The Complex Range of Gain
Falls Mainly In the s Plane

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Abstract

This technical report promotes graphically based met\wdsfor determining the gain margin and phase margin of linear time-invariant single-input, single-output control systems. The gain margin, or amount of gain that can be increased before the closed-loop system becomes unstable, can be determined from a graph showing the angle of each closed-loop system eigenvalue in the complex plane as a logarithmic function of real gain. By identifying the gain interval for which all eigenvalues have angles within the stable region, the gain margin can be calculated. At any constant real gain, the phase margin, or range of phase angle corresponding to a stable closed-loop system, can be determined from a graph of the angle of each closed-loop system eigenvalue in the complex plane as a function of gain angle. The proposed methods do not require frequency calculations, and serve as alternate means for stability-robustness studies. Furthermore, the phase margin determination highlights the importance of root sensitivity, with the practical design guideline of not selecting control gains near break-points.

Classical Presentation of Relative Stability

In the design of control systems one is interested in the relative stability of a system as well as its absolute stability. Although as Bellman has stated "there is no stability to the definition of stability," a system can be considered as being absolutely stable if a transient oscillation decays and ultimately vanishes. A system on the bonier of absolute instability are prone to oscillations that continue for a long time. The overshoot and settling time for step inputs may be excessive degrading the performance of the system. Furthermore, systems operating near marginal stability may be driven to instability from sensor noise, disturbances, and modeling errors. Thus, a system has to be relatively as well as absolutely stable in practice, making robustness a paramount design consideration.

In the time domain, relative stability of a linear time-invariant (LTI), single-input, single-output (SISO) control system is measured by parameters such as the maximum overshoot and the settling time. In the frequency domain, the resonance peak can be used to indicate relative stability. Alternatively, relative stability can be determined by means of the Nyquist plot of the loop transfer function g(s). The proximity of the Nyquist contour (g(jco) polar plot) to the critical point, (-1, jO), yields an indication the closed-loop system's degree of stability.

A typical Nyquist plot or open-loop frequency locus for a minimum phase transfer function g(s) is shown in Figure 1. (It is assumed that g(jco) is a minimum-phase transfer function, so that the portion of the Nyquist contour that corresponds to s=jco, 0<co<oo, is sufficient for stability analysis.) Application of the Nyquist encirclement test shows that the closed-loop system is absolutely stable. If the loop gain is low, the Nyquist plot of g(jco) intersects the negative real axis at a point that is quite far to the right of the critical point. As the gain is increased, the intersection point of the Nyquist contour and the negative real axis moves closer to the critical point, eventually passing through it (corresponding to marginal stability) and then encircling it (indicating instability).

The gain margin is used to quantify the distance between the Nyquist contour intersection of the real axis and the critical point. In general, if the intersection occurs at a distance lg(jo)p)l from the origin, then multiplying the gain by a factor l/lg(jo)p)l produces instability. The factor l/lg(jo>p)l is the gain margin, and the special frequency, Op, is the phase crossover frequency, *i.e.,* the frequency at the phase-crossover where Zg(jo>p) = 180*.. In control engineering, it is more common to express this factor in decibels (dB) with a positive gain margin indicating a stable system. The gain margin GM in decibels is given by

$$GM = 20log \left(\frac{1}{|g(j\omega)p)|}\right) = -20log |g(j\omega_p)|$$

(1)

Thus, **the gain margin** is the number of decibels by which the magnitude of the open-loop **frequency response falls short** of unity when the phase angle is 180*.
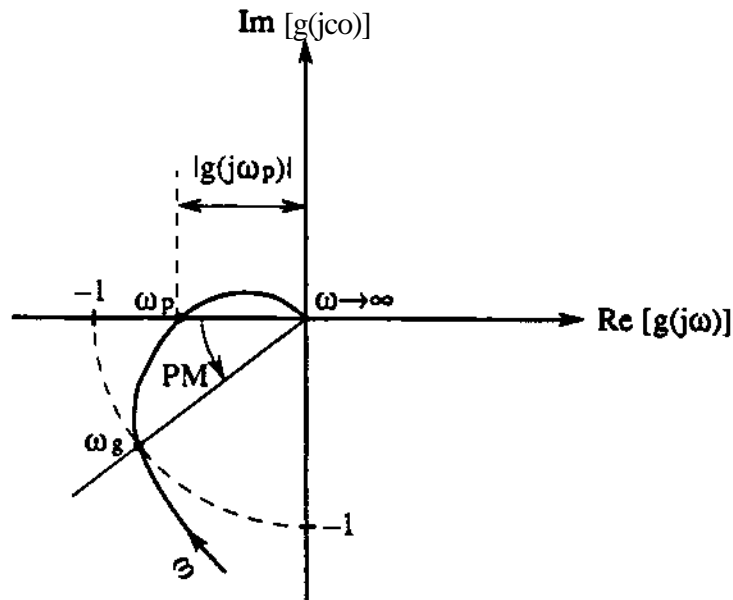

Figure 1. Nyquist Diagram for Calculation of Gain and Phase Margin.

Since the gain margin is the factor by which the gain can be multiplied before the closed-loop system becomes unstable, it follows that it can also be determined from the root locus. The gain margin can be expressed as

$$GM = 20 log \left(\frac{k^*}{k}\right)$$

(2)

where k* is **the** gain corresponding to marginal stability, ż.e., the gain at the crossing of the jco axis in the root locus plot. If the root locus does not cross the stability boundary for any gain, the gain margin is infinite.

The phase margin is also a measure of relative stability. It is the angle by which the phase of the open-loop frequency response falls short of -180° when the magnitude is unity. Thus, the phase margin, denoted as PM in Figure 1, is the additional phase lag required to just make the system unstable. The phase margin is the phase at the frequency, O)g, the gain crossover frequency, where the magnitude or "gain" of g(jco) is unity (0 dB).

A positive phase margin indicates **a stable system.** **(In Figure** 1, the phase margin is **measured clockwise** from the negative real axis, with a positive phase margin denoting a **stable system.)**

The phase and gain margins can be viewed as safety factors in the design specifications. A useful rule-of-thumb that is generally applicable to control systems is **that** for adequate closed-loop stability the gain margin should be greater than 6 dB and the **phase** margin should be between $30^{\#}$ and $60^{\#}$ (Ogata, 1990). (The 6 dB limit corresponds to **the** quarter amplitude decay response obtained with the gain settings given by the Ziegler-Nichols ultimate-cycle method (Palm, 1986).) Some control engineers offer more restrictive measures, suggesting GM $\geq$ 8 dB and PM £ $40^{\#}$ or even 50*. These values should be viewed as rough, albeit often useful, working guides. In general, it is not desirable to make the margins too large since this corresponds to low gain systems yielding sluggish designs that may result in large steady-state errors (Palm, 1986).

