A Reduced Hessian Method for Large Scale Constrained Optimization

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A REDUCED HESSIAN METHOD FOR LARGE SCALE CONSTRAINED OPTIMIZATION

by

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ABSTRACT

We propose a quasi-Newton algorithm for solving large optimization problems with nonlinear equality constraints. It is designed for problems with few degrees of freedom, and is motivated by the need to use sparse matrix factorizations. The algorithm incorporates a correction vector that approximates the cross term Z^TWYp_Y in order to estimate the curvature in both the range and null spaces of the constraints. The algorithm can be considered to be, in some sense, a practical implementation of an algorithm of Coleman and Conn. We give conditions under which local and superlinear convergence is obtained.

Key words: **successive quadratic programming, reduced Hessian methods, constrained optimization, quasi-Newton method, large-scale optimization.**

Abbreviated title: **A Reduced Hessian Method**

1. Introduction.

We consider the nonlinear optimization problem

$$\min_{x\in \mathbb{R}^n} f(x) \tag{1.1}$$

$$\text{subject to } \; c(x) = 0, \tag{1.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions. We are particularly interested in the case when the number of variables n is large, and the algorithm we propose, which is a variation of the successive quadratic programming method, is designed to be efficient in this case. We assume that the first derivatives of f and c are available, but our algorithm does not require second derivatives.

The successive quadratic programming (SQP) method for solving (1.1)-(1.2) generates, at an iterate x_k, a search direction d_k by solving

$$\min_{d} \; g\{x_k)^T d + \backslash d^T W\{x_k)d \tag{1.3}$$

$$\text{subject to } \; c(x_k) + A(x_k)^T d = 0, \tag{1.4}$$

where g denotes the gradient of /, W denotes the Hessian of the Lagrangian function X(x, A) = $f(x) + X^T c(x)j$ and A denotes the n x m matrix of constraint gradients

$$A(x) = [\nabla c_1(x), \ldots, \nabla c_m(x)]. \tag{1.5}$$

A new iterate is then computed as

$$x_{k+1} = x_k + a_k d_k, \tag{1.6}$$

where a_k is a steplength parameter chosen so as to reduce the value of the merit function. In this study we will use the $t\backslash$ merit function

***„(*) =/(*) + /i|K*)lli. (1-7)**

where /i is a penalty parameter; see for example Conn (1973), Han (1977) or Fletcher (1987). We could have used other merit functions, but the essential points we want to convey in this article are not dependent upon the particular choice of the merit function.

The solution of the quadratic program (1.3)-(1.4) can be written in a simple form if we choose a suitable basis of R^n to represent the search direction d_k. For this purpose, we introduce a nonsingular matrix of dimension n, which we write as

$$[Y_k \; Z_k\backslash \tag{1.8}$$

where Y_k G $R^{n x m}$ **and** Z_k € $R^{nx}(^{n}\sim^{m})$, and assume **that**

$$A^T_k Z_k = 0. \tag{1.9}$$

(FVom now on we abbreviate $A(x_k)$ as A_k, **$g(x_k)$** as **g_{kj} etc.)** Thus Z_k is **a basis for the tangent space of the constraints.** We can now express **d*,** the solution **to (1.3)-(1.4), as**

$$d_k = Y_k p_Y + Z_{kPz}, \tag{1.10}$$

for some vectors p_Y € R^m and p_z € $R^{n''*m}$. Due to (1.9) the linear constraints (1.4) become

$$c_k + Al \, Y_k p_Y = 0. \tag{1.11}$$

If we assume that A_k has full column rank then the nonsingularity of $[Y_k \; Z_k]$ and equation (1.9) imply that the matrix AjY_k is nonsingular, so that p_Y is determined by "(1.11):

$$p_Y = -[itfr*]^{-1}c*. \; ' \tag{1.12}$$

Substituting this in (1.10) we have

$$d_k = -Y_k[A_k^T Y_k]^{-1} c_k + Z_k p_Z. \tag{1.13}$$

Note that

$$Y_k[A_k^T Y_k]^{-1} \tag{1.14}$$

is a right inverse of A_k^T, and that the first term in (1.13) represents a particular solution of the linear equations (1.4).

We have thus reduced the size of the SQP sub-problem which can now be expressed exclusively in terms of the variables p_Z. Indeed, substituting (1.10) into (1.3), considering $Y_k p_Y$ as constant, and ignoring constant terms, we obtain the unconstrained quadratic problem

$$\min_{p_Z} \quad (Z_k^T g_k + Z_k^T W_k Y_k p_Y)^T p_Z + \tfrac{1}{2} p_Z^T (Z_k^T W_k Z_k) p_Z. \tag{1.15}$$

Assuming that $Z_k^T W_k Z_k$ is positive definite, the solution of (1.15) is

$$p_Z = -(Z_k^T W_k Z_k)^{-1}[Z_k^T g_k + Z_k^T W_k Y_k p_Y]. \tag{1.16}$$

This determines the search direction of the SQP method.

We are particularly interested in the class of problems in which the number of variables n is large, but n — m is small. In this case it is practical to approximate $Z_k^T W_k Z_k$ using a variable metric formula such as BFGS. On the other hand, the matrix $Z_k^T W_k Y_k$, of dimension $(n - m) \times m$ may be too expensive to compute directly when m is large. For this reason several authors simply ignore the "cross term" $Z_k^T W_k Y_k p_Y$ in (1.16) and compute only an approximation to the reduced Hessian $Z_k^T W_k Z_k$; see Coleman and Conn (1984), Nocedal and Overton (1985), and Xie (1991). This approach is quite adequate when the basis matrices Y_k and Z_k in (1.8) are chosen to be orthonormal (Gurwitz and Overton (1989)).

However, for large problems computing orthogonal bases can be expensive, and it is more efficient to obtain Y_k and Z_k by simple elimination of variables (cf. Fletcher (1987)). Unfortunately, in this case ignoring the cross term $Z_k^T W_k Y_k p_Y$ can make the algorithm inefficient, as is illustrated by an example given in a companion paper (Biegler, Nocedal and Schmid (1993)). The central point is that if the basis matrices Y_k and Z_k are not orthogonal, the range space component $Y_k p_Y$ may be very large and ignoring the contribution from the cross term in (1.16) can result in a poor step.

Therefore in this paper we suggest ways of approximating the cross term $[Z_k^T W_k Y_k] p_Y$ by a vector w_k,

$$[Z_k^T W_k Y_k] p_Y \approx w_k, \tag{1.17}$$

without computing the matrix $Z_k^T W_k Y_k$. We will see that this can be done without substantially increasing the cost of the iteration, and we will show that the rate of convergence of the new algorithm is 1-step Q-superlinear, as opposed to the 2-step superlinear rate for methods that ignore the cross term (Byrd (1985) and Yuan (1985)). The null space step (1.16) of our algorithm will be given by

$$p_z = -(ZlW_kZ_k)^{-l}[Zlg_k + Ob^{**}], \qquad (1.18)$$

where $0 < (_k \leq 1$ is a damping factor to be discussed later on.

To describe our first strategy for computing **the vector w_k we consider a quasi-Newton method in which the rectangular matrix $Z\%W_k$ is approximated by a matrix <?&, using** Broyden's **method.** We then obtain w_k by multiplying this **matrix by** %py, i.e,

$$w_k = S_k Y_k p_Y.$$

How should *Sk* **be updated?** Since W*+i = Vj$_x$i(xfc+i, A*+i), **we** have that

$$Z_k^T W_{k+1}(x_{k+1} - x_k) \ll Zj^*[V_xL(^*k+u^*k+i) - V_xZ(x^*, A^*_{+1})], \qquad (1.19)$$

when £fc+i is close to £&• We use this relation **to** establish **the following secant equation:** we demand that *Sk+i* satisfy

$$S^*+i(^{**}+i - ^{**}) = ZfiV.\wedge Xib+i, A_{A\cdot+1}) - V_xL(x_k, Aib+i)]. \qquad \mathbf{(1.20)}$$

One point in this derivation requires clarification. In the left hand side of (1.19) we have *Z%Wk+ij* and not £*+iW*+i. We could have used *Zk+* in (1.19), avoiding an inconsistency of indices, but this is not necessary since we will show that using Z* instead of Zjk+i in (L20) results in algorithms with all the desirable properties. This fact will not be surprising to readers familiar with the analysis of SQP methods - see for example Coleman and Conn (1984) or Nocedal and Overton (1985).

Let us now consider how to approximate the reduced Hessian matrix *Z%WkZk*. Using (1.6) and (1.10) in (1.20) we obtain

$$[S_{k+1}Z_k]\alpha_k p_z = -\alpha_k S_{k+1}(Y_k p_Y) + Z_k^T[\nabla_x L(x_{k+1}, \lambda_{k+1}) - \nabla_x L(x_k, \lambda_{k+1})].$$

Since 5*+i approximates ZjW*, this suggests the following secant equation for 2?*+i, the quasi-Newton approximation to the reduced Hessian $Z_k^T W_k Z_k$:

$$B_{k+1}s_k = y_k, \qquad \mathbf{(1.21)}$$

where s_k and y_k are defined by

$$Sk = <^*_k p_z,$$

and

$$y_k = Zl[V_xL(x_{k+u} Ajt+i) - V_xL(x^*, Aib_{+l})] - \overline{w}_k, \qquad (1.22)$$

with

$$\overline{w}_k = \alpha_k S_{k+1}(Y_k p_Y). \qquad (1.23)$$

We will update B_k by the BFGS formula (cf. Fletcher (1987))

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \quad y_k \; s_k \tag{1.24}$$

provided $s£y_k$ is sufficiently positive.

We would like to highlight a subtle, but important point. We have defined two correction terms, w_k and $\overline{w}_k$. Both are approximations to the cross **term** $(Z^T WY)p_Y$. The first term, w_k, which is needed to define the null space **step (1.18) - and thus the** new iterate x_k+i - makes use of the matrix 5*. The second term, $\overline{w}_{kj}$ which is used in (1.22) to define the BFGS update of B_k, is computed using the new Broyden matrix Sjt+i, and takes into account the steplength a_k. We will see below that it is useful to incorporate the most recent information in $\overline{w}_k$.

The Lagrange multiplier estimates A* needed in the definition (1.22) of y_k will be defined by

$$X_k = -[Y?A_k]^{-l} Y?g_k. \tag{1.25}$$

This formula is motivated by the fact that, at a solution x_m of (1.1)-(1.2), we have $—g_m = A_m X_m$, and since KJA^K,]"*[1] is a right inverse of J4^\

$$A. = -\backslash YjA.]^{-l}Y?g..$$

Using the same right inverse (1.14) in the definitions of p_Y and A* will allow us a convenient simplification in the formulae presented in the following sections. We stress, however, that other Lagrange multiplier estimates can be used, and that the best choice in practice might be the one that involves the least computation or storage.

We can now outline the sequential quadratic programming method analyzed in this paper.

Algorithm I

1. **Choose constants tj € (0,1/2) and r, r' with 0 < r < T^* < 1. Set k : = 1 and choose a starting point xi and an (n-m) x (n-ra) symmetric and positive definite starting matrix $B\backslash$.**

2. **Evaluate $f_k, g_{ky} c_k$ and A_k, and compute Y_k and Z_k.**

3. **Compute p_Y by solving the system**

$$(AlY_k)p_Y = -c_k. \qquad \text{(range space step)} \tag{1.26}$$

4. **Compute an approximation** w_k **to $(Z^\wedge W_k Y_k)p_Y$.**

5. Choose **the damping** parameter $(_k$ € (0,1] and compute p_2 from

$$B_k p_z = -[Zlg_k + (_k w_k], \qquad \text{(null space step)} \tag{1.27}$$

Define the search direction by

$$d_k = Y_k p_Y + Z_{kPz}. \tag{1.28}$$

6. Set $c*k$ = 1 and choose the weight fx_k of the merit function (1.7).

7. Test the line search condition

$$\phi_{\mu_k}(x_k + \alpha_k d_k) \le \phi_{\mu_k}(x_k) + \eta\alpha_k D\phi_{\mu_k} ixkid_k), \tag{1.29}$$

where D^^x^dk) is the directional derivative of the merit function <j> in the direction d_k.

8. If (L29) is not satisfied, choose a new a* € $[Ta_k,T'a_k]$ and go to (7); otherwise set

$$Xk+i = x_k + a_k d_k. \tag{1.30}$$

9. Evaluate A+i,0Ji-+ii^c*+i, A_k+_X, and compute Y/-$_{+1}$ and Z*+i.

10. Compute the Lagrange multiplier estimate

$$A*+i = -\{Y\wedge Ak+\wedge Y\wedge g\wedge. \tag{1.31}$$

Define Wk (as will be discussed in §3), and compute

$$*k = otkPz, \tag{1.32}$$

and

$$y* = \bar{} ti[[V_xL(xk+u\backslash k+i) - VxL(x_{k9} Aib+0] - \bar{u}J*. \tag{1.33}$$

If the update criterion (to be discussed in §3.3) is satisfied, compute 2?*$_+$i by the BFGS formula (1.24); else set J9*+i = B_k.

