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**Learning Theory
and
Descriptive Set Theory**

by
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Learning Theory and Descriptive Set Theory

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In 1965, E. Mark Gold and H. Putnam observed independently that recursive functionals can be viewed as mechanical scientists trying to investigate a given hypothesis in the limit [1], [11]. Gold then essentially characterized the hypotheses that mechanical scientists can successfully decide in the limit in terms of arithmetic complexity. These ideas were developed still further by Peter Kugel [4]. In this paper I will extend this approach to obtain characterizations of identification in the limit, identification with bounded mind-changes, and identification in the short run, both for computers and for ideal agents with unbounded computational abilities. The characterization of identification with n mind-changes entails, as a corollary, an exact arithmetic characterization of Putnam's n -trial predicates, which closes a gap of a factor of two in Putnam's original characterization [11].

It will be shown that solvability results concerning identification problems can be viewed as estimations of complexity for second-order relations; arithmetic complexity when the scientist is effective, and Borel complexity otherwise. This very general perspective illuminates the relationships between the various learning-theoretic paradigms, since function identification, language identification, and logical theory identification all drop out as special cases.

The hierarchy-theoretic approach of Gold and Putnam has an additional advantage. Traditional negative arguments in learning theory have usually required that the learner succeed regardless of the order in which observations are presented [1]. All of the standard characterization theorems assume this requirement [7] [8] [9] [10]. One technical reason for the assumption is that the negative sides of these characterization theorems are established using variants of a diagonal argument known as the *locking sequence lemma* [10]. By relativizing the recursion-theoretic approach of Gold and Putnam, we can obtain more general characterizations of identifiability that apply no matter what the scientist knows *a priori* about data ordering. The operative notion of relativization is not relativization to an oracle as is usual in recursion theory, but rather relativization to background knowledge, as is more common in measure theory and statistics.

The learning-theoretic interpretation of the apparatus of the theory of recursive functional has some pedagogical advantages. For example, it will be shown how the basis theorems of mathematical logic answer questions about how the difficulty of empirical science relates to the computational difficulty of the theory under investigation. The learning-theoretic interpretation of recursion theory also raises questions about the invariance of classical results (such as the basis theorems) under changes in background knowledge.

1. Data Presentations, **Hypotheses, and Background Knowledge**

I assume that evidence is expressible in a recursive language, so evidence sentences may be encoded recursively by natural numbers. We view the data presentation received by an empirical scientist as potentially infinite, so a *data presentation* will be viewed as an co-sequence of natural numbers. Hypotheses will also be viewed as discrete objects encoded by natural numbers.

There are many features that scientists would like hypotheses to have, including simplicity, unity, empirical adequacy, and so forth. To avoid interminable debates about the precise nature of these requirements, we will simply assume that there is some well-defined notion of *hypothesis adequacy* holding between data presentations and hypotheses. Since the aim of inquiry will be to determine whether a hypothesis is adequate, we may simply identify a hypothesis with the set H of all infinite data presentations for which it is adequate. So a *hypothesis* is just some arbitrary subset of ω^{10} .

Background knowledge restricts the scientist's *a priori* uncertainty as to the data presentation he will ultimately see in the limit. So we may also think of *background knowledge* K as some arbitrary subset of $(\omega^{\mathbb{R}})$.

2. Paradigms of Hypothesis Assessment

A problem of hypothesis assessment is a situation in which a scientist is given a hypothesis and is asked to assess its adequacy on the basis of increasing data fed from some infinite data presentation. An *assessment method* is a function that takes a finite data segment to some guess about the adequacy of the hypothesis in question. The guess 1 means that the hypothesis in question is adequate, 0 means that it is not, and * represents unwillingness to produce a clear guess.

$$\phi: \omega^* \rightarrow \{1, 0, *\}.$$

Now we will consider a sequence of notions of reliable success for hypothesis assessment methods. Let $\chi_H(t)$ be the characteristic function of H.

ϕ decides H over K with certainty \Leftrightarrow
 $\forall t \in K \exists n$ s.t. $\langle N|t|n \rangle = \chi_H(t)$ and $\forall m < n, \langle K|t|m \rangle = *$.

ϕ verifies H over K with certainty \Leftrightarrow
 $\forall t \in K, t \in H \Leftrightarrow (\exists n$ s.t. $\phi(t|n) = 1$ and $\forall m < n, \phi(t|m) = *)$.

ϕ refutes H over K with certainty \Leftrightarrow
 $\forall t \in K, t \notin H \Leftrightarrow (\exists n$ s.t. $\langle K|t|n \rangle = 1 [0]$ and $\forall m < n, \langle M|t|m \rangle \neq *)$.

ϕ decides H over K in n mind-changes starting with 1 [0, *] \Leftrightarrow
 $\forall t \in K \exists n \in \omega \forall m > n, \phi(t|m) = \wedge(t)$ and
 $\phi(t|0) = 1 [0, *]$ and
 $mc(\wedge, t) < n$.

where

$mc(\phi, t|0) = 0$ and
 $mc(\phi, t|n+1) = mc(\phi, t|n)$ if $\phi(t|n) = \phi(t|n+1)$ and
 $mc(\phi, t|n+1) = mc(\phi, t|n) + 1$ otherwise.

ϕ decides H over K in n mind-changes starting with 0 [1, *] \Leftrightarrow
 $\phi(\phi) = 0 [1, *]$ and
 $\forall t \in K \exists n \in \omega \forall m > n, \langle t|m \rangle = \chi_H(t)$ and
 $mc(\phi, t) < n$.

ϕ decides H over K // f?e // m/f \Leftrightarrow
 $\forall t \in K \exists n \in \omega \forall m \geq n, \langle t|m \rangle = \chi_H(t)$.

ϕ verifies H over K // i Me limits
 $\forall t \in K \forall i \in \omega (\exists n \forall m \geq n \langle t|m \rangle = 1)$.

ϕ refutes H over K // n f?e // n/f \Leftrightarrow
 $\forall t \in K \forall i \in \omega (\exists n \forall m \geq n \langle t|m \rangle \neq 0)$.

H is [effectively] [decidable] [verifiable] [refutable] over K [with certainty] [within mind-changes] [starting with 0 [1, *]] [// the limit] \Leftrightarrow

[3 [total recursive] s.t. 4] [decides] [verifies] [refutes] H over K [with certainty] [with n mind-changes] [starting with 0 [V]] [in the limit]

3. Characterizations of Reliable Assessment

Let $K \subseteq \omega^\omega$ represent background knowledge. For each $\sigma \in \omega^*$, let the *K-fan* $K_\sigma = \{t \in K: \text{s.t. } t \text{ extends } \sigma\}$. R is a **type $\langle k, j \rangle$ relation** $\Leftrightarrow R \subseteq (\omega^\omega)^k \times \omega^j$. Let R be a type $\langle k, j \rangle$ relation. R is **K-basic open** $\Leftrightarrow \exists$ K-fans $F_1, \dots, F_k \subseteq \omega^\omega, \exists S_1, \dots, S_j \subseteq \omega$ s.t. $R = F_1 \times \dots \times F_k \times S_1 \times \dots \times S_j$. R is **K-open** $\Leftrightarrow R$ is a union of basic open relations of type $\langle k, j \rangle$. R is **K-closed** $\Leftrightarrow \bar{R}$ is K-open.