In this report, the concepts of gain margin and phase margin are interpreted using an alternate paradigm, namely the feedback block diagram of Figure 2 where the forward gain is given by k=lklei^{zk}. The gain margin corresponds to the range Ikl can be adjusted, assuming Zk=0, for the closed-loop system to be stable. Similarly, the phase margin corresponds to the range that Zk can be adjusted for a given Ikl such that the closed-loop system is stable. This perspective does not involve the calculation of the gain or phase crossover frequencies, nor does it require Nyquist or Bode plots for illustrating the gain and phase margins.
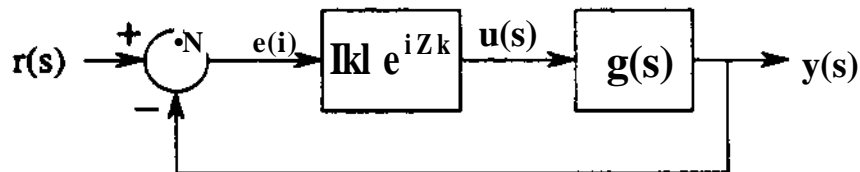

Figure 2. Feedback Block Diagram with Forward Complex Gain.

The gain k in equation (2) is related to the gain margin under the assumption that the gain is real. (In any physical system the gain is real.) An advantage of employing equation (2) is that it provides a means to determine the gain margin from relations linking the gain and measures of stability. The most popular graphically-based tool employing gain is the

Evans root locus, in which gain is an implicit **variable. To** exploit **the** relation of equation (2) **we seek a graphical** tool that expresses the information of the root locus in conjunction with **the gain continuum** (as opposed to discrete tick marks representing gain on the root locus). **Since a key** assumption in root locus theory is that the gain is real, root locus analysis is limited to calculation of gain margin and not phase margin. By generalizing the forward gain to be a complex quantity with magnitude and phase angle, it is possible to generate **a** graphical tool to determine phase margin. This report promotes the use of graphically-based tools for gain margin and phase margin determination.

Gain Margin from Angle-Gain Plot

An alternative graphical representation of the standard root locus plot is to present the magnitude and angle (phase) of the closed-loop system eigenvalues in separate graphs that show the explicit dependency of the forward real gain k=lkleJ^{z0}=lkl. These plots, called the magnitude and angle gain plots, respectively, have been proposed (Kurfess and Nagurka, 1991a, 1991b) as a useful pair of plots for control system analysis and design. In fact, they follow from a natural progression of perspectives of the standard root locus in an analogous fashion that the Bode plots are an alternate representation of the Nyquist diagram. By directly exposing the influence of gain magnitude on the system eigenvalues, the gain plots enable the designer to select values of gain corresponding to stable behavior that meet desired performance specifications, such as achieving the natural frequency and damping ratio of interest. From the angle-gain plot, the range of gains for which the closed-loop system is stable can be determined by inspection. The magnitude-gain plot is also an important design aid since the slopes of the loci are related to root sensitivity magnitudes (Kurfess and Nagurka, 1991c). Again, by inspection, the designer can select gains corresponding to robust operating regimes.

The expression given by equation (2) is especially well suited for determination of gain margin from the angle-gain plot. In particular, k*, the gain corresponding to marginal stability, can be determined directly from the angle-gain plot by noting the angle of any eigenvalue that maps to a location on the imaginary axis, *i.e.,* Zs = ±90*. Thus, given a design value of magnitude k, equation (2) can be used to calculate the gain margin. From equation (2),

$$GM = -20 \log |k| + 20 \log |k^*| \tag{3}$$

which suggests that the gain margin is logarithmically proportional to Ikl, as shown in Figure 3. The slope of the line is -20 dB/gain and the intercept is GM = 20 log Ik*^{l} dB at lkl=l and GM=0 dB at lkl=lk*$^!$.

Example 1

The opciv-loop transfer function of this example is given by

$$g_1(s) = \frac{1}{V + \ll r}$$

(4)

It is embedded in a closed loop system of Figure 2 with k=lkleJ°. The root locus is shown in Figure 4. As the gain is increased the real eigenvalue moves deeper in the left half plane along the real axis whereas the complex conjugate pair of eigenvalues crosses the imaginary axis and enters the right half plane. This behavior is readily observable in the associated set of gain plots, namely the magnitude and angle gain plots of Figure 5a,b, respectively. By inspection of the angle-gain plot, marginal stability is reached at k=k*=128. From equation (2), the gain margin GM = -20 log k + 42.1 dB. At k=100 the GM = 2.1 dB indicating that the closed-loop system with unity forward gain can be increased 2.1 dB before the stability margin is reached. At k=200 the GM = -3.88 dB, *i.e.,* the gain must be decreased 3.88 dB for stability to be reached. The results for several gain magnitudes are summarized in Table 1. Also shown is the phase crossover frequency, calculated from a frequency analysis. In the proposed approach there is no need to compute this intermediate frequency.
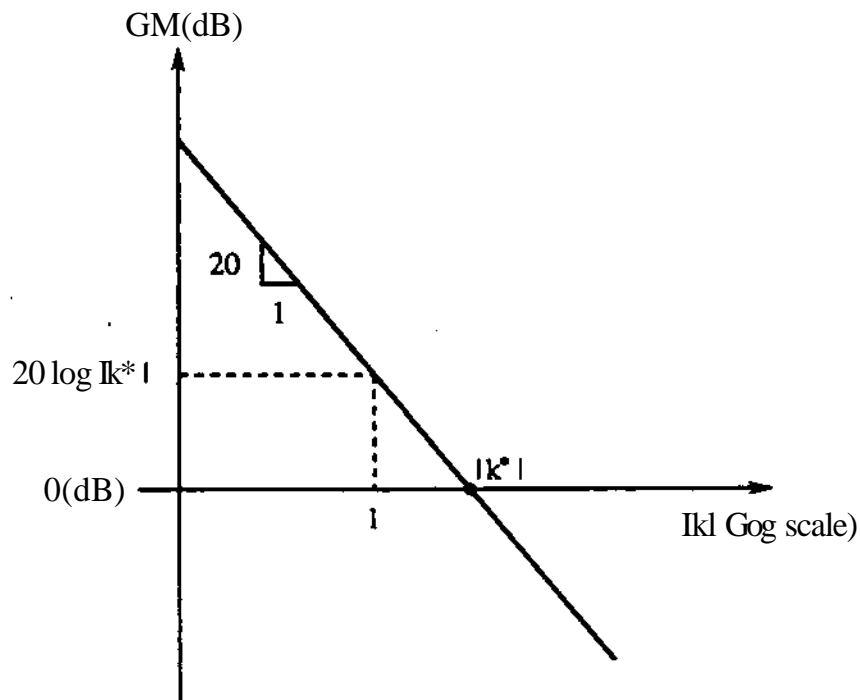
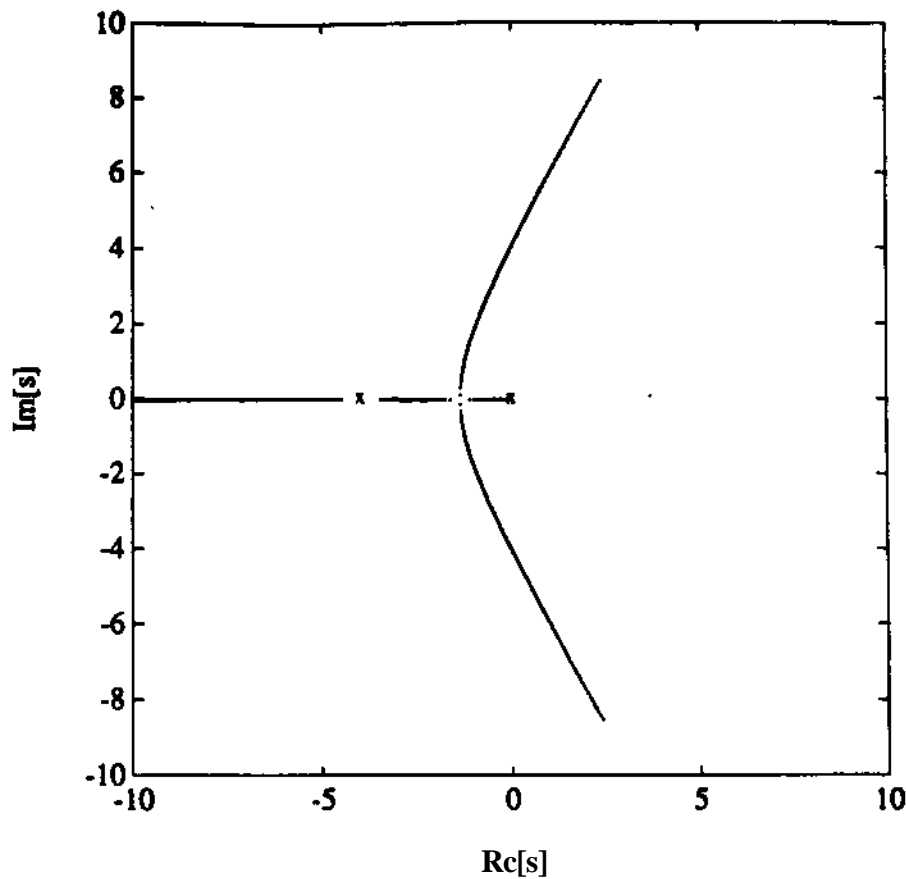Figure 3. Gain Margin *vs* Gain Magnitude Showing Logarithmic Relation.