11. Set $k := k + 1_f$ and go to (3).

The algorithm has been left in a very general form, but in the next sections we discuss all its aspects in detail. In §2 we consider the choice of the basis matrices Y_k and Zfc. In §3 we describe the calculation of the correction terms w_k and $\bar{w}_k$, the conditions under which BFGS updating takes place, the choice of the damping parameter £*, and the procedure for updating the weight fi_k in the merit function. §4 and §5 present an analysis of the local behavior of the algorithm, and show that the rate of convergence is at least R-linear. §6 presents a superlinear convergence result, and some final remarks in §7 conclude the paper.

We now make a few comments about our notation. Throughout the paper, the vectors p_Y and p_z are **computed at x_k, and could be denoted by p_Y^ and p_z^^{k}K but we** will normally omit the superscript for simplicity. The symbol || • || denotes the $I2$ vector norm or the corresponding induced matrix norm. When using the $l\backslash$ or /c*> norms we will indicate it explicitly by writing || • ||i or || • ||oo. A solution of problem (1.1) is denoted by x*, and we define

$$e_k = x_k - x_m \quad \text{and} \quad a_k = \max\{||ej;||, ||e*+i||\}. \tag{1-34}$$

Here, and for the rest of the paper, VI(i, A) indicates the gradient of the Lagrangian with respect to x only.

2. The Basis Matrices,

As long as *Zk* spans the null space of Aj[, and *[Yk Zk]* is **nonsingular, the choice of *Yk* and *Zk*** is **arbitrary. However, from the viewpoint of numerical stability and robustness** of the algorithm **it** is **desirable** to define *Yk* and *Zk* **to** be **orthonormal,** i.e.

$$\begin{aligned} Z(x)^T Z(x) &= I_{n-m} \\ Y(x)^T Y(x) &= I_m \\ Y(x)^T Z(x) &= 0. \end{aligned}$$

One way of **obtaining** these matrices is by **forming the QR** factorization **of *A*. However, for** large **problems** computing this **QR factorization** is often **too expensive. Therefore** many researchers, **including** Gabay (1982), **Gilbert (1991),** Fletcher **(1987), Murray and** Prieto (1992), and Xie (1991), consider other, non-orthogonal, choices of *Y* and *Z*. For example, if we partition x into m basic or dependent variables **(which without** loss **of generality are assumed to be the first m variables) and *n—m* nonbasic or control variables, we induce the partition**

$$A(x)^T = (C(x) \; N(x))_t \qquad (2.1)$$

where the *mxm* basis matrix *C(x)* is assumed to be nonsingular. We now define *Z(x)* and *Y(x)* to be

$$Z(x) = \begin{bmatrix} -C(x)^{-1}N(x) \\ I \end{bmatrix} \quad Y(x) = \begin{bmatrix} I \\ 0 \end{bmatrix}. \qquad (2.2)$$

When *A(x)* is large and sparse, a sparse LU decomposition of *C(x)* can often be computed efficiently, and this approach will be considerably less expensive than the QR factorization of *A*. Note that from the assumed nonsingularity of *C(x)* both *Y(x)* and *Z(x)* vary smoothly with x_r provided the same partition of the variables is maintained. In our implementation of the new algorithm (Biegler, Nocedal and Schmid (1993)) we choose Y_k and *Zk* by (2.2).

There is a price to pay for using non-orthogonal bases. If the matrix *C* is ill-conditioned (and this can be difficult to detect) the step computation may be inaccurate. Moreover, even if the basis is well conditioned the range space step Ykp_Y could be large and ignoring the cross term can cause serious difficulties. This phenomenon is illustrated in **a 2-dimensional example** given in Biegler, **Nocedal and Schmid (1993). It** is **shown in** that example that **if the** cross **term** *ZjWkYkPr* is ignored, the ratio \\xk + rf*||/||£*|| can be arbitrarily large, even close to the solution. It is also shown that these inefficiencies disappear if the cross term is approximated as suggested in the following sections.

In the rest of the paper we allow much freedom in the choice of the basis matrices. They can be given by (2.2), can be orthonormal, or can be chosen in other ways. The only restrictions we impose are that $A^\wedge Zk = 0$ is satisfied, that the nxn matrix $[Y_k\ Z_k]$ is nonsingular and well-conditioned, and that this matrix varies smoothly in a neighborhood of the solution.

3. Further Details of the Algorithm

In this section we consider how to calculate approximations w_k and $\bar{v}3_k$ to $\{Z^\wedge W_k Y_k) p_Y$ to be used in the determination of the search direction p_z and in updating 2?&, respectively. We also discuss when to skip the BFGS update of the reduced Hessian approximation, as well as the selection of the damping factor £* and the penalty parameter /z&.

To calculate approximations to $(Z^T WY) p_Y$ we propose two approaches. First, we consider a finite difference approximation to $Z\% W_k$ along the direction $Y_k p_Y$. While this approach requires additional evaluations of reduced gradients at each iteration, it gives rise to a very good step. The second, more economical approach, defines Wk and $\overline{w}_k$ in terms of a Broyden approximation to $Z^\wedge Wk$, as discussed in §1, and requires no additional function or gradient evaluations. Our algorithm will normally use this second approach, but as we will later see, it is sometimes necessary to use finite differences.

3.1. Calculating w_k and JD_k Through Finite Differences.

We **first calculate the** range space step p_Y **at** x_k **through equation (1.26).** Next we compute the reduced gradient of the Lagrangian **at** $x_k + Y_k p_Y$ and define

$$w_k = Zl[VL(x_k + Y_k p_Y, \backslash_k) - VL(x_{ki}\ A^\wedge)]. \qquad (3.1)$$

After the step to the new iterate $Xk+i$ has been taken, we define

$$W_k = Z\backslash[VL(x_k + a_k Y_k p_Y, \backslash_k +_x) - VL(x_k, A^*_{+1})], \qquad (3.2)$$

which requires **a** new evaluation of gradients if $\alpha^* \wedge 1$.

We note that this finite difference approach is very similar **to** the **algorithm of Coleman and Conn (1982, 1984). Starting at a point** z_{kj} the **Coleman-Conn algorithm (with** steplength $a_k = 1$) is given by

$$Z_k p_z = -Z(z_k) Bj;^l Z(zk)^T g(zk) \qquad (3.3)$$

$$Y_k p_Y = -Y(z_k)[A(z_k)^T Y(z_k))^{-l} c(z_k + Z_{kPz}) \qquad (3.4)$$

$$z_{k+1} = Zk + ZkPz + Y_k p_Y. \qquad (3.5)$$

Let us now consider Algorithm I, and to better illustrate its similarity with the Coleman and Conn method, let us assume that instead of (3.1), Wk is defined by

$$w_k = Z(x_k + Y_k p_Y)^T g(x_k + Y_k p_Y) - ZJLz_k)^T g(x_k),$$

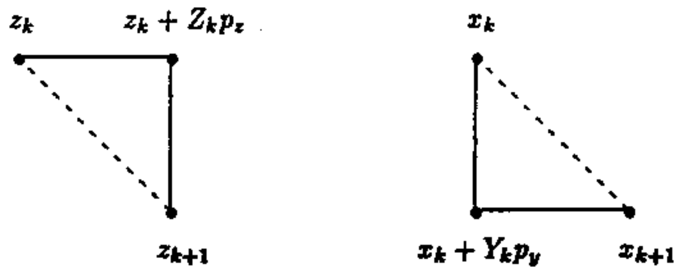
which differs from (3.1) by terms of order $O(\|p_Y\|)$. Then Algorithm I with $\alpha_k = 1$, is given by

$$Y_k p_Y = -Y(x_k)[A(x_k)^T Y(x_k)]^{-1} c(x_k) \tag{3.6}$$

$$\begin{aligned} Z_k p_Z &= -Z(x_k) B_k^{-1}[Z(x_k)^T g(x_k) + w_k] \\ &= -Z(x_k) B_k^{-1}[Z(x_k + Y_k p_Y)^T g(x_k + Y_k p_Y)]. \end{aligned} \tag{3.7}$$

$$x_{k+1} = x_k + Y_k p_Y + Z_k p_Z. \tag{3.8}$$

The similarity between the two approaches is apparent in Figure 1, especially if we consider the intermediate points in the Coleman-Conn iteration to be the starting and final points, respectively.



Coleman-Conn step — The step of Algorithm 1

Figure 1

In the Coleman-Conn algorithm, the approximation B_k to the reduced Hessian $Z_k^T W_k Z_k$ is obtained by moving along the null space direction $Z_k p_z$, and making a new evaluation of the function and constraint gradients. To be more precise, Coleman and Conn define

$$y_k^* = Z_k^T[\nabla L(x_k + Z_k p_z, \lambda_k) - \nabla L(x_k, \lambda_k)],$$

and $s_k = Z_k^T[x_{k+1} - x_k]$, and apply a quasi-Newton formula to update B_k. Algorithm I, using finite differences, amounts essentially to the same thing. To see this, note that if formula (3.2) is used in (1.33) then

$$y_k = Z_k^T[\nabla L(x_{k+1}, \lambda_{k+1}) - \nabla L(x_k + \alpha_k Y_k p_Y, \lambda_{k+1})],$$

which represents a difference in reduced gradients of the Lagrangian along the null space direction $Z_k p_z$,

Byrd (1990) and Gilbert (1989) showed that the sequence $\{x_k + Z_k p_z\}$ (but not the sequence $\{x_k\}$) generated by the Coleman-Conn method converges one-step Q-superlinearly.

If Algorithm I always computed the correction terms *Wk* and *Wk* by finite differences, its cost and convergence behavior would be similar to those of the Coleman-Conn method. However, we will often be able to avoid these additional gradient evaluations by using the more economical approach discussed next.

3.2. Using Broyden's Method to Compute *Wk* and IW*•

We can approximate the rectangular matrix *ZjWk* by **a** matrix *Sk* updated by Broyden's method, and then compute *Wk* and *Wk* by post-multiplying this matrix by Y*p_y or **by a** multiple of this vector. As discussed in §1 it is reasonable to impose the secant equation (1.20) on this Broyden approximation, which can therefore be updated **by** the formula (cf. Fletcher (1987))

$$S_k \quad c \;, \frac{(\bar{y}_k - S_k\bar{s}_k)\bar{s}_k^T}{\bar{s}_k^T\bar{s}_k}, \tag{3.9}$$

where

$$\bar{y}_h = Z\%[VL(z_k+_u\backslash_M) - VL(x^*, A^{**},)] \tag{3.10}$$

and

$$\bar{s}_k = {}^{**}+i - {}^{**}\bullet \tag{3.11}$$

We now define

$$Wk = SkYkp_Y \quad \text{and} \quad IE^* = a_kSk+iYkp_Y. \tag{3.12}$$

It should be noted that this approach requires the storage of the *(n — m)xn* matrix *Sk For problems where n — m is small this expense is far less than the storage of a full Hessian approximation to *Wk*. On the other hand, if n - m is not very small it may be preferable to use a limited memory implementation of Broyden's method. Here the matrices *Sk* are represented implicitly, using, for example, the compact representation described in Byrd, Nocedal and Schnabel (1992). The advantage of the limited memory implementation is that it only requires the storage of a few n-vectors to represent *S*.**

There is no guarantee that the Broyden approximations *Sk* will remain bounded, and therefore we need to safeguard them. At the beginning of the algorithm we choose a positive constant *T* and define

$$w_k := \begin{cases} w_k & \text{if } \|w_k\| \leq \frac{\Gamma}{\|p_Y\|^{1/2}}\|p_Y\| \\ w_k\frac{\Gamma\|p_Y\|^{1/2}}{\|w_k\|} & \text{otherwise.} \end{cases} \tag{3.13}$$

The correction *Wk* will be safeguarded in a different way. We choose a sequence of positive numbers {7*} such that EgL^jt < 00, and set

$$\bar{w}_k = \begin{cases} \bar{w}_k & \text{if } \|\bar{w}_k\| < \gamma_k\|p_Y\|/\gamma_k \\ \bar{w}_k\frac{\|p_Y\|}{\gamma_k\|\bar{w}_k\|} & \text{otherwise.} \end{cases} \tag{3.14}$$

As the iterates converge to the solution, p_v -» 0, so that from (3.12) and from the boundedness of *Yk* we see that these safeguards allow the Broyden updates *Sk* to become unbounded, but in a controlled manner. We will show in §4 and §5 that with safeguards (3.13) and (3.14) Algorithm I is locally and R-linearly convergent, and that this implies that the Broyden updates *Sk* do, in fact, remain bounded, so that the safeguards become inactive asymptotically.