The **finite Borel hierarchy relative to K** is defined as follows¹, where R is assumed to be of type $\langle k, j \rangle$.

$$R \in \Sigma_1^{B, K} \Leftrightarrow R \text{ is K-open.}$$

$$R \in \Pi_1^{B, K} \Leftrightarrow R \text{ is K-closed.}$$

$$R \in \Sigma_{n+1}^{B, K} \Leftrightarrow R = \{\langle t, x \rangle \in (\omega^\omega)^k \times \omega^j: \exists n \in \omega \ P(t, x, n)\},$$

where P is type $\langle k, j+1 \rangle$ and $P \in \Pi_n^{B, K}$.

$$R \in \Pi_{n+1}^{B, K} \Leftrightarrow R = \{\langle t, x \rangle \in (\omega^\omega)^k \times \omega^j: \forall n \in \omega \ P(t, x, n)\},$$

where P is type $\langle k, j+1 \rangle$ and $P \in \Sigma_n^{B, K}$.

$$R \in \Delta_n^{B, K} \Leftrightarrow R \in \Sigma_n^{B, K} \cap \Pi_n^{B, K}.$$

Now we proceed to the arithmetic hierarchy. Let R be a relation over $(\omega^\omega)^k \times \omega^j$. Turing machine M is a **positive test** for R over $K \Leftrightarrow \forall t \in K^k, \forall x \in \omega^j, R(t, x) \Leftrightarrow M$ halts with 1 after receiving x as input and after scanning some finite segment of each t occurring in t . R is **K-RE** $\Leftrightarrow R$ has some positive test over K . R is **K-Co-RE** $\Leftrightarrow \bar{R}$ has some positive test over K . The **arithmetic hierarchy relative to K** is defined as follows:

$$R \in \Sigma_1^{0, K} \Leftrightarrow R \text{ is K-RE}$$

$$R \in \Pi_1^{0, K} \Leftrightarrow R \text{ is K-Co-RE}$$

¹This definition is a special case of [6], p. 20. We don't need the extra generality here. It is usual to use bold-face sigma's and pi's and to use the superscript 0. The unavailability of bold Greek symbols has led to the use of superscript B in this paper.

$$R \in \Sigma_n^{0,K} \iff R = \{ \langle t, x \rangle \in (o^k \times X) : \exists n \in \mathbb{N} P(t, x, n) \},$$

where P is type $\langle k, j+1 \rangle$ and $P \in \Pi_n^{0,K}$.

$$R \in \Pi_n^{0,K} \iff R = \{ \langle t, x \rangle \in (o^k \times X) : \forall n \in \mathbb{N} P(t, x, n) \},$$

where P is type $\langle k, j+1 \rangle$ and $P \in \Pi_n^{0,K}$.

$$H \in \Delta_n^{0,K} \iff \dots$$

Theorem 1 (Gold and Putnam):

Let $K, H \in \mathcal{O}$.

(a) H is [effectively] decidable over K with certainty $\iff H \in \Delta_1^{B,K} [\Delta_1^{0,K}]$.

(b) H is [effectively] verifiable over K with certainty $\iff H \in \Sigma_1^{B,K} [\Sigma_1^{0,K}]$.

(c) H is [effectively] refutable over K with certainty $\iff H \in \Pi_1^{B,K \cup O,K} [L_1^{0,K}]$.

(d) H is [effectively] decidable over K in the limit $\iff H \in \Delta_2^{B,K} [L_2^{0,K}]$.

(e) H is [effectively] verifiable over K in the limit $\iff H \in \Sigma_2^{B,K} [\Sigma_2^{0,K}]$.

(f) H is [effectively] refutable over K in the limit $\iff H \in \Pi_2^{B,K \cup O,K} [L_2^{0,K}]$.

Proof: The effective parts of (c) and (d) are established in [1], and the effective part of (c) is shown in [11]. The ineffective cases follow by the same arguments with references to computability suppressed, (a) and (b) are trivial.

4. Some Applications

4.1: The Empirical Irony of Cognitive Science

Consider the hypothesis that a given system under observation has a computable input-output behavior. This amounts to the hypothesis $H_{rec} \bullet \{t: t \text{ is a recursive sequence}\}$. Suppose we don't know what to expect out of the system, so $K = \mathcal{O}$. Each singleton $\{t\}$ is closed and since H_{rec} is

countable, $H_{\text{rec}} \cap K \in \Sigma_n^{B, K}$, so a non-effective scientist can verify H_{rec} in the limit. But $\overline{H_{\text{rec}}}$ is not verifiable in the limit, by a simple diagonal argument (each finite segment of a non-computable sequence can be extended by a computable sequence). So $H_{\text{rec}} \in \Sigma_2^{B, K} - \Pi_2^{B, K}$. A standard fact of recursion theory is that $H_{\text{rec}} \in \Sigma_3^{0, K} - \Pi_3^{0, K}$. Hence H_{rec} is not even verifiable in the limit by a computer. The "irony" is this. If human nature is computable, human scientists cannot verify this fact even in the limit because they are computable. But if human nature is not computable, human scientists cannot verify the non-computability of human nature in the limit either, because no system could, ideal or otherwise.

4. 2: Basis Theorems and Hypothesis Complexity

A complete hypothesis specifies everything that will ever be seen, in the correct order. Hence, a complete hypothesis is a singleton $\{t\}$. In logic, the arithmetic complexity of $\{t\}$ is known as the *implicit* complexity of t and the arithmetic complexity of t (viewed as a set of ordered pairs) is known as the *explicit* complexity of t . From a learning theoretic perspective, the implicit complexity of t is just the complexity of investigating the complete hypothesis $\{t\}$, whereas the explicit complexity of t is just the computational difficulty of generating the n th prediction specified by t .

A natural question is: how computationally complex can t be before empirical investigation of $\{t\}$ becomes hopeless for computable methods? It turns out that the basis theorems of mathematical logic already provide surprising answers to this question in many cases. It is indeed striking that the basis theorems were conceived of and proved with no applications to scientific methodology in mind.

Theorem 2: If $\{t\}$ is effectively refutable and $\text{rng}(t)$ is finite then t is recursive.

Proof: [3, p. 79] and Theorem 1 (c) ■

Theorem 3: If $\{t\}$ is effectively verifiable in the limit and $\text{rng}(t) \subseteq \{1, 0\}$ then t is recursive.

Proof: Kreisel's basis theorem [3, p. 108] and Theorem 1 (e). ■

The boundedness of the range of t turns out to be crucial. Effective scientists can decide some extremely non-effective complete hypotheses with just one mind-change if the range of t is not finite.

Theorem 4: There is a non-arithmetic t s.t. $\{t\}$ is effectively refutable.

Proof: [3, p. 107] and Theorem 1 (c). •

The basis theorems are not relativized to background knowledge K, so the above results hold only when $K = \infty$. It would be interesting to know the status of the basis theorems when K is not trivial.