Figure 4. Root Locus Plot of Example 1.

Phase Margin from Angle-Angle Plot

By relaxing the constraint that the gain be purely real, it is possible to graphically depict the phase margin in a plot showing the angle of each closed-loop eigenvalue *vs.* the angle of the gain. The perspective of viewing the forward gain as a complex quantity is counter to the traditional presentation. It can result in counter-intuitive behavior, *e.g.,* the possibility of generating complex system eigenvalues that do not occur as complex conjugate quantities.

For a complex forward gain of given magnitude, it is possible to compute and display **the root** loci showing the closed-loop eigenvalue trajectories in the complex plane as implicit functions of the gain angle. However, sketching rules are not available and there is limited, if any, useful information for the designer. An alternative graphical tool is to depict **the** angle of each closed-loop system eigenvalue vs. the angle of the gain, Zk, for a given real gain Ikl. We have called this graph the angle-angle plot. The phase margin can be determined by inspection of this graph by identifying the smallest angle of k for which any eigenvalue crosses the instability boundary, *i.e.,* ±90* in the complex plane.
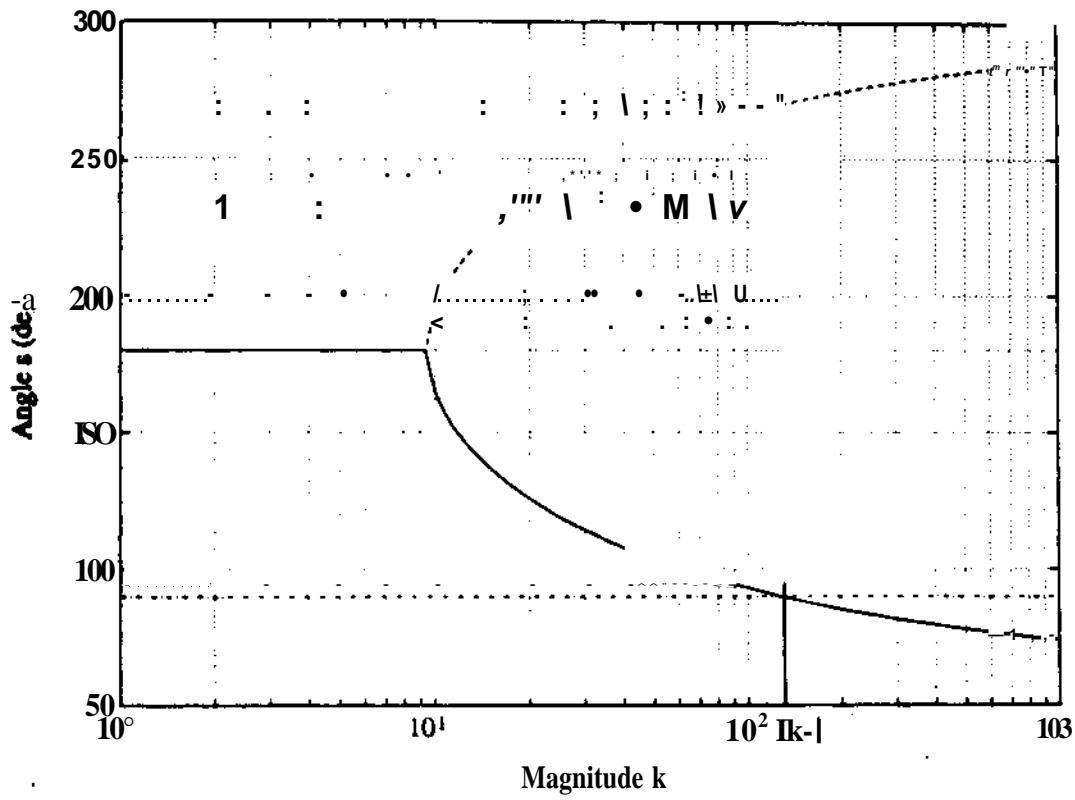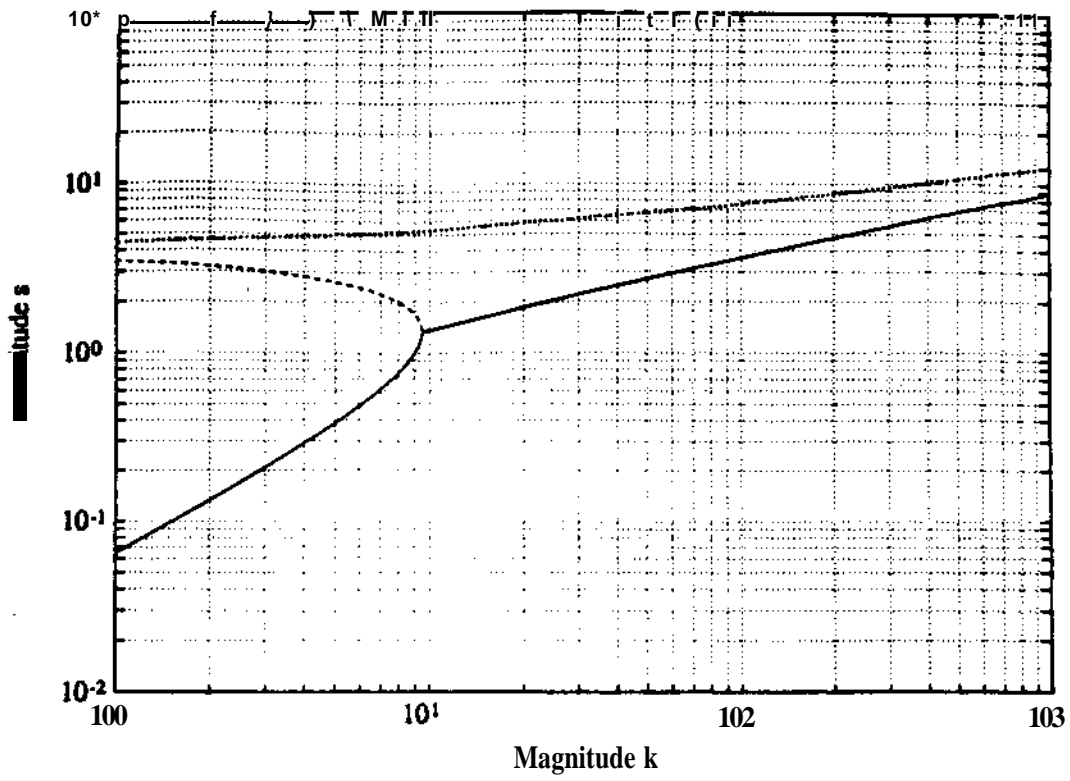
Figure 5. (a) Magnitude Gain Plot and (b) Angle Gain Plot of Example 1.