Our Broyden approximation to the correction terms *Wk* and Z;* was motivated by recent work of Gurwitz (1993). She approximates *Z^WkZk* by the BFGS formula with

$$s_k = Z_k^T[x_{k+1} - x_k]$$

and

$$y_k = Z_k^T[\nabla L(x_{k+1}, \lambda_{k+1}) - \nabla L(x_k, \lambda_{k+1})],$$

and approximates *Z][WkYk* by a matrix *Dk* using Broyden's formula (3.9) with

$$\bar{s}_k = Y_k^T[x_{k+1} - x_k]$$

$$\bar{y}_k = Z_k^T[\nabla L(x_{k+1}, \lambda_{k+1}) - \nabla L(x_k, \lambda_{k+1})] - B_k p_z.$$

Since **the** updates **may not always be defined, Gurwitz proposes to sometimes skip the update of 27* or *Dk,* and shows 1-step Q-superlinear convergence *if and only if one* of the updates is taken at each iteration. We will argue below that it is preferable to update an approximation to *Z%Wk,* as is done in Algorithm I, instead of an approximation to *ZjWkYk,* as proposed by Gurwitz.**

A related method was derived by Coleman and Fenyes (1992). Their partitioned Lower Half Update (LHU) simultaneously updates approximations to *ZjWkZk* and *Z^WkYky* by means of a new variational problem. The resulting updating formula requires the solution of a cubic equation, and its roots can correspond to cases where updates should be avoided (e.g. *s%yk* ≤ 0). The drawback of this approach is that choosing the correct root is not always easy.

Finally, an earlier proposal, due to Tagliaferro (1989), consists of approximating the matrix

$$[Z_k^T W_k Z_k \quad Z_k^T W_k Y_k]$$

using a formula that **can** be viewed **as an** extension to the **PSB** update. One **disadvantage of this approach is that the matrices generated by this updating procedure may become very ill-conditioned.**

3.3. Update Criterion.

It is well known that the BFGS update (1.24) is well defined only if the curvature condition *sjyk* > 0 is satisfied. This condition can always be enforced in the unconstrained case by performing an appropriate line search; see for example Fletcher (1987).

However when constraints are present the curvature condition $s^\wedge y_k > 0$ can be difficult to obtain, even near the solution.

To see this we first note from (1.33), (1.28), (1.32) and from the Mean Value Theorem that

$$
\begin{aligned}
Vk &= Z_k^T \left[\int_0^1 V^2_{xx} L\{x_k + r a_k d_k, \text{Ajt+i}) d\tau \right] a_k d_k - \bar{w}_k \\
&\equiv Z_k^T \tilde{W}_k \alpha_k d_k - \bar{w}_k \\
&= Z\tilde{l} W_k Z_k s_k + a_k Z l \tilde{W}_k Y_{kPy} - Wk, \qquad (3.15)
\end{aligned}
$$

where we have defined

$$\tilde{W}_k = \int_0^1 V^2_{xx} L(x_k + r a_k d_k, X_k^\wedge) dT. \qquad (3.16)$$

Thus

$$*Ivk = si\ (zl\tilde{W}kZk)\ s_k + a_k sl\ (zf\tilde{W}kYk)\ \text{PY} - 4\text{W} \qquad (3.17)$$

Near the solution, the first term on the right hand side will be positive since $Z\%\tilde{W}_k Z_k$ can be assumed positive definite. Nevertheless the last two terms are of uncertain sign and can make sjy* negative. Several reduced Hessian methods in the literature set VJ_k equal to zero for all fc, and update B_k only if py is small enough compared with s_k that the first term in the right hand side of (3.17) dominates the second term (see Nocedal and Overton (1985), Gurwitz and Overton (1989), and Xie (1991)).

Skipping the BFGS update may appear to be a crude heuristic, but we argue that it gives rise to a sound algorithm. First of all, the last two terms in (3.17) normally converge to zero faster than the first term, so that the right hand side of (3.17) will often be positive and BFGS updating will take place frequently. Furthermore, if the right hand side of (3.17) is negative, the range space step $Y_k py$ is relatively large, resulting in sufficient progress towards the solution. These arguments will be made more precise in §5.

We conclude that skipping the BFGS update is desirable in some circumstances and we now present a strategy for deciding when to do so. Recall that c_k, defined by (1.34), converges to zero if the iterates converge to x«.

Update Criterion I*

Choose a constant $7_M > 0$ *and a sequence of positive numbers* {7*} *such that* ££^7* < 00

(this is the same sequence {7*} *that was used in (3.14))-*

- *If* $\bar{w}_k$ *is computed by Broyden's method, and if both* $sjy_k > 0$ *and*

$$\text{IIPvll} \leq ll\|Pz\| \qquad (3.18)$$

hold at iteration k, then update the matrix B_k *by means of the BFGS formula (1.24) with* s_k *and* y_k *given by (1.32) and (1.33). Otherwise, set* $B_{k+1} = B_k$.

- *If* $\overline{w}_k$ *is computed by finite differences, and if both* $s_k^T y_k > 0$ *and*

$$\|p_Y\| \leq \gamma_{fd}\|p_Z\|/\sigma_i^{1/2} \tag{3.19}$$

hold at iteration k, then update the matrix Bk by means of the BFGS formula (1.24) with Sk and yk given by (1.32) and (1.33). Otherwise, set J9jb+1 = *Bk.*

Note that a* requires knowledge of the solution vector z«, and is therefore not computable. However we will later see that a* can be replaced by any quantity which is of the same order as the error e*, for example the optimality conditions (||Zjff*|| + ||cjfc||). Nevertheless for convenience we will leave a* in (3-19).

We now closely consider the properties of the BFGS matrices *Bk* when Update Criterion I is used. Let us define

$$\cos\theta_k = \frac{s_k^T B_k s_k}{\|s_k\|\,\|B_k s_k\|}, \tag{3.20}$$

which, as we will see, is a measure of the goodness of the null space step Z^*p_2. We begin by restating a theorem from Byrd and Nocedal (1989) regarding the behavior of costf* when the matrix *Bk* is updated by the BFGS formula.

Theorem 3.1 ***Let*** *{Bk) be generated by the BFGS formula (1.24) where, for all* £ ≥ 1,

sk 4 0 *and*

$$\frac{y_k^T s_k}{s_k^T s_k} \geq m > 0 \tag{3.21}$$

$$\frac{\|y_k\|^2}{y_k^T s_k} \leq M. \tag{3.22}$$

Then, there exist constants fix, fa, (3$ > 0 *such that, for any k* ≥ 1, *the relations*

$$\cos\theta_j \geq \beta_1 \tag{3.23}$$

$$\beta_2 \leq \frac{\|B_j s_j\|}{\|s_j\|} \leq \beta_3 \tag{3.24}$$

hold for at least \^k] *values of j* € [l,fc].

This theorem refers to the iterates for which BFGS updating takes place, but since for the other iterates i?*+i = £*, the theorem characterizes the whole sequence of matrices *{Bk}*. Theorem 3.1 states that, if sjy* is always sufficiently positive, in the sense that conditions (3.21) and (3.22) are satisfied, then at least half of the iterates at which updating takes place are such that costf; is bounded away from zero and *BJSJ.* =' 0(||$j||). Since it will be useful to refer easily to these iterates, we-make the following definition.

Definition 3.1 *We define J to be the set of iterates for which BFGS updating takes place and for which (3.23) and (3.24) hold. We call J the set of "good iterates", and define J_k = Jd {1,2,...,*}.*

Note that if the matrices B_k are updated only a finite number of times, their condition number is bounded, and (3.23)-(3.24) are satisfied for all k. Thus in this case all iterates are good iterates.

We now study the case when BFGS updating takes place an infinite number of times. Let us assume that all functions under consideration are smooth and bounded. If at a solution point a:* the reduced Hessian ZjW_mZ+ is positive definite, then for all x_k in a neighborhood of x_m the smallest eigenvalue of $Zj\tilde{W}_kZ_k$ is bounded away from zero ($\tilde{W}_k$ is defined in (3.16)). We now show that in such a neighborhood Update Criterion I implies (3.21)-(3.22).

Let us first consider the case when $\overline{w}_k$ is computed by Broyden's method. Using (3.17), (3.18) and (3.14), and since 7* converges to zero, we have

$$\begin{aligned} s_k^T y_k &\geq C\|s_k\|^2 - O(\gamma_k^2\|s_k\|^2) - O(\gamma_k\|s_k\|^2) \\ &\geq m\|s_k\|^2 \end{aligned} \tag{3.25}$$

for some positive constants *C,m*. Also, from (3.15), (3.18) and (3.14) we have that

$$\begin{aligned} \|y_k\| &\leq O(\|s_k\|) + O(\gamma_k^2\|s_k\|) + O(\gamma_k\|s_k\|) \\ &\leq O(\|s_k\|). \end{aligned} \tag{3.26}$$

We thus see from (3.25)-(3.26) that there is a constant *M* such that for all *k* for which updating takes place

$$\frac{\|y_k\|^2}{y_k^T s_k} \leq M,$$

which together with (3.25) shows **that (3.21)-(3.22) hold when Broyden's method is used.**

$U\overline{wk}$ is computed by the finite difference formula (3.2), we see from (1.33) **and** the Mean Value theorem that there is a matrix $\hat{W}_k$ such that

$$\begin{aligned} y_k &= Z_k^T[\nabla L(x_{k+1}, \lambda_{k+1}) - \nabla L(x_k + \alpha_k Y_k p_Y, \lambda_{k+1})] \\ &\equiv Z_k^T \hat{W}_k Z_k s_k. \end{aligned}$$

Reasoning as before we see that (3.25) and (3.26) also **hold** in this case, **and that (3.21)-(3.22) are** satisfied in the case when finite differences **are used.** These **arguments show that,** in **a** neighborhood **of** the solution **and whenever** BFGS updating **of** *Bk* **takes place,** $s_k^T y_k$ is sufficiently positive, as stipulated by (3.21)-(3.22).

3.4. Choosing fi_k and (k-

We will now see that by appropriately choosing the penalty parameter /x& **and** the damping parameter Ok for ti;*, the search direction generated by Algorithm I is always a descent direction for the merit function. Moreover, for the good iterates J, it is a direction of strong descent.

Since *dk* satisfies the linearized constraint (1.11) it is easy to show (see eq. (2.24) of Byrd and Nocedal (1991)) that the directional derivative of the *l* merit function in the direction *dk* is given by

$$D4>v_k ixk\backslash d_k) = gjd_k - Hk\backslash\backslash c_k\backslash\backslash i. \tag{3.27}$$

The fact that the same right inverse of *Aj* is used in (1.26) and (1.31) implies that

$$g_k^T Y_k p_Y = \lambda_k^T c_k. \tag{3.28}$$

Recalling the decomposition (1.28) and using (3.28) we obtain

$$\begin{aligned} D\phi_{\mu_k}(x_k \backslash d_k) &= gJZ_k p_Z - \mu_k \|c_k\|_1 + \lambda j c_k \\ &= (Zfg_k + CkVfk)^T Pt \sim \text{OfcfjfPz} \sim /\text{»*ll«fc||i} + **^T\text{«*}\bullet \end{aligned} \tag{3.29}$$

Now from (1.32) and (1.27) we have that

$$B_k s_k = -ak(Zjg_k + \&\text{»*})\bullet \tag{3.30}$$

Substituting this in (3.20) we obtain

$$\cos\theta_k = \frac{-(Z_k^T g_k + \zeta_k w_k)^T p_Z}{\|Z_k^T g_k + \zeta_k w_k\| \, \|p_Z\|}. \tag{3.31}$$

Recalling the inequality A^c$_{fc}$ ≤ ||A/t||oo||cjk||i, and using (3.31) in (3.29) we obtain, for all k,

$$D\phi_{\mu_k}(x_k{}', d_k) \leq -\backslash\backslash Zlg_k + \text{ati>/t|| llftllcos}\wedge - (kwlpz - (tt_k - \backslash\backslash X_k \backslash\backslash co)\backslash\backslash c_k \backslash\backslash i. \tag{3.32}$$

Note also from (3.30) and (1.32) that

$$\frac{\text{INI}}{\backslash\backslash BkSk\backslash\backslash} = \frac{\text{llftll}}{\backslash\backslash Zlgk + \zeta_k w_k\|}. \tag{3.33}$$

We **now** concentrate on the good iterates *J,* as given in Definition 3.1. If *j* € *J,* we have from (3.33) **and** (3.24) that

$$\text{fyzJQi} + \text{Ci}\wedge\text{ll} \leq \text{lp}\wedge\text{H} \overset{1}{<} jUj_{9j} + 0\text{«»il|}. \tag{3.34}$$

Using this and (3.23) in (3.32) we obtain, for *j* € *J,*

$$\begin{aligned} D\phi_{\mu_j}(xj;dj) &\leq \frac{1}{\beta_3}\backslash\backslash Zf_{Sj} + \text{Ci}\wedge\text{lPcos}0j - awjA^{J)} \sim (H - UAUUcjh. \\ &\leq -|\text{ll}\wedge\text{ill}|^2 - {}'\wedge p\text{-}(gJz_{jWj)} - <_{jW}j_P? - {}_{(w} - \backslash\backslash XJ\backslash U\backslash\backslash CJ\backslash U, \end{aligned}$$

where we have dropped the non-positive term *—(.] cos 0J\\WJ\\^2/ fa*. Since we can assume that p_3 > 1 (it is defined as an upper bound in (3.24)), we have

Wvixjidj) <_-jJZfgjW^2 + facosOjlgjZjWjl - QwJpP] - {N - IIAjlU)!!^.

