Stochastic Evaluation and Optimization of Flexibility in Multiproduct Batch Plants

by

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ABSTRACT

This paper deals with the problem of evaluating and optimizing a stochastic measure that integrates flexibility and reliability in multiproduct batch plants. It is assumed that uncertainties in the product demands are given in terms of statistical distributions, and that expected failure rates of the equipment are given in terms of discrete probabilities. Based on recent work by the authors (Straub and Grossmann, 1989), an efficient computational method is proposed to evaluate the expected stochastic flexibility in multiproduct batch plants. It is also shown that this method can be incorporated in the optimization for determining the sizes and parallel equipment that maximize the expected stochastic flexibility under a capital investment constraint. By varying the specification of the latter, trade-offs can be established between investment cost and flexibility.

Introduction

In recent years the study of batch plants has received significant attention. This is due to the growth in demand for specialized chemicals, which are often complex to produce and have low volumes of production. The ability of batch chemical plants to efficiently process different products has also contributed to this increased attention.

There are two basic types of batch plants: the multiproduct and the multipurpose plants. In the multiproduct plant the products are produced using the same sequence of processing steps. In this configuration the products are produced one at a time since all products go through the last stage, and the last stage can only produce one product at a time. In the multipurpose plant the products do not necessarily follow the same sequence of processing steps. Thus, with this plant it is possible to produce products simultaneously.

The optimal design of batch plants has been discussed by Sparrow et al. (1975) who addressed the optimal sizing of multiproduct batch plants using heuristics and branch and bound methods. Grossmann and Sargent (1979) have studied the same problem but formulated it as an MINLP to determine the optimal solution. Knopf et al. (1982) extended the MINLP formulation in order to also size the semi-continuous equipment. Suhami and Mah (1982) formulated an MINLP model to design a restricted class of multipurpose chemical plants a problem that subsequently was studied by Vaselenak et al. (1987) and Faqir and Karimi (1988). Other aspects of batch plants have also been considered. For example, Modi and Karimi (1989) discuss the use of intermediate storage between stages while Birewar and Grossmann (1989) discuss scheduling aspects in determining the optimal design. A general review on design methods for batch plants is given in Reklaitis (1989).

One aspect of batch plants that has received much less attention is the flexibility of the plant; that is the ability of the plant to meet production requirements given that there are uncertainties in demands and technical specifications such as size factors and processing times. Some of the first authors to investigate this problem were Reinhart and Rippin (1986). They discuss the effect of uncertainties in both product demands and in technical specifications. They define the flexibility as the probability of meeting the demand requirements, but assuming that there is only one piece of equipment in each stage, thus not accounting for failures in equipment. Wellons and Reklaitis (1989) have also investigated the flexibility of batch plants by determining the optimal staged expansions over time to account for increases in product demand.

In this paper we will present a framework to evaluate and optimize the expected stochastic flexibility of multiproduct batch plants that operate with single product campaigns. It will be assumed that the demands of the products are uncertain and characterized by continuous probability distribution functions. In addition, the availability of the various pieces of equipment will be considered uncertain with discrete probabilities for failure. Methods will first be presented to evaluate the probability of meeting the uncertain demand requirements (stochastic flexibility). As a next step we will incorporate the uncertainty in the availability of the equipment (traditionally the domain of reliability) to evaluate the expected value of the stochastic flexibility. Finally, the optimization of these flexibility measures will be considered for the design of multiproduct batch plants.

Review **of Expected Stochastic Flexibility**

The metric which will be used to characterize multiproduct batch plants with discrete (equipment availability) and continuous uncertainties (product demands) is the Expected Stochastic Flexibility, E(SF), which has recently been proposed by Straub and Grossmann (1989). The stochastic flexibility, SF, is a probabilistic measure of a system's ability to tolerate continuous uncertainties for a given discrete state. The discrete uncertainties are taken into account when determining the expected value of the stochastic flexibility.

The concept of the stochastic flexibility is shown in Figure 1. The triangle represents the feasible region of operation for the system in the space of the continuous uncertainties, 6_1 and 0_2. Each of the continuous uncertainties is described by a probabilistic distribution. In this case 8_1 and e_2 are independent parameters characterized by normal distributions, which gives rise to a joint distribution whose contours are circles. The stochastic flexibility is the cumulative probability of the joint distribution that lies within the feasible region. Thus, mathematically, the stochastic flexibility is the integral of the joint distribution over the shaded region.

