# A Global Optimization Algorithm for Linear Fractional and Bilinear Programs

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EDRC 06-143-92

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October 1992

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**Abstract**

In this paper a new deterministic method for the global optimization of mathematical models that involve the sum of linear fractional and/or bilinear terms is presented. Linear and nonlinear convex estimator functions are developed for the linear fractional and bilinear terms. Conditions under which these functions are nonredundant are established. It is shown that additional estimators can be obtained through projections of the feasible region that can also be incorporated in a convex nonlinear underestimator problem for predicting lower bounds for the global optimum. The proposed algorithm consists of a spatial branch and bound search for which several branching rules are discussed. Illustrative examples and computational results are presented to demonstrate the efficiency of the proposed algorithm.

## Introduction

Many engineering design problems can be formulated through mathematical programming models (Reklaitis and Ravindran, 1983; Papalambros and Wilde, 1988; Grossmann, 1990). These models, however, often involve nonconvex functions and therefore when conventional techniques are used they can get trapped in local solutions. Recently there has been a significant effort in the area of global optimization. Stochastic and deterministic methods have been developed; for recent extensive reviews see Schoen (1991) and Horst (1990). Deterministic methods have the advantage that they can provide rigorous guarantee of global optimality of the solution but require some assumptions about the mathematical structure of the model. Since many nonlinear optimization models in engineering design do exhibit a special structure, there is a clear incentive to consider the solution of these problems with deterministic methods.

An important class of nonconvex optimization problems with special structure correspond to nonlinear programming problems with bilinear or linear fractional terms. Al-Khayyal (1992) presented a review of the models and applications of bilinear programming. The bilinear and linear fractional terms are factorable functions for which McCormick (1976) has presented a general approach for deriving underestimator functions that can be incorporated in global optimization algorithms. Al-Khayyal and Falk (1983) proposed an algorithm for bilinear programs with linear constraints in which linear estimators over the bilinear terms are used. Swaney (1990) addressed the asymptotic behavior that can occur in this type of algorithm when a solution does not lie at an extreme point. Algorithms for bilinear programming models have also recently been developed by Sherali and Alameddine (1990). These authors presented a linearization reformulation technique that embeds the method proposed by Al-Khayyal and Falk (1983) and predicts stronger bounds for the global optimum. However, the main limitation is that the size of the linear programming underestimator problems grows exponentially with the number of constraints in the original problem.

Falk and Palocsay (1991) proposed an algorithm for optimizing the sum of linear fractional functions subject to linear constraints. The algorithm consists of a sequence of linear programming problems in which bounds on feasible subsets are added. These bounds are tightened iteratively to reduce the search space. These authors also developed convergence properties for this algorithm by extending the approach presented by Dinkelbach (1967). However, the rate of convergence of this method can be slow. Konno et al. (1991) addressed the minimization of the sum of two linear fractional functions over a polytope. This is done using parametric linear programming algorithms. Floudas and Visweswaran (1990) presented an algorithm based on a Benders based decomposition approach that can be used to solve bilinear and/or fractional programming problems. In this method a sequence of subproblems and relaxed dual subproblems are solved. Although the advantage of this method is that the subproblems correspond to linear programs, one potential difficulty is that the number of relaxed dual subproblems that have to be solved at each iteration may grow exponentially with the number of variables interacting in different nonconvex terms.

In this paper a new deterministic method for the global optimization of mathematical models that involve the sum of linear fractions and/or bilinear terms is presented. The proposed method is a generalization of the work presented by Quesada and Grossmann (1992) for minimizing the sum of linear fractional functions that arises in the global optimal design of Heat Exchanger Networks. The unique feature of the proposed method is that bilinear and linear fractional terms are substituted by both linear and nonlinear convex estimator functions that can be derived using the approach presented by McCormick (1976). Conditions under which the estimator functions for different types of terms are nonredundant are determined. A convex nonlinear underestimator problem is then proposed that predicts lower bounds for the global optimum. These bounds can be further strengthened by the inclusion of additional estimators that are obtained through projections of the feasible space. For the particular case of bilinear terms, the additional estimators are equivalent to the reformulation technique of Sherali and Alameddine (1991). To find the global optimum, a spatial branch and bound search is conducted in which the lower bounds are obtained from the nonlinear underestimator problem. Modifications to the branching rules proposed by Sherali and Alameddine (1991) are used in this search.

The paper is organized as follows. Firstly, the case of a nonconvex objective function and a convex feasible region is considered. Here the properties of the different estimators functions, the formulation of the convex NLP underestimator problem and the basic algorithm are presented. Also, the performance of the algorithm is illustrated through a small example. The algorithm is then extended to the case of nonconvex feasible regions for which the necessary modifications are described. Finally, numerical results are given for a variety of problems that have been reported in the literature.

**Mathematical model**

The following mathematical programming problem that is considered in this paper involves a nonconvex objective function with linear fractional, bilinear and convex terms, and is defined over a bounded convex feasible region,

min f = £ £ q, JjJ - I IcyPfl, + h(p, q, z)
i€l j€J J 9j l€l f j€J' J J

st g(p, q, z) < d (PO)

ze ZcR n

pL<P <P U

q L<q <q u, (q^O.Je J)

**p e K+ , q € K+**

The functions h, h:R liU"'l+IJI +m -> R\ and g, g:R III*lrl*lal"l,IJ"l -> $R^m$, are assumed to be convex and differentiable. The set Z is bounded; Cy are real coefficients of the linear fractional terms or bilinear terms; d is an m-vector. The variables $p_t$ and qy are bounded and non-negative and the 4
lower bound for the variables $q_{jf}$ j e J, in the denominator of the fractional terms is strictly positive. For simplicity in the presentation the sets I and $I^1$, and J and $J^f$ are assumed to be disjoint, although this assumption can be easily relaxed.

If any variable $p_4$ or qj (J G'J) is not restricted to be positive, it can be substituted by two new variables such that:

$$Pt = Pi^{*}-Pf \quad W$$
$$p_i^u \geq p_i^+ \geq 0, \quad -p_i^L \geq p_i^- \geq 0$$

$$q_j = Qj^+ - Of \quad (2)$$
$$q_j^u \geq q_j^+ \geq 0, \quad -q_j^L \geq q_j^- \geq 0$$

The variable cy for j *e* J is required to be strictly one signed to avoid singularities. In case that the variable is negative it can be transformed by setting,

$$cy= -q,' \quad (3)$$

To facilitate the analysis and the development of the algorithm, problem (PO) will be reformulated by introducing additional variables and relabeling the variables p and q by the variables x, y and r, with the following equations:

| | | | | |
|---|---|---|---|---|
| Pt=Xio | q,=yj | ^ = ilj | i e l . j e J | (4) |
| $Pi=r_{i0}$ | q,=yj | Piq, = Xij | i e F . J e J ' | (5) |

Also, for convenience the following sets are defined for the positive (P) and negative (N) terms in the objective of (PO):

$P_R$ = {(i,j,k,m) | i G I, j G J, $c_y$ > 0, k = j, m = 0}

$P_B$ = {(i,j,kjn) | i G $I^1$, j G J', $c_{tJ}$ > 0. k = 0, m = j}

$N_R$ = {(i,j,k,m) | i G I, j G J. $_{CiJ}$ < 0, k = j, m = 0}

$N_B$ = {(ij,k,m) | i G I . J G $J^1$, Cy< 0, k = 0, m = j}

with P = $P_R$ u $P_B$ and N = $N_R$ u $N_B$.

By using (4) and (5) and the above definition of sets, problem (PO) can be written in the following compact form:

min f = £ I c«,r r X £ Cy x«,+ h(x, y, rf z)
iel j€J tel' JeJ*

st. yj rik ≥ Xtm (i, j , k, m) e P

yjítk^x^ (i.j, k,m)e N

g(x,y,r,z)≤d (PI)

x^L ≤ x ≤ x^U

yL≤y ≤y«

i*-≤r ≤r^u

ZE Z

where the new inequalities have been relaxed according to the sign of the cost coefficient (c^ > 0 for (i, j, k, m) € P and $c_y$< 0 for (i, j, k, m) € N). Also, the bounds for x, y and r are obtained from (4) and (5) using the bounds for p and q in (PO). Problems (PI) and (PO) are equivalent and the algorithm is presented based on formulation (PI).

## Estimator functions

Following the treatment of McCormick (1976) (see Appendix A), the bilinear terms that appear in the constraints of (PI) can be replaced by linear convex estimator functions (A. 11 and A. 12) yielding the following constraints:

| | | |
|---|---|---|
| $x_{4m} \le y_j^u r_{1k} + r_{ik}^L y_j - y_j^u r_{ik}^L$ | (i.J,k,m)6P | **(6a)** |
| $x_{4m} \le y_j^L r_{ik} \quad r_{ik}^u y_j - y_j^L r_{ik}^u$ | (i,j,k.m)eP | (6b) |
| $x_{4m} \ge y_j^L r_{ik} + r_{ik}^L y_j - y_j^L r_{ik}^L$ | (i.j.k,m)eN | (7a) |
| $xi_m$ £ $yj^u r_{lk}$ + rtk^j - $yj^u r_{lk}^u$ | (i,j.k.m)eN | (7b) |

Note from the above that it is only necessary to include over or underestimators depending of the sign of the cost coefficient, Cy.

Additionally, as shown in Appendix A, it is possible to develop nonlinear convex underestimator functions according to (A. 15) for the constraints with (i, j, 1, m,) e P which yields:

$r_{ik}$ ^ + $x_{4m}^L$( ^ - ^ i ) (i,j,k,m)6P (8a)

$$r_{lk} \geq \frac{x_{lm}}{y_j^L} + x_{lm}^u( jJ-JJc) \qquad (tj.k,m)eP \qquad (8b)$$

The following properties can be established for the linear and nonlinear estimator functions in (6) and (8) for the linear fractional terms in (PO) with positive cost coefficients, c^.

<u>Property 1</u>. When *T^* = ^?r (or *T^* = ^ r) . (i, j, k, m) € $P_R$, the linear overestimator (6a) (or (6b)) is a linearization of the nonlinear underestimator (8a) (or (8b)).

**Proof.** See Appendix B.

<u>Corollary 1.</u> The nonlinear underestimator (8a) (or (8b)) is stronger than the linear overestimator (6a) (or (6b)) when $r_{lk}^L$ = ^*r* (or $r_{lk}^u$ = -*£-), (i, j, k, m) e $P_R$.

**Proof.** See Appendix B.

Tlie following property, however, establishes that the linear overestimators in (6) are not necessarily redundant.

<u>Property 2.</u> When $r_{,k}^L$ > ^?r (or $r_{\&}^u$ < -^£- )t (it jt k, m) e $P_R$, the linear overestimator (6a) (or 6b) is nonredundant.

