picks out that vertex in terms only of labels.

Kolmogorov and Uspensky used their condition (a) for similar purposes. That condition is formulated in our setting as follows: For every graph $G \in \mathcal{S}$, if u and v are vertices of G both adjacent to some vertex w of G , then $lb(u) \wedge lb(v)$. We call a connected \mathbb{F}^* -labeled graph satisfying condition (a) a *Kolmogorov complex over C* or, briefly, a K-complex. Condition (a) implies our principle of unique location, but is not implied by it. The first claim is easily established; for the second claim one sees directly that $*-A-A-A-A$ is an example of a K-graph that is not a K-complex.

2. Preservation under rule application. We require that T is a partial function from S to S . As matters stand, programs and K-graph machines cannot be "identified"; the reason is this: not every rule, when applied to a K-graph, yields a K-graph. As a trivial example, consider $R = *-A-B \Rightarrow *-C-B$ and $K = B-C-*-A-B-A$. Then $R[K] = B-C-*-C-B-A$ which is certainly not a K-graph, even though K is a K-graph and R is a rule.

It is straightforward to modify any given machine program H to a program TV such that TZ and TZ' yield the same result when II transforms a K-graph into a K-graph, but $*R'$ makes the machine diverge on K-graph inputs that II transforms into graphs not satisfying the principle of unique location. Given this modification, TV defines a unique partial function from S to S . — Kolmogorov and Uspensky required a particular structure on rules preserving condition (a). They use a partial function $\langle / \rangle : VA \rightarrow VC$ to determine those vertices that, in our definition, are in $VA \cap VC$. As one has to be careful only about the symbols adjacent to vertices in the image of 0, they imposed in effect the following condition (?) for a given rule $(A, C, 0)$:

$$(\forall G \in \mathcal{S}) [(y = 4 \triangleright (x) \text{ kvye} C) \wedge [lb(v) = * \text{ or } (3w)(wx \in A \ \& \ lb\{w\} = \triangleright \langle \langle \rangle \rangle)]]].$$

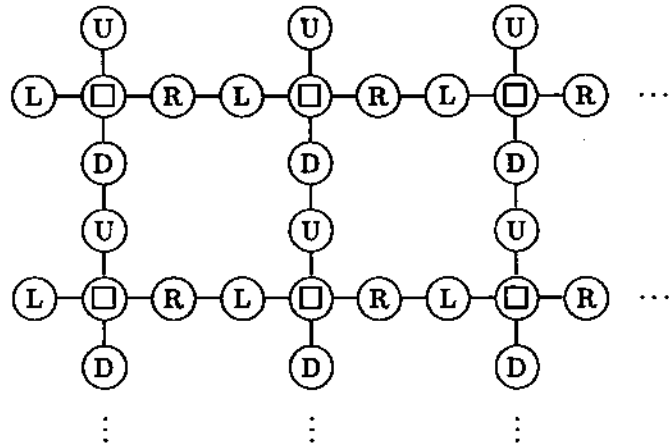
Consequently, if a program \mathcal{R} satisfies (β) , then $\mathcal{M} = \langle \mathcal{S}, \mathcal{F} \rangle$ is a K-graph machine, where \mathcal{S} is the set of all K-graphs on \mathcal{L} and \mathcal{F} is the unique function on \mathcal{S} defined by \mathcal{R} .

3. Turing's conditions. K-graph machines clearly satisfy the determinacy condition, but also the boundedness and locality conditions—when those are suitably interpreted: the number of “immediately recognizable” symbolic configurations is given by the number of distinct antecedents and consequents of the machine’s program; operations are quite properly viewed as modifying observed configurations, and observed labeled vertices lie always within a fixed “radius” around the central vertex. (The radius can be read off from the program, e.g., it can be taken to be the maximal length of paths in any K-graph of the program.) We make some additional remarks about the principle of local action, as it might be thought that—even in the case of string machines—locality is violated! The reason being, that in an “implementation” of those machines, e.g., on a standard Turing machine, the total tape content is affected when using a rule that replaces a string by either a longer or a shorter one. However, this seems to be pertinent only if the tape has a rigid extrinsic coordinate system as given, for example, by the set of integers \mathbf{Z} . When a different presentation of Turing machines is chosen, as suggested for example in (Gandy 1980), or when the underlying structure is flexible to insertions, as in our set-up, the concern disappears.¹⁰ It is precisely the use of an *intrinsic coordinate system*, guaranteed through the principle of unique location, that makes for the locality of the replacement operations.