5. Characterizations of Hypothesis Assessment with Bounded Mind-Changes

It remains to characterize empirical decidability with n mind-changes. Putnam [11] discusses a closely related notion under the rubric of *n-trial predicates*, but his characterization leaves a gap of a factor of two between its upper and lower bounds.² I will provide an exact characterization in terms of the following, finitary versions of the Borel and Arithmetic hierarchies. The topological version will be indexed with C and the computational version will be indexed with c, for "mind-changes"¹¹.

$$\begin{aligned}
 & \forall C, K \quad B.K \\
 & He^2_{C, K} \ll He^s_{B, K} \ll \text{HisK-open} \\
 & He^n_{C, K} \ll He^n_{i'} \gg \text{HisK-closed} \\
 & H \in \Sigma^n_{n+i} \Leftrightarrow H \text{ is of form } S \cup O, \text{ where } S \in \Sigma^n_{n'} \text{ and } O \text{ is K-open.} \\
 & He^n_{n+i} \Leftrightarrow H \text{ is of form } S \cup C, \text{ where } S \in \Sigma^n_{n'} \text{ and } C \text{ is K-closed} \\
 & H \in \Delta^n_{i'} \Leftrightarrow H \in \Sigma^n_{n'} \cap \Pi^n_{n'} \\
 & C = \bigcup_{n \in \omega} \Delta^n_{n'}
 \end{aligned}$$

The only difference between the c hierarchy and the C hierarchy is that we replace K-open sets with K-RE sets. The inductive clauses are the same as before.

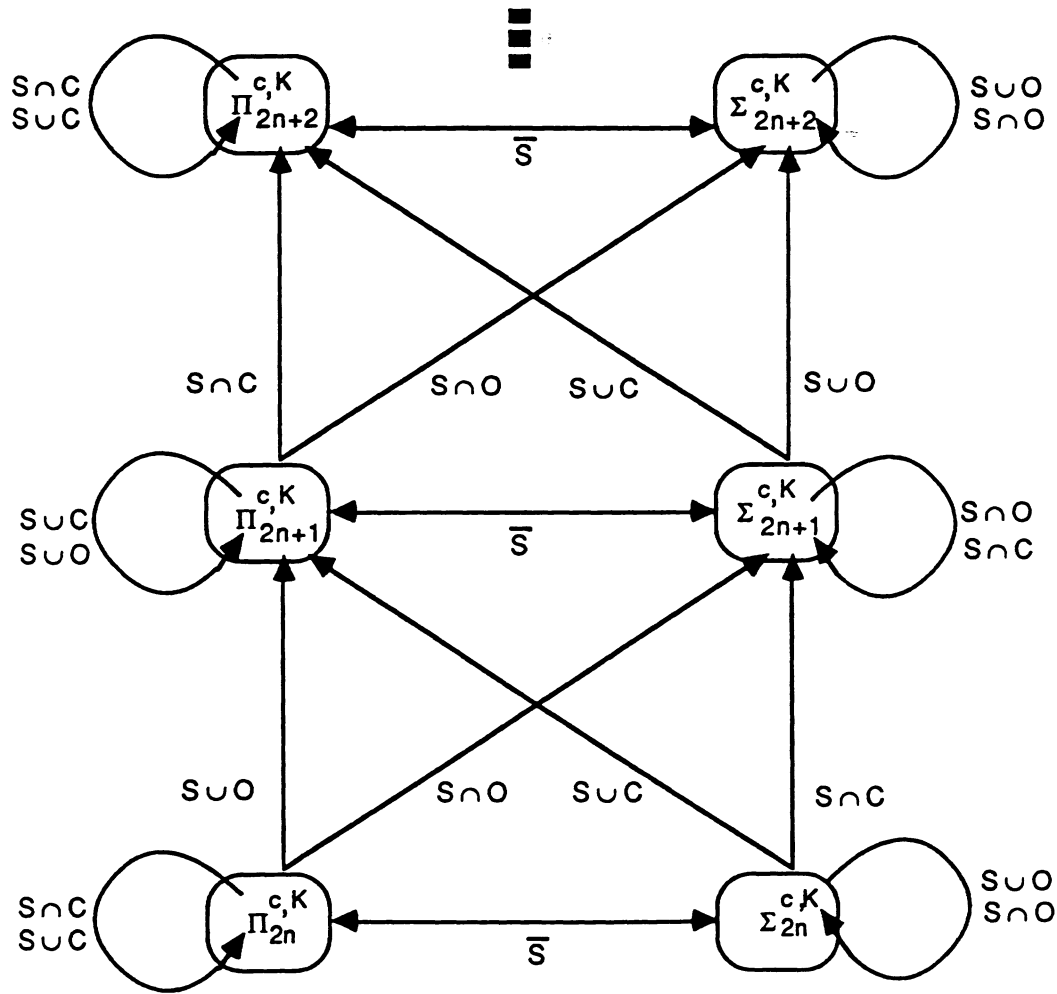
$$\begin{aligned}
 & H \in \Sigma^n_{i'} \Leftrightarrow H \in \Sigma^n_{i'} \Leftrightarrow \text{HisK-RE} \\
 & He^n_{i'} \ll He^n_{i'} \Leftrightarrow \text{HisK-Co-Re}
 \end{aligned}$$

²Putnam did not state the exact characterization as a question, and did not require an exact characterization for the purposes of his paper.

Following Putnam, let $S \subseteq \text{Pow}(\omega^\omega)$ and let S^* denote the result of closing S under finite unions and intersections.

Straightforward calculations verify the following closure laws. Observe that laws operative at a level in the hierarchy depend upon whether that level is even or odd.

Proposition 5: Let O be K-open [K-RE] and let C be K-closed [K-Co-RE]. If H is in the indicated class, then $H \cap C$, $H \cup C$, $H \cap O$ and $H \cup O$ are in the indicated classes:



■

Now we may characterize limiting empirical decidability with at most n mind-changes.