**Proof.** See Appendix B.

For the bilinear terms in (PO) with positive cost coefficients, c^, the following properties can be established.

<u>Property 3.</u>: When $x^{\wedge 1}$- = *Tfjy}** (or *x^* = *To^yf),* (i, j, k, m) € $P_B$, the linear overestimator (6a) (or (6b)) is a secant of the nonlinear underestimator (8a) (or (8b)).

**Proof.** See Appendix B

<u>Corollary 2.</u> The linear overestimator (6a) (or (6b)) is stronger than the nonlinear underestimator (8a) (or (8b)) over the feasible region when *x^* = *r^^* (or $x^{\wedge 11}$ = $r^\wedge uy)^{11}$), (i, j, k, m) e $P_B$.

**Proof.** See Appendix B

<u>Property 4</u>. When $x_{im}^{L} > r^f$ (or $X|_m^u < r_{ik}^u yj^u$), (i, j, k, m) € $P_B$, the nonlinear underestimator (8a) (or (8b)) is nonredundant.

Proof. See Appendix B

A geometrical interpretation of properties 1 to 4 is as follows. Consider the case of a fractional term ((i, $j_f$ k, m) e $P_R$, Property 1 and 2) and the projection of the feasible region into the space of the variables involved in the nonconvex term (variables x and y. Figure la). The estimators functions (6, 8) have the property that they yield an exact approximation of the nonconvex term when one of the variables involved in the function is at its lower or upper bound (see Appendix A). For the fractional term the nonlinear estimators (8) are expressed in terms of the variables involved in that term, x and y, and provide an exact approximation at the boundary defined by the bounds of these variables. As for the linear underestimator in (6), note that the contours of the fractional term ($r = \frac{}{y}$) correspond to straight lines that pass through the origin and are given by the dashed lines in Figure la. Therefore, the linear estimator functions (6) give exact approximations at the boundaiy defined by the bounds of the variables r and y. As shown in corollary 1, when the bound over the individual nonconvex term is redundant (e.g. $r^1 - = \frac{xL}{y}$*r) there is no part of the feasible region in which the linear estimator will be stronger. However, when this bound is not redundant (e.g. $i^4 - > \frac{xL}{y}$JJ) the linear estimator provides an exact approximation at points in which the nonlinear cannot.

For the bilinear term, the linear estimators (6) are the ones that are expressed in terms of the variables involved in the nonconvex term, y and r and therefore they provide an exact approximation at the bounds of these variables as seen in Fig. lb. Also, according to Property 4,. the nonlinear estimator in (8) will only be nonredundant if there is a strong bound over the individual bilinear term (eg. $x^L > rV$ in Fig. lb).

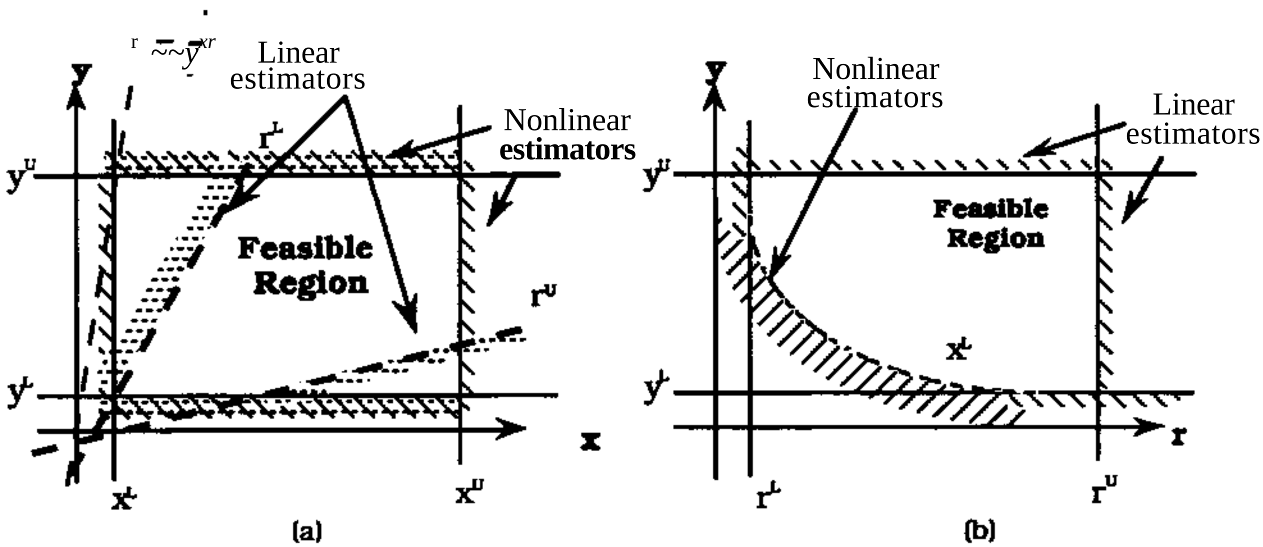


Figure 1. Projected feasible region for (a) fractional term and (b) bilinear term

**Variable bounds**

It is important to have tight lower and upper bounds for the variables x, y and r involved in the nonconvex terms in (PI) since these bounds determine the quality of the approximation given by the estimator functions in $(6)_f$ (7) and (8). Firstly, the bounds determine the size of the search space for the global optimum. Secondly, from the value of these bounds it is possible to determine in advance whether a given estimator function will be redundant.

In the initialization step of the algorithm, lower and upper bounds for the variables $x_{10}$ and yj in the fractional terms (i 6 I, J e J) and for the variables $r_{i0}$ and yj in the bilinear terms (i € $I^1$, j € $J^f$) are calculated. This is accomplished by solving the corresponding minimization and maximization subproblems for each variable:

| | | |
|---|---|---|
| $y_{,}^{L}$ = {min yj \| g(x, y, r, z) £ d} | je JuJ[1] | (9a) |
| $y_{,}^{u}$ = {maxy, \| g(x, y. r, z) $\le$ d} | je JuJ[1] | (9b) |
| $x_{,0}^{L}$ = {min $x_{im}$ \| g(x, y. r, z) $\le$ d} | **l€ I** | (9c) |
| $XK)^{U}$ = (maxxta \| g($x_f$ y, r, z) $\le$ d} | **ie I** | (9d) |
| $r_{10}^{L}$ = {min $r_{tk}$ \| g(x, $y_f$ $r_f$ z) $\le$ d} | **ie r** | (9e) |
| $r_{iO}^{u}$ = {max r^ \| g(x, y, r, z) $\le$ d} | **ie r** | (9f) |

where the bounds in (PI) are used for the variables x, y, r and z in the above subproblems. These subproblems have a unique solution since the feasible region is convex. Also, since they have the same feasible region and are independent, they can be solved in parallel.

Furthermore, for the fractional terms it is possible to calculate lower and upper bounds for each individual term $r_{iJt}$ i e I, j G J by solving the subproblems,

$$ry^{1}- = \{ \min\wedge, \text{ st } g(x, y, r, z) \leq d, z \text{ G } Z) \qquad \text{ie } I, j \; e \; J \qquad (10a)$$

$$r_{tj}^{u} = ( \text{ma}\wedge, \text{ st } g(x, y, r, z) \leq d, z \text{ G } Z\} \qquad \text{iG } I, j \text{ G } J \qquad (10b)$$

The subproblems in (10) have a unique solution (Bazaara and Shetty (1979)) and in the case that the feasible region is given by linear constraints they can be reformulated as LP problems through the transformation proposed by Charnes and Cooper (1962). Having generated these bounds it is possible to determine whether the linear overestimators (6) for a fractional term are nonredundant.

Note that for the bilinear terms, it is not possible to obtain valid bounds over each individual nonconvex term Xy, i G $I^1$, j G J\ since the resulting subproblems are nonconvex. However, the nonlinear underestimators (8) may still be useful when nonredundant bounds over the individual bilinear terms are known a priori, or, as it will be shown later, when a nonconvex feasible region is considered and bilinear terms are present in the constraints.

**Projections**

The linear and nonlinear estimator functions in (6), (7) and (8) presented in the previous section use fixed lower and upper bounds of the variables over which the approximation is obtained. Additional estimators that tighten the convex representation can be generated by considering bounds corresponding to hyperplanes of the boundary of the feasible region and which are projected in the space of the variables involved in the estimator function. In particular, from the solution of the subproblems (9), (10) solved in the initialization step in which bounds for the individual variables are determined, the following inequalities projected in the space (yj, x^„) can be obtained:

$$0 \geq \sum_{\ell} \lambda_{\ell}\, g_{\ell}^{lin}(y_j, x_{im}, \zeta^*) \qquad (11)$$

where X,are the Lagrange multipliers and $g_\ell^{ln}(y_j, x_{im}, \zeta^*)$ Is a linearization of the constraint $g_\ell$ given in terms of yj and $x_{im}$ with the remaining variables C fixed at their optimal value C*:

$$g_\ell^{lin}(y_j, x_{im}, \zeta^*) = g_\ell(y_j^*, x_{im}^*, \zeta^*) + \nabla_{y_j, x_{im}} g_\ell(y_j^*, x_{im}^*, \zeta^*) \begin{bmatrix} y_j - y_j^* \\ x_{im} - x_{im}^* \end{bmatrix} \qquad (12)$$

The projections in (11) give rise to linear inequalities of the following form:

$$y_j \leq a + b x_{im} \qquad (13a)$$

$$y_j \geq a' + b' x_{im} \qquad (13b)$$

from which the following nonlinear underestimators that are similar to (8) can be generated:

$$r_{ik} \geq \frac{x_{im}}{y_j} \geq \frac{x_{im}}{a+bx_{im}} + x_{im}^L\left(\frac{1}{y_j} - \frac{1}{a+bx_{im}}\right) \qquad (i,j,k,m) \in P_R \qquad (14a)$$

$$r_{ik} \geq \frac{x_{im}}{y_j} \geq \frac{x_{im}}{a'+b'x_{im}} + x_{im}^U\left(\frac{1}{y_j} - \frac{1}{a'+b'x_{im}}\right) \qquad (i,j,k,m) \in P_R \qquad (14b)$$

The following property can be established for these estimators:

<u>Property 5</u> The nonlinear inequality (14a) (or (14b)) is a valid convex underestimator when $b < 0$ (or $b' < 0$), and is nonredundant with respect to the nonlinear underestimator in (8a) (or (8b)).

Proof. See Appendix B.

It can happen that when the projected inequalities in (13) are obtained using the solution of the bounding subproblem, only a simple bound over the variable is obtained (e.g. b=0; $a = y_j^u$) instead of a linear inequality. In this case, if desired, it is possible to solve an additional problem fixing the projection variable at a given value within the bounds (i.e. $x_{im} = x_{im}'$ with $x_{im}^L < x_{im}' < x_{im}^U$).