It is now clear that if

$$20\ ccsBAgJZjwA - Qwjp\{^{j)} \leq p\backslash\backslash cj\backslash\backslash_u \qquad (3.35)$$

for some constant />, and if

$$N > \text{Halloo} + 2p, \qquad (3.36)$$

then for all j € J,

$$D\phi_{\mu_j}(xfidi) \prec \frac{\beta_1}{}\,-jJlZjgjW^2 - PUCJUL \qquad (3.37)$$

This means that if (3.35) and (3.36) hold, then for the good iterates, *j* € J, the search direction *dj* is a strong direction of descent for the *i* merit function in the sense that the first order reduction is proportional to the KKT error.

We will choose Ok so that (3.35) holds for *all* iterations. To see how to do this we note from (1.27) that

$$p_z = -B_k^{-1} Z_k^T g_k - \zeta_k B_k^{-1} w_k,$$

so **that for** *j* = ***k*** **(3.35)** can be written as

$$\zeta_k[2\cos\theta_k|g_k^T Z_k w_k| + w_k^T B_k^{-x} 2\ g_k + \zeta_k u\#\ B_k^{-}\ w_k] \leq \rho\|c_k\|_1. \qquad (3.38)$$

It is clear that this condition is satisfied for a sufficiently small and positive value of Ot-Specifically, at the beginning of the **algorithm we choose a constant *p* > 0 and, at every** iteration &, define

$$G = \min\{1, \hat{G}\} \qquad (3.39)$$

where Ôt is the largest value that satisfies (3.38) as an equality.

The penalty parameter *fj.k* must satisfy (3.36), **so we define it at every iteration of** the **algorithm** by

$$a\cdot\_ / \ll *-! \quad \text{if } \mathbf{W\text{-}i} \geq \text{Halloo} + 2p$$
$$*^k\sim\backslash \quad \text{IIAtlloo} + 3/> \quad \text{otherwise.} \qquad \mathbf{(3.40)}$$

The damping factor Œ and the updating formula for the penalty **parameter** *fik* have been defined so as to give strong descent for the good iterates J. We now show that they ensure that the search direction is also a direction of descent (but not necessarily of strong descent) for the other iterates, *k* £ J. Since (3.35) holds for all iterations by our choice of Ok, we have in particular . *

$$-\zeta_k w_k^T p_z \leq \rho\|c_k\|_1.$$

Using this and (3.40) in (3.32), we have

$$D\phi_{\mu_k}(x_k;d_k) \le -\|Z_k^T g_k + \zeta_k w_k\| \, \|p_Z\| \cos\theta_k - \rho_k \|c_k\|_1. \tag{3.41}$$

The directional derivative is thus non-positive. Furthermore, since $w_k = 0$ whenever Œ = 0 (regardless of whether w_k is obtained by finite differences or through Broyden's method), it is easy to show that this directional derivative can only be zero at a stationary point of problem (1.1)-(L2).

3.5. The Algorithm

We can now give a complete description of the algorithm that incorporates all the ideas discussed so far, and that specifies the only remaining question, namely when to apply finite differences and when to use Broyden's method to approximate the cross term. The idea is to consider the relative sizes of p_Y and p_z. Update Criterion I generates the three regions *R*, *R2* and *R3* illustrated in Figure 2. The algorithm starts by computing it;* through Broyden's method and by calculating p_Y and p_2. If the search direction is in *R* or #3, we proceed. Otherwise we recompute w_k by finite differences, use this value to recompute p_z, and proceed. The reason for applying finite differences in this fashion is that **in the middle** region *Ri* neither Broydens' method*is **good enough, nor is the convergence sufficiently tangential, to** give **a** superlinear **step. Therefore we need to resort to finite differences to obtain a good estimate of w_k. The motivation behind this strategy** will **become clearer when we study the rate of convergence of the algorithm in** §6.

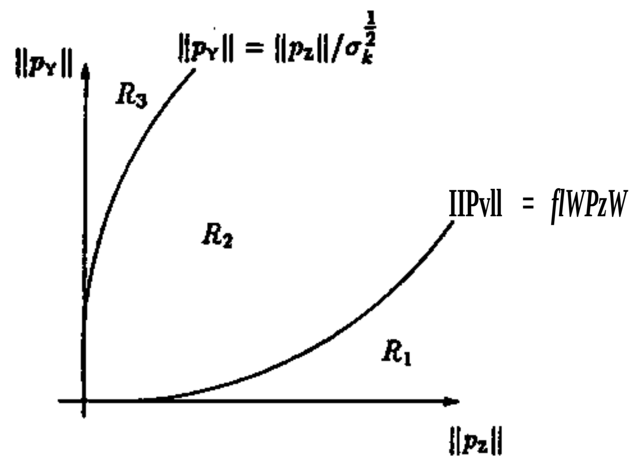


Figure 2

Note from Updating Criterion I that the BFGS update of *Bk* is skipped if the search direction is in R_3. A precise description of the algorithm follows.

Algorithm II

1. Choose constants 77 £ (0,1/2), p > 0 and T,T' with 0 < r < r' < 1, and positive constants F and $j_{{d}$ for conditions (3.13) and (3.19), respectively. For conditions (3.14) and (3.18), select a summable sequence of positive numbers {7*}. Set k := 1 and choose a starting point $x\backslash$, an initial value $fi\backslash$ for the penalty parameter, an $(n - m)$ x $(n - m)$ symmetric and positive definite starting matrix $B\backslash$ and an $(n - m)xn$ starting matrix $S\backslash$.

2. Evaluate *fk^ffk^k* &nd *Ak,* and compute *Yk* and *Zk*-

3. Set *findiff* = *false* and compute p_Y by solving the system

$$(AlYk)pY = -c*. \qquad \text{(range space step)} \qquad (3.42)$$

4. Calculate Wt using Broyden's method, from equations (3.12) and (3.13).

5. Choose the damping parameter 0; from equations (3.38) and (3.39) and compute p_z from

$$B_kPz = -[Zlg_k + CkWk)- \qquad \text{(nuU space step)} \qquad (3.43)$$

6. **If (3.19)** is **satisfied and** (3.18) is ***not* satisfied, set *findiff* = *true* and recompute w* from equation (3.1).**

«

7. **If *findiff* = *true* use this new value of 10* to choose the damping parameter Ct from equations (3.38) and (3.39)** and **recompute p_z from equation (3.43).**

8. **Define** the search **direction** by

$$d_k = Y_kp_Y + Z_kpz, \qquad \mathbf{(3.44)}$$

and set a& = 1.

9. Test the line search condition

$$<f>»_k(*k+<**<**) \leq <t>v_k(*k) + VOtkD^ixk;d_k). \qquad \mathbf{(3.45)}$$

10. **If** (3.45) is not satisfied, choose a new a* € [raj^r'a*] and go to 9; otherwise set

$$s*+i = x_k + a_kd_k. \qquad (3.46)$$

11. Evaluate A+i^+iiC/fe+iMfc+ii and compute l^+i and Z*+i.

12. Compute the Lagrange multiplier estimate

$$*k+i = -[n^T+i^*+i]^{-1}n^T+i^i. \qquad (3.47)$$

and update /Zjt so as to satisfy (3.40).

13. Update $Sk+i$ using equations (3.9) to (3.11). If *findiff* = *false* calculate $\overline{w}_k$ by Broyden's method through equations (3.12) and (3.14); otherwise calculate $\overline{w}_k$ by (3.2).

14. If $s\text{^}yk \leq 0$ or if (3.19) is not satisfied, set $B_k+\backslash = B_k$. Else, compute

$$H = a^*p_z, \tag{3.48}$$

$$y_k = Z\%[VL(x_k+_u \text{ Ajb+O} - \text{VI}(**, \text{Afc+OJ} - \quad \text{TO}^*, \tag{3.49}$$

and compute B_k+i by the BFGS formula (1.24).

15. Set $k := k + 1$, and go to 3.

We mentioned in §3.1 that, when using finite differences, there are various ways of defining w_k and TcJ*, but for concreteness we now assume in steps 6 and 13 that they are computed by (3.1) and (3.2), respectively. We should also point out that the curves in Figure 2 may intersect, creating a fourth region, and in practice we should stipulate a new set of conditions in this region. We will not discuss these conditions here and leave this to the paper that considers the implementation of the algorithm (Biegler, Nocedal and Schmid (1993)).

In the next sections we present several convergence results for Algorithm II. The analysis, which does not assume that the BFGS matrices B_k or the Broyden matrices S_k axe bounded, is based on the results of Byrd and Nocedal (1991), who have studied the convergence of the Coleman-Conn updating algorithm. We also make use of some results of Xie (1991), who has analyzed the algorithm proposed by Nocedal and Overton (1985) using non-orthogonal bases Y and Z. The main difference between this paper and that of Xie stems from our use of the correction terms w_k and $\overline{w}_k$, which are not employed in his method.

4. Semi-Local Behavior of the Algorithm.

We first show that the merit function <j> decreases significantly at the good iterates J, and that this gives the algorithm a weak convergence property. To establish the results of this section we make the following assumptions.

Assumptions 4.1 The sequence $\{x_k\}$ generated by Algorithm II is contained in a convex set D with the following properties.

(I) The functions $/ : R^n \to R$ and $c : R^n \to R^m$ and their first and second derivatives are uniformly bounded in norm over D.

(II) The matrix $A(x)$ has full column rank for all x 6 D, and there exist constants 70 and fio such that

$$\|Y(x)[A(x)^T Y(x)]^{-1}\| \leq \gamma_0, \quad \|Z(x)\| \leq A>, \tag{4.1}$$

for all $x \in D$.

(III) For all $k \geq 1$ for which B_k is updated, (3.21) and (3.22) hold.

(IV) The correction term *Wk* is chosen so that there is a constant $n > 0$ such that for all k,

$$\text{IKH} \leq \text{KIMI}^1/^2. \tag{4.2}$$

Note that condition (I) is rather strong, since it would often be satisfied only if *D* is bounded, and it is far from certain that the iterates will remain in a bounded set. Nevertheless the convergence result of this section can be combined with the local analysis of §5 to give a satisfactory semi-global result. Condition (II) requires that the basis matrices *Y* and *Z* be chosen carefully, and is important to obtain good behavior in practice. Note that (4.1) and (3.42) imply that

$$\text{linpvll} \leq 7\text{o}|\text{M}|. \tag{4.3}$$

Condition (III) is justified in the last paragraphs of §3.3, where it is shown that (3.21) and (3.22) are satisfied whenever BFGS updating takes place in a neighborhood of a solution point. Condition (III) and Theorem 3.1 ensure that at least half of the iterates at which BFGS updating takes place are good iterates.

We have left some freedom in the choice of w* since (4.2) suffices for the analysis of this section. Relation (4.2) holds for the finite difference approach, since (3.1) implies that *Wk = O(YkPy),* **and** since (I) ensures that {‖c*‖} is uniformly bounded (see (5.21)). Furthermore, the safeguard (3.13) and (4.3) immediately imply **that** (4.2) is satisfied when the Broyden approximation is used.

The following result concerns the good iterates J, as given in Definition 3.1.