Theorem 6:

(a) H is [effectively] decidable over K in n mind-changes starting with 0 $\Leftrightarrow \exists r \in \mathbb{N} \exists \delta \in \Sigma_n^{c,K} \exists \Delta_n^{c,K} \exists \Pi_n^{c,K}$

(b) H is [effectively] decidable over K in n mind-changes starting with 1 $\Leftrightarrow \exists H \in K e \Sigma_n^{c,K} \exists \Delta_n^{c,K}$

(c) H is [effectively] decidable over K in n mind-changes starting with * $\Leftrightarrow \exists H \in K e \Delta_n^{c,K} \exists \Pi_n^{c,K}$

(d) H is [effectively] decidable over K in n mind-changes $\Leftrightarrow \exists H \in K e \Sigma_n^{c,K} \exists \Delta_n^{c,K} \exists \Pi_n^{c,K}$

Proof: (b) \Rightarrow Suppose that [recursive] Φ decides H over K in n mind-changes starting with 0. Define $O(t, n) \Leftrightarrow \text{mind}(\Phi, t) \geq n$ and define $C(t, n) \Leftrightarrow \text{mind}(\Phi, t) \leq n$. $O(t, n)$ is K-open [K-RE] and $C(t, n)$ is K-closed [K-Co-RE]. First, let's consider the case when n is even. Then since Φ always starts with conjecture 0 and never uses more than n mind-changes over K, we have:

$\forall t \in K, \exists n \in \mathbb{N} H \Leftrightarrow$

Φ changes its mind some odd number of times $\leq n-1 \Leftrightarrow$

$(O(t, 1) \& C(t, 1)) \vee (O(t, 3) \& C(t, 3)) \vee \dots \vee (O(t, n-1), O(t, n-1)) \vee \dots$

$[O(t, 1) \& C(t, 1)] \vee [O(t, 1) \& O(t, 3) \& C(t, 3)] \vee [O(t, 1) \& O(t, 3) \& O(t, 5) \& C(t, 5)] \vee \dots$

$\vee [O(t, 1) \& O(t, 3) \& O(t, 5) \& \dots \& O(t, n-3) \& O(t, n-1) \& C(t, n-1)] \Leftrightarrow$

$O(t, 1) \& [C(t, 1) \vee [O(t, 3) \& [C(t, 3) \vee \dots \vee [O(t, n-1) \& C(t, n-1)]]]]$ (by factoring).

which is a $\Sigma_n^{c,K} \exists \Delta_n^{c,K} \exists \Pi_n^{c,K}$ property of t.

Now for the case in which n is odd. Since Φ starts out with conjecture 0 and never uses more than n mind-changes over K, we have

$\forall t \in K, \exists n \in \mathbb{N} H \Leftrightarrow$

Φ does not change its mind some even number of times $\leq n \Leftrightarrow$

$\neg C(t, 0) \& \neg(O(t, 2) \& C(t, 2)) \& \dots \& \neg(O(t, n-1), C(t, n-1)) \vee \dots$

$O(t, 1) \& [C(t, 1) \vee O(t, 3)] \& [C(t, 3) \vee O(t, 5)] \& [C(t, 5) \vee O(t, 7)] \& \dots \& [C(t, n-3) \vee O(t, n-1)] \Leftrightarrow$

$O(t, 1) \& [C(t, 1) \vee O(t, 3)] \& [C(t, 1) \vee C(t, 3) \vee O(t, 5)] \& [C(t, 1) \vee C(t, 3) \vee C(t, 5) \vee O(t, 7)] \& \dots \&$

$[C(t, 1) \vee C(t, 3) \vee \dots \vee C(t, n-5) \vee \dots \vee C(t, n-3) \vee O(t, n-1)] \Leftrightarrow$

$O(t, 1) \& [C(t, 1) \vee [O(t, 3) \& [C(t, 3) \vee \dots \& [C(t, n-3) \vee O(t, n-1)]]]]$ (by factoring)

which is a $\Sigma_n^{c,K} \exists \Delta_n^{c,K} \exists \Pi_n^{c,K}$ property of t.

\Leftarrow Suppose that $H \cap K \in \Sigma_n^{C, K}[\Sigma_n^{C, K}]$. Then $H \cap K$ may be expressed in the form

$O_1 \cap [C_2 \cup [O_3 \cap [C_4 \cup \dots [C_{n-1} \cup O_n]]]$ if n is odd, or of form

$O_1 \cap [C_2 \cup [O_3 \cap [C_4 \cup \dots [C_{n-1} \cap O_n]]]$ if n is even.

In either case, define ϕ to conjecture 0 until O_1 is verified by the data, after which ϕ says 1 until C_2 is refuted by the data, after which ϕ says 0 until O_3 is verified by the data, after which ϕ says 1 until.... ϕ will succeed with at most n mind-changes.

(a) follows from (b) by duality.

(c) \Rightarrow Suppose that ϕ decides H over K with n mind-changes starting with $*$. Define

$$\psi_0(\sigma) = \begin{cases} 0 & \text{if } \phi(\sigma) = * \\ \phi(\sigma) & \text{otherwise} \end{cases} \quad \psi_1(\sigma) = \begin{cases} 1 & \text{if } \phi(\sigma) = * \\ \phi(\sigma) & \text{otherwise} \end{cases}$$

ψ_0 succeeds in n mind-changes starting with 0 and ψ_1 succeeds in n mind-changes starting with 1. By (a), $H \cap K \in \Sigma_n^{C, K}[\Sigma_n^{C, K}]$ and by (b), $H \cap K \in \Pi_n^{C, K}[\Pi_n^{C, K}]$.

\Leftarrow Suppose $H \cap K \in \Delta_n^{C, K}[\Delta_n^{C, K}]$. Then by (a) and (b), we have [effective] methods ψ_1, ψ_0 that succeed in n mind-changes starting with 1 and with 0, respectively. Define

$$\phi(\sigma) = \begin{cases} * & \text{if } \sigma \text{ is empty} \\ \psi_1 & \text{if } \psi_1(\sigma) = \psi_0(\sigma) \\ \phi(\sigma-) & \text{otherwise} \end{cases}$$

which [effectively] decides H with n mind-changes starting with $*$.

(d) \Leftarrow follows from (a) and (b).

\Rightarrow Let [effective] ϕ decide H over K in n mind-changes. Let σ be empty. If $\phi(\sigma) = 1$ then $H \in \Pi_n^{C, K}[\Pi_n^{C, K}]$, by (a). If $\phi(\sigma) = 0$ then $H \in \Sigma_n^{C, K}[\Sigma_n^{C, K}]$, by (b). If $\phi(\sigma) = *$, then $H \in \Delta_n^{C, K}[\Delta_n^{C, K}]$, by (c).

■

As an application of Theorem 6, we have

Proposition 7:

$$(a) \forall K, C \in \Sigma_2^{A, B, K} \text{ and } C \in \Sigma_2^{0, K}$$

Let K be the set of all sequences that converge either to 0 or to 1. Then

$$(b) \forall n \exists c \in \Sigma_{n+1}^{c, K} \text{ and } \exists i \text{ such that } \forall t \in \Sigma_{n+1}^{c, K} \text{ if } t_i = c \text{ then } t \in K$$

$$(c) \forall n \exists \Sigma_n^{c, K} \cup \Pi_n^{c, K} \subset \Delta_{n+1}^{c, K} \text{ and } \exists \Sigma_n^{c, K} \cup \Pi_n^{c, K} \subset \Delta_{n+1}^{c, K}$$

$$(d) (\Sigma_1^{B, K})^* = C \text{ and } (\Sigma_1^{0, K})^* = c$$

Proof: (a) To show that inclusion is proper let $K = \{t: t \text{ converges to 1 or to 0}\}$ and let $H = \{t: t \text{ converges to 1}\}$. Consider the assessment method Φ that produces 1 if the current data entry is 1 and who produces 0 otherwise. Φ is clearly effective and Φ decides H in the limit over K . So $H \cap K \in \Sigma_2^{A, B, K}$, by Theorem 1. Let Ψ be an assessment method that starts with 0. We can fool Ψ more than n times about H over K . Feed all 1's until Ψ changes its mind to 1. Now feed all 0's until Ψ changes its mind to 0, etc, until $n+1$ mind-changes occur. (If the awaited mind-change does not occur, then we produce a data presentation in K on which Φ converges to the wrong answer). After the $n+1$ mind-changes occur, continue to output 1 forever, so the data presentation constructed is in K . So, by Theorem 6 we have that $\forall n \exists H \cap K \in \Sigma_n^{A, B, K}$. Hence, $H \cap K \notin C$.