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An angle-angle plot can be generated for a given value of the gain magnitude. This is analogous to the angle-gain plot which corresponds to a single value of the gain angle, namely $Zk=O$. A popular example of a zero angle variational magnitude analysis is the Evans root locus plot where Ikl only is varied. Thus, it is possible to produce a family of angle-angle plots for different values of Ikl. Once Ikl is chosen, a phase analysis is quite important since gain margin analysis alone does not suffice for determining stability and robustness (Kuo, 1991).

The angle-angle plot presents phase margin information directly. The phase margin is read from the plot as the Zk that forces any of the eigenvalues to cross the $\pm90"$ boundaries. The phase margin is available without the use of frequency domain information, *i.e.,* there is no need to compute the 0 dB crossover frequency for the forward loop transmission. In addition to phase margin, the angle-angle plot shows phase margin sensitivity. The sensitivity is observed from the slopes of the curves. Large derivatives indicate that the phase margin is sensitive to angle variations. Sensitivity information is important when considering augmenting the system with other systems such as low pass filters, or when including modeling errors into the control design (Doyle and Stein, 1981).

Example 1 Revisited

We again consider the system given by equation (4). For the phase margin analysis we are especially interested in $Ikl = 9.48$ corresponding to the break-point of the eigenvalues on the root locus, and gain magnitudes near it *(e.g.,* $Ikl = 9$ and 10). The break-point gain has been isolated to illustrate the corresponding significant changes in the phase margin sensitivity. Table 1 shows the results, and includes an entry for $0)g$, the 0 dB gain crossover frequency obtained via standard Bode plot techniques.

Figures 6a,b,and c are the angle-angle plots for the cases $Ikl = 9$, 9.48, and 10, respectively. From the figures, the phase margin is the angle of k that causes an eigenvalue to cross the $90^{\#}$ line. An interesting attribute is the linear asymptotic behavior of the curves as Zk increases. A second intriguing feature is the large slope of the curve near $Z k=0^{\#}$ for the angle-angle plot for the case $Ikl = 9.48$. This phenomenon is expected since the sensitivity of the system is infinite at the break-point. Because of the infinite sensitivity, gains placing eigenvalues near the break-points should be avoided when designing control systems. As demonstrated in the example, choosing a gain that is slightly different from the break-point gain, can significantly reduce the sensitivity of the phase margin.

Table 1. Results of Gain and Phase Margin Analyses for Example 1.