Lemma 4.1 *If Assumptions 4-1 hold and if/ij = ft is constant for all sufficiently large j, then there is a positive constant* 7_M *such that for all large j € J_f*

$$\phi_\mu(x_j) - \phi_\mu(x_{j+1}) \geq 7\text{M}\ [\backslash\backslash zjgjf + \text{IM}|,]. \tag{4.4}$$

Proof. Using (3.37) we have for all *j € J*

$$D\phi_{\mu_j}(x_j; d_j) \leq -b_2 \left[\|Z_j^T g_j\|^2 + \|c_j\|_1 \right], \tag{4.5}$$

where 62 = min(/?i//?3,/>). Note that the line search enforces the Armijo condition (3.45),

$$\phi_{\mu_j}(x_j) - \phi_{\mu_j}(x_{j+1}) \geq -\eta\alpha_j D\phi_{\mu_j}(xj;\ dj). \tag{4.6}$$

It is then clear from (4.5) that (4.4) holds, provided the a,-, *j* € J, can be bounded from below. Suppose that *aj* < 1, which means that (4.6) failed for a steplength ā:

$$\phi_{\mu_j}(x_j + \bar\alpha d_j) - \phi_{\mu_j}(x_j) > \eta\bar\alpha D\phi_{\mu_j}(x_j; d_j), \tag{4.7}$$

where

$$r\tilde{a} \leq aj \tag{4.8}$$

(see step 10 of Algorithm II). On the other hand, expanding to second order we have

$$*_{Mi}(*i + \&dj) - tvMi) \leq \&D\wedge(x_{j;}dj) + \tilde{a}^2 b_x\backslash\backslash dj\backslash\backslash\backslash \tag{4.9}$$

where $b\backslash$ depends on /ij. Combining (4.7) and (4.9) we have

$$(r,- \wedge\tilde{a}D\wedge xjidj) < \tilde{a}XWdjW^2. \tag{4.10}$$

Next we show that, for j 6 «/,

$$\|<*;\|^2 \leq HWZfgjW^2 + \text{IkiHi}], \tag{4.11}$$

for some constant 63. To do this we make repeated use of the following elementary result,

$$a, b \geq 0 \quad \Rightarrow \quad a^2 + 2ab + b^2 \leq 3a^2 + 3b^2. \tag{4.12}$$

Using (3.44), (4.12), (4.1) and (4.3) we have

$$\begin{aligned} |K-|^2 &\leq \|Z_j p_Z^{(j)}\|^2 + 2\|Z_j p_Z^{(j)}\|\,\|Y_j p_Y^{(j)}\| + \|Y_j p_Y^{(j)}\|^2 \\ &\leq 3\left[\|Z_j p_Z^{(j)}\|^2 + \|Y_j p_Y^{(j)}\|^2\right] \\ &\leq 3\ [fito\&tf + 7o^2 l\|c;\|^2]\ \bullet \end{aligned} \tag{4.13}$$

Also by (3.34), (4.12), (4.2) and noting that $\|\bullet\| \leq \|\bullet\|_1$ we have that for $j \in J$

$$\begin{aligned} \|p_Z^{(j)}\|^2 &\leq \frac{1}{\beta_2^2}\left[\|Z_j^T g_j\|^2 + 2\zeta_j\|Z_j^T g_j\|\,\|w_j\| + \zeta_j^2\|w_j\|^2\right] \\ &\leq \frac{3}{\beta_2^2}\left[\|Z_j^T g_j\|^2 + \zeta_j^2\|w_j\|^2\right] \\ &\leq \frac{3}{\beta_2^2}\left[\|Z_j^T g_j\|^2 + \kappa^2\|c_j\|_1\right], \end{aligned}$$

since Ci $\leq$ 1- Since $\|CJ\|_I$ is uniformly bounded on D, we see from this relation and (4.13) that (4.11) holds, where

$$b_3 = \max\{9\wedge/\wedge^2, 3(3/c^2\wedge/\wedge + \wedge \sup \|c(x)\|\|)\}.$$

Combining (4.10), (4.5) and (4.11), and recalling that $rj < 1$ we obtain

$$\bar{\alpha} > \frac{(1-\eta)b_2}{b_1 b_3}. \tag{4.14}$$

This relation and (4.8) imply that the steplengths aj are bounded away from zero for all $j \in$ «/, and since by assumption $fij = ft$ for all large j we conclude that (4.4) holds with 7/x = 1762 mim{1,(1 - 77)7-62/(6163)}.

D

It is now easy to show that the penalty parameter settles down, and that the set of iterates is not bounded away from stationary points of the problem.

Theorem 4.2 *If Assumptions 4.1 hold, then the weights {fi$_k$} are constant for all sufficiently large k and*

$$\liminf_{\text{fc-KX)}}(\|Z \quad \| + \|c_k\|) = 0.$$

Proof, First note that by Assumptions 4.1 (I)-(II) and (3.47) that {||A/,||} is bounded. Therefore, since the procedure (3.40) increases /z* by at least *p* whenever it changes the penalty parameter, it follows that there is an index k_0 and a value /x such that for all *k>ko,fi*$_k$ = *fx>* _||A|bH + 2p.

If BFGS updating is performed an infinite number of times, by Assumptions 4.1-(III) and Theorem 3.1 there is an infinite set *J* of good iterates, and by Lemma 4.1 and the fact that *<f>n(xk)* decreases at each iterate, we have that for *k* > fco>

$$\begin{aligned}\phi_\mu(x_{k_0}) - \phi_\mu(x_{k+1}) &= \sum_{j=k_0}^{k}(\phi_\mu(x_j) - \phi_\mu(x_{j+1})) \\ &\geq \sum_{j\in J\cap[k_0,k]} (\phi_\mu(x_j) - \phi_\mu(x_{j+1})) \\ &\succ \text{7M} \sum_{j\text{€}Jn[ko,k)} [\|Z_j^T g_j\|^2 + \|c_j\|_1].\end{aligned}$$

By Assumption 4.1-(I) *<f>^(x)* **is** bounded below **for all x 6** ***D*****, so the last sum is finite, and thus the term inside the square brackets converges to zero. Therefore**

$$\lim_{\substack{j\in J \\ j\to\infty}} (\|\text{35ill} + \text{INIi}) = 0. \qquad \textbf{(4.15)}$$

If BFGS updating is performed a finite number of times then, as discussed after Definition 3.1, all iterates are good iterates, and in this case we **obtain the stronger result**

$$\lim_{\text{it—MX>}} (\|Z_k^T g_k\| + \|c_k\|_1) = 0.$$

□

5. Local Convergence

In this section we show that if x_m is a local minimizer that satisfies the **second order optimality conditions,** and if the penalty parameter /x* is chosen large enough, then x_m is **a point of attraction for** the sequence of iterates **{z*} generated** by Algorithm **II.** To **prove this result** we will make the following assumptions. In what follows *G* denotes the reduced Hessian of the Lagrangian function, i.e.

$$G_k = ZlV^2_{xx}L(x_k,\backslash_k)Z_k. \qquad (5.1)$$

Assumptions 5.1 The point x_m is a local minimizer for problem (1.1)-(1.2) at which the following conditions hold.

(1) The functions / : R^n —• R and c : R^n —• R^m are twice continuously differentiable in a neighborhood of x_{my} and their Hessians are Lipschitz continuous in a neighborhood of a;*.

(2) The matrix $A(x_m)$ has full column rank. This implies that there exists a vector A* 6 R^m such that

$$VL(x_m, A.) = g(x_m) + A(x_m)X_m = 0.$$

(3) For all q € $R^n\sim^m$, q ^ 0, we have $q^T G_m q > 0$.

(4) There exist constants 70, *0o* and 7_C such that, for all a: in a neighborhood of x_m,

$$\backslash\backslash Y(x)[A(x)^T Y(x)]\text{-}^{l}\backslash\backslash \leq 70, \quad ||Z(x)|| \leq \quad fa \tag{5.2}$$

and

$$||[r(x)Z(i)]\text{-}^{1}|| \leq {}_{7c}. \tag{5.3}$$

(5) *Z{x)* and A(x) are Lipschitz continuous in a neighborhood of x_{mj} i.e. there exist constants 7_2 and 7A such that

$$\backslash\backslash X(x) - X(z)\backslash\backslash \leq fx\backslash\backslash x\text{-}z\backslash\backslash, \tag{5.4}$$

$$\backslash\backslash Z(x)\text{-}Z(z)\backslash\backslash \leq f_z\backslash\backslash x\text{-}z\backslash\backslash, \tag{5.5}$$

for all z,z **near** x_m.

Note that (1), (3) and (5) imply that for all (z, A) sufficiently near (i.,A.), **and for all** q **e** $\mathbf{R^B\sim^m}$,

$$m\backslash\backslash q\backslash\backslash^2 \leq q^T G(x,X)q \leq M\backslash\backslash qf, \tag{5.6}$$

for some positive constants *m,M.* We also note that Assumptions 5.1 **ensure that** the conditions (3.21)-(3.22) required by Theorem 3.1 hold whenever BFGS updating takes place in a neighborhood of x_m, as argued at the end of §3.3. Therefore Theorem 3.1 can be applied in **the** convergence analysis.

The following **two** lemmas **are** proved by Xie (1991) **for** very general choices **of** *Y* and *Z.* Their result generalizes Lemmas 4.1 and 4.2 of Byrd and Nocedal (1991); see also Powell (1978).

Lemma 5.1 *If Assumptions 5.1 hold, then for all x sufficiently near* x_m

$$\textbf{Till* ~ *.||} \leq \textbf{IK*)||} + \textbf{HZ(*)}^{\mathbf{r}}\textbf{5(*)ll} \leq \textbf{Tall* - *.||}, \tag{5.7}$$

for some positive constants 71,72-

This result states that, near x_m, the quantities *c(x)* and $Z(x)^T g(x)$ may be regarded as a measure of the error at *x.* The next lemma states that, for a large enough weight, the merit function may also be regarded as a measure of the error.

Lemma 5.2 *Suppose that Assumptions 5.1 hold at x«. Then for any // >* HA*!^ *there exist constants* 73 > 0 *and* 74 > 0, *such that for all x sufficiently near* x_m

7311* - ar.lt^2 ≤ *M*)* ~ *,.(*.) ≤74 [||Z(x)^{T}ir(x)||^2 + HcWlU] . **(5.8)**

Note that the left inequality in (5.8) implies that for a sufficiently large value of the penalty parameter, the merit function will have a strong local minimizer at z*. We will now use the descent property of Algorithm II to show convergence of the algorithm. However, due to the non-convexity of the problem, the line search could generate a **step** that decreases the merit function but that takes us away from the neighborhood of z*. To rule this out we make the following assumption.

Assumption 5.2 The line search has the property that, for all large A;, ^((1 — *0)xk* + 0£*+i) ^ *4>ii{^{x}k)* for all *0* € [0,1]. In other words, *Xk+* is in the connected component of the level set *{x* : <£$_M$(z) ≤*<t>n(xk)}* that contains a*.

There is no practical line search algorithm that can guarantee this condition, **but** it is likely to hold close to z*. Assumption 5.2 is made by Byrd, Nocedal **and Yuan (1987) when** analyzing the convergence of variable metric methods **for unconstrained problems, as well as by Byrd and Nocedal (1991) in** the **analysis of Coleman-Conn updates for equality constrained optimization.**

Lemma 5.3 ***Suppose that the iterates generated by Algorithm II are contained in a convex region D satisfying Assumptions 4-1 //<*** iterate x^ is sufficiently close to a solution point x* that satisfies Assumptions 5.1, and if the weight /x^ is large enough, then the sequence of iterates converges to z*.***

Proof. By Assumptions 4.1 (I)-(II) and (3.47) we know that {||A*||} is bounded. Therefore the procedure (3.40) ensures that the weights /*& are constant, say /i* = /1 for all large *k*. Moreover, if an iterate gets sufficiently close to z«, we know by (3.40) and by the continuity of A that *i* > ||A,||. For such value of /*, Lemma 5.2 implies that the merit function has a strict local minimizer at z*. Now suppose that once the penalty parameter has settled, and for a given c > 0, there is an iterate z/^ such that

$$||x_{k_0} - x \quad {}^{2} \quad 727470$$

where 70 is such that || • ||i ≤ 7o|| • ||- Assumption 5.2 shows that for any A: ≥fco, *%k* is in the connected component of the level set **of z^ that contains** z^, **and** we can assume that *e* is small enough that Lemmas 5.1 and 5.2 hold in this level set. Thus since *<f>n(xk)* < *<Pt*(^{x}ko)* f°^{r} *k* ^ ^0* and since we can assume that ||£j£flfjfcJl < 1, we have from Lemmas 5.1 and 5.2, for any $k \geq k_0$

$$
\begin{aligned}
\|x_k - x_*\| &\le \gamma_3^{-\frac{1}{2}}(\phi_\mu(x_k) - \phi_\mu(x_*))^{\frac{1}{2}} \\
&\le \gamma_3^{-\frac{1}{2}}(\phi_\mu(x_{k_0}) - \phi_\mu(x_*))^{\frac{1}{2}} \\
&\le \left(\frac{\gamma_4}{\gamma_3}\right)^{\frac{1}{2}} \left[\|Z_{k_0}^T g_{k_0}\|^2 + \|c_{k_0}\|_1\right]^{\frac{1}{2}} \\
&\le \left(\frac{\gamma_4}{\gamma_3}\right)^{\frac{1}{2}} \left[\|Z_{k_0}^T g_{k_0}\|^2 + \hat{\gamma}_0\|c_{k_0}\|\right]^{\frac{1}{2}} \\
&\le \left(\frac{\gamma_2\gamma_4\hat{\gamma}_0}{\gamma_3}\|x_{k_0} - x_*\|\right)^{\frac{1}{2}} \\
&\le \epsilon.
\end{aligned}
$$

This implies that the whole sequence of iterates remains in a neighborhood **of** radius € of x*. If € is small enough we conclude by (5.8), by the monotonicity of *{<t>p(xk)}* and Theorem 4.2 that the iterates converge to x_m.

a

The assumptions of this lemma, which is **modeled after a result in Xie (1991), are restrictive - especially the assumption on the penalty parameter. One can relax these assumptions and obtain a stronger result, such as Theorem 4.3 in Byrd and Nocedal (1991), but the proof would be more complex, and is not particularly relevant to Algorithm II since it is only based on the properties of the merit function. Therefore instead of further analyzing the local convergence properties of the new algorithm, we will proceed to study** its **rate of convergence.**

5.1. R-Linear Convergence.

For the **rest of the paper we assume that the iterates generated by Algorithm II converge to £*, which implies that for all large fc, /i* =** ***fi*** **> ||A*||. The analysis that follows depends on how often BFGS updating is applied, and to make this concept precise we** define ***U*** to **be the set of iterates at which BFGS updating takes place,**

$$U = \{* : Bk+i = BFGS(B_k, s_{k\%}\ y_k)h \qquad (5.9)$$

and let

$$U_k = Un\{l,2 \ldots k\}. \qquad (5.10)$$

The number of elements in *Uk* will be denoted by *\Uk\-*

Theorem 5.4 *Suppose that the iterates* {**} *generated by Algorithm II converge to a point* x_m *that satisfies Assumptions 5.1. Then for any k € U and any j >_k*

$$\|x_j - x_*\| \le Cr^{|U_k|}, \qquad (5.11)$$

for some constants C > 0 *and* 0 ≤ r < 1.