(b) To show that the inclusion is proper, define $\#0(t) =$ the number of 0's occurring in t .

$$P_n(t) = \begin{cases} \left(\bigvee_{k=1}^{\lfloor n/2 \rfloor} \#0(t) = 2k \right) & \text{if } n \text{ is even} \\ \left(\left(\bigvee_{k=1}^{\lfloor (n-1)/2 \rfloor} \#0(t) = 2k \right) \vee \#0(t) \geq n+1 \right) & \text{if } n \text{ is odd} \end{cases}$$

P_n is readily seen to be effectively decidable in n mind-changes starting with 0. A simple diagonal argument shows that P_n cannot be decided with fewer mind-changes starting with 0. Now apply Theorem 6.

$S \subseteq \omega$ is an *n-trial predicate* \Leftrightarrow
 there is some total recursive f such that for each $n \in \omega$, $\lim_{k \rightarrow \omega} f(n, k)$ converges to $\chi_S(n)$ with at most n mind-changes.

Putnam's n -trial predicates can be viewed as a special kind of empirical hypothesis whose adequacy depends only on the first datum observed. Define $H_S = \{t: t_1 \in S\}$

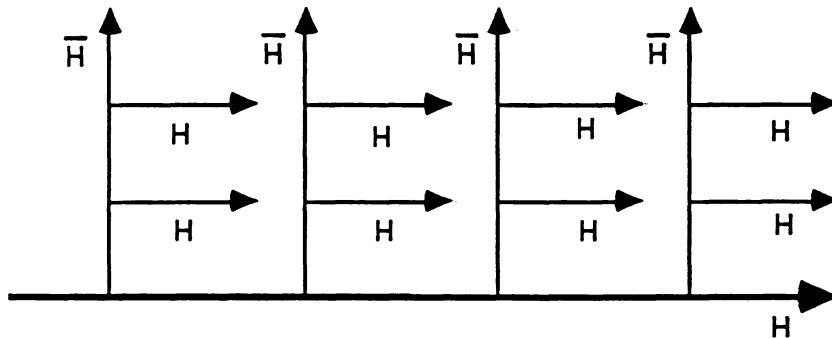
Proposition 9: S is an n -trial predicate $\Leftrightarrow H_S \in \Sigma_n^c \cup \Pi_n^c$

Proof: Theorem 7. ■

6. Feathers and Demons

The "hard side" of Theorem 7 was to show that a scientist fails in n mind-changes if the C -complexity of the hypothesis under examination is too high. I will now provide a complementary characterization that makes negative arguments more transparent than positive arguments.

Suppose that hypothesis H is distributed in K in the following manner.



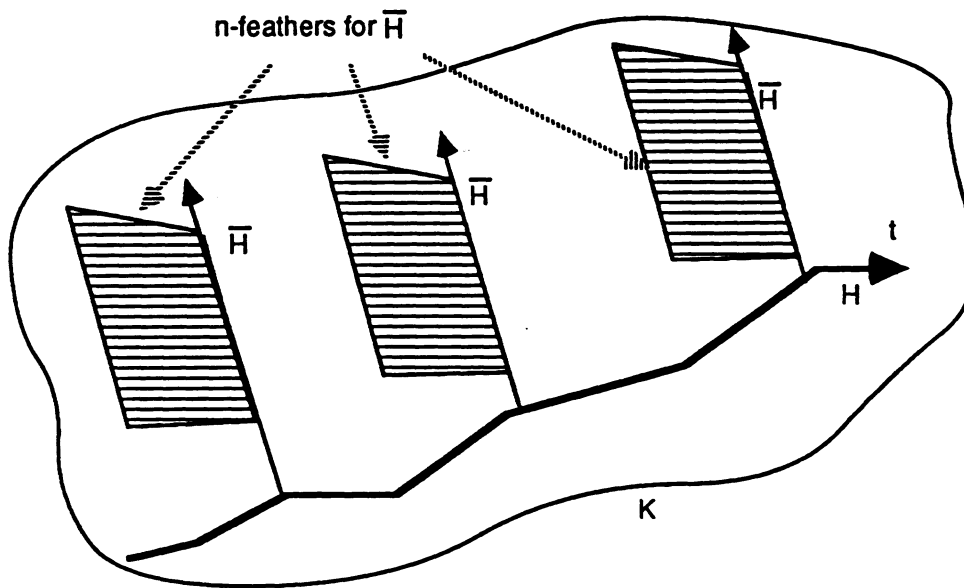
Given the depicted situation, a demon can easily fool an arbitrary scientist who starts with 0 three times, and a scientist who starts with 1 can be fooled twice. The demon leads the scientist down the bold path until ϕ says 1 (which must happen, else the demon stays with the bold path and ϕ fails in the limit). As soon as ϕ says 1, the demon proceeds up the next available path for \bar{H} . Now ϕ must eventually say 0, at which time the demon veers to the right down the next available path for H .

K may be thought of as an infinite "feather" whose "shaft" is the bold path, and whose alternating "barbs" are the other paths. We may define feathers more generally as follows:

K is a 1-feather for H with shaft $t \Leftrightarrow t \in K \cap H$.

K is an $n+1$ -feather for H with shaft $t \Leftrightarrow$
 $t \in K \cap H$ and
 $\forall n \exists t \in K$ s.t.
 $t|_n = t'|_n$ and
 K is an n -feather with shaft t for \bar{H}

K is an n -feather for $H \Leftrightarrow \exists t$ s.t. K is an n -feather for H with shaft t .



We may now define the *feather dimension* of K for H :

$Dim_H(K) = n \Leftrightarrow K$ is an n -feather for H and K is not an $n+1$ -feather for H .

Theorem 9:

- (a) H is decidable over K in n mind-changes starting with 0 $\Leftrightarrow K$ is not an $n + 1$ -feather for H .
- (b) H is decidable over K in n mind-changes starting with 1 $\Leftrightarrow K$ is not an $n + 1$ -feather for \bar{H} .
- (c) H is decidable over K in n mind-changes starting with $*$ \Leftrightarrow
 K is not an $n + 1$ -feather for H and K is not an $n + 1$ -feather for \bar{H}
- (d) H is decidable over K in n mind-changes \Leftrightarrow

K is not an $n+1$ -feather for H or K is not an $n+1$ -feather for \overline{H}

Corollary:

$H \cap K \in \mathcal{V}^{f,K} \ll K$ is not an $n+1$ -feather for H

$H \cap K \in \mathcal{V}^{C,K} \Leftrightarrow K$ is not an $n+1$ -feather for H .