Ikl	$|k|/|k^*|$	GM (dB)	PM (deg)	(Op (rad/s)	ci)$_g$ (rad/s)
1.00	0.00781	42.14	88.21*	4.0000	0.0625
9.00	0.0703	23.06	74.29*	4.0000	0.5220
9.48	0.0741	22.61	73.49*	4.0000	0.5803
10.0	0.0781	22.14	72.63*	4.0000	0.6108
100	0.781	2.14	7.29*	4.0000	3.5212
128	1.00	0.00	0.00*	4.0000	4.0000
200	1.56	-3.87	-12.06*	4.0000	4.9445
1000	7.81	-17.86	-44.19*	4.0000	9.4672
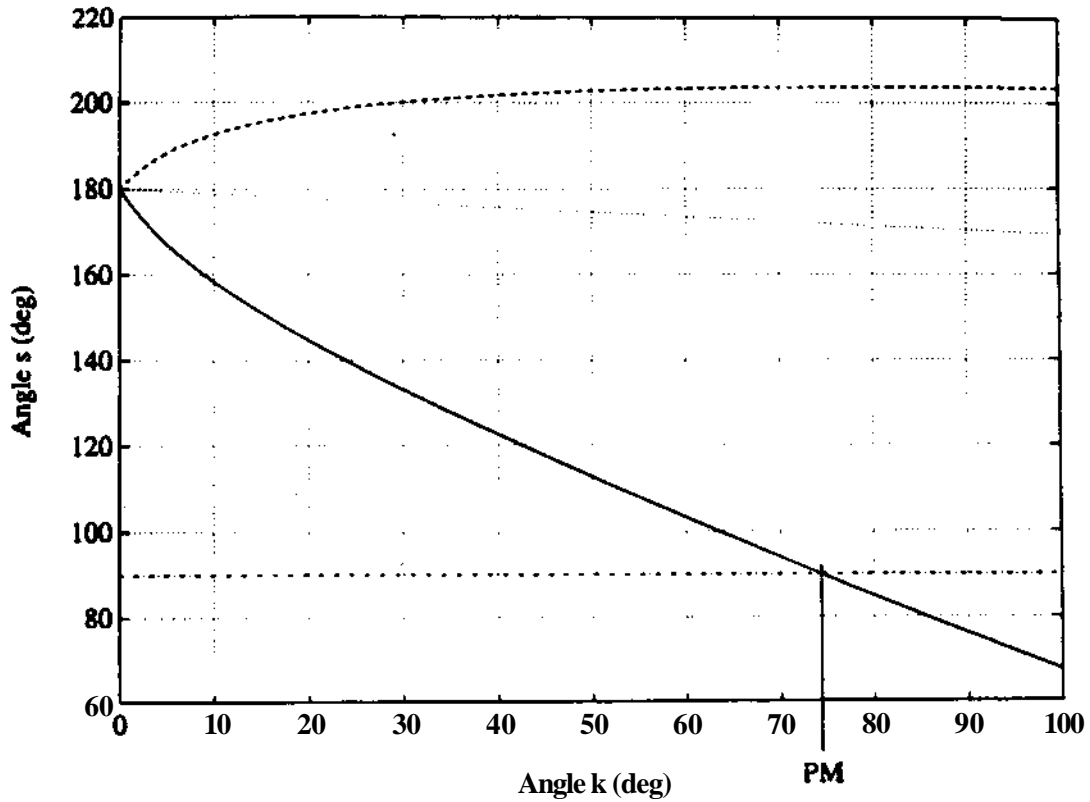
Figure 6a. Angle Angle Plot of Example 1 for Ikl = 9.
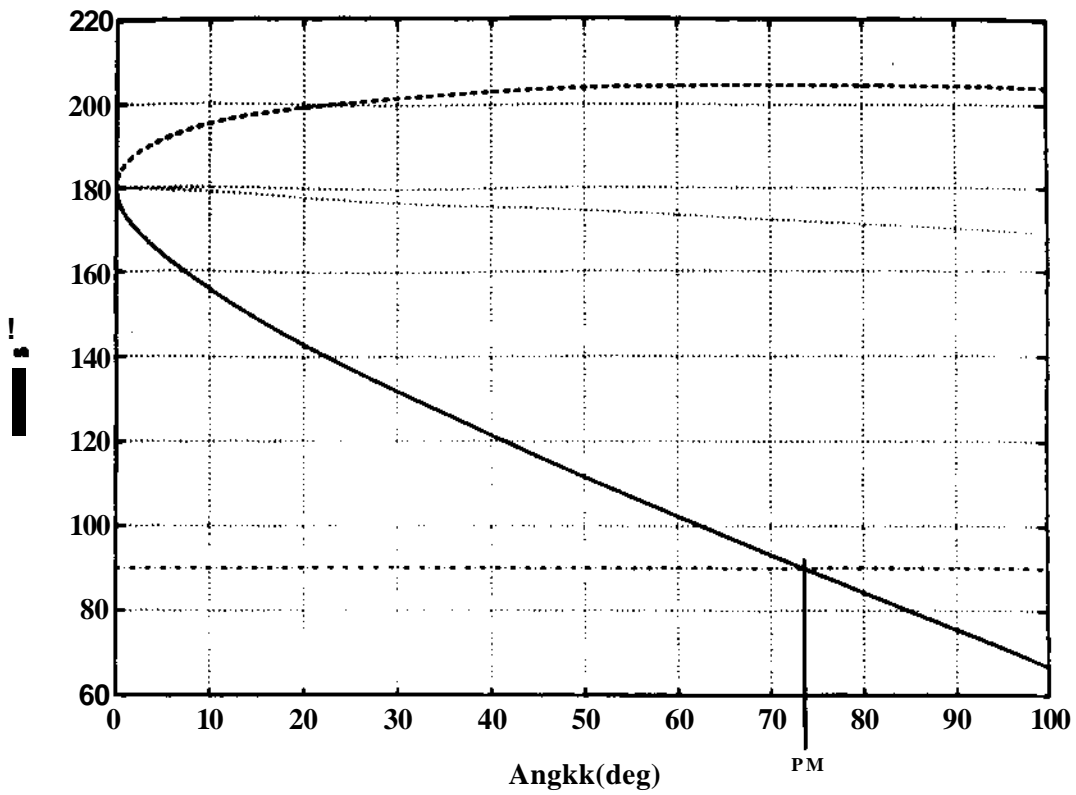
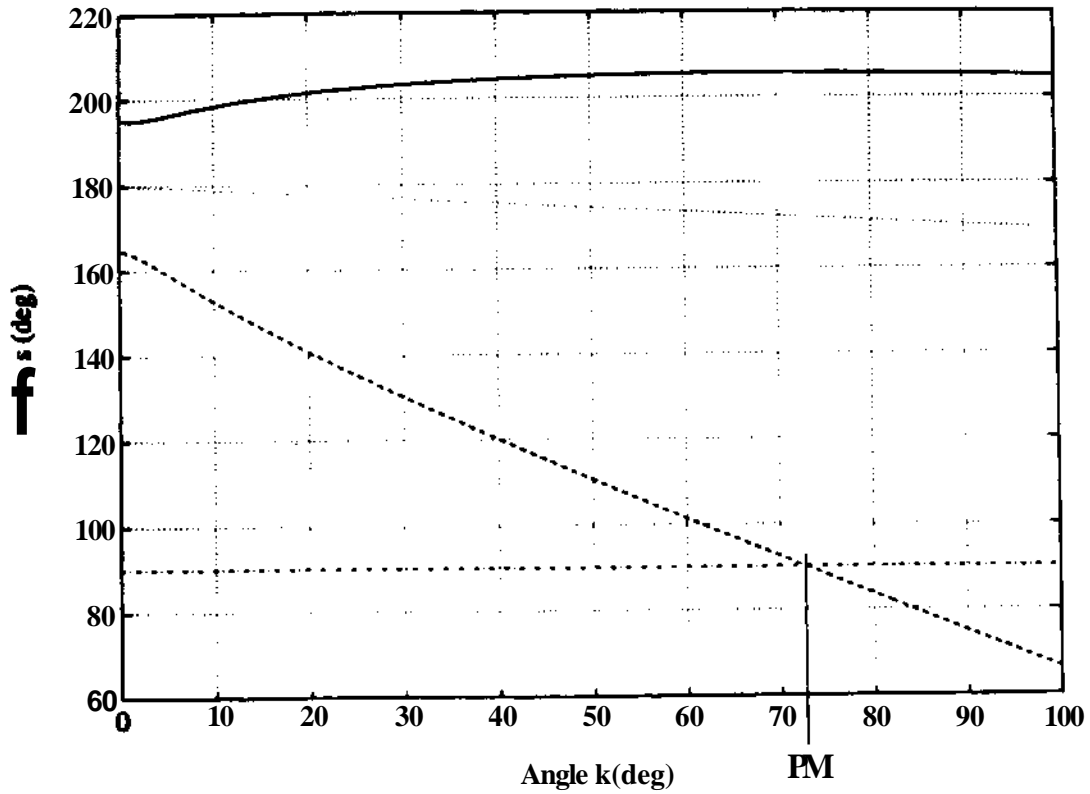**Figure 6b. Angle Angle Plot of Example 1 for |k| = 9.48.**

Figure 6c. Angle Angle Plot of Example 1 for |k| = 10.

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Example 2

This **example** (adapted from Cannon, 1967) considers a third order system with a transmission zero employing integral control.

$$g_2(s) = \frac{s+1}{s\,(s-1)(s+10)^2} \tag{5}$$

This system is open-loop unstable. The root locus plot of Figure 7 shows that there is a regime for which the closed-loop system is stable. The corresponding gain plots are shown in Figures 8a,b. It is possible to determine the stable range as $132 < k < 1227$ directly from the angle gain plot. From equation (2) the gain margin can be computed at any design value of k. For example, at k = 500, the lower gain margin is 20log(132/500) * **—11.5** dB and the upper gain margin is 20log(1227/500) * 7.8 dB. The implication is that the gain can vary by —11.5 dB and +7.8 dB about 500 before instability occurs. The phase margin can be determined by observation from the angle-angle plot, shown in Figure 9a as 17.5" for |k|=500 and expanded in Figure 9b.