The discrete uncertainty involves changes in the state of a design which result in different feasible regions. The effect of a design change is shown in Figure 2. Here normal operation is represented by State 1 and results in the outer triangle. State 2 represents the process in which some equipment has failed. In this case the size of the feasible region gets smaller since the process has less capacity.

The expected stochastic flexibility is calculated by summing up the product of the probability for each discrete state and its corresponding stochastic flexibility. In this way, the E(SF) represents, qualitatively, the probability of feasible operation that we can expect on average over a large time period.

As will be shown in this paper considerable advantages can be taken from the special structure of the model of the multiproduct batch plants to simplify the calculation for the expected stochastic flexibility.

Model

The model used to characterize the operation of the batch plant is reported by Grossmann and Sargent (1979). The plant has M stages. Each stage j, j=1,...M, contains Nj identical pieces of equipment. The number of units Nj and their corresponding sizes Vj are the design variables. In this design the products are processed one at time and follow the same sequence of processing stages.

The design chosen needs to be able to produce the required demand for each of the *HP* products. At the design stage the demand for product Qj i=1,...NP is unknown but can be specified by a probability distribution. Thus for the case with only uncertainties in the demands, the SF of a design measures the probability of satisfying these demands.

The constraints that define the system are shown below. The first constraint states that the cycle time Ty for each product i must be equal to the largest processing time in any stage (Sparrow et al, 1975):

$$T_{Li} = \max_j \left\{ \frac{t_{ij}}{N_j} \right\} \quad j=1\ldots M \qquad \wedge$$

Here ty is the time to process one batch of product i in stage j. It is assumed that ty is not a function of the batch size. The cycle time represents the time between the production of successive batches of the final product i (care must be taken when referring to product i, since all intermediates are also referred to as product i). This constraint can be rationalized in the following manner. Assume that we have a time period T in which to operate. Neglecting other stages, each stage j will be able to produce (T/tjj)*Nj batches of product «. For example, if product 1 has ti«j=3, ^2=6 and t-|3=4 and we have T=15 the first stage will be able to produce 5 batches, the second 2.5 and the third 3.75 with N=1 in all stages. Obviously if the stages are connected together (regardless of the order) the second stage is rate limiting, determining the rate at which the final product is produced. To determine this rate for product i we can set

$$T^* \left\{ \frac{N_j}{t_{ij}} \right\} = 1 \quad \text{batch} \qquad (2)$$

and solve for T which gives {ty/Nj}. Choosing the largest time as in (1) then gives the time necessary to produce one batch.

Next, the maximum batch size for each product i is given:

$$B_i = \min_j \left\{ \frac{V_j}{S_{ij}} \right\} \quad j=1,\ldots M \qquad (3)$$

Here Vj is the size of the equipment in stage j and Sy ,the size factor, is the capacity required to process a unit mass of product. This constraint states that the batch size cannot exceed the volume divided by the size factor for all stages j.

The production constraint is given by

$$T \quad B_i \quad " \qquad (4)$$

where H is the horizon and represents the total time available to process all products. The quotient Qj/Bj is equal to the total number of batches that will be processed for product i. This quantity multiplied by the time, Ty , to produce successive batches gives the total time to process product i. In using this constraint we are assuming single product campaigns where we are neglecting the effect of the heads and tail in the schedule of each product i.

Finally, we have a cost function in terms of the number of parallel units, Nj, and the equipment sizes VJ:

$$\mathrm{Cost} = \sum_{J} \alpha \quad (5)$$

where CCJ and pj are constants, with 0< pj < 1.

There are several interesting aspects of this model. Most notable is the fact that the feasible region is linear in the space of the demands Qj, once we have specified the design variables N_j and V_j. This follows from the fact that from equations (1) and (3) the ratios

$$\gamma_i = \frac{T_{L\,i}}{B_i} = \max_j \left\{\frac{t_{ij}}{N_j}\right\} / \min_j \left\{ \quad_{ij} \right\} \qquad i=1,\dots,NP$$

are constant. In this way the horizon constraint in (4) reduces to the linear inequality:

$$\sum_{i} \gamma_i Q_i \leq H \quad (6)$$

An example of a feasible region for 2 products is given in Figure 3.