The inequalities in (11) can also be projected in the space $(y_{jf}\ r_{i0})$, i e I\ j e $J^f$ for the bilinear terms in (PO) leading to inequalities of the form

$$r_{i0} \leq a + by, \qquad (15a)$$

$$r_{10} \text{ £ } a + b'yj \qquad (15b)$$

These can also be used to generate additional estimator functions through the linear estimators (6) and (7). In the bilinear case the estimators have the following form:

$$x^*_m \leq yjr^* \leq yfa + (a + byj)\ yj - yjHa + by,) \quad (i, j, k, m)\ G\ P_B \qquad (16a)$$
$$Xi_m \leq yjIIfc \leq yj^u r_{lk} + (a^f + by,)_{yj} - {}_{yj}{}^u(a^f + b'_{yj}) \quad (i, j, k, m)\ e\ P_B \qquad (16b)$$

$$x_{tm} \geq y_j r_{tk} \geq yj\wedge k + (a - byj)\ y, - y,^L(a - byj) \quad (i_t\ j, k_f\ m)\ e\ N_B \qquad (17a)$$
$$x_{tm} \geq yjIIfc \geq yj^u r_{lk} + (a^* - b\wedge)\ y, - {}_{yj}{}^u(a' - h'\wedge) \quad (i, j, k, m)\ €\ N_B \qquad (17b)$$

where b<0 and $b^f$<0 yield convex estimators. These type of inequalities are in fact equivalent to the ones proposed by Sherali and Alameddine(1990). The difference is that they are only generated when quadratic convex terms are obtained and here only the bilinear term is linearized. With the approach presented in this paper, only a small number of this type of constraints are generated since it is possible to identify nonredundant linear functions in an explicit form.

The bilinear terms in the constraints of problem (PI) can come from bilinear or fractional terms in the objective function of problem (PO). When the original term was a fractional one, projections of the type of equation (16) are not possible to be generated since the variable $r_4j$, i e I, j € J, does not exist in the constraint set g(x, y, r, z) $\leq$ 0. Hence, it is not possible to project the bounds of y, over $r_{tJ}$. However, projections can be performed for the bounds of r^, l e l j e J, when the following fractional bounding problems (10) are solved:

$$\min (\text{or max}) \frac{x_{tQ}}{y_j} = r_t j$$
$$\text{st. } g(x, y, r, z) \leq 0 \qquad (PR)$$
$$0 \leq x^L \leq x \leq x^u$$
$$0 < y^L \leq y \leq y^u$$

The nonlinear problem in (PR) has a unique solution. Moreover, as Charnes and Cooper (1962) have shown. If the constraints g(x, y, r, z) are linear, (PR) can be transformed into a linear programming problem. To achieve this, the transformation variable $t_j = \frac{1}{y_j}$ is introduced yielding the formulation:

min (or max) tpqo
st. g(x, y, r, z) £ 0 (PRl)

$0 \leq x^L \leq x \leq x^u$
$0 < y^L \leq y \leq y^u$

The linear constraints and bounds in (PRl) are multiplied by tj and the resulting products of variables are denoted as (x\ y\ r\ z') which yields the LP problem:

min (or max) $x_{i0}'$
st g'(x\ y\ i\ z\ t) $\leq$ 0 (PR2)

$x^1$ £ 0, $y^1$ > 0

Here g* includes the transformed original constraints and the additional linear constraints generated from the bounds. The solution of the LP problem (PR2) is used to generate projections of the variable x^ over ^ In a similar form as In (11),

$$0 \leq \sum_{\ell} \lambda_\ell g_\ell (x_{i0}'. t_J) \quad (18)$$

In this form the projection that is generated has the form:

xw,$^1$ £ d + e t, (or XJO$^1$ £ $d^1$ + e't,) (19)

or expressing it In terms of the original variables,

r,j2d+£ (orr$_{lj}$≤d'+^) (20)

In this way, additional estimator functions can be obtained by using equation (20) in the linear underestimators (7),

$$x_{,m} \geq r^{\wedge}yj \geq (dy, + e) + y^{\wedge}r^{*} - (d + \overset{e}{y})_{yj}{}^{L} \qquad (i, j, k, m) \in N_R \qquad (21a)$$

$$X!_{,,,} \geq r_{tj}y, \geq (dy, + el + yj^{u}r_{lk} - (d^{f} + \overset{e'}{y})\,yj^{u} \qquad (i, j, k, m) \in N_R \qquad (21b)$$

These estimators are convex when $e < 0$ ($e^f < 0$).

Property 6. The nonlinear estimators in (16), (17) and (21) provide an exact approximation at the boundary defined by the projected cut.

Proof. See Appendix B.

The estimators in (14), (16), (17) and (21) are only a subset of all the projected estimators functions that can be generated. In this paper at most one projected estimator for each variable in the nonconvex term is used. The estimators (16) and (21) can be particularly relevant because when the cost coefficient, c^, is negative the nonlinear estimators (8) cannot be used. This is illustrated in example 3.

**Convex Nonlinear Underestimator Problem**

Having derived the linear and nonlinear bounding approximations (6), (7), (8), (14), (16), (17) and (21) for the nonconvex terms in (PI), a convex nonlinear underestimator problem (NLPJ for problem (PI) can be defined as follows. Valid bounds over the variables are generated from (9) and (10) to define the set $Q = \{ x, y, z: x^L \leq x \leq x^{U}, y^L \leq y \leq y^u, r^L \leq r \leq r^U\}$ and the nonconvex terms in problem (PI) are substituted by the convex approximating functions. The projections for the upper and lower bounds of the variables are denoted by linear functions $(\cdot) = a \pm b(\cdot)$ where the convexity conditions are satisfied (eg. Property 5). This then leads to the following convex nonlinear programming problem:

$$\min f_L = \sum_{i\in I}\sum_{j\in J} c_{ij}\, r_{ij} - \sum_{i\in I'}\sum_{j\in J'} C_{ij}\, x_{ij} + h(x, y, r, z)$$

St. Xta, £ yjhTik + r ^ j - y^r^1- (if j, k, m) e N

$x_{im} \geq y_j^u r_{ik} + r_{ik}^u y_j - y_j^u r_{ik}^u$ $\quad (i,j,k,m) \in N$

$x_{im} \leq y_j^L r_{ik} + r_{ik}^u y_j - y_j^L r_{ij}^u$ $\quad (i, J, k, m) \in P$ **According to**

$x_{im} \leq y_j^u r_{ik} + r_{ik}^L y_j - y_j^u r_{ik}^L$ $\quad (i, J, k, m) \in P$ **Properties 1 to 4.**

$r_{ik} \geq \dfrac{x_{im}}{y_j^L} + x_{im}^u\left(\dfrac{1}{y_j} - \dfrac{1}{y_j^L}\right)$ $\quad (i,j,k,m) \in P$

^ ^ ^ ^ - y ?[1] ' (i.J.k.m)eP

$r_{ik} \geq \dfrac{x_{im}}{\phi(x_{im})} + x_{im}^L\left(\dfrac{1}{y_j} - \dfrac{*}{\phi(x_{im})}\right)$ $\quad (i.j.k,m) \in P_R$

$r_{ik} \geq \dfrac{x_{im}}{\phi(x_{im})} + x_{im}^u\left(\dfrac{1}{y_j} - \dfrac{1}{\phi(x_{im})}\right)$ $\quad (i, j, k, m) \in P_R$ **(NLPJ**

$x_{im} \leq y_j^L r$& + <Wyj) yj - y ^ t y ) $\quad$ (*. j. k. m) e $p_B$

$x_{im} \leq$ yj"$r_{ik}$ + <)>(yj) yj - yj"<I>(y)) $\quad$ (l. j, k, m) e $P_B$

$x_{im} \geq y_j^L r$& +¥yj) yj - yjN>(yj)) $\quad$ (i, j. k, m) e $N_B$

$x_{im} \geq$ yj"rik + <t>(yj) yj - yj"<I>(yj) $\quad$ (i, j. k, m) € $N_B$

$x_{im} \geq$ (dy$_j$ ·«• e) + y j ^ - (d + ^ yj$^L$ $\quad$ (i.J. k, m) e $N_R$

$x_{im} \geq$ (d'y$_j$ + e$^f$) + y$_j$"$r_{ik}$ - (d* + H y," $\quad$ (i, J, k, m) G $N_R$

$g(x, y, r, z) \leq d$

$(x, y, r) \in Q,\ z \in Z$

**<u>Property 7</u>. Any feasible point (x, y, r, z) in problem $NLP_L$ provides a valid lower bound to the objective function of problem (PI). Furthermore, the optimal solution $f_L^f$ of (NLPJ provides a valid lower bound to the global optimum (f*) of problem (PI).**

**Proof. See Appendix B.**

**<u>Corollary</u> 3. If the optimum solution $f_L^*$ from $NLP_L$ is equal to the objective function value f from (PI) it corresponds to the global optimum of (PI).**

**Proof. See Appendix B.**

**Partitioning Scheme (Branch and Bound)**

An optimal solution of problem $NLP_L$ provides a valid lower bound to the global optimum of problem (PI) (Property 7). At the same time this solution is a feasible solution to the problem (PI) since problem $NLP_L$ includes all the original constraints. Hence, only an evaluation of the original objective function is required to obtain a valid upper bound. Also, the solutions of the bounding subproblems (9) and (10) provide a feasible solution that can be used to generate an upper bound of the global solution. When the lower and upper bound are equal, the global solution has be obtained (Corollary 3). If there is a gap between these bounds, a partition of the feasible region must be performed. The estimator functions can then be updated in each subregion to yield tighter lower bounds over each subregion. A spatial branch and bound search is performed to successively partition the feasible region along the coordinate directions of the variables. When the lower bound for a particular subregion is greater or equal than the best upper bound available, the subregion is discarded. Also during this search procedure feasible solutions are obtained with which the best upper bound can be updated.

For partitioning the feasible region, it is necessary to select a variable over which the division of the space will be performed and its corresponding value. The first rule considered here is the same one that the one used by Sherali and Alameddine (1990), only that now bilinear and fractional terms are present in the objective function.