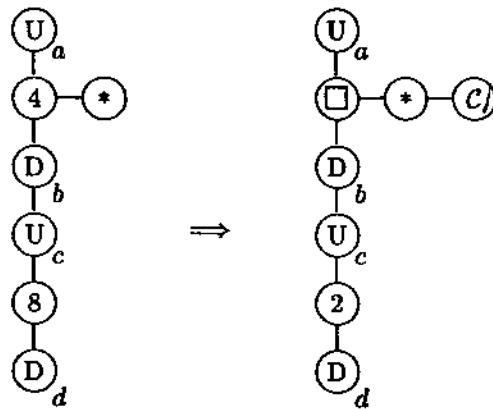
4 Subsumption and Simulation

K-graph machines capture the general starting-point of Turing’s analysis in a most natural way. Consider, for example, encoding the squares in Turing’s “child’s arithmetic book” as follows:

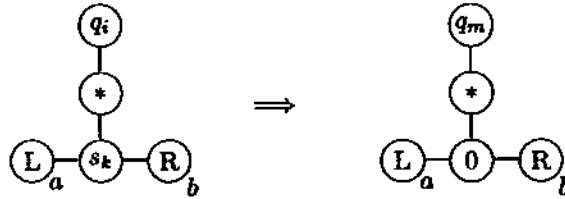
¹⁰These two ways of dealing with the issue are two sides of the same coin.



One can attach numerals to the squares (like writing them in the book), and directly encode, for example, the elementary school algorithm for column addition via K-graph operations. A rule from such a machine is shown below. (The rule collapses two digits and enters a special "carry" configuration so that successive rule applications will move the central vertex to the top of the next column and place a '1' there before returning to continue collapsing the current column.)



It is also immediately evident that Turing machines are K-graph machines: consider the following formulation of the rule $aqiSk/3 \Rightarrow ag_m0/?$ from section 2:



Such considerations can be given in a similarly direct way for string machines and their generalizations to higher dimensions, thus in particular for the generalized Turing machines described in (Kleene 1952). A host of other models of computations, including the *Kolmogorov machines* as defined in (Uspensky and Semenov) or the *register machines* as introduced by (Shepherdson and Sturgis), can be shown to be subsumed under K-graph machines. Joining these observations with the main result of this section, we have an *absolutely uniform way of reducing computations of a particular model to Turing machine computations*: We have only to verify that the computation model is subsumed under K-graph machines. So let us turn to the substantive task of this section, namely, to simulate K-graph machine computations by a Turing machine, i.e., for an arbitrary K-graph machine $M = \langle S, T \rangle$ over the alphabet C we construct a Turing machine¹¹ M over the alphabet $\{0,1\}$ that simulates M . The simulation requires (i) that we give linear representations of K-graphs, and (ii) that we show for every $s \in S$, if $cr = s, S_1, \dots, S_n$ is a computation of M , then $r = T_0, T_1, \dots, T_m$ is a computation of M . Here T_0 represents s , and T_m represents a K-graph isomorphic to S_n ; furthermore, there exists a subsequence $T_{t_1}, T_{t_2}, \dots, T_{t_{w-1}}$ of r such that for $1 < j < n - 1$, T_{t_j} represents a K-graph isomorphic to S_j . The conditions for infinite computations are similar.