Figures 10a,b are the traditional Bode magnitude and phase plots, respectively, for the system at k=500. The phase and gain margins are marked in both of these plots- The difficulty with computing these quantities via the Bode plots is the necessity to determine the gain cross-over frequency, $\omega_g \approx 4.24$ rad/s, the lower phase cross-over frequency, ω_{pi} * 1.3 rad/s, and the upper phase cross-over frequency $\omega_{pU} \ll 7.7$ rad/s. These frequencies, may be considered intermediate or indirect variables in the calculations and are not necessary when the analyses are conducted in the *gain domain.* Furthermore, the determination of the gain margin from the Bode magnitude plot may be confusing as the upper gain margin is determined from a measurement below that of the lower gain margin.
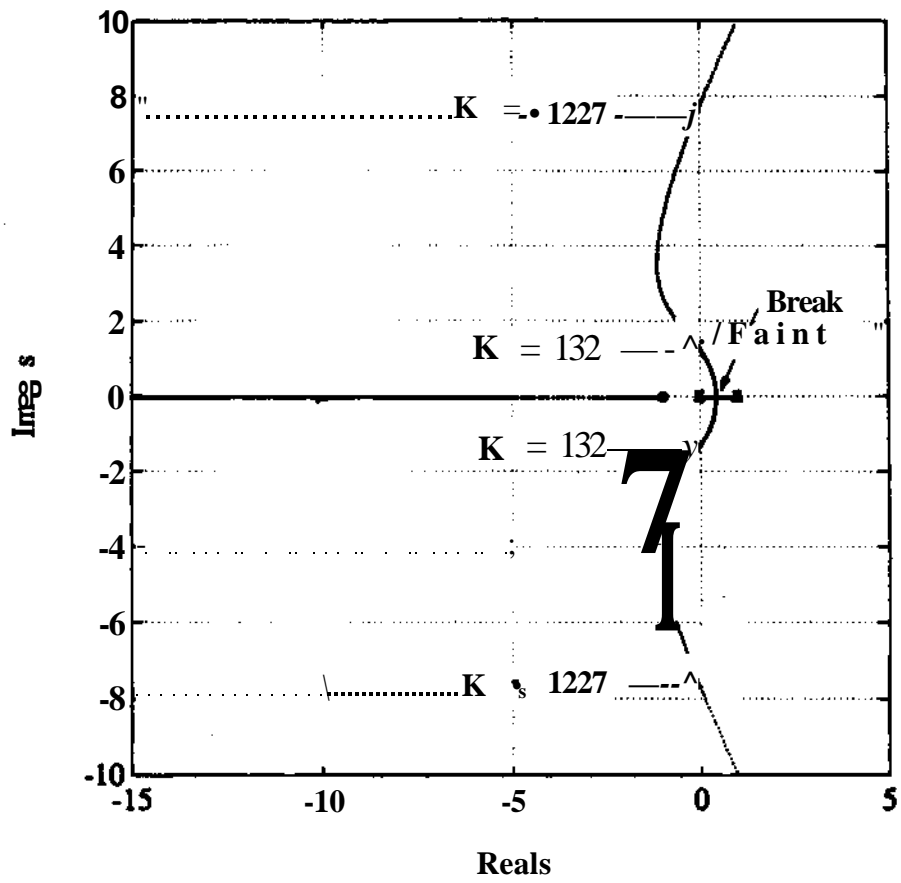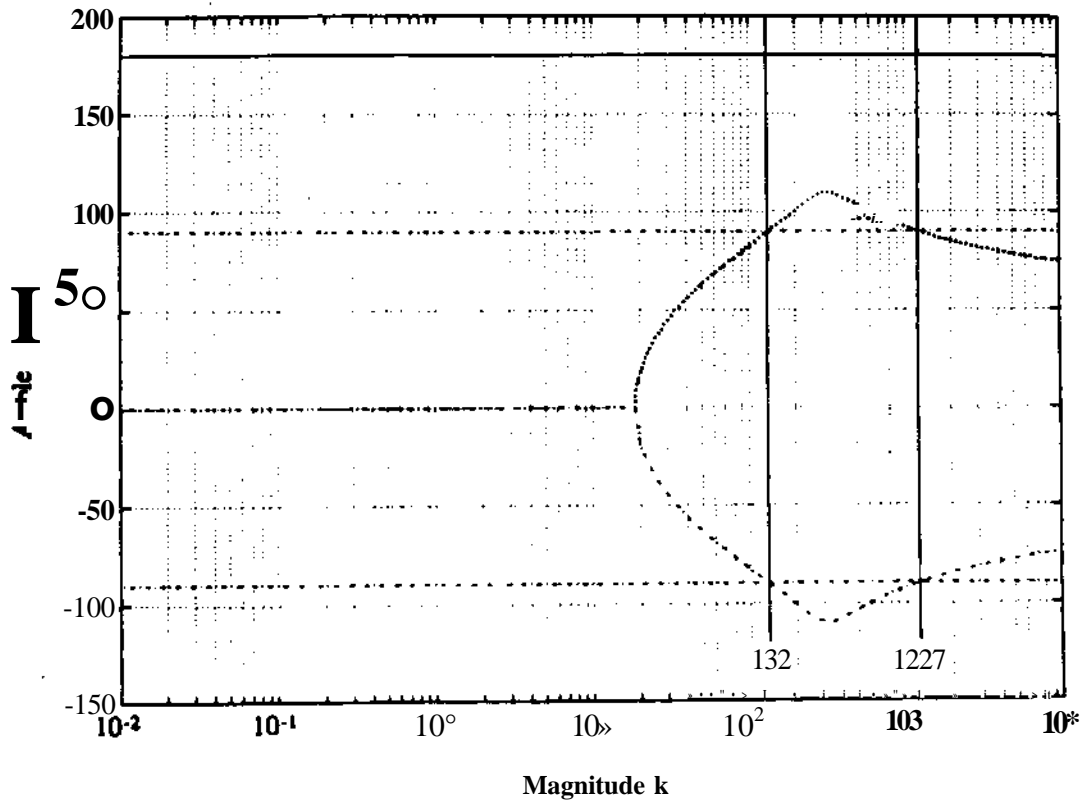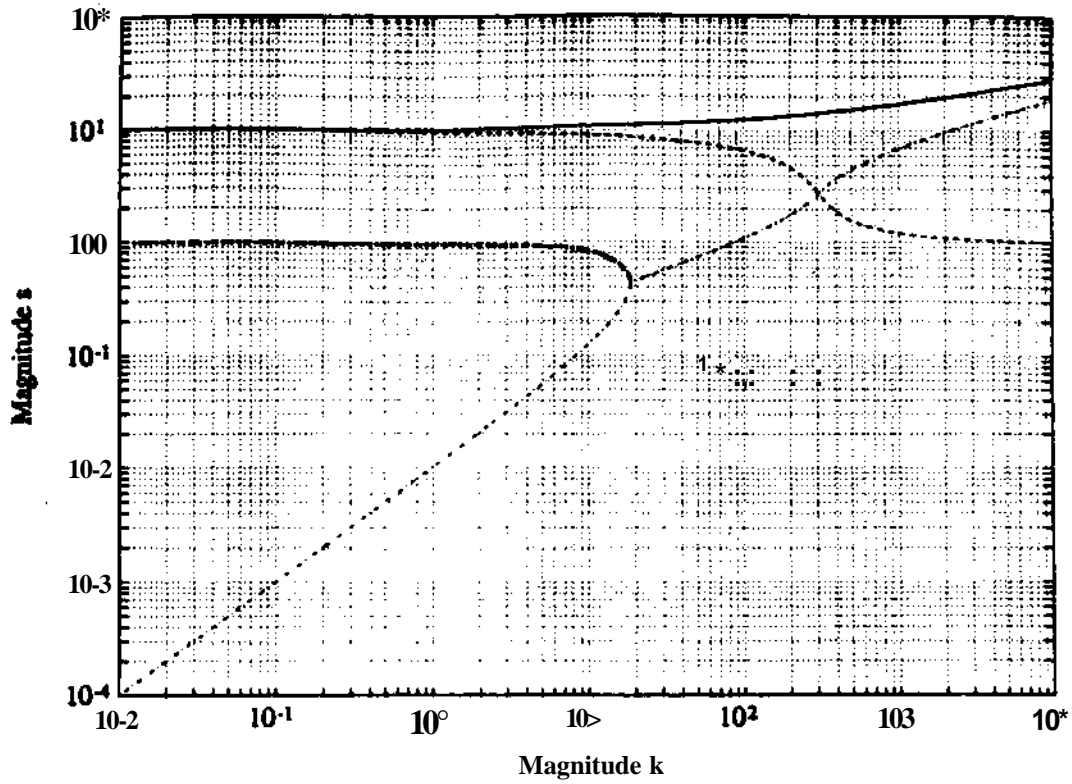
Figure 7. Root Locus Plot of Example 2.