**Rule 1:** $(v_f\ w, v \backslash v^M) \in \arg\max \wedge . \wedge \ \{C_t, (r_{ik} - \frac{x_{im}}{y_j})\}$

if $v' = O$ $\quad \beta_v = \sum_j \text{Abs}\{c_{vj}(r_{v0} - \frac{x_{vj}}{y_j})\}$

$\quad\quad i \quad\quad Y_{vr} \}$

if $\beta_v > \beta_w$ select $x_{v0}$ else select $y_w$

if $v'' = O$ $\quad \beta_v = \sum_j \text{Abs}\{c_{vj}(r_{vj} - \frac{x_{v0}}{y_j})\}$

$$\beta_w = \sum_i \text{Abs}\{c_{tw}(r_{tw} - \frac{x_{t0}}{y_w})\}$$

if $\beta_v > \beta_w$ select $x_{v0}$ else select $y_w$

In this way the nonconvex term for which the approximation differs the most is selected, and the variable that is involved in this term and can affect the most the other approximations is selected. A second rule that can also be used is as follows:

**Rule 2:** (vl$_f$ wl$_f$ v1\ vl") €arg *max^M.**** {Cij(r$_{lk}$ - $\frac{x_{tm}}{y_j}$)}

co = a Abs{Cv!$_{wl}$(r$_{vlvl}$ - $\frac{x_{vl\,vl}}{y_{wl}}$)}

**select a (v,w,v$^f$\v$^M$) for which Abs(r$_w$~ ^p$^1$ )> a>** *Jvr*

proceed as in Rule 1.

where 1 £ a > 0. This rule reduces to the first one when a = 1. There is a tradeoff between the two rules. Rule 1 may be attractive when the size of 111, IFI, IJI or IJ$^1$1 is small because the variables that are present in more terms are selected and a smaller number of partition variables are used. In practice rule 2 can be more useful since there are cases in which there are more than one term for which the difference in the approximation can be large. In this situation it may be useful to select a variable on which partitioning has not been performed previously because this allows for the tightening of its bounds.

**Algorithm**

StepO. Initialization.

(a) Set f= eo, F=0, select tolerances e and *y*

(b) For each variable x^, i e I, y$_{jv}$ j e J, and r^, ie I\ y$_{jf}$ i€ I$^1$, and the terms ^, i e I, j e J, determine:

-lower and upper bounds by solving the bounding programs in (9) and (10).

-Optional: obtain projections of the variables (as in (13), (15) and (19)) using either the solutions of the previous bounding problems or problems at a fixed level for the projection variable.

-Evaluate the original objective function for each of these feasible solutions. If f < f set f=f and store the solution as the incumbent solution (*).

(c) Store the bounds in Q°, and set F=Fu{0}

Step 1. Convex underestimator problem

(a) Solve problem NLP$_L$for Q° to obtain f$_L$°.

(b) Evaluate the original objective function f°. If f°< f set f =f° and store the solution as the incumbent solution f).

Step 2. Convergence

(a) For the subregions j in F, if $f_L$J- f ≥ ef delete subregion j from F (F=F\{J}).

(b) If F=0 the e-global solution is given by the incumbent solution.

Step 3. Partition

(a) Take the last region k in F ($Q^k$) and apply the selection rule (Rule 1 or Rule 2).

(b) Subdivide subregion $Q.^k$ in subregions $£l^{k+1}$ and $Q^k+^2$ by adding the respective bound or inequality. Delete subregion $Cl^k$ from F and store subregions $Q^{k+1}$ and $Q^{k+2}$ in F (F=(F\{k}) ∪{k+1,k+2}).

Optional: Update the bounds in subregions k+1 and k+2 for the variables involved in the nonconvex term with the partition variable.

Step 4. Convex underestimator problems

(a) Solve problem $NLP_L$ for $Q^{k**}$ and $Q^{k*2}$ to obtain $f_L^{k+1}$ and $f_L^{k+2}$.

(b) Evaluate the original objective function for each of these feasible solutions. If f< f, set f =f and store the solution as the incumbent solution (*).

-Optional: When the difference between the objective function of the convex underestimator problem $NLP_L$ and the incumbent solution f is smaller than a given tolerance ((f* - fj/f* < y), solve the original nonconvex problem (P) for $Q^{k+1}$ and/or $Cl^k+^2$ using its convex solution as the initial point. If f< f set f =f and store the solution as the incumbent solution O.

(c) If $f_L^{k+1} < f_L^{k+2}$ invert $ft^{k+1}$ and $Q^k+^2$ in F. Go to step 2.

It should be noted that when there is a strong interdependence between the variables involved in the nonconvex terms, it can be useful to update the bounds of some of the variables when a partition is performed (Step 3 optional). In particular the bounds of the variables that are involved in nonconvex terms with the partition variables can be updated. Also, the original nonconvex problem (PI) can be solved over the corresponding subregion when the difference between the upper and lower bound are small to accelerate the convergence.

The proposed algorithm can be used for the global optimization of linear fractional programming problems, bilinear programming problems or problems that involve fractional and bilinear terms in the objective function in which the feasible region is convex. When the feasible region is described by linear constraints the bounding problems solved in the initialization step O.b are LP problems.

Property 8. The branch and bound algorithm will either terminate in a finite number of partitions at a global optimal solution, or generate a sequence of bounds that converge to the global solution.

Proof. See Appendix B.

**Illustrative Example**

Example 1 Bilinear objective

The formulation of the NLP underestimator problem and the performance of the algorithm is illustrated by solving the following example proposed by Al-Khayyal and Falk (1983).

$$\begin{aligned} &\min f = -x + xy - y \\ &\text{st.} \; -6x + 8y \le 3 \\ &\quad 3x - y \le 3 \\ &\quad 0 \le x, y \le 5 \end{aligned} \qquad \text{(ALK)}$$

The feasible region with the original objective function is plotted in Figure 2. First, valid bounds are obtained for x and y by solving the corresponding LP's which yields:

$$0 \le x, y \le 1.5 \qquad (22)$$

From the solution of these LP problems two projected inequalities can be obtained:

$$x \le \frac{3+y}{3} \qquad (23)$$

$$y \le \frac{3+6x}{8} \qquad (24)$$

The NLP underestimator problem is then given by:

$$\min f = -x + w - y$$

st. $-6x + 8y \leq 3$

$3x - y \leq 3$

$w \geq 0$

$w £ 1.5 y + 1.5 x - 2.25$ (ALK[1])

$w \geq [\frac{3+y}{3}] y + 1.5 x - 1.5 [\frac{3+y}{3}]$

$w £ 1.5 y + [\frac{3+6x}{8}] x - 1.5 [\frac{3+6x}{8}]$

$0 \leq x, y \leq 1.5$

The solution of this initial $NLP_L$ is $f_L^{\circ} = -1.2569$ and the actual objective for this solution is f = -0.892 which is the incumbent solution. The initial underestimator problems for the Al-Khayyal and Falk approach and the one proposed here are shown in Figure 3.

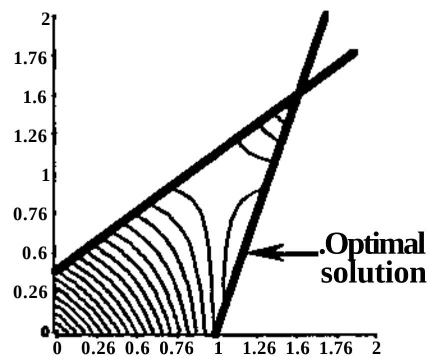


Figure 2. Feasible region and contours for example (ALK)

Since there is a gap between the lower bound and the incumbent solution a partition is made selecting the variable x. The solutions of problem $NLP_L$ at the two subregions are $f_L$*= -0.995 ($x \leq 0.803$) and $f_L^2$ = -1.1761 (x £ 0.803) and the incumbent solution is f= -1.0048. Therefore, the first subregion can be eliminated. A new partition is performed in the second subregion with the variable y yielding $f_L^3$ = -1.0833 ($y \leq 0.783$) and $f_L^4$= -1.0807 (y £ 0.783) with the incumbent being f= -1.0833. Hence, the global solution is found at x= 1.167, y=0.5, f=-1.0833 The computational results for this example are given in Table 1.

The total time required to solve the problem with the proposed algorithm, including the

Table 1. Computational results for example 1 (or problem ALK).

| method | Size[1] | Initial $f_L$ | Initial r | Subregions | Solution |
|---|---|---|---|---|---|
| Al-Khayyal | (3,4) | -3.000 | -0.750 | >103 | -1.0833 |
| Sherali | (5.21) | -1.500 | -0.9375 | 11 | -1.0833 |
| proposed | (3,6) | -1.2569 | -0.892 | 5 | -1.0833 |

[1] (n,m) n= no. of variables, m = no. constraints.

time for the bounding problems, is 0.75CPU sees, on a IBM/R6000-530 using MINOS 5.2 for solving the LP and NLP problems.

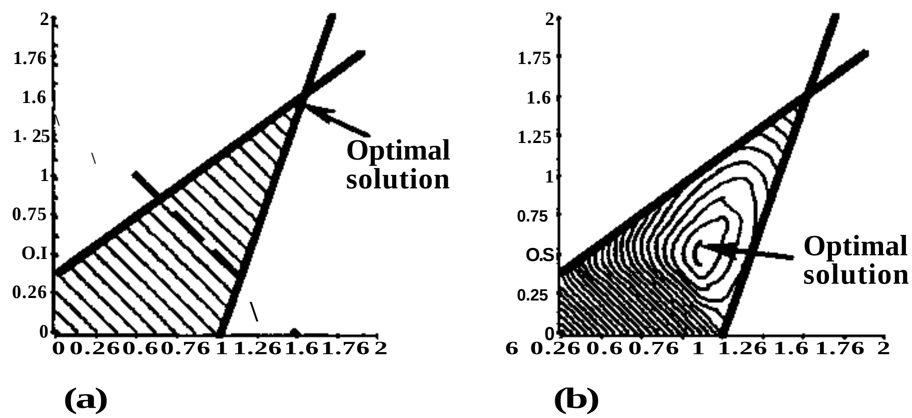


Figure 3. Feasible region and contours for underestimators by (a) Al-Khayyal and Falk, (b) proposed method.

This example has also been modified (Example la) to have an objective function that involves bilinear and fractional terms as in problem (PO):

$$\min f = -x + xy - y + \frac{\hat{}}{x}$$

$$\text{st. } -6x + 8y \le 3$$

$$3x - y \le 3 \qquad \text{(ALK1)}$$

$$x + y \ge 1$$

$$0 \le x, y \le 5$$

Applying the algorithm, the initial convex $NLP_L$ has a solution of $f_L^\circ$ = -1.00 with x=1.00 and y=0.00, and the incumbent solution is f* = -1.00. Hence, the global optimal solution is obtained in this case without having to perform a spatial branch and bound search.

## Nonconvex Feasible Regions

The solution method presented for the nonconvex objective function defined over a convex feasible region in problem (PO) can be extended with some modifications to the case of nonconvex feasible regions. In this section an outline of the necessary modifications are given. Here the constraints that describe the feasible region can involve convex, linear fractional and bilinear terms in the same form than the nonconvex objective function in problem (PO). The problem considered is as follows (variable bounds are omitted):

mingo

st. g^O /e L (P2)

**where g,= S i v f - I I <VPiQj + Wp, q, z), /=O,1.....L**
**i€i jej J 9j ter j€j- J**

Problem (P2) can be reformulated in the same form as (PO) yielding,

mingo

st g,≤0 *ie* L (P3)

**yj rik ≥ xim** (l, j, k, m) €P

**yj rik ≤ xim** (l, j, k, m) eN

where g/=£ £ Cj, ry- £ Z Ct/xy + $h^k$($x_f$ y, $r_t$ z), /=O,1....L
i€l j€J i€i' jej'

The same type of transformations and estimator functions that were presented previously are used for every function g, (/ *e* {0}u L)

An important difference between having the nonconvex terms only in the objective function or in the constraint is that there are alternative representations of the nonconvex constraints. Depending of the individual bounds of each particular nonconvex term, different representations may be tighter or nonredundant.