Before addressing (ii) in the proof of theorem 6, we discuss the linear representation of K-graphs. We assume that C is linearly ordered and let $<$ be the lexicographical ordering on finite sequences of symbols from C induced by that linear ordering. A second ordering on finite sequences a, θ of symbols is defined by $a < \theta$ iff a is shorter than θ or a and θ are of the same length and $a < \theta$. The *address* $Ad(v)$ of a vertex $v \in V$ is the $<$ -minimal element of $Lbs(v)$; by connectedness such an address exists, and by the principle of unique location Ad is injective. We assume that each vertex of U is a natural number, and define for an arbitrary edge $uv \in E$ the *location description* $LD(uv)$ by:

$$LD(uv) = \begin{cases} (U, lb(u), lb(v)) & \text{if } Ad(v) < Ad(u) \\ (v, lb(v), u, lb(u)) & \text{if } Ad(v) > Ad(u) \end{cases}$$

When we refer to an edge uv we assume from now on that $Ad(u) < Ad(v)$. We define now an ordering on the location descriptions $LD(E) = \{LD(uv) \mid uv \in E\}$ for a given K-graph K as follows: $LD(u_1v_1) < LD(u_2v_2)$ iff $Ad(u_1) < Ad(u_2)$

¹¹In contrast to our earlier discussion, we are going to use a Turing machine whose tape is extendable only to the right.

or $u_1 = u_2$ and $\text{Ad}(v_1) < \text{Ad}(v_2)$. $\text{GR}(K)$, the canonical graph representation of K , is the C-ordered sequence of location descriptions for K .

To obtain a *tape representation* of this sequence in the alphabet $\{0,1\}$, we first assign natural numbers to the symbols in C : 0 is used for the least element (in the original linear ordering of \mathcal{L}), 1 for the next, etc. Every vertex $v \in V$ is already a natural number, and we assume without loss of generality that a state S with n vertices consists of $\{1, \dots, n\}$. Finally, natural numbers are represented in a modified binary form obtained from the standard one by replacing every 1 by 11 and every 0 by 10; elements of a sequence of encoded natural numbers are separated by exactly two 0's; elements of a sequence of such sequences are separated by exactly three 0's. After these preparatory considerations, we can establish:

Theorem 6 *Any K-graph machine A_i can be simulated by some Turing machine M .*

Proof. The program of the Turing machine M that is to simulate M transforms any state S into $F(S)$, returns to its initial state, and, if possible, further transforms the resulting state $T(S)$; the machine halts when none of the rules defining T can be applied. (It should be obvious that this yields the kind of simulation indicated above.) In the following more detailed description, we always use S to refer to the graph currently coded on the tape, even though there are stages when some edges are removed and others are added.

The first step in constructing M 's program is to modify the encoding of M 's program T with rules $i^?_0, \dots, i^?_r$.¹² For a given rule, the antecedent A and the consequent C are encoded by $\text{GR}(A)$ and $\text{GR}(C)$. As we are only concerned with the isomorphism class represented by each rule, we are free to encode each rule R_i ($0 \leq i < r$) by an isomorphic rule. Let N be the maximum number of vertices occurring in any rule and replace each rule R_i by an isomorphic R'_i such that for every vertex v of R^i , $N \cdot i < v \leq N \cdot (i + 1)$. K will indicate now the modified program $i^?_0, \dots, i^?_r$. The tape is to contain as input S , all of whose vertices v are replaced however by $u + M$ (where $M = N - r$) so that all (vertex-) numbers are greater than M . Obviously, $\text{CRD}(S \setminus AR) = 0$ for any R . This has also the following advantage (as will be clear from our further discussion): by looking at a vertex one can determine immediately whether it occurred in the original state or whether it was written by a rule; in the latter case one can decide which rule wrote it.

¹²An alternative to this simulation is the following: We can also represent a rule on the tape simply by writing $\text{GR}(A)$ followed by a separator (say, '0000') and then by $\text{GR}(C)$. To represent the program we represent each rule in order, separating them by, say, '00000'. Then we can describe a Turing machine U that simulates any K-graph machine conceptually in exactly the same way in which a universal Turing machine simulates any other Turing machine M . In the latter case the input to the machine is a Gödel number of M and an input to M . Here, U will take as input the coded program for the K-graph machine followed by the initial state. Then U will carry out systematically the coded program.

Now M examines each rule in the given order via a subroutine CHECK-R that determines whether R can be applied to S . If R is applicable, then CHECK-R tries to modify S to an isomorphic S' , such that AR is a subgraph of S' . If R is not applicable, CHECK-R may nevertheless have changed some vertices on the tape, but the modified graph is isomorphic to S . Note that this way of proceeding is essentially "dual" to that in the definition of K-graph machines: we replace vertices occurring in the original state rather than those in the rule being applied.