Proof. Using (4.4) and (5.8) we have for ż € *J*,

$$M*i) \sim M*i+i) \succeq \frac{\gamma_\mu}{\gamma_4}\, lM*i) \sim *M(*01 \bullet \qquad (5.12)$$

Let us define r = (1 - 7^/74) $^{\frac{1}{4}}$. Then for i € •/

$$0_M(x,-+i) - tf_M(*.) \le r^4\, [*,,(*,\bullet) - <\wedge(x,)]. \qquad (5.13)$$

We know that the merit function decreases at each step, and by (5.8) we have, for $j \ge k$ and *keU,*

$$\begin{aligned} \|x_j - x_*\| &\le \gamma_3^{-\frac{1}{2}}(\phi_\mu(x_j) - \phi_\mu(x_*))^{\frac{1}{2}} \\ &\le \gamma_3^{-\frac{1}{2}}(\phi_\mu(x_k) - \phi_\mu(x_*))^{\frac{1}{2}}. \end{aligned}$$

We continue in this fashion, bounding the right hand side by terms involving earlier iterates, but using now (5.13) for all good iterates. Since by Theorem 3.1 at least half of the iterates at which updating takes place are good iterates, i.e. *\Jk* ≥ *\\Uk*, we have

$$\begin{aligned} \|x_j - x_*\| &\le \gamma_3^{-\frac{1}{2}}\left[r^{4|J_k|}(\phi_\mu(x_1) - \phi_\mu(x_*))\right]^{\frac{1}{2}} \\ &\le \gamma P^{1}\left[r^{2|U_k|}(\phi_\mu(x_1) - \phi_\mu(x_*))\right]^{\frac{1}{2}} \\ &\le [\gamma P^{1}(\phi_\mu(x_1) - \phi_\mu(x_*))^{\frac{1}{2}}]\, r^{|U_k|} \\ &\equiv C r^{|U_k|}. \end{aligned}$$

□

This result implies that if *{\Uk\/k}* is bounded away from zero, then **Algorithm II** is R-linearly **convergent. However,** BFGS **updating could take place only a finite number of times, in which case this ratio would converge to zero. It is also possible for BFGS updating to take place an infinite number of times, but every time less often, in such a way that *\Uk\/k* —• 0. We therefore need to examine the iteration more** closely.

We make use **of the** matrix function *rp* defined by

$$\psi(B) = tr(B) - \ln(de*(B)), \qquad (5.14)$$

where *tr* denotes the trace, and *det* the determinant. It can be shown that

$$\text{lncond}(U) < \qquad tf(fl), \qquad (5.15)$$

for any positive definite matrix *B* (Byrd and Nocedal (1989)). We also make use of the weighted quantities

$$\bar{y}_k = G;^{1/2} y_{fc}, \qquad h = G\,i\,'\,V \qquad (5.16)$$

$$\tilde{B}_k = GZ^{ll2}B_kG:"\backslash \quad (5.17)$$

$$cosdk = \text{—}\backslash \; \frac{}{\text{l|£fc«*IIIMI}}, \quad (5.18)$$

and

$$\tilde{q}_k = \frac{\tilde{s}_k^T \tilde{B}_k \tilde{s}_k}{\tilde{s}_k^T \tilde{s}_k}. \quad (5.19)$$

One can show (see eq. (3.22) of Byrd and Nocedal (1989)) **that if *Bk* is updated by the BFGS formula then**

$$\psi(\tilde{B}_{k+1}) = \psi(\tilde{B}_k) + \frac{\|\tilde{y}_k\|^2}{\tilde{y}_k^T \tilde{s}_k} - 1 - \ln \frac{\tilde{y}_k^T \tilde{s}_k}{\tilde{s}_k^T \tilde{s}_k} + \ln \cos^2 \tilde{\theta}_k + \left[1 - \frac{\tilde{q}_k}{\cos^2 \tilde{\theta}_k} + \ln \frac{\tilde{q}_k}{\cos^2 \tilde{\theta}_k}\right]. \quad (5\_20)$$

This expression characterizes the behavior of the BFGS matrices B_k, and will be crucial to the analysis of this section. However before we can make use of this relation we need to consider the accuracy of the correction terms. We begin by showing **that** when finite differences are used to estimate w_k and tZ^, these are accurate to second order.

Lemma 5*5 *// at the iterate x_k, the corrections w_k* ***and** *Wk* ***are computed by the finite difference formulae (3.1)-(3.2), and if** *Xk* **is sufficiently close to a solution point** x_m **that satisfies Assumptions** 5.1_f **then***

$$»k = O(\|py\|), \quad (5.21)$$

$$|K - ZjW.Y_kp^*\backslash\backslash = Otelfev\|) \quad (5.22)$$

and

$$\backslash\overline{w}_k - ZjW.YkPvW = O(a_k\backslash\backslash p_Y\backslash\backslash). \quad (5.23)$$

Proof. Recalling that Vi(x, A) = $g(x) + A(x)\backslash$, we have from (3.1) that

$$\begin{aligned} & Z_k^T[\nabla L(x_k + Y_k p_Y, \lambda_k) - \nabla L(x_k, \lambda_k)] \\ = \; & Z_k^T[\nabla L(x_k + Y_k p_Y, \lambda_*) - \nabla L(x_k, A.)] + Zl[(A(x_k + Y_{kPy}) - A_k)(X_k - A.)] \\ = \; & Z_k^T \left[\int_0^1 V^2_{xx}L\{x_k + rY_{kPY}, A.)rfr\right] Y_k p_Y + Zl[\{A(x_k + Y_{k?Y}) - A_k)(X_k - A.)] \\ \equiv \; & Z_k^T \overline{W}_k Y_k p_Y + Z_k^T[(A(x_k + Y_k p_Y) - A_k)(\lambda_k - X.)]. \end{aligned} \quad (5.24)$$

Let us assume that x_k is in the neighborhood of $x+$ were (5.2)-(5.5) hold. Then $\backslash\backslash\backslash_k—\backslash^*\backslash\backslash$ = $O(\|e_k\|) = O(cr_k)$, where a_k is defined by (1.34). Therefore the last term in (5.24) is $O(\|p_Y\|\sigma^*)$, which proves (5.21). Also a simple computation shows that

$$[Z\bar{f}\overline{W}_k - ZJ\bar{W}.]5'^*PY = O(\text{^}llPvll).) \quad (5.25)$$

Using these facts in (5.24) yields the desired result (5.22). To prove (5.23), we only note that *ak* $\leq$ 1, and reason in the same manner.

a

Next we show that the condition number of the matrices *Bk* is bounded, and that at the iterates *U* at which BFGS updating takes place the matrices *Bk* are accurate approximations of the reduced Hessian of the Lagrangian.

Theorem 5.6 *Suppose that the iterates {xk} generated by Algorithm* **II** *converge to a solution point* x_m *that satisfies Assumptions 5.1. Then* {||JB*||} *and* {\\B^[1]1|} *are bounded, and for all k* 6 *U*

$$||(5, - ZjW_mZ+)p_z\\ = o(||<f*||). \qquad (5.26)$$

Proof, We will only consider iterates *k* for which BFGS updating of *Bk* takes place. We have from (3.49), (3.46), (3.44), (3.16) and (3.48)

$$\begin{aligned} y_k &= Z_k^T[\nabla L(x_{k+1}, \lambda_{k+1}) - \nabla L(x_k, \lambda_{k+1})] - \bar{w}_k \\ &= Z_k^T \left[\int^1 \nabla^2_{xx} L(x_k + ra_kd_k, A*_{+1})dr\right] a_kd_k - \bar{w}_k \\ &= \alpha_k Z_k^T \tilde{W}_k(Z_kp_z + Y_kp_Y) - \bar{w}_k \\ &= Z_k^T\tilde{W}_kZ_ks_k + <*k(Zl\tilde{W}k - ZjW_m)Yk_{PY} + (ctkZjWA'kPy - \bar{to}*). \qquad (5.27) \end{aligned}$$

Since ***Wk*** **can be computed by** Broyden's **method** or **by** finite **differences, we need to consider these two cases separately.**

Part I. **Let us first assume that** $\bar{w}_k$ **is determined by Broyden's method. A simple computation shows that** ||Zj$\tilde{W}$* - *ZjW*$_m$\\ = *O*(a_k), **and from (3.14) we have that** $\bar{w}_k$ = 0(IIPYII/7*)- **Using this and Assumptions 5.1 in (5.27) we have**

$$\begin{aligned} y_k &= Z_k^T\tilde{W}_kZ_ks_k + (\sigma_k + 1 + 1/\gamma_k)O(\alpha_k\|p_Y\|) \\ &= [Zl\tilde{W}kZk - G_m)sk + G_ms_k + (ak + 1 + 1/7*)O("*IIPY||)- \qquad (5.28) \end{aligned}$$

Recalling (5.16) and noting that $\tilde{y}j\tilde{s}k$ = *yjsk* **we** have

$$\tilde{y}_k^T\tilde{s}_k = s_k^T(Z_k^T\tilde{W}_kZ_k - G_*)s_k + \|\tilde{s}_k\|^2 + (\sigma_k + 1 + 1/\gamma_k)O(\alpha_k\|p_Y\|)\|\tilde{s}_k\|,$$

since \\$\tilde{s}$k\\ **and** ||5jt|| **are of the same order. Therefore**

$$\begin{aligned} \frac{\tilde{y}_k^T\tilde{s}_k}{\|\tilde{s}_k\|^2} &= 1 + \frac{s_k^T(Z_k^T\tilde{W}_kZ_k -}{\|\tilde{s}_k\|^2} \cdot + (\sigma_k + 1 + 1/\gamma_k)O\left(\frac{\|\alpha_kp_Y\|}{\|\tilde{s}_k\|}\right) \\ &= 1 + 0\{a_k) + \{a_k + 1 + l/_7ftX^{||a}_{>} (jj\wedge) \bullet \qquad (5.29) \end{aligned}$$

Similarly from (5.28) and (5.16) we have

$$\begin{aligned} \tilde{y}l\tilde{y}_k \leq\ & \\(Zj\tilde{W}_kZ_k - G.)s_k\\^2\\G:^l\\ + 2\\(Zj\tilde{W}_kZ_k - G.)s_k\\ ||G;^{1/2}|| \, ||j_A|| + ||5ik||^2 \\ & +2(<r_{fc} + 1 + l/7*)0(||a*pv||)|K?:*|| (ll\wedge H + \\\{Zl\tilde{W}_kZ_k - G.)s_k\|\|G_*^{-1/2}\|) \\ & +(\sigma_k + 1 + 1/\gamma_k)^2O(\|\alpha_kp_Y\|)^2, \end{aligned}$$

and thus

$$\frac{\|\tilde{y}_k\|^2}{\|\bar{s}_k\|^2} \le 1 + O(\sigma_k) + (\sigma_k + 1 + 1/\gamma_k)(1+\sigma_k)O\left(\frac{\|\alpha_k p_Y\|}{\|\bar{s}_k\|}\right)$$
$$+(\sigma_k + 1 + 1/\gamma_k)^2 O\left(\frac{\|\alpha_k p_Y\|^2}{\|\bar{s}_k\|^2}\right). \tag{5.30}$$

At this point we invoke the update criterion, and note from (3.18) that if BFGS updating of B_k takes place at iteration k, then $\|\alpha_k p_Y\| \le \gamma_k^2\|s_k\|$ where $\{\gamma_k\}$ is summable. Using this, the assumption that σ_k converges to zero, and (5.29) we see that for large k

$$\frac{\bar{y}_k^T\bar{s}_k}{\|\bar{s}_k\|^2} = 1 + O(\sigma_k + \gamma_k), \tag{5.31}$$

and using (5.30)

$$\frac{\|\bar{y}_k\|^2}{\|\bar{s}_k\|^2} = 1 + O(\sigma_k + \gamma_k).$$

Therefore

$$\frac{\|\bar{y}_k\|^2}{\bar{y}_k^T\bar{s}_k} = \frac{\|\bar{y}_k\|^2}{\|\bar{s}_k\|^2}\,\frac{\|\bar{s}_k\|^2}{\bar{y}_k^T\bar{s}_k} = 1 + O(\sigma_k + \gamma_k). \tag{5.32}$$

We now consider $\psi(\bar{B}_{k+1})$ given by (5.20). A simple expansion shows that for large k, $\ln(1 + O(\sigma_k + \gamma_k)) = O(\sigma_k + \gamma_k)$. Using this, (5.31) and (5.32) we have

$$\psi(\bar{B}_{k+1}) = \psi(\bar{B}_k) + O(\sigma_k + \gamma_k) + \ln\cos^2\bar{\theta}_k + \left[1 - \frac{\bar{q}_k}{\cos^2\bar{\theta}_k} + \ln\frac{\bar{q}_k}{\cos^2\bar{\theta}_k}\right]. \tag{5.33}$$

Note that for $x \ge 0$ the function $1 - x + \ln x$ is non-positive, implying that the term in square brackets is non-positive, and that $\ln\cos^2\bar{\theta}_k$ is also non-positive. We can therefore delete these terms to obtain

$$\psi(\bar{B}_{k+1}) \le \psi(\bar{B}_k) + O(\sigma_k + \gamma_k). \tag{5.34}$$

Before proceeding further we show that a similar expression holds when finite differences are used.