$H \cap K \in \wedge^{n^1} \Leftrightarrow K$ is not an $n+1$ -feather for H and K is not an n -feather for \overline{H}

$H \cap K \in \mathcal{U}_n^{C,K} \ll \mathcal{U}_n^{C,K} \ll K$ is not an $n+1$ -feather for H or K is not an n -feather for \overline{H} .

Proof: (a) & (b) \Rightarrow Prove the contrapositive by means of Theorem 6 and the usual demonic argument.

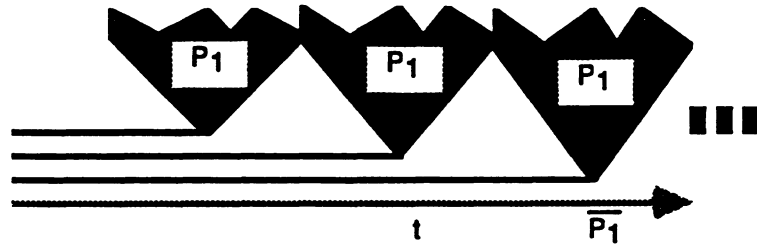
\Leftarrow Argument by induction on n . Base case for (a): Suppose that K is not a 1-feather for H . Then $H = 0$. Let $t \in K$. Then $t \in \overline{H}$. So the trivial method $\langle \sigma \rangle = 0$ succeeds in 0 mind changes. The base case for (b) is similar.

Now suppose (a) and (b) for each $n^f < n$. Hence, if K is not an $n+1$ feather for \overline{H} then there is a method \mathcal{V}^1_H, K that decides H over K in n^1 mind-changes starting with 1, and if K is not an $n+1$ feather for H then there is a method \mathcal{V}^0_H, K that decides H over K in n^1 mind-changes starting with 0. Now define:

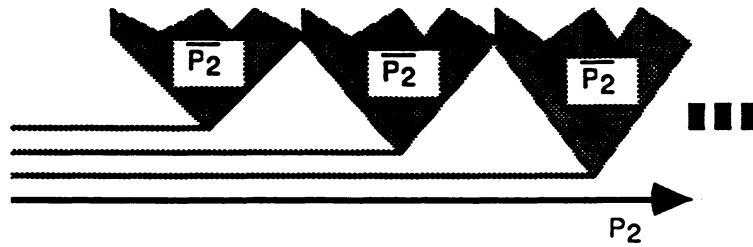
$\text{trunc}(H, K, n, a) = \begin{cases} \text{(the shortest } XQG \text{ s.t. } \text{Dim}^v(Ka) \leq n, \text{ if there is one} \\ a \text{ otherwise} \end{cases}$

$$\Phi_{n+1}^0(\sigma) = \begin{cases} 0 \text{ if } a \text{ is empty} \\ 0 \text{ if } \text{Dim}H(Kc) > n \text{ and } \text{Dim}^v(Ko) > n \\ \Psi_{H, K, \text{trunc}(H, K, n, \sigma)}^1(\sigma) \text{ if } \text{Dim}^v(K\sigma) \leq n \\ \Psi_{H, K, \text{trunc}(H, K, n, \sigma)}^0(\sigma) \text{ otherwise (i.e. if } \text{Dim}^v(Ka) \leq n \end{cases}$$

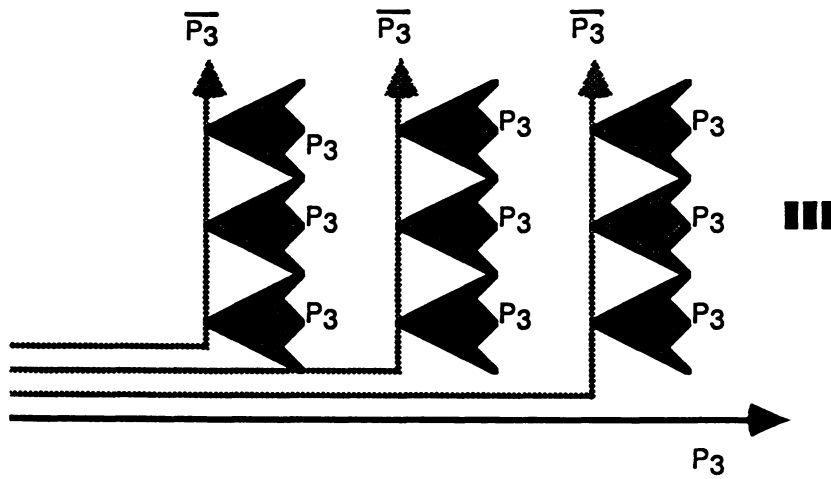
Suppose that K is not an $n+2$ -feather for H . Let $t \in K$. Since dimension never increases on evidence, there are two cases to consider.



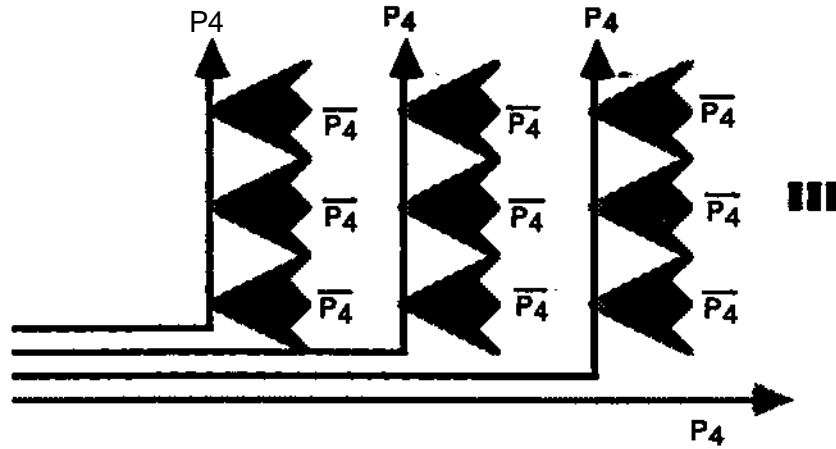
K is clearly a 2-feather for $\overline{P_1}$ and $P_1 \in \Sigma_1^{C,K}$ since P_1 is open. Next, let $P_2 = \overline{P_1}$. Now we have a 2-feather for P_2 , and $P_2 \in \Pi_1^{C,K}$.



We know from the Corollary of theorem 3.8 that to build a more complicated feather, we add some open set to P . The following sort of set will suffice.



$P_3 - P_2$ is clearly K -open, since it is depicted as a union of fans. So P_3 is of the form $O \cup C$, and is therefore in $\Pi_2^{C,K}$. Let P_4 be the complement of P_3 .



Now we are again free to add a dimension to P_4 by augmenting it with an open set. By successive complementations and open set additions, we can build feathers of arbitrary finite dimension.

7. Paradigms of Discovery

in problems of hypothesis assessment, the scientist is assigned some hypothesis whose adequacy is to be investigated on the basis of increasing data. In discovery problems, the scientist is required to invent an adequate hypothesis on the basis of increasing data. Most results in learning theory concern discovery rather than assessment. Interest has centered on grammatical inference, recursive function identification, and the induction of first-order theories from presentations of structure diagrams. Each of these applications is a special case of the following setting.