CHECK-R starts with the head leftmost on the tape and proceeds by moving to the right, searching for appropriate edges one at a time. Let min be the least and max the greatest vertex of R . In the i th step, we are looking for an edge in S which matches the i th edge uv of AR . U is the second vertex of some edge (already matched to an edge in S) and occurs consequently in S . We search for w in S such that the edge uw is in S and such that $lb(w) = lb(v)$. If such a vertex is found, we distinguish two cases: (i) if $min \leq w \leq max$ and $w \neq v$, then w was written by R and is the image of some vertex in AR other than v , and the search for the correct vertex has to be continued; (ii) if $w < min$ or $w > max$ or $w = v$, then w is the correct vertex, and M substitutes v for w everywhere on the tape and proceeds to search for the next matching edge in S .

In sum, if M reaches the end of the tape before finding such a vertex, M fails for this R ; if CHECK-R fails for every rule i ?, M halts; if CHECK-R finds a matching edge for every edge in AR , R is applicable. (Note that the principle of unique location allows us to avoid backtracking in case the algorithm fails for a particular vertex.)

If a rule R is applicable, M is going to modify its tape appropriately. For each rule R there is a subroutine APPLY-R which applies R to the current state: all edges which contain any vertex from $VA_R \setminus VC_R$ are erased and all those from CR are inserted—leftmost onto the tape (in the order of their appearance in the canonical encoding $GR(CR)$). The tape contains now a representation of $R[S]$ recall that the next state $F(S)$ is defined as the connected component of $R[S]$ containing the central vertex. However, the remaining computation will not be interfered with by unconnected edges left on the tape. Finally, the head is returned to the left end of the tape, and the state is set to M 's initial state. •

We want to determine now, in a rough way, the number of steps M needs to transform a K-graph S with n vertices into $T(S)$. Assume that the language for M contains l symbols, and that M 's program has r rules of size at most N (i.e., at most N vertices occur in AUC). For a given state S with n vertices, the maximal degree of each vertex is $l+1$; otherwise, the principle of unique location would be violated. Thus at most $n(l+1)/2$ edges have to be represented. The largest (vertex-) number to be represented is n , which has length $2l \log n$ in our modified binary notation. The largest LD has length of order $\log(2n+2Z)$. Thus the representation of S is of length $O(n \log n)$.

The renumbering step must traverse the entire tape. Since we wish to increase each vertex by at least M , we take M' to be the least power of two greater than or equal to M and add M' to each vertex. This operation requires shifting all of the cells right of the vertex being updated up to $\log M'$ cells to the right. This requires rewriting up to $O(n \log n)$ cells for each vertex. Since this operation is done to all occurrences (of which there may be up to $l + 1$ many) of each of the n vertices, it is an $O(n^2 \log n)$ operation. (The rewriting itself can be done in a single pass over the number and requires $\log n$ steps.)

The further rewriting operations required for finding the applicable rule all involve replacing numbers greater than M' by numbers smaller than M' ; thus, no shifting is involved in these operations, since we allow extra 0's to occur between integers. Attempting to match a given edge in a rule to one on the tape might require looking at the entire tape and is an $O(n \log n)$ operation; that may have to be done for every edge in every antecedent.

If we succeed in finding an applicable rule, we apply it; i.e., we transform the tape by erasing all edges from A_R and inserting all edges from C_R at the beginning of the tape. This may require shifting all $O(n \log n)$ symbols by at most the length of the largest $\text{GR}(A_R)$. Hence only $O(n \log n)$ many steps are required for rule selection and application. But once this has been accomplished, the entire transformation is complete, so the complexity of the simulation is $O(n^2 \log n)$.

Now let us consider simulating a full computation of M . If we let $k = \max\{|C_R| - |A_R| \mid R \in \mathcal{R}\}$, then for any $S \in \mathcal{S}$, $|\mathcal{F}(S)| \leq |S| + k$; here $|K|$ is the cardinality of the set of vertices of K . Let h be a natural number such that $hn^2 \log n$ is the complexity of the "step-simulation" for \mathbf{M} of M we just discussed. Assume, in a first example, that \mathcal{M} runs in constant time, say, in m steps. Then the length of the computation of \mathbf{M} for input of size n is bounded by

$$s = hn^2 \log n + h(n+k)^2 \log(n+k) + \cdots + h(n+km)^2 \log(n+km).$$

Clearly,

$$n^2 \log n \leq s \leq mh(n+km)^2 \log(n+km) = O(n^2 \log n)$$

so $s = O(n^2 \log n)$.