Consider the following nonconvex constraint:

$$\sum_{i=1}^{n} z_i \le x_1 x_2 \qquad (25)$$

which can also we written as follows

$$\frac{z}{x_1} \le x_2 \qquad (z \le x_1 x_2) \qquad (26)$$

$$z = \sum_{i=1}^{n} z_i \qquad (27)$$

With (26) and (27) it is possible to develop nonlinear overestimator functions over the fractional term, $\frac{z}{x_1}$. If the bounds are nonredundant, linear underestimators can also be obtained. These estimators give exact approximations at the lower and upper bounds of the variables involved in the estimators function ($x_1$, $x_2$ and z).

The constraint in (25), however, can also be expressed as,

$$\sum_{i=1}^{n} \frac{z_i}{x_1} \le x_2 \qquad (28)$$

and nonlinear underestimators can be generated over each fractional term, $\frac{z_i}{x_1}$. These estimators provide exact approximations at the lower and upper bounds of each variables $z_i$ and which are nonredundant. There are points for which each variable $z_i$ is at either of its bounds and $x_1^L < x_1 < x_1^u$, $x_2^L < x_2 < x_2^u$ and $z^L < z < z^u$. For this point, representation (26) cannot yield an exact approximation while the representation (28) is exact.

Independently on whether or not the nonconvex constraints in problem (P2) are rearranged, several modifications are required in the algorithm presented earlier to handle nonconvex feasible regions. Firstly, the solution of the nonlinear convex problems are not necessarily feasible solutions and they cannot always be used to update the upper bound. For this reason the original nonconvex problem (P2) can be solved to ensure that an upper bound is available during the search. In this work the solution of the first convex problem $NLP_L$ is used as an initial point. To update the upper bound additional nonconvex problems can be solved during the search.

Secondly, to generate the lower and upper bounds of the variables there are different options. One is to simply consider the subset of convex constraints to generate these bounds. Alternatively, the nonconvex terms present in the constraints can be substituted by the linear estimator functions (6)-(7). This allows to consider the possible interactions between all the variables. The bounding subproblems can be solved in parallel since they are independent. In the case that they are solved in a sequential form the bounds that are obtained can be used in the subsequent subproblems if these variables are involved in estimator functions of nonconvex terms since better approximations will be obtained.

Finally, a modification of the branching rule is also necessary. The coefficients of the nonconvex terms in the constraints are not sufficient for comparing the approximations since the constraints can be scaled up or down by constant factors affecting in this form the selection of the term. This problem can be avoided by including the Lagrange multipliers of the constraints along with the coefficient of the nonconvex term in the selection rule. In this form the selection rule when nonconvex terms are present in the constraints and objective function is as follows:

**Rule 3:** (v, w, v\ v'') e arg max$_{(1Jtkjn)}$ {X'q,' (r$_{ik}$ - ^) } /€ L uO

proceed as Rule 1 or Rule 2

This rule has the advantage that when different expressions of the same constraint (as in 26 and 28) are included in the formulation, only the constraint that is active $\{X_e > 0)$ is considered when selecting the partition variable.

**Illustrative Example**

Example 2 Nonconvex Feasible Region

The following example has a feasible region that is disjoint (see Fig 4). The problem is given by:

$\min f = -x_r\, 2x_2 - yi - 2y_2$

st $x\hat{}i \leq 10$

$xay_2 \hat{} 10$

$2xj + 2y_2 \geq 13$

$15x_2 + 20y_2 \geq 87$

$\frac{y_1}{x_1} \leq 2x_2$

$0.5 \leq xj \leq 5$

$0.5 \leq x_2 \leq 4$

$0.5 \leq y! \leq 10$

$0.5 \leq y_2 \leq 3$

The first convex underestimator problem has a solution of $f_L° = -23.8$. Using this solution as a starting point, the original nonconvex problem is solved to obtain an upper bound. This yields the incumbent solution with an objective function of f= -23.06. The variable yi is selected for partitioning (y!= 8.944) since it is over its terms that the approximations are not exact. The two new subregions have solutions that are greater of equal than the incumbent solution and the global solution is Xi=1.118, $x_2$=4, y!=8.944 and $y_2$=2.5 with f=-23.06.

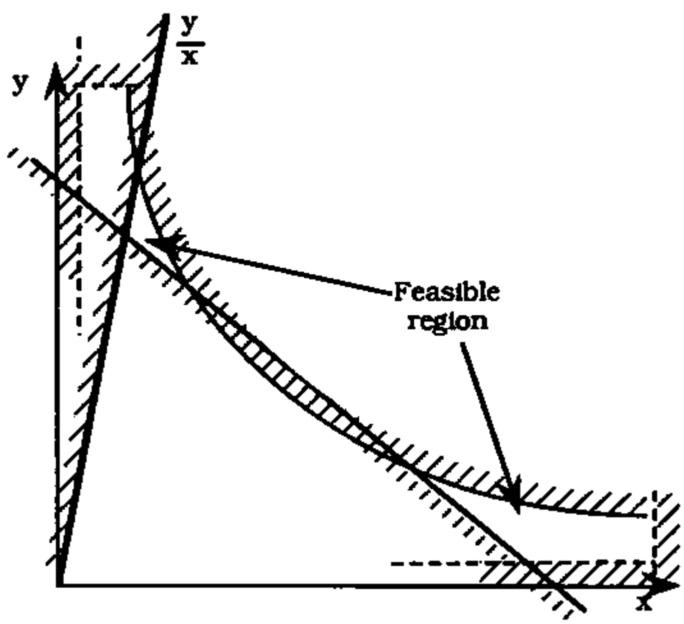


Figure 4. Disjoint feasible region for example problem.

**Computational Results**

In this section the global optimization of linear fractional, bilinear and polynomial problems is considered with the proposed algorithm. Size and characteristics of these problems are given in Table 2. Examples 3, 4 and 5 are fractional programs given in Falk and Palocsay (1991). Examples 6, 8, 9 and 10 correspond to bilinear problems, example 7 is a polynomial problem that can be reformulated as a bilinear problem and example 11 involves both bilinear and fractional terms. The computational results are given in Table 3. As can be seen 4 out of the 12 examples only required the solution of one convex NLP underestimator problem. Except for the example 11, the computational requirements were very modest. The bounding subproblems, nonconvex subproblems and the convex NLP subproblems were solved using MINOS 5.2 through GAMS on a IBM/R6000-530. The total time requirements for the underestimator subproblems was less than 1 cpu sec. except for problem 11 that required 10.3 sec. For the initialization step the total time requirements were also less than 1 cpu sec. Details on the examples are given below.

In some of these examples (Examples 9 and 10) it was not possible to generate

Table 2. Size and characteristics of example problems.

| Example | Total no. variables | Nonconvex var. | Nonconvex terms | Constraints | Feasible Region |
|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | 2 | C |
| 1a | 2 | 2 | 2 | 3 | C |
| 2 | 4 | 4 | 4 | 6 | N |
| 3 | 6 | 4 | 2 | 5 | C |
| 4 | 7 | 4 | 2 | 9 | C |
| 5 | 6 | 4 | 2 | 6 | C |
| 6 | 4 | 4 | 4 | 6 | C |
| 7 | 4 | 3 | 2 | 3 | C |
| 8 | 2 | 2 | 1 | 1 | N |
| 9 | 4 | 4 | 2 | 4 | N |
| 10 | 9 | 6 | 2 | 6 | N |
| 11 | 10 | 7 | 5 | 7 | N |
| 12 | 20 | 14 | 7 | 16 | C |

C-convex, N=nonconvex

additional estimators through the projections and there was a difference between the lower and upper bound at the initial node. For these examples the estimators used here correspond to the convex envelope of that nonconvex terms. Nevertheless, the algorithm presented here and the modifications to the branching rule (Rule 2 and 3) allowed fast convergence by using a different variable to partition on.

Table 3. Computational results for examples.

| Example | **Initial lower bound** | Initial upper bound | **Global solution** | Number of subregions |
|---|---|---|---|---|
| 1 | **-1.2569** | **-0.892** | **-1.0833** | 5 |
| 1a | -1 | -1 | -1 | 1 |
| 2 | **-21.8** | **-21.06** | **-21.06** | 3 |
| 3 | -5.0529 | **-4.9919** | -5.0 | 3 |
| 4 | **-2.47** | **-2.47** | **-2.47** | 1 |
| 5 | **1.5953** | 1.625 | **1.6231** | 3 |
| 6 | -13 | -13 | -13 | 1 |
| 7 | -6 | -2 | -4.5 | 3 |
| 8 | **-6.66** | **-6.66** | -6.66 | 1 |
| 9 | **2.8284** | **2.966** | **2.966** | 5 |
| 10 | -500 | **-400** | **-400** | 3 |
| 11 | **7049** | **2834** | **7049** | 53[1] |
| 12 | **126.91** | **131.87** | 127.01 | 3 |

[1] Not solved to optimality. Tolerance 5%

**Example 3.** Linear fractional objective. The formulation for this example is given by:

$$\max f = \frac{37x_1 + 73x_2 + 13}{13x_1 + 13x_2 + 13} + \frac{63x! - 18x2 + 39}{13x_1 + 26x_2 + 13}$$

st $\quad 5x! - 3x2 = 3$ **(FA1)**

$1.5 \leq x_2 \leq 3$

Introducing the additional variables, $y_4$ and z,, and constraints to express the objective function as a sum of linear fractions of single variables yields.

$$\min f = -\frac{y_1}{z_1} - \frac{y_2}{z_2}$$

St. $\quad yi = 37x! + 73x_2 + 13$

$Zi = 13X! + 13x_2 + 13$

$y_2 = 63x! - 18x_2 + 39$ **(FA1')**

$Z2 = 13x! + 26x_2 + 13$

$5x! - 3x_2 = 3$

$1.5 \leq Xi \leq 3$

$y, z \geq 0$

Since this is a minimization problem and all the coefficients are negative, it is not possible to use the nonlinear underestimators (8) of the fractional terms. Bounds are generated for all the variables and it Is possible to identify a projection for the first fractional term that can be used to generate an additional estimator

$$r_1 \leq \frac{-59.948}{z_1} + 4.5764 \qquad (29)$$

The first convex underestimator problem has a solution of $f_L^\circ$ = -5.0529 with the ratio terms r=(-3.8525, -1.228), The exact objective function is f = -4.9919 and it is the incumbent solution. The approximation for the first term is exact, so the second term is used for partitioning the feasible space. Two subregions are considered using the value of the incumbent solution to partition. The first one has a solution of $f_L^1$ = -4.98 ($r_2 \geq$ 1.1394) and it can be discarded, the second one has a solution of $f_L^2$ = -5.0002 ($r_2 \leq$ 1.1394). This subregion is 0.004% within global optimality. At this point the algorithm can stop or if an extra partition is done the global solution with an objective function of f=-5.0 is found exactly with x^ 3.0 and X2 =4.0. 11 should be noted that Falk and Palocsay (1991) required 20 iterations to solve this problem.