Hypothesis assessment methods do not have to read or to produce hypotheses, so hypotheses could be viewed abstractly as uncountable sets of infinite sequences. This will not do when discovery procedures are computers. Instead, we will assume simply that hypotheses are stated in a discrete, finitary language with a decidable syntax. Hence, hypotheses, like data sentences, may be encoded by natural numbers. As before, we will assume that the goal of inquiry is some relation of **adequacy** $A \subseteq (O^* \times C)$ holding between infinite data presentations and hypotheses. A may entail consistency with the total data, explanatory completeness over the total data, simplicity, unity, or any other desideratum that depends only upon the hypothesis and the total data. We will let $A_i = \{t: A(t, i)\}$.

A discovery method will be a map from finite segments of data presentations to hypotheses, i.e. $\phi: O^* \rightarrow C$. We will consider the following concepts of successful discovery:

ϕ **identifies A-adequate hypotheses over K with certainty** \Leftrightarrow
 $\forall t \in K \exists n$ s.t. $A(t, \phi(t|n))$ and $\forall m < n, \phi(t|m) = *$.

ϕ **identifies A-adequate hypotheses in n mind-changes** \Leftrightarrow
 $\forall t \in K \exists n \in \omega \forall m > n, \phi(t|m) = \chi_H(t)$ and
 $mc(\phi, t) \leq n$.

ϕ **identifies A-adequate hypotheses over K in the limit** \Leftrightarrow
 $\forall t \in K \exists n \forall m \geq n \phi(t|m) = \phi(t|n)$ & $A(t, \phi(t|n))$.

A-adequate hypotheses are [effectively] identifiable over K

$\left[\begin{array}{l} \text{with certainty} \\ \text{with } n \text{ mind changes} \\ \text{in the limit} \end{array} \right] \Leftrightarrow$

\exists [total recursive] ϕ s.t. ϕ identifies A-adequate hypotheses over K

$\left[\begin{array}{l} \text{with certainty} \\ \text{with } n \text{ mind changes} \\ \text{in the limit} \end{array} \right]$

8. Characterizations of Reliable Discovery

Each of these senses of success requires that an adequate hypotheses be found for each data presentation in K. It is therefore trivial that A must cover K in the following sense if success is to be possible.

A covers K $\Leftrightarrow \forall t \in K \exists i \in \omega$ s.t. $A(t, i)$.

Now we may characterize identification and identification in the limit.

Theorem 10:

(a) **A-adequate hypotheses are [effectively] identifiable over K with certainty** \Leftrightarrow

$\exists A' \subseteq A$ s.t. A' covers K and $A' \in \Sigma_1^{B, K} \left[\begin{array}{l} \Sigma_1^0, K \\ \Sigma_1 \end{array} \right] \Leftrightarrow$
 $\exists A' \subseteq A$ s.t. A' covers K and $A' \in \Delta_1^{B, K} \left[\begin{array}{l} \Delta_1^0, K \\ \Delta_1 \end{array} \right] \Leftrightarrow$

(b) **A-adequate hypotheses are [effectively] identifiable over K in the limit** \Leftrightarrow

$\exists A' \subseteq A$ s.t. A' covers K and $A' \in \Sigma_2^{B, K} \left[\begin{array}{l} \Sigma_2^0, K \\ \Sigma_2 \end{array} \right] \Leftrightarrow$

$$\exists A^* \in \mathcal{A} \text{ s.t. } A^* \text{ covers } K \text{ and } A^* \in \mathcal{A}^2 \stackrel{B,KT \quad O,K}{1^* \quad 2 \quad J} \Leftrightarrow$$

Proof: (a) \Rightarrow Let effective discovery method Φ identify A -adequate hypotheses over K . Define

$$A^*(t, i) \Leftrightarrow \exists n \text{ s.t. } \Phi(t|n) = i \text{ and } \forall m < n, \Phi(t|m) = \perp$$

A^* covers K and $A^* \in \mathcal{A}$ since Φ identifies A -adequate hypotheses over K with certainty. By definition, $A^* \in \mathcal{A}^1 \stackrel{B,KLO,K}{**j \quad W. \quad J}$. But we also have

$$\forall t \in K, \neg A^*(t, i) \Leftrightarrow \exists k \neq i (\Phi(t|n) = k \text{ and } \forall m < n, \Phi(t|m) = \perp)$$

$$\text{So } A^* \in \mathcal{A}^1 \stackrel{B,K[O,K]}{L^1 \quad J}$$

\Leftarrow Suppose that $A^* \in \mathcal{A}$ covers K and $A^* \in \mathcal{A}^1 \stackrel{B,K[O,K]}{L^1 \quad J}$. Let $\text{VERIFY}[t, i]$ be an [effective] positive test for A^* . Now we can define the following [effective] discovery method, which identifies A -adequate hypotheses over K with certainty.

DISCOVER(a):

set $n := \text{length}(a)$

if $\forall i \leq n, \text{VERIFY}(a_i, i), \text{VERIFY}(a_2, i), \dots, \text{VERIFY}(a_n, i)$ all produce $*$ or 0, [in n computational steps], return $*$

else, return the first i such that the first non- $*$ output of TEST on a is 1.

(b) \Rightarrow Let effective discovery method Φ identify A -adequate hypotheses over K in the limit. Define

$$A^*(t, i) \Leftrightarrow \exists n \forall m \geq n \Phi(t|m) = i.$$

A^* covers K and $A^* \in \mathcal{A}$ since Φ identifies A -adequate hypotheses over K . By definition, $A^* \in \mathcal{A}^1 \stackrel{B,K[O,K]}{L^1 \quad J}$. Since Φ converges to some i on each $t \in K$, we also have that

$$\forall t \in K, \neg A^*(t, i) \Leftrightarrow \exists n \forall m \geq n \Phi(t|m) \neq i.$$

$$\text{so } A^* \in \mathcal{A}^2 \stackrel{B,K[O,K]}{L^2 \quad J}$$

=> Suppose that $A \subseteq A$ covers K and $A \in \Lambda^{B, K, L, O, K} L^{2^*} J$. Let $REFUTE_f(n, i)$ be an [effective] positive test for the relation $\forall m \leq n \exists i(m) \ll i$. Now we can define the following [effective] discovery method, which identifies **A-adequate hypotheses over K** in the limit. Let $\langle j \rangle_i, \langle j \rangle_2$ be the ordered pair encoded by j under some fixed, recursive bijection from \mathbb{C}^0 to \mathbb{C}^2 .

LIM-DISCOVER(a):

Set $n := \text{length}(a)$

Produce $\langle j \rangle_2$ where j is the least $k < n$ such that $REFUTE[\langle cx, \langle j \rangle_i, \langle j \rangle_2]$ does not return 1 [within n computational steps] if there is such a j .

Else, produce \

LIM-DISCOVER identifies **A-adequate hypotheses over K** in the limit.