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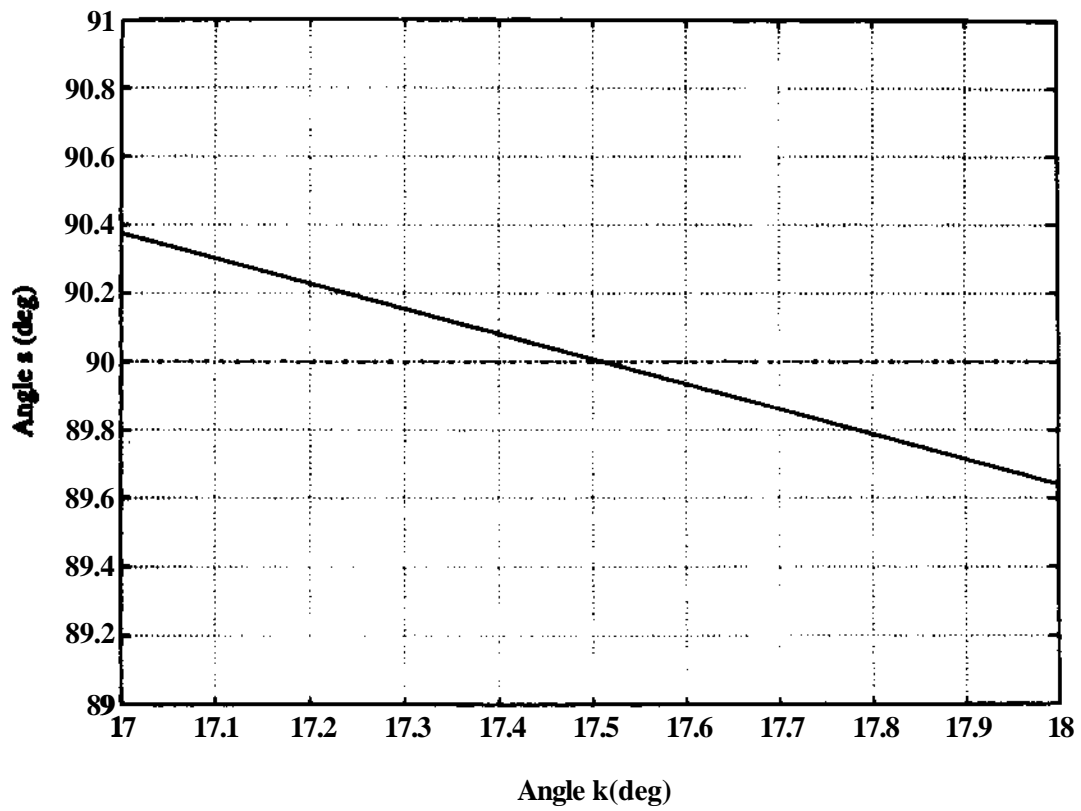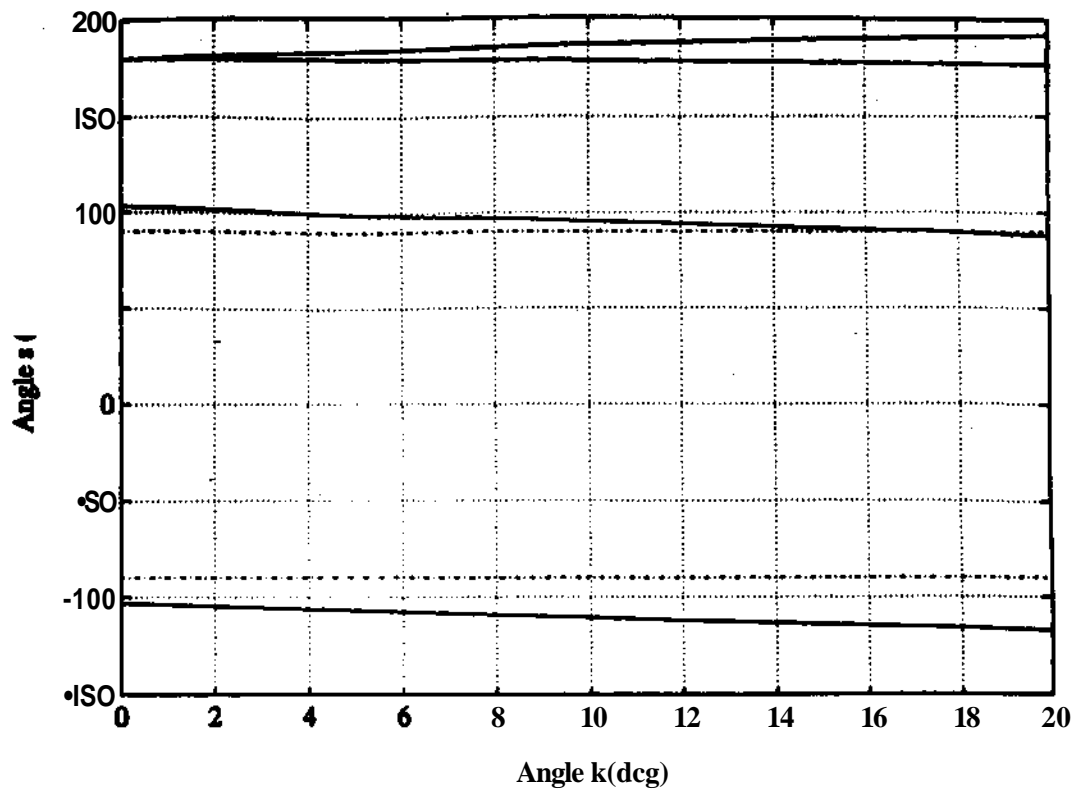**Figure 8. (a) Magnitude Gain Plot and (b) Angle Gain Plot of Example 2.**