**Example 4.** Linear fractional objective. For the next example a similar transformation is necessary. The problem is given by:

$$\min f = -\frac{y_1}{z_1} - \frac{y_2}{z_2}$$

$$\begin{aligned} \text{st} \quad & y_1 = 3x_1 + x_2 - 2x_3 + 0.8 \\ & z_1 = 2x_1 - x_2 + x_3 \\ & y_2 = 4x_1 - 2x_2 + x_3 \\ & z_2 = 7x_1 + 3x_2 - x_3 \\ & x_1 + x_2 - x_3 \leq 1 \\ & -x_1 + x_2 - x_3 \leq -1 \\ & 12x_1 + 5x_2 + 12x_3 \leq 34.8 \\ & 12x_1 + 12x_2 + 7x_3 \leq 34.8 \\ & -6x_1 + x_2 + x_3 \leq -4.1 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{aligned} \qquad (FA2)$$

The global optimal solution is $x_1$ =1, $x_3$ = 0 and f*= -2.47.

**Example 5.** Linear fractional objective. The last problem of this series corresponds to a minimization problem and the formulation after the addition of extra variables is given by:

$$\min\ f = \frac{y_1}{z_1} + \frac{y_2}{z_2}$$

$$\text{st}\quad y! = -Xj + 2x_2 + 2$$
$$Z_X = 3Xj - 4X2 +^5$$
$$y_2 = 4xj - 3X2 + 4 \qquad \textbf{(FA3)}$$
$$Z2 = -2x_2 + X2 + 3$$
$$X! + X2 \wedge 1.5$$
$$*1 \le X_2$$
$$0 \le Xj \le 1$$
$$0 \le Xa \le 1$$

for this example a solution within e=0.07% of the global optimal x^O, $x_2$=0.284 and f*= 1.6231 is obtained.

**Example 6.** Bilinear objective. The next example is a bilinear problem taken from Visweswaran and Floudas (1990b) and the formulation is given by,

$$\min f = xj - x_2 - yi - x\wedge + xjy_2 + X2yi - X2y_2$$
$$\text{st.}\ Xj + 4x_2 \le 8$$
$$4x_2 + x_2 \wedge 12$$
$$3x_2 + 4x_2 \pounds 12$$
$$2yi + y_2 \wedge 8 \qquad \text{(FL1)}$$
$$y, + 2y_2 \le 8$$
$$y! + y_2 \le 5$$
$$Xi.x_{2t}y_{lf}\ \ y_2 \ge 0$$

The global optimal solution corresponds to X!=3, $x_2$=0, yi=4, $y_2$=0 and f*=-13.

**Example 7.** Polynomial objective. The next problem is a polynomial problem taken from Floudas and Visweswaran (1991) and the formulation is:

$$\min f = -6y + 4.5y^2 - y^3$$
$$0 \le y \le 3 \qquad \text{(FL3)}$$

The feasible region is convex and the only nonconvex term is the one with the cubic term. The problem can be reformulated as:

$$\min f = -6y + 4.5y^2 - x_3$$
$$\text{st} \quad x_3 \le x_2 y$$
$$x_2 \le x_1 y \qquad \text{(FL3}^1\text{)}$$
$$x_1 = y$$
$$0 \le x_1 \le 3,\ 0 \le x_2 \le 9,\ 0 \le x_3 \le 27,\ 0 \le y \le 3$$

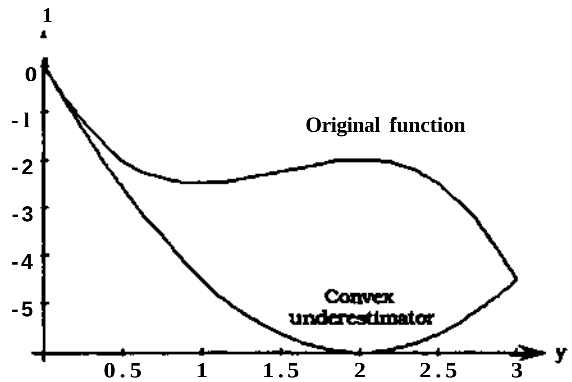


Figure 5. First underestimator for polynomial problem.

The original objective function and the approximation are plotted against the original variable in Fig. 5. The global optimal solution is at y=3 with f*=-4.5.

**Example 8.** Bilinear constraint. Consider the small example presented by Sahinidis and Grossmann (1991) where the formulation includes a bilinear constraint,

$$\min f = -x - y$$
$$\text{st} \quad xy \le 4$$
$$0 \le x \le 6 \qquad \text{(FL2)}$$
$$0 \le y \le 4$$

The global optimal solution is at x=6, y=2/3 and f*=-6,66.

**Example 9.** Bilinear constraint. The next small example is taken from Lo and Papalambros (1990). Here the model is given by:

$$\min \ 2x_1 + x_2$$
$$\text{st} \ \frac{1}{x_1} - x_2 \le 0$$

$$x_1y_1 + x_2y_2 \geq 0$$
$$x_1 = y_1$$
$$x_2 = y_2$$
$$0.1 \leq x, y \leq 2.5$$

The global solution is x^O.517, Xa=1.932 with f*=2.966.

**Example 10.** Bilinear constraint. The following example is taken from Lasdon et al. (1979) and Swaney (1990)

min 6xj + I6X2 + IOX4 -9*5 + 10x7 * 15x8

St X! + $X_2$ - $X_3$ - Xg =0

0.03X! +O.OlX$_2$ -X3X9 - X3Xg= 0

$x_3$ + X4 - Xe = 0

Xe + $x_7$ - Xe = 0

$x_3x_9$ + 0.02X4 - 0.025X5 ≤ 0

$x_6x_9$ + 0.02$x_7$ - 0.015xe £ 0

x £ 0, X! ≤ 300, $x_2$ ≤ 300, $x_3$ ≤ 100, X4 ≤ 100, X5 ^100, Xg≤ 200, $x_7$ ≤ 200, Xe ≤ 200

0.01 ≤X9≤0.03

The global solution is located at $x_3$=0, Xe=100 and X9=0.01 with f*=-400.

**Example 11.** Bilinear and fractional constraints. The last example is the mathematical model for an alkylation plant and is taken from Liebman et al. (1986). The model is:

min 5.04xj + 0.035x2 + 10$x_3$ + 3.36xs - O.O63X4X7

st $x_2$ = 1.22x4 -$x_5$

X9 + 0.222$x_{10}$ - 35.82 = 0

3$x_7$-$x_{10}$ -133 = 0

$x_7$ < 86.35 + 1.098XQ - 0.038 $XQ^2$ + 0.325(x6 - 89)

$x_4x_9$ + 1000$x_3$ - 9800(^ = 0;

$x_2$ + X5 -XiXa = 0

1.12 + 0.13167X8 -0.00667$X8^2$ -$f^1$ ≥ 0

1 ≤ X! <, 2000, 1 < $x_2$ < 16000, 0 ≤ $x_3$ ≤ 120, 1 ≤ X4 £ 5000, 0 ≤ X5 ≤ 2000

85 ≤ xs ≤ 93, 90 ≤ $x_7$ ≤ 95, 3 ≤ Xs ^ 12, 1.2 ≤ X9 ^ 4, 145 £ $x_{10}$ ≤ 162

Tighter bounds for the variables are obtained by solving a bounding subproblems. In this formulation it is possible to rewrite some of the constraints, in particular it proves useful to rewrite the sixth constraint in the following forms:

$$\frac{x_2}{x_1} + \frac{x_5}{x_1} - x_8 = 0$$

$$\frac{x_2}{x_8} + \frac{x_5}{x_8} - x_1 = 0$$

The first convex problem has a solution of $f_0^L = -2824$ and using these solution as an initial point for solving a nonconvex problem an upper bound of $f^U = -1161$ with $x_5 = 2000$ is obtained and corresponds to the global solution. After 53 nodes the solution is proven optimal within a 5% tolerance.

**Example 12.** Linear fractional objective. This formulation corresponds to a heat exchanger network in which the objective is to minmiize the total area. Arithmetic mean is used for calculating the temperature driving force.

$$\min f = \frac{50Q_{11}}{\Delta T_{11}} + \frac{70Q_{12}}{\Delta T_{12}} + \frac{5Q_{21}}{\Delta T_{21}} + \frac{20Q_{22}}{\Delta T_{22}} + \frac{2Q_{H1}}{\Delta T_{H1}} + \frac{50Q_{C1}}{\Delta T_{C1}} + \frac{100Q_{C2}}{\Delta T_{C2}}$$

st.

$$Q_{11} + Q_{12} = T_{C11} - 300$$

$$Q_{12} + Q_{22} = T_{C12} - T_{C11}$$

$$Q_{H1} = 400 - T_{C12}$$

$$Q_{C1} = 1.5(T_{H11} - 310)$$

$$Q_{11} = 1.5(T_{H12} - T_{H11})$$

$$Q_{12} = 1.5(410 - T_{H12})$$

$$Q_{C2} = 1.5(T_{H21} - 300)$$

$$Q_{21} = 1.5(T_{H22} - T_{H21})$$

$$Q_{22} = 1.5(420 - T_{H22})$$

$$\Delta T_{11} = \frac{T_{H12} - T_{C11} + T_{H11} - 300}{2}$$

$$\Delta T_{12} = \frac{410 - T_{C12} + T_{H12} - T_{C11}}{2}$$

$$\Delta T_{21} = \frac{T_{H22} - T_{C11} + T_{H21} - 300}{2}$$

$$\Delta T_{22} = \frac{420 - T_{C12} + T_{H22} - T_{C11}}{2}$$

$$\Delta T_{Hi} = \frac{450 - 400 + 450 - T_{£12}}{2}$$

$$\Delta T_{ci} = \frac{310 - 285 + T_{HI\,1} - 295}{2}$$

$$\Delta T_{C2} = \frac{300 - 285 + T_{H21} - 295}{2}$$

$$Q_f \; \Delta T . T \hat{} O$$

The global optimal solution is $Q_{2j} = 100$, $Q_{ci} = 150$ and $Q_{Q2} = 80$ with $f = 127.01$.