Part II. Let us now consider the iterates k for which updating takes place and for which $\bar{w}_k$ is computed by finite differences. In this case (3.19) holds. Again we begin by considering (5.27),

$$y_k = Z_k^T\tilde{W}_k Z_k s_k + \alpha_k(Z_k^T\tilde{W}_k - Z_*^T W_*)Y_k p_Y + (\alpha_k Z_*^T W_* Y_k p_Y - \bar{w}_k).$$

Using (5.23) the last term is of order $O(\alpha_k\|p_Y\|)$, and so is the second term. Thus

$$\begin{aligned} y_k &= Z_k^T\tilde{W}_k Z_k s_k + O(\sigma_k\alpha_k\|p_Y\|) \\ &= (Z_k^T\tilde{W}_k Z_k - G_*)s_k + G_* s_k + O(\sigma_k\alpha_k\|p_Y\|). \end{aligned} \tag{5.35}$$

Noting once more that $\bar y_k^T \bar s_k = y_k^T s_k$ and recalling the definition (5.16) we have

$$\bar y_k^T \bar s_k = s_k^T (Z_k^T \bar W_k Z_k - G_*) s_k + \|\bar s_k\|^2 + O(\sigma_k \alpha_k \|p_Y\| \|\bar s_k\|),$$

since $\|\bar s_k\|$ and $\|s_k\|$ are of the same order. Therefore

$$\begin{aligned} \frac{\bar y_k^T \bar s_k}{\|\bar s_k\|^2} &= 1 + \frac{s_k^T (Z_k^T \bar W_k Z_k - G_*) s_k}{\|\bar s_k\|^2} + O\left(\sigma_k \frac{\|\alpha_k p_Y\|}{\|\bar s_k\|}\right) \\ &= 1 + O(\sigma_k) + O\left(\sigma_k \frac{\|\alpha_k p_Y\|}{\|\bar s_k\|}\right). \end{aligned} \tag{5.36}$$

Similarly from (5.35) and (5.16) we have

$$\begin{aligned} \bar y_k^T \bar y_k \le\ & \|(Z_k^T \bar W_k Z_k - G_*) s_k\|^2 \|G_*^{-1}\| + 2\|(Z_k^T \bar W_k Z_k - G_*) s_k\|\,\|G_*^{-1/2}\|\,\|\bar s_k\| + \|\bar s_k\|^2 \\ & + \sigma_k O\left(\|\alpha_k p_Y\|\,\|G_*^{\frac{1}{2}}\|\,[\|\bar s_k\| + \|(Z_k^T \bar W_k Z_k - G_*) s_k\|\,\|G_*^{-1/2}\|]\right) \\ & + \sigma_k^2 O(\|\alpha_k p_Y\|^2) \end{aligned}$$

and thus

$$\frac{\|\bar y_k\|^2}{\|\bar s_k\|^2} \le 1 + O(\sigma_k) + \sigma_k O\left(\frac{\|\alpha_k p_Y\|}{\|\bar s_k\|}\right) + \sigma_k^2 O \quad . \tag{5.37}$$

We now invoke Update Criterion I, and note from (3.19) that if BFGS updating of B_k takes place at iteration k, then $\|p_Y\| \le \gamma_d \|p_Z\|/\sigma_k^{1/2}$. Using this, (5.36) and the fact that σ_k converges to zero, we see that for large k

$$\frac{\bar y_k^T \bar s_k}{\|\bar s_k\|^2} = 1 + O(\sigma_k^{1/2}),$$

and using (5.37)

$$\frac{\|\bar y_k\|^2}{\|\bar s_k\|^2} = 1 + O(\sigma_k^{1/2}).$$

Therefore

$$\frac{\|\bar y_k\|^2}{\bar y_k^T \bar s_k} = \frac{\|\bar y_k\|^2}{\|\bar s_k\|^2} \frac{\|\bar s_k\|^2}{\bar y_k^T \bar s_k} = 1 + O(\sigma_k^{1/2}). \tag{5.38}$$

We now consider $\psi(\bar B_{k+1})$ given by (5.20). Noting that $\ln(1 + O(\sigma_k^{1/2})) = O(\sigma_k^{1/2})$ for all large k, we see that if updating takes place at iteration k

$$\psi(\bar B_{k+1}) = \psi(\bar B_k) + O(\sigma_k^{1/2}) + \ln\cos^2 \bar\theta_k + \left[1 - \frac{\bar q_k}{\cos^2 \bar\theta_k} + \ln \frac{\bar q_k}{\cos^2 \bar\theta_k}\right]. \tag{5.39}$$

Since both $\ln\cos^2 \bar\theta_k$ as well as the term inside the square brackets are non-positive, we can delete them to obtain

$$\psi(\bar B_{k+1}) \le \psi(\bar B_k) + O(\sigma_k^{1/2}). \tag{5.40}$$

We now combine the results of Parts I and II of this proof. Let us subdivide the set of iterates U for which BFGS updating takes place into two subsets: U^* corresponds to the iterates in which $\overline{W}k$ is computed by Broyden's method, and U'' to the iterates in which finite differences are used. We also define U'_k = $U'H$ {1,2, ...A:} and $U\%$ = $U''C\backslash$ {1,2, ...&}.

Summing over the set of iterates in Uk, using (5.34) and (5.40), and noting that 2?j+i = Bj for j £ Uk, we have

$$\psi(\bar{B}_{k+1}) \le \psi(\bar{B}_1) + d \underset{j\in U''_k}{£} {}^\circ T + {}^\wedge E_{j\in U'_k} {}^a i + {}^C * \underset{j\in U'_k}{£} TJ. \qquad (5\text{-}41)$$

for some constants $Ci, C2, Cz$. By (5.11) and since $\backslash U''_j\backslash \le \backslash Uj\backslash$,

$$\begin{aligned} \sum_{j\in U''} c r_j V^2 &\le \sum_{j\in U''} C^{1/2} r^{|U_j|/2} \\ &\le \sum_{j\in U''} C^{1/2} r^{|U''_j|/2} \\ &= \sum_{i=1}^{\backslash U''\backslash} C^{1/2} r^{i/2} \\ &< \infty. \end{aligned}$$

Similarly

$$\underset{j\in U'}{E} *i < ».$$

and since {7*} is summable we conclude from (5.41) that {$tp(\bar{B}_k)$} is bounded above, fiy (5.14) $if>(\bar{B}k)$ = SLi('t — !»'•)» where /, are the eigenvalues of $\bar{J}3^*_f$ and it is easy to see that this implies that both $\backslash\backslash B_k\backslash\backslash$ and $H^{\wedge 1}$!! are bounded.

To prove (5.26), we sum relations (5.33) and (5.39), recalling that <r*, 7* and alj^2 are summable, to obtain

$$\psi(\bar{B}_{k+1}) \le C + \sum_{j\in U_k} \left(\ln \cos^2 \bar{\theta}_k + \left[1 - \frac{\tilde{q}_k}{\cos^2 \bar{\theta}_k} + \ln \frac{\tilde{q}_k}{\cos^2 \bar{\theta}_k} \right] \right),$$

for some constant C. Since $tf>(\bar{B}k+i) > 0$, and since both In $\cos^2 0^*$ and the term inside the square brackets are non-positive we see that

$$\lim_{fc\to\infty} \text{In} \cos^2 \S k = 0,$$

and

$$\lim_{\substack{k\to\infty \\ k\in U}} \left[1 - \frac{\wedge}{\cos^2 \theta_k} + \text{in} \frac{2tJ}{\cos^2 \theta_k} \right] = 0.$$

Now, for $x \geq 0$ the function $1 - z + \ln a$: is concave and has its unique maximizer at $x = 1$. Therefore the relations above imply that

$$\lim_{\substack{k \to \infty \\ k \in U}} \cos \tilde{6}k = \lim_{\substack{k \to \infty \\ k \in U}} 4^* = 1. \tag{5.42}$$

Now from (5.18)-(5.19)

$$\begin{aligned} \frac{\|G_*^{-1/2}(B_k - G_*)p_z\|^2}{\|G_*^{1/2}p_z\|^2} &= \frac{\|(\tilde{B}_k - I)\tilde{s}_k\|^2}{\text{1I**1I*}} \\ &= \frac{\|\tilde{B}_k\tilde{s}_k\|^2 - 2\tilde{s}l\tilde{B}_k h + \tilde{s}_k^T\tilde{s}_k}{\tilde{s}_k^T\tilde{s}_k} \\ &= \frac{\tilde{q}_k^2}{\cos\tilde{\theta}_k^2} - 2\tilde{q}_k + 1. \end{aligned}$$

It is clear from (5.42) that the last term converges to 0 **for** k 6 £/, which implies that (5.26) holds.

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This result immediately implies that the iterates **are R-linearly convergent, regardless of how often updating takes place.**

Theorem 5*7 ***Suppose that the iterates*** {x&} ***generated by Algorithm II converge to a solution point*** x_m ***that satisfies Assumptions 5.1. Then the rate of convergence is at least R-linear.***

Proof. Theorem 5.6 implies that the condition number of the matrices {£*} is bounded. Therefore *all* the iterates are good iterates, and reasoning as in the proof of Theorem 5.4 we conclude that for all j

$$\text{II*;} - *.\| \leq \text{Cr»''},$$

for some constants $C > 0$ and $0 \leq r < 1$.

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We can now show that the Broyden matrices ***Sk*** **are bounded. This is easy to do, since we have established that the iterates** *{xk}* **converge R-linearly. We make use of the well-known bounded deterioration property for Broyden's method (cf. Lemma 8.2.1 in Dennis and Schnabel (1983)),** which **states that under Assumptions 5.1**

$$\text{HSb+i} - zTw.\| \leq \|s_k - zjw.\| + c\sigma^*,$$

for some constant $C > 0$. Due to the R-linear convergence of {x^}, we obtain:

$$\begin{aligned} \|S_{k+1} - Z_*^T W_*\| &\leq \|S_1 - Z_*^T W_*\| + C\sum_{i=1}^{k}\sigma_k \\ &< \infty, \end{aligned}$$

which shows that the matrices S_k remain bounded. We then see from (3.12) that the Broyden corrections w_k and $\bar{w}_k$ satisfy

$$w_k = O(\|p_Y\|) \quad \bar{w}_k = O(\|p_Y\|), \tag{5.43}$$

and it is clear that the safeguards (3.13) and (3.14) become inactive for all large k. Therefore the algorithm will not modify the information supplied by Broyden's method, asymptotically, and this is important in establishing superlinear convergence.

6. Superlinear Convergence

Without the correction terms w_k and TD_k, and using appropriate update criteria, Algorithm II is 2-step Q-superlinearly convergent. This was proved by Nocedal and Overton (1985) assuming that Y_k and Z_k are orthogonal bases, and assuming that a good starting matrix B_1 is used. This result has been extended by Xie (1991) for more general bases and for any starting matrix $B_1 > 0$. In this section we will show that if the correction terms are used in Algorithm II, the rate of convergence is 1-step Q superlinear. This result is possible by Update Criterion I and by the selected application of finite difference approximations, which allow BFGS updating to occur more frequently.

To establish superlinear convergence we need to ensure that the steplengths α_k have the value 1 for all large k. When a smooth merit function, such as Fletcher's differentiate function (Fletcher (1973)) is used, it is not difficult to show that near the solution unit steplengths give a sufficient reduction in the merit function and will be accepted. However the non-differentiable l_1 merit function (1.7) used in this paper may reject steplengths of one, even very close to the solution. This so-called "Maratos effect" requires that the algorithm be modified to allow unit steplengths and to achieve a fast rate of convergence. We will not consider this modification here, so as not to complicate the already lengthy analysis of this paper and since it does not affect the main structure of the algorithm or its essential properties. In the companion paper (Biegler, Nocedal and Schmid (1993)), which is devoted to a numerical investigation of Algorithm II, we describe how to incorporate the non-monotone line search (or watchdog technique) of Chamberlain et al (1982) that allows unit steplengths to be accepted for all large k. The analysis of the modified algorithm would be similar to that presented in §5.5 of Byrd and Nocedal (1991).