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Figure 9. (a) Angle Angle Plot of Example 2 for lkl = 500, (b) Expanded View.
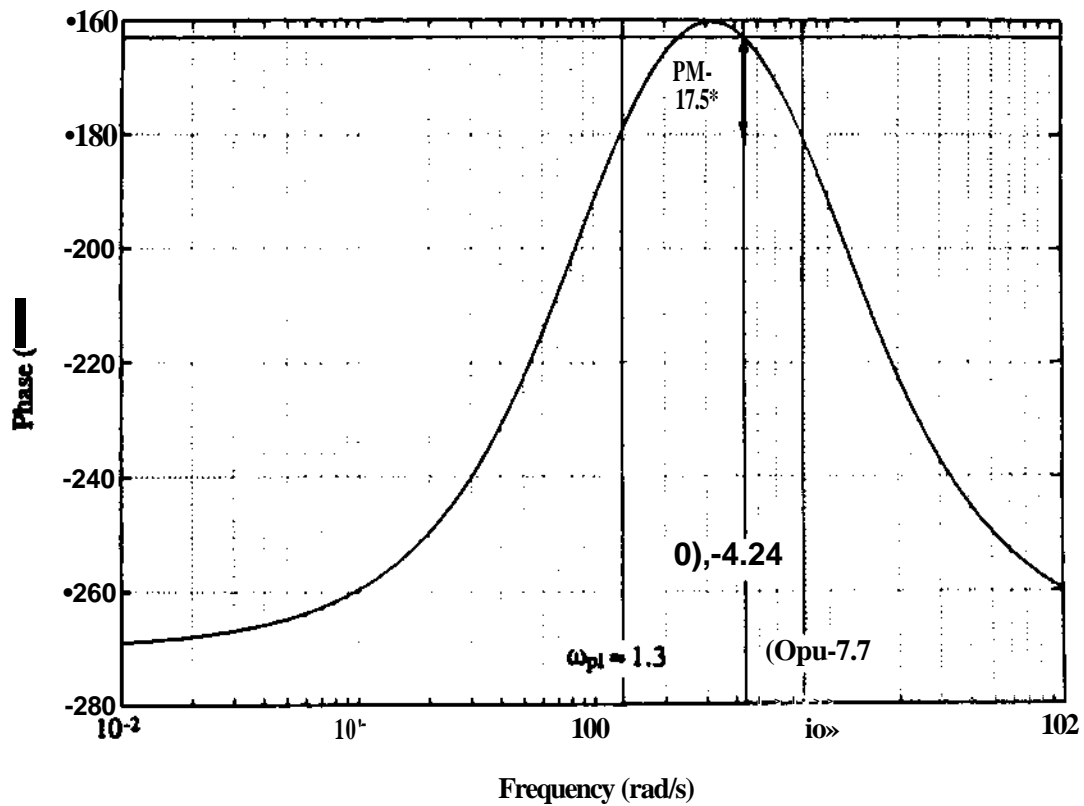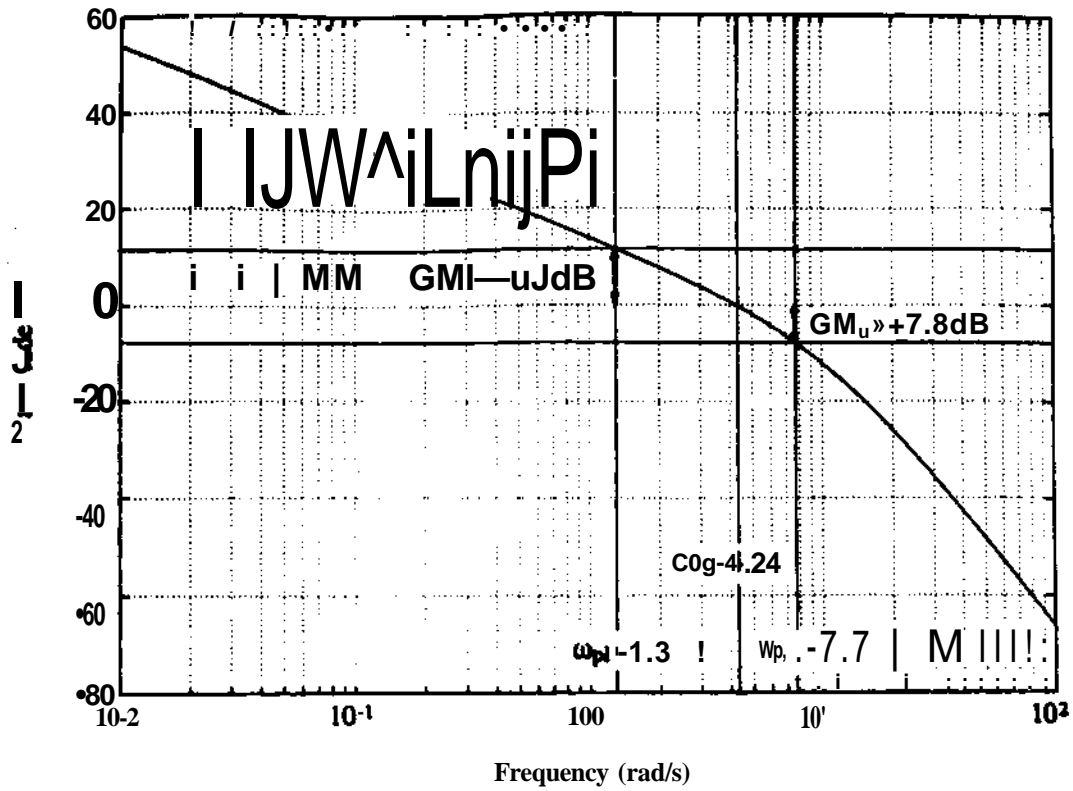
Figure 10. (a) Bode Magnitude Plot and (b) Phase Plot of Example 2 for Ikl = 500.

Closing

This **note presents** graphically-based methods for studying relative stability of LIT SISO systems. **The** angle-gain plot and the angle-angle plot are proposed for finding the gain margin and **phase** margin, respectively. The angle-gain plot recasts the information of the standard root locus in a form that exposes the explicit functional dependence of forward gain magnitude on the angle of each closed-loop system eigenvalue; the angle-angle plot explicitly relates the forward gain angle to system eigenvalue angles. Furthermore, the sensitivity of the phase margin is available and augments classical design techniques. The proposed methods, recommended for analysis and design of classical control systems, are useful geometric tools that do not require frequency analysis. The proposed framework employs independent gain and phase axes in plots naturally suited for determining gain and phase margin, respectively.

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