## Conclusions

An algorithm for the global optimization of linear fractional and bilinear programming problems has been proposed that relies on the solution of nonlinear convex underestimator problems which result from substituting the nonconvex terms by linear and nonlinear estimator functions. Conditions under which these functions are nonredundant have been established. It has also been shown that additional valid estimator functions can be obtained through projections from subproblems for tightening the variable bounds. Thirteen examples reported in the literature have been solved using the proposed method, showing that strong lower bounds are obtained in most of the cases. This greatly reduces the enumeration of nodes in the spatial branch and bound search with which the computational requirements are kept small. Efforts are currently under way to test the performance of the algorithm in larger problems.

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**Appendix A. Estimators for Factorable Functions**

A concave overestimating function of a product of functions Is given by (McCormick (1983)),

fix) g(y) ≤ minlf" Cg(y) + $g^L$ C^x) - f» $g^L$. $f^L$- $C_g$(y) + $g^u$Cf(x) - *P*- g"] (A. 1)

where f, $g^1$^ and $g^L$ are positive bounds over the functions fix) and g(y) such that:

**0 ≤ $f^L$ ≤ f(x) ≤** f' (A.2)

0 5 $g^L$ 5 g(y) £ $g^u$ (A.3)

and Cjcx) and Cgty) are concave functions such that for all x and y in some convex set:

C,(x) Sf(x) (A.4)

Cg(y) £g(y) (A.5)

In a similar way as in (A. 1), the convex underestimating function of a product of functions is given by.

fix) g(y) £ maxlf $c_g$(y) + g" c^x) - **f** g". $f^L$- cg(y) + gLcrfx) - *f- g-]* (A.6)

and q(x) and $c_g$(y) are convex functions such that for all x and y in some convex set:

**$c_f$(x) ≤ f(x)** (A.7)

**$c_g$(y) ≤ g(y)** (A.8)

In the case of bilinear functions (flx)=x and g(y)=y) the individual concave and convex bounding functions of each individual term are given by the function itself:

C(x) = x = c«(x) (A.9)

**$C_g$(y)** = y = $c_g$(y) (A. 10)

Thus, from (A. 1) and (A.6) the following under and over estimator functions are obtained:

xy £ max[ x*y + y*oc -xy-. x"y +y"x - **$x^u y^u$]** **(A. 11)**

xy £ *maxl yhf* + y"x -x$V^1$, x"y +y*oc - x'y-) (A.12)

For fractional linear terms (f(x)=x and $g(y)=\frac{1}{y}$). it is possible to generate a convex underestimator function because the individual convex bounding functions are given by,

$$c_f(x)=x \quad (A.13)$$

$$c_g(y) = \frac{1}{y} \quad (A.14)$$

From (A.11) the underestimator function can be expressed as:

$$\frac{x}{y} \geq \max\left[\frac{x}{y^L} + x^u\left(\frac{1}{y} - \frac{1}{y^L}\right), \frac{x}{y^u} + x^L\left(\frac{1}{y} - \frac{1}{y^u}\right)\right] \quad (A.15)$$

The estimator functions (A. 11), (A. 12) and (A. 15) have the property that they match the original function when one of the variables is at a bound. This is because the individual convex and concave bounding functions in (A.9), (A. 10), (A. 13) and (A. 14) are the functions themselves.

## Appendix B. Mathematical Properties and Proofs

Property 1. When $r^\wedge =$ ^"J- (or $r*^{11}$ = ^fc-), (i, j, k, m) e $P_R$, the linear overestimator (6a) (or (6b)) is a linearization of the nonlinear underestimator (8a) (or (8b)).

Proof Consider the linear overestimator (6a)

$$Xi_m \wedge Yj\wedge ik + \dot{r}dty - {}_{yj}{}^{u}r_{lk}L \quad (B.1)$$

Rearranging (B.I) leads to:

$$r_{1k} \wedge - \wedge + r_{1k} L \quad (B.2)$$

Using the condition that $r_{ik}^{L} = \wedge j - f$ equation (B.2) yields

$$r\,i\,k \wedge \wedge + x \wedge H \wedge r - \text{-gjjfcpr}) \quad (B.3)$$

The nonlinear underestimator (8a) gives rise to the constraint

$$r_{1k} \wedge + Xim^{L}(\wedge - \wedge j) \quad (B.4)$$

The first term of equations (B.3) and (B.4) is the same. Now compare the nonlinear term NTty) = fc" - T^ir) fr<>$^m$ equation (B.4) with the linear term {^ - T^sta') from equation (B.3). Both terms are equal at $y)^u$. Furthermore, a linearization of the nonlinear term at $yj = yj^u$ yields the linear term:

$$NT(y_J«) + V_M NT(y_j-)(y_J-y_j-) = .\wedge \quad (yj-yj^u) = \wedge - \wedge f\,T \quad (B.5)$$

Thus, (6a) is a linearization of (8a) and in a similar form it can be proven that for $r^\wedge$ = -^t", (6b) is a linearization of (8b). I

Corollary 1. The nonlinear underestimator (8a) (or (8b)) is stronger than the linear overestimator (6a) (or (6b)) when $r_{ik}^{L}$ = ?*k (or $r_{ik}^{u}$ = -*j£), (i, $j_f$ k, m) e $P_R$.

Proof. From Property 1 and the fact that the nonlinear underestimators in (8a) are convex in yj, any linearization is a supporting hyperplane (see Fig. 1). I

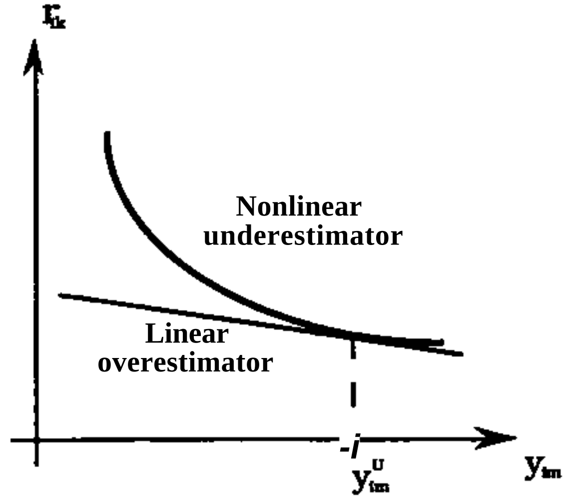


**Figure 1. Comparison between linear and nonlinear estimators (6a) and (8a) of the linear fractional terms in (PO).**

**Property 2.** When $r_{lk}^L > \frac{x_{lt}^L}{y_j} r$ (or $r^\wedge < \wedge r^u$), (i, j, k, m) € $P_R$, the linear overestimator (6a) (or 6b) is nonredundant.

**Proof.** Consider a feasible point with x^ and *yf* such that $\frac{x_{tm}^+}{y_j} = r_{lk}^L$ f with x^n* > $Xj_m^L$ and yj* < $yj^u$. Evaluating the linear overestimator (6a) at (x^/, $yj^+$) and rearranging it as in (B.2) yields:

$$r_{ik} \geq \frac{x_{tm}^+}{y_j^u} - \frac{r_{lk}^L y_j^+}{y_j^u} + r_{lk}^L = \frac{r_{lk}^L y_j^+}{y_j^u} - \frac{r_{lk}^L y_j^+}{y_j^u} + r_{lk}^L \qquad (B.6)$$

The linear overestimator for that point then reduces to,

$$r_{ik} \geq r_{ik}^L \qquad (B.7)$$

The nonlinear underestimator (8a) for this point is,

$$r_{ik} \wedge - \wedge Sr + x_{tm}^L \left(\frac{1}{y_j^+} - \frac{1}{y_j^u}\right) \qquad (B.8)$$

and using the relation $\frac{x_{TM}^+}{y_j} = r_{lk}^L$, for expressing (B.8) in terms of $r_{lk}^L$ yields

$$r_{ik} \geq \qquad (B.9)$$

Defining a=Aj and p = $\wedge r$ . the equation (B.9) can be expressed as

$$r_{,k}£ [a + p] r\wedge - apr_{,k}^L = r_{,k}^L [ a + p (1 - a) ] \qquad (B.10)$$

Since $0 \leq a < 1$ and $0 \leq P < 1$,

$$1 = \alpha + (1 - \alpha) > \alpha + \beta(1 - \alpha) = \phi \qquad (B.11)$$

the nonlinear underestimator reduces to.

$$r_{lk} \geq \phi r_{lk}^L, \qquad \phi < 1 \tag{B.12}$$

Comparing (B.7) and (B.12), it follows that the linear overestimator (6a) is stronger at the point $(x_{lm}^*, y/)$ and then nonredundant. I

Property 3.: When $x^\wedge = Tufyf^*$ (or $x^\wedge{}^{*1} = ru^\wedge j^{11}$), (i, j, k, m) $e$ $P_{Bf}$ the linear overestimator (6a) (or (6b)) is a secant of the nonlinear underestimator (8a) (or (8b)).

Proof. The nonlinear underestimator (8a)

$$r_{lk} \geq \frac{x_{lm}}{y_j^u} + x_{lm}^L\left(\frac{h}{y_j} - \frac{\wedge u}{y_j}\right) \tag{B.13}$$

can be expressed as:

$$x_{lm} - y_j^u r_{lk} - x_{lm}^L + \frac{x_{lm}^L y_j^u}{y_j} \leq 0 \tag{B.14}$$

Using $x_{lm}^L = r_{lk}^L y_J^L$

$$x^*_m - yj^u r_{lk} + r_{ik}L \wedge\wedge -_{yj}{}^L) \leq 0 \tag{B. 15}$$

The linear overestimator (6a) is given by

$$Xto \leq y_,^u r_{lk} + r \wedge j - y \wedge r \wedge \tag{B. 16}$$

that can be expressed as:

$$xim - yj^u r_{lk} + r_{lk}^L (yj^u - yj) \leq 0 \tag{B.17}$$

The difference between both equations (B.15) and (B.17) is in the last term $r(\frac{y_i^u y_i^L}{y_j} - yj^L)$ versus $(yj^u - y_,)$. Both term are equal at the extreme values $yj = y_,^L$ and $yj = yj^u$. Since the nonlinear term is convex and the linear one matches its value at the extreme points the latter is a secant of the nonlinear estimator. I

Corollary 2. The linear overestimator (6a) (or (6b)) is stronger than the nonlinear underestimator (8a) (or (8b)) over the feasible region when $x^\wedge = r^{\wedge\wedge}$ (or $x^\wedge = T^{\wedge\wedge}$), (i, $j_f$ k. m) € $P_B$.