8. Learning Theory Results as Relative Complexity Classifications

The following examples illustrate how the standard paradigms of language learnability and function identification drop out as special cases of the approach adopted here. From our perspective, standard results in learning theory may be thought of as strong relative complexity classifications for relations of type $\infty^0 \times \infty$.

Function Identification:

The problem of identifying set Rec of total recursive functions:

Adequacy relation: $Af_{un}(t, i) \Leftrightarrow \phi = t$

Background knowledge: $K \subseteq Rec$

One of the first negative results about function identification is that the collection of all recursive functions is identifiable in the limit, but not effectively so. The positive result follows from the fact that $Af_{un}(t, i) \Leftrightarrow \exists j = t \Leftrightarrow \forall n \exists i(n) = t_n$. Since the relation $\exists i(n) = t_n$ is $\in \Lambda_i^{B, Rec}$, $Af_{un} \in \Pi_1^{B, Rec}$. The situation is different in the computable case: $Af_{un}(t, i) \ll fr = t \Leftrightarrow \forall n \exists i(n) = t_n \Leftrightarrow \forall n \exists k \exists j(n) \exists i = t_n$. Gold's negative result together with Theorem 10 tells us that this characterization is optimal, i.e. that $Af_{un} \in \Pi_2^{0, Rec} \wedge \dots$. Indeed, Gold's result tells us that there is no $A \subseteq Af_{un}$ covering Rec such that $A \in \Lambda^{0, Rec}$.

Language Identification by RE Index:

The problem of identifying language class $L \subseteq RE$:

Adequacy relation: $A_{RE}(t, i) \Leftrightarrow W_i = mg(t)$
 Background knowledge: $K_L = \{t: \exists S \in L \text{ s.t. } mg(t) = S\}.$

Here, the basic theorem is that no collection of languages L' containing all finite languages and one infinite language is identifiable in the limit, even by an ineffective learner [10]. In our generalized notation, this is the claim that A_{RE} hypotheses are not identifiable over $K_{L'}$, which together with Theorem 10 implies that there is no $A' \subseteq A_{RE}$ covering $K_{L'}$ such that $A' \in \Sigma_2^{B, K_{L'}}$. A general upper bound meeting Gold's lower bound is easy to calculate.

$$\begin{aligned} W_i = mg(t) &\Leftrightarrow \\ \forall n [\phi(n) \downarrow \Leftrightarrow \exists k \text{ s.t. } n = t_k] &\Leftrightarrow \\ \forall n \forall k \exists k' [\dots] &\in \Pi_2^{0, K_{RE}} \subseteq \Pi_2^{0, K_{L'}}. \end{aligned}$$

$$\text{Hence, } A_{RE}(t, i) \in \Pi_2^{0, K_{RE}} - \Sigma_2^{B, K_{RE}}.$$

Another standard example is the collection L_{fin} of all finite languages.

$$\begin{aligned} \forall t \in K_{L_{fin}}, W_i = mg(t) &\Leftrightarrow \\ \exists k \forall k' \geq k \forall n [\phi(n) \downarrow \Leftrightarrow n \in mg(t|k')] &\in \\ \forall n [\phi(n) \downarrow \Leftrightarrow \exists k \text{ s.t. } n = t_k] &\Leftrightarrow \\ \forall n \forall k \exists k' [\dots] &\in \Sigma_2^{0, K_{L_{fin}}}. \end{aligned}$$

Language Identification by recursive index:

The problem of identifying language class $L \subseteq RE$:

Adequacy relation: $A_R(t, i) \Leftrightarrow \phi_i = \chi_{mg(t)}$
 Background knowledge: $K_L = \{t: \exists S \in L \text{ s.t. } mg(t) = S\}.$

Let L' be as in the last example. Gold showed that L' is not identifiable by an effective learner even when the data presentations are assumed to be primitive recursive. Let $Prim$ be the set of all primitive recursive sequences. Let $K' = K_{L'} \cap Prim$. Then Gold's result shows that there is no $A \subseteq A_R$ covering $K_{L'} \cap Prim$ such that $A \in \Sigma_2^{0, K_{L'} \cap Prim}$. Once again it is easy to compute an upper bound that matches Gold's lower bound:

$$\phi_i = \chi_{\text{rng}(t)} \Leftrightarrow$$

$$\forall n [(\exists m \ n = t_m) \Rightarrow \exists k \ \phi_i(n) \downarrow^k = 1] \text{ and } [\forall m' \ n \neq t_{m'} \Rightarrow (\exists k' \ \phi_i(n) \downarrow^{k'} = 0)] \Leftrightarrow$$

$$\forall n \forall m \exists m' \exists k \exists k' [\dots] \in \Pi_2^{0, \text{KRE}} \subseteq \Pi_2^{0, \text{KL} \cap \text{Prim}}$$

9. Characterizations of Reliable Discovery with Bounded Mind-Changes

Given the results so far, it is natural to guess that A-adequate hypotheses are identifiable over K in n mind-changes just in case $\exists A' \subseteq A$ s.t. A' covers K and $A' \in \Delta_n^{C, K}$. But this conjecture is quite mistaken.

Proposition 11: $\exists A, K$ s.t. $A \in \Pi_1^{C, K}$ but $\forall n$ A-adequate hypotheses are not identifiable over K in n mind-changes and \bar{A} -adequate hypotheses are effectively identifiable over K in one mind-change.

Proof: Let K be the set of all recursive functions, and let A be A_{rec} of the above example. ■

The problem is that discovery depends not only on the topology of each hypothesis, but also on how the data presentations of different hypotheses are interleaved together. This interleaved structure of the adequacy relation can be captured exactly if we generalize the notion of n-feathers slightly.

K is a 1-feather for i mod A with shaft t $\Leftrightarrow t \in K \cap A_i$.

K is an n+1-feather for i mod A with shaft t \Leftrightarrow

$t \in K \cap A_i$ and

$\forall n \exists t' \in K \exists k \in \omega$ s.t.

$t \upharpoonright n = t' \upharpoonright n$ and

K is an n-feather with shaft t for $K \text{ mod } A$ with shaft t', and $t' \notin A_k$.

K is an n-feather for i mod A $\Leftrightarrow \exists t$ s.t. K is an n-feather for i mod A with shaft t.

K is an exact n-feather for i mod A \Leftrightarrow

K is an n-feather for i mod A and $\forall m > n$, K is not an m-feather for i mod A.

Theorem 12:

A-adequate hypotheses are identifiable over K in n mind-changes starting with * \Leftrightarrow

$\forall i$, K is not an n-feather for i.

Proof: Analogous to the proof of Theorem 6. ■

Example: Recall the case of learning finite languages by RE index. It is easy to see that K_{Lin} is an n -feather for A_{RE} for each n , so the finite languages are not identifiable under any bounded number of mind changes.

10. Conclusion

Complete characterizations have been presented for effective and ineffective hypothesis assessment, in the short run, in the long run, and with bounded mind changes. Complete characterizations have also been presented for effective and ineffective discovery in the limit, and for non-effective discovery with bounded mind-changes. It remains to characterize effective discovery with bounded mind changes.

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