If \mathcal{M} runs in higher order time, however, the step-complexity of \mathbf{M} is not preserved. Assume, for example, that \mathcal{M} runs in mn^c -many steps, for some m and c . Then the complexity of \mathbf{M} for input of size n is bounded by

$$hn^2 \log n + h(n+k)^2 \log(n+k) + \cdots + h(n+kmn^c)^2 \log(n+kmn^c) = O(n^{3c} \log n)$$

This analysis illustrates the quite obvious point that the complexity of computing a given function depends on the machine used to carry out the computation. An interesting example is multiplication: it can be computed in linear

time by RAM's and SMM's (Schönhage), but it is also known that Turing machines cannot multiply in linear time (Cook and Aanderaa; Paterson e.a.); we do not know, whether K-graph machines can. This question and related ones raise many interesting issues about complexity, particularly whether one model allows a more fundamental analysis of the complexity of algorithms than another.

5 Concluding Remarks

For Turing the ultimate justification for his restrictive conditions lies in the *necessary limitation* of human memory, and that can be directly linked to physical limitations also for machines; cf. (Mundici and Sieg), section 3. Church in his review of Turing's paper seems to have mistaken Turing's analysis as an analysis of machine computations. Church's apparent misunderstanding is common: see, e.g., (Mendelson 1990). So it is worthwhile to point out that machine computability was analyzed only much later by Gandy (1980). Gandy followed Turing's three-step-procedure of analysis, axiomatic formulation of general principles, and proof of a reduction theorem, but for "discrete deterministic mechanical devices", not computers.

Gandy showed that everything computable by a device satisfying his principles, a *Gandy machine*, can already be computed by a Turing machine. To see clearly the difference between Turing's analysis and Gandy's, note that Gandy machines incorporate parallelism: they compute directly Conway's game of life and operate, in parallel, on bounded parts of symbolic configurations of possibly unbounded size. The boundedness conditions for Gandy machines and the *principle of local causation* are motivated by physical considerations. We have been concerned, in contrast, with an explication and generalization of Turing's arguments for his thesis, that all mechanical processes can be Simulated by (Turing) machines. We are coming back to this starting-point of our considerations through three remarks.

First, Turing analyzed *mechanical* processes of a human computer. The reduction of string machines or of K-graph machines to letter machines over a two-element alphabet does not show that *mental* processes cannot go beyond mechanical ones; it only shows that Turing machines can serve as a "normal form" for machines, because of the simplicity of their description.¹³ The question, whether (different kinds of) machines are adequate mathematical models for mental processes, is left completely open. That is an empirical issue!

Second, the formulation of the boundedness and locality conditions for mechanical processes and the design of general machine models allow us to give

¹³For this reason Turing machines are most suited for theoretical investigations. This state of affairs is analogous to that involving logical calculi: natural deduction calculi reflect quite directly the structure of ordinary arguments, but have a somewhat involved metamathematical description; in contrast, axiomatic logical systems are not suited as frameworks for direct formalizations, but—due to their simple description—are most suitable for metamathematical investigations.

uniform reductions. A natural generalization of K-graph machines, not giving up these broad conditions, captures parallel computations of "discrete deterministic mechanical devices", not computers. A future paper of ours gives such a generalization, based on the presentation of machines by Gandy (1980) discussed above.

Third, support for Turing's thesis is best given in two distinct steps: (i) mechanical processes satisfying boundedness and locality conditions can be recognized "directly", without coding or other effective transformations, as computations of a general model; (ii) computations of the general model can be simulated by Turing machines. The plausibility of Turing's thesis rests exclusively on the plausibility of the *modified central thesis* (i); after all, (ii) is a mathematical fact. Our modification of Turing's *central thesis* states that mechanical processes are easily seen to be computations of K-graph machines; in our view, this is a most plausible claim.

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