Proof. This follows trivially from Property 3 and the fact that the nonlinear underestimator (8a) is convex (see Fig. 6). I

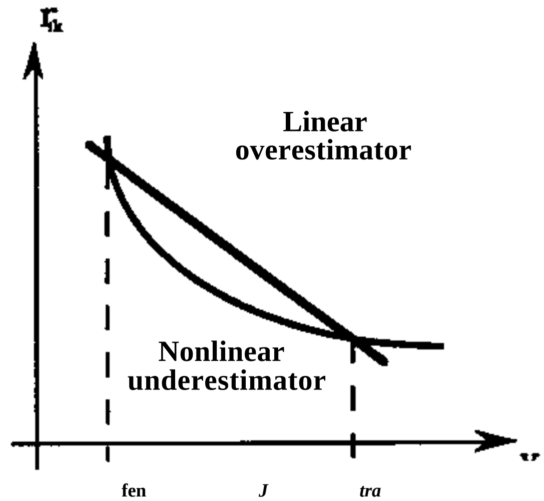


**Figure 6. Comparison between linear and nonlinear estimators (6a) and (8a) of the bilinear terms in (PO).**

**Property 4.** When $x_{im}^L > r_{ik}^L y_j^L$ (or $x_{im}^u < r_{ik}^u y_j^u$), (i, j, k, m) € $P_B$, the nonlinear underestimator (8a) (or (8b)) is nonredundant.

**Proof.** Consider a feasible point $(r_{ik}^+, y_j^+)$ such that $r_{ik}^+ y_j^+ = x_{im}^L$, $r_{ik}^u > r_{ik}^+ > r_{ik}^L$ and $x_{im}^u > x_{im}^+ > x_{im}^L$. The nonlinear underestimator (8a) for this point gives:

$$r_{ik}^+ \ge \frac{x_{im}}{y_j^u} + x_{im}^L\left(\frac{1}{y_j} - \frac{1}{y_j^u}\right)$$

yielding

$$\frac{x_{im} - x_{im}^L}{y_j^U} \le r_{ik}^+ - \frac{x_{im}^L}{y_j^+} = \frac{r_{ik}^+ y_j^+ - x_{im}^L}{y_j^+} = 0 \qquad \text{(B.19)}$$

which in turn implies

(B.20)

Now consider the linear overestimator (6a) for the same point:

$$x_{im} \le y_j^u r_{ik}^+ + r_{ik}^L y_j^+ - y_j^u r_{ik}^L = \frac{y_j^u x_{im}^L}{y_j^+} + \frac{r_{ik}^L x_{im}^L}{r_{ik}^+} - \frac{y_j^u r_{ik}^L r_{ik}^+}{r_{ik}^+} \qquad \text{(B.21)}$$

which yields.

$$x_{im} \le x_{im}^L\left(\frac{y_j^u}{y_j^+} + \frac{r_{ik}^L}{r_{ik}^+} - \frac{y_j^u r_{ik}^L}{y_j^+ r_{ik}^+}\right) \qquad \text{(B.22)}$$

Defining $\alpha = \frac{y_j^u}{y_j^+} > 1$ and $\beta = \frac{r_{ik}^L}{r_{ik}^+} < 1$. equation (B.22) reduces to:

$$x_{im} \le x_{im}^L\left(\beta + \alpha(1 - \beta)\right) = x_{im}^L\left(1 - 1 + \beta - \alpha(\beta - 1)\right) = x_{im}^L\left(1 + (\beta - 1)(1 - \alpha)\right) \qquad \text{(B.23)}$$

Since (p- 1) < 0 and (1- a) < 0, x^ ≤ O $X4_m^L$ with O > 1. The comparison with the other linear overestlmator (6b) is equivalent and also yields $x_{lm}$ ≤ Ox^1-, which in turn implies that the nonlinear underestimator (8a) is stronger at (r^, $yj^+$). I

Property 5 The nonlinear inequality (14a) (or (14b)) is a valid convex underestimator when b < 0 (or $b^1$ < 0), and is nonredundant with respect to the nonlinear underestimator in (8a) (or (8b)).

Proof For the first part of the proof constraint (13) can be expressed as,

$$\frac{1}{yj} \quad \frac{1}{a + bx_{lm}} - \qquad \text{(B.24)}$$

Multiplying by the lower bound constraint (x^ - $Xj_m^L$ £ 0) yields the valid inequality,

$$(x_{lm} - x_{lm}^L)\left(\frac{1}{y_j} - \frac{}{a + bx_{lm}}\right.$$

Rearranging yields:

$$\frac{x_{lm}}{y_j} \geq \frac{x_{lm}^L}{y_j} + (Xlm_{\prime\prime} \wedge {}^M \frac{1}{a + bx_{lm}}) \qquad \text{(B.26)}$$

which corresponds to the nonlinear underestimator (14).

The Hessian matrix of the underestimating function in (14) is given by

$$\begin{bmatrix} -\frac{2b(a + bx«m^L)}{(a + bx_{lm})^3} & 0 \\ 0 & \frac{2x_{lm}^L}{yj^3} \end{bmatrix} \qquad \text{(B.27)}$$

The term ($a+bxj_m$) is positive over the feasible region since,

$$a+bx_{,m} \wedge y, > 0 \qquad \text{(B.28)}$$

and hence,

$$-\frac{21b(a + bx_{lm}^L)}{(a + b\,x_{lm})^3} > 0 \quad \text{if } b < 0 \qquad \text{(B.29)}$$

Also for $x_{lm}^L$>0,

$$\frac{2x_{lm}^L}{y_j^3} > 0 \qquad \text{(B.30)}$$

If x^M) equation (B.26) reduces to the convex inequality (b<0).

$$1_{lk} \leq \overline{a + bx_{lm}} \qquad \text{(B.31)}$$

Therefore, if b<0 the Hessian matrix (B.27) is positive definitive and the function is convex.

Now consider a feasible point (x^*, yf) in the strict interior such that y,* = a + *hx*^* and $XJ^L < yj^+ < yj^u$ and *«m$^L$ < Xim* < ^m$^{11}$- Equation (14) for the nonlinear underestimator with projection reduces to.

$$\frac{x_{im}^+}{y_j^+} \geq \frac{x_{im}^+}{a + bx_{im}^+}.$$

and is therefore an exact approximation of the linear fractional term. Since yj+ does not lie in the boundary defined by the bounds of the variables x^ and yj the nonlinear underestimator (8a) yields,

$$\frac{x_{im}^L}{yj^+} + \frac{Xlm^* - Xfan^L}{yj^u} < \frac{Xim^L}{yj^+} + \frac{x_{im'} - Xjm^L}{yj^+} - \frac{x_{lm'}}{yj^+} \qquad (B.33)$$

which Is a strict inequality. The other nolinear underestimator (8b) for this point yields,

$$\frac{x_{im}^+}{y_j^L} + x_{im}^u\left(\frac{1}{y_j^+} - \frac{1}{y_j^L}\right) < \frac{x_{im}^+}{y_j^L} + x_{im}^+\left(\frac{1}{y_j^+} - \frac{1}{y_j^L}\right) \leq \frac{x_{im}^+}{y_j^+} \qquad (B.34)$$

which is a strict inequality.

Hence, the projected nonlinear underestimator (14) is stronger than the nonlinear underestlmators (8) for the point ($x_{lm}$^, y,*). I

Property 6.The additional estimators in (16), (17) and (21) provide an exact approximation at the boundary defined by the projected cut.

Proof. In the same spirit as the proof for Property 5, select a point for which the projected inequality (20) is a strict inequality and for which the variables are not at their bounds.

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Property 7. Any feasible point (x, y, r, z) in problem $NLP_L$ provides a valid lower bound to the objective function of problem (PI). Furthermore, the optimal solution $f_L^f$ of (NLPJ provides a valid lower bound to the global optimum (f*) of problem (PI).

Proof. Any feasible point (x, y, r, z) for problem (NLPJ is also a feasible solution to problem (PI) since the inequalities g(x, y, r, z) £ 0 are identical in both problems. Since the approximating functions in (NLPJ represent a relaxation of the bilinear inequalities in (PI), they have the effect of underestimating the objective function C of problem (PI). Thus it follows that at the given feasible point $f_L \leq f$.

**For the global optimum (x*. y*, r\ z*) of problem (PI) it then follows that f* ≥ f_L*_t where f_L* is the objective of $NLP_L$ evaluated at that point. Since f_L\ the optimal solution of $NLP_L$ is unique due to its convexity, f_L* ≥ f_L\ and thus f* ≥ f_L*.** I

**Corollary 3. If the optimum solution f_L' from $NLP_L$ is equal to the objective function value f from (PI) it corresponds to the global optimum of (PI).**

**Proof Iff is not the global optimal solution of problem (P) then there exists a global solution *f** < f. But by Property 7, f_L* ≤ f* which contradicts the assumption that $f_L'$=f is a solution to $NLP_L$.**

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**Property 8. The branch and bound algorithm will either terminate in a finite number of partitions at a global optimal solution, or generate a sequence of bounds that converge to the global solution.**

**Proof Given the branch and bound procedure, there are two possibilities. In the first one, at a given node the lower bound $f_L$ of the underestimator $NLP_L$ is identical to the original objective function in which case the algorithm terminates in a finite number of partitions.**

**In the second possibility an infinite sequence of partitions is generated. This in turn implies that there is a subregion that is being infinitely partitioned. Let the sequence of solutions be denoted by (^k l and £=(x, y, r, z). By the termination criteria it is known that,**

**f ^k -'f_L ^k > 0** **(B.35)**

**Since the upper bound is at least as strong as the evaluation of the actual objective function for the current solution £^k.**

**flC^k*) - W) * &^k' - *L^k > 0** **(B.36)**

**there must exist at least one nonconvex term, +, for which its feasible region is infinitely partitioned. By the partition rule 1,**

$$\left(\frac{x_{lm}}{y_j - r_{jk}}\right) \leq \left(\frac{x_{lm}^+}{y_j^+ - r_{jk}^+}\right) \qquad (a37)$$

**Summing up over all the nonconvex terms, t, it follows that**

**t ( ^ - $r_{lk}^+$) S f (Cl - f_L(C^k l *ZF-*- k** > 0** **(B.38)**

The variables for the nonconvex term (•) have some bounds defining an interval. Since the partition is of the same nature as the one used by Al-Khayyal and Falk, the variables in the sequence must converge to one of the bounds. Moreover, the series has to converge to a point. When one of the bounds of a variable are not changing, this variable is selected for the partition in the algorithm. When one of the variables is at its bounds the representation is exact, $x\lambda_{pr}^{+} = r_{lk}^{+}$. Therefore,

$$0 \geq f(C^{k}) - f_L(C^{k}) \geq P^{k} - f_L^{k} \geq 0 \qquad \text{(B.39)}$$

which means that equality between the lower bound $f_L$ and the original function f must hold. Since by Property 7, $f_L^{k}$ is a lower bound to the global optimal solution, it corresponds to the global solution. ∎