In the remainder of this section we assume that the iterates generated by Algorithm II converge R-linearly to a solution and that unit steplengths are taken for all large k. We begin by showing that the damping parameter ϕ_k, used in (3.43) to ensure that descent directions are always generated, has the value of 1 for all large k.

We have shown in Theorem 5.6 that $\|S_k^{-1}\|$ is bounded above. Also (5.21), (5.2) and (3.42) show that, when finite differences are used, $w_k = O(\|p_Y\|) = O(\|c_k\|)$ and by (5.43) we see that this is also the case when Broyden's method is used. Using these facts, and noting that $\|\cdot\| \leq \|\cdot\|_1$, we see that there is a constant C such that the left hand side of (3.38) can be bounded by

$$C_k[2\cos\theta_k\|g_{Z_k}\|\|w_k\| + w_k^T B_k^{-1} Z_k^T g_k + (w_k^T B_k^{-1} w_k) \leq \frac{1}{4}[C_k C(\|e_k\| + C_k\|c_k\|)]\|c_k\|_1$$

since g{Zk = O(||efc||). As the iterates converge to the solution, and since Qk ≤ 1> the term inside the square brackets is less than the constant p given in (3.38), showing that & = 1 for all large k. This, and the remarks made at the end of §5 show that all the safeguards included in Algorithm II become inactive asymptotically.

We can we show that the Broyden matrices satisfy the condition of Dennis and Moré (1974) for superlinear convergence. Note from Algorithm II that a Broyden update of Sk is always performed, regardless of whether a BFGS update of Bk takes place or **not.** The following result is a straightforward modification of a well-known property for Broyden's method.

Lemma 6.1 *Suppose that the iterates generated by Algorithm II converge R-linearly to a point x_m that satisfies Assumptions 5.1 and that the matrices Sk remain bounded. Then*

$$lim_{k\to\infty}\frac{H(s_{fc} - zjw.)d_k\|}{\|d_k\|} = 0. \qquad (6.1)$$

Proof. The proof is essentially given in Griewank (1986), and is also very similar to the analysis in Dennis and Schnabel (1983, pp. 183-4), but we will give it here for the sake of completeness. Using the Broyden formula (3.9) we have

$$\begin{aligned} Sk+i-ZjW. &= S_{fc} - Z_*^T W_* + \frac{(\bar{y}_k - S_k\bar{s}_k)\bar{s}_k^T}{\bar{s}_k^T\bar{s}_k} \\ &= c <*T_W , \frac{(ft - ZjW_m s_k)sl}{\$j.Sk} , \frac{(Z_*^T W_* - S_k)\bar{s}_k\bar{s}_k^T}{\bar{s}_k^T\bar{s}_k} \\ &= (S_k - ZjW.)(I - sksl/slh) + (ft - Z_*^T W_*\bar{s}_k)\bar{s}_k^T/\bar{s}_k^T\bar{s}_k. \end{aligned}$$

Defining Ek = S* - $Z?W*$, applying Lemma 8.2.5 of Dennis and Schnabel (1983), recalling (3.10)-(3.11) and using the Mean Value theorem, we obtain

$$\begin{aligned} \|E_{k+1}\|_F &\le \|E_k(I - \bar{s}_k\bar{s}_k^T/\bar{s}_k^T\bar{s}_k)\|_F + O(\sigma_k) \\ &\le \|E_k\|_F - \frac{\|E_k\bar{s}_k\|^2}{2\|E_k\|_F\|\bar{s}_k\|^2} + O(\sigma_k). \end{aligned}$$

Rearranging this expression yields

$$\frac{\|E_k\bar{s}_k\|^2}{\|\bar{s}_k\|^2} \le 2\|E_k\|_F\,[\|E_k\|_F - \text{HJEMIP} + 0(c_k)\|, \qquad (6.2)$$

and since the elements of S_k remain bounded we have for some A that for all $k \ge \bar{k}$, $\|E_k\| \le$ A/2 and

$$\sum_{k=\bar{k}}^{\infty} \|\bar{s}_k\|^2 \quad - \quad \cdots \quad \sum_{k=\bar{k}} O(\sigma_k)].$$

Since $\{<Jk\}$ converges R-linearly, the last term is summable, which implies that

$$\lim_{k\to\infty} \frac{\|H\wedge * \|^a}{\ldots} = U.$$

Noting that $Sk = <*kdk$ gives the desired result.

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This lemma shows that Sk is an accurate approximation to ZjW_m along d*, and Theorem 5.6 shows that, when updating takes place, Bk is an accurate approximation to $Zj\backslash V_m Z^*$ along p_Y. We will make use of these two facts and the following lemma, which is an application of the well-known result of Boggs, Tolle and Wang (1982).

Lemma 6.2 *Suppose that the iterates generated by Algorithm II converge R-linearly to a point x_m that satisfies Assumptions 5.1, and that $ctk = 1$ for all large k. If, in addition*

$$\lim_{k\to\infty} \frac{\|B_{kPz} + w_k \, T \, Z?W.d_k\|}{\|d_k\|}, \tag{6.3}$$

then the rate of convergence is 1-step Q-superlinear.

Proof. Nocedal and Overtoil (1985, Theorem 3.2) show that if an algorithm **of the form**

(6.4)

$$x_{k+1} = Xk + d^*,$$

converges to a point x_m that satisfies Assumptions 5.1, and if

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$$\lim_{k\to\infty} \frac{\|(\tilde{S}_k - zjw.)d_k\|}{\|d_k\|} = 0,$$

then the rate of convergence is superlinear. Algorithm II clearly satisfies the second equation in (6.4), $Ajdk = —c\wedge$. Now since $dk = YkP\backslash + ZkPz$ we have

$$[Y_k \; Z_k]{\sim}U_k = | \pounds \; J, \tag{6.6}$$

and let us write $Wk = Tkp_Y$ for some matrix 7^*. Then recalling that $(k = 1$ **for all large** fc, we have from (3.43) that

$$[T_k \; B_k][Y_k \; Z_k]^{-1} d_k = -Z_k^T g_k.$$

Thus we can define $\tilde{S}k = [Tk\ Bk][Yk\ Zk]^{-1}$, and the condition (6.5) for superlinear convergence is

$$\lim_{k\to\infty} \frac{\|([T_k Bk])'kZ_k]-*-ZTW_m)d_k\|}{\|d_k\|} = 0.$$

However, using (6.6) and $w_k = T_k p_Y$ we have that $[T_k\ B_k][Y_k\ Z_k]\sim^l d_k = T_k p_Y + B_k p_z =$ wjt + $B_k pz<$, giving the desired result.

a

We can now prove the final result of this section. The analysis is complicated by the fact that BFGS updating may not always take place, and by the fact that the correction terms are sometimes computed by finite differences and sometimes by Broyden's method. We therefore consider the following three sets of iterates, based on Update Criterion I and illustrated in Figure 2.

- $R_1 = \{j \mid \|p_Y^{(j)}\| \le \gamma_j^2 \|p_Z^{(j)}\|\}$,
- $R_2 = \{j \notin R_1 \mid \|p_Y^{(j)}\| \le \|p_Z^{(j)}\|/\sigma_j^{1/2}\}$,
- $R_3 = \{j \mid \|p_Y^{(j)}\| > \|p_Z^{(j)}\|/\sigma_j^{1/2}\}$,

and note that both $\sim f_k$ and a_k are summable.

Theorem 6.3 *Suppose that the iterates generated by Algorithm II converge R-linearly to a point x_m that satisfies Assumptions 5.1, and that $a_k = 1$ for all large k. Then the rate of convergence is 1-step Q-superlinear.*

Proof. Since $d_k = Y_k p_Y + Z_k p_z$ we have

$$\begin{bmatrix} p_Y \\ p_Z \end{bmatrix} = [Y_k\ Z_k]^{-1} d_k.$$

Therefore assumption (5.3) implies that

$$\text{llpvl!} = O(\|d_{fc}\|), \qquad \text{HPZII} = 0(\|<y). \tag{6.7}$$

Now

$$\begin{aligned} \|B_k p_Z + w_k - Z_*^T W_* d_k\| &\le \|B_k p_Z - Z_*^T W_* Z_k p_Z\| + \|w_k - Z_*^T W_* Y_k p_Y\| \\ &\le \backslash\backslash B_{kPz} - ZjW_m Z_* p_Z\| + \|w_k - Z_*^T W_* Y_k p_Y\| \\ &\quad +0(\backslash\backslash e_k\backslash\backslash\backslash\backslash_{Pz}\backslash\backslash). \end{aligned}$$

Since by (6.7) the last term is of order $o(\|p_z\|) = o(\|<\{*\|)$, the objective of the proof is to show that

$$11{*}{*}1{*} - ZjW.Z_{mPz}\backslash\backslash + |K - ZTW_m Y_{kPy}\backslash\backslash = o(\|d_{fc}\|), \tag{6.8}$$

for this together with (6.3) will give the desired result. We consider the three regions $R\backslash, Ri$ and $R\%$ separately. Algorithm II is designed so that in 7?2, to_k must be computed by finite differences. On the other hand since p_z is recomputed in step 7, after which we can be in any of the three regions, we see that in $R\backslash$ and #3 Wk may be computed by finite differences or by Broyden.

If A: € R_u we have that $||p_Y|| = o(||p_z||) = o(||4||)$• We also know from (5.43) that $Wk = O(||PY||)$ when the correction is computed by Broyden's method, and by (5.21) this relation also holds when Wk is computed by finite differences. Therefore for k € Ru

$$\|w_k - Z_*^T W_* Y_k p_Y\| = o(\|d_k\|). \tag{6.9}$$

Furthermore, since updating always takes place in Ru (5.26) holds i.e.

$$\backslash\backslash B_{kPz} - Z?W.Z_m p_z\backslash\backslash = o(||4||). \tag{6.10}$$

We have thus established (6.8) for all k ***e*** $R\backslash$.

Let us now suppose that k 6 #2* in which case Wk is computed by finite differences. Using (5.22) we have that

$$|K - ZjW.YkPvW = o(\backslash\backslash p_Y\backslash\backslash) = od|*||)_f \tag{6.11}$$

where the last step follows from (6.7). Since updating always takes place in #2* equation (6.10) also holds in this case, and we conclude that (6.8) holds for all k € /?2-

Finally we consider the case when k 6 $R3$. Now p_z satisfies

$$p_z = o(\|p_Y\|) = o(\|d_k\|). \tag{6.12}$$

If *k* € *Rz* and the correction term w_k is computed by Broyden's **method as** $Wk = S_k Y_k p_Y$ **(see (3.12)) we have**

$$\begin{aligned} \backslash\backslash w_k\text{-}ZjW.Y_k p^*\backslash\backslash &= \|(S_k - Z_*^T W_*) Y_k p_Y\| \\ &\leq \|(S_k - Z_*^T W_*) d_k\| + \|(S_k - Z_*^T W_*) Z_k p_z\|. \end{aligned}$$

Using (6.1), (6.12) and the boundedness of Sk we see that the right hand side is of order $o(||*||)$, so that (6.11) holds. On the other hand, if w_k is computed by finite differences, **we have directly from** (5.22) that (6.11) holds. In addition (6.12) and **the boundedness of *Bk* shows that (6.10) holds for** all A: € $R3$, regardless **of whether** finite **differences or** Broyden's method are used.

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7. Final Remarks

We have presented a new reduced Hessian algorithm for large scale equality constrained optimization. The motivation for this work has been practical: an earlier reduced Hessian code of ours, designed for large problems, was often subject to instabilities, and we have aimed to develop a more robust algorithm that resembles the full-space SQP method, but is less expensive to implement. In a forthcoming paper (Biegler, Nocedal and Schmid (1993)) we discuss our computational experience with the new method. That paper describes how to handle inequality constraints and discusses numerous important details of implementation not considered here. These include the choices of all constants and tolerances, the strategy for coping with the case when the basis matrix C in (2.1) changes, and the procedure for computing the damping parameter Cfc, which was only outlined in (3.39). We also discuss in that paper how to apply the updating criterion away from the solution. We believe that the new algorithm can be very useful for solving large problems, especially those with few degrees of freedom.

We have only focused on convergence results that helped us in the design of the algorithm and that revealed its main properties. The analysis was complicated by two factors. We did not assume that the BFGS matrices B_k or the Broyden matrices S_k were bounded, which required careful consideration of their behavior. This analysis paid off by suggesting safeguards that are useful in practice and ensure a superlinear rate of convergence. The other complicating factor was the fact that the frequency of BFGS updating can vary drastically: it can take place at every iteration, never, or in various patterns. As was found earlier by Xie (1991), it is necessary to develop the theory in sufficient generality to cover all of these cases, and this significantly increased the complexity of some of the results.

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8. *

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