Evaluation of Stochastic Flexibility

In order to evaluate the SF of a particular design we need to integrate the joint probability distribution function of the demands Qj, i=1 ,...NP, over the corresponding feasible region. One way to do this is to apply the inequality reduction scheme of Straub and Grossmann (1989). But given that there is effectively only one constraint (6), the inequality reduction scheme simplifies a great deal. To explain how to evaluate the SF first consider the following integral, for the case of two demands:

$$SF = \int_{JO}^{Q_1^{MAX}} \int_{JO}^{Q_2^{MAX}(Q_1)} j(Q)\,dQ2\,dQi \quad (7)$$

Here Q_1^{max} is obtained by setting $Q_2 = 0$ in the horizon constraint (6) resulting in

$$Q_1^{max} = \frac{H}{YI} \quad (8)$$

The upper bound on the inner integral can be written in terms of Q^ through the horizon constraint:

$$Q_2^{max} = (H - Q_1 * \gamma_1)/\gamma_2 \quad (9)$$

Equations (8) and (9) then define the bounds on the integral. In general the form of j(Q) will be complex enough to force the integral to be evaluated numerically. An appropriate method is to use a two dimensional Gaussian Quadrature, see Straub and Grossmann (1990). The only difficulty with this method is that with a larger number of products the integral becomes increasingly expensive to solve.

There is, however, another way to determine the SF that takes advantage of the structure of the problem. Note that the SF also represents the probability that the horizon constraint will be satisfied,

$$SF=Pr\{XQiY£H\} \quad i \tag{10}$$

Now assume that we let H_A equal the left hand side of the horizon constraint,

$$H_A=ZQiYi \quad i \tag{11}$$

Then the SF is simply the probability that H_A ^ H. If we further assume that the demands for products Qj, i=1,...NP, are characterized by normal distributions, fj(Qj), then since H_A is a linear function of Qj, i=1,..NP, H_A is also distributed normally with the following mean and variance (see Appendix)

$$\mu_{H_A}=\sum_i \gamma_i \mu_{Q_i} \tag{12}$$

$$\sigma^2_{H_A}=\sum_i \gamma_i^2 \sigma^2_{Q_i} \tag{13}$$

where JXQJ and $a^2{}_{Qj}$ are the mean and variance of the density functions fj(Qj), i=1,...,NP.

This transformation allows us to formulate the SF as a one dimensional integral, regardless of how many products there are:

$$SF=(\overset{H}{} f(H_A)dH_A \quad JO \tag{14}$$

Example 1

In order to clarify the concept of the SF a small example will be presented that involves 2 products that are processed in 3 stages. The data are shown in Table 1. This example is taken from Grossmann and Sargent (1979). The horizon time for this problem is 6000 hours. The first design that will be evaluated is the solution from Grossmann and Sargent (1979), that is shown in Table 2(a). This design was obtained by optimizing the volumes and the number of units for the mean values of the demands. The SF of this design was evaluated using 9 point Gaussian quadrature on the integral in (14). In the actual implementation of the quadrature scheme the lower bound was set equal to |!HA .3* <JHA, for values of H_A smaller than this the integral of $f(H_A)$ over the region 0 to liHA -3* <JHA is negligible in comparison to the integral over the region HHA "3* G HA t0 H- Evaluation of the design results in SF=0.498 with a design cost of $106,769. This result means that this design can only operate feasibly in 49.8% of the situations that are likely to be encountered. Consider in contrast the alternative design in Table 2(b) which has somewhat larger volumes. The cost of this design is $110,000, a 3.3% increase in comparison to the previous design. This design results in a SF=0.815, or an increase of 63.7% in the SF.

Evaluation of the Expected Stochastic Flexibility

Another type of uncertainty that may occur in a batch plant is equipment failure. Unlike the continuous uncertainties in product demand, equipment failure is characterized by a discrete probability distribution. Because of the discrete nature of the distributions we can define different states in the plant. The total number of discrete states for equipment availability is $2^{\sum_j N_j}$ however since the units in each stage are identical we can greatly reduce the number of states we need to analyze. For example in a plant with one stage and two identical pieces of equipment we have 3 conditions, both available, one available and none available. We characterize the different states in the following manner. First recall that Nj is the total number of units that exist in stage j. A state k is defined by the number of active or available units in each stage, n_j^k, where $0 \le n_j^k \le N_j$. A state will be characterized by the set $S_k=\{n_1^k, n_2^k, \ldots, n_M^k\}$, which corresponds to the number of active units in each stage. Furthermore, we will say that a state l is a substate of k if $n_j^l \le n_j^k$ for all j.

Since we have Nj+1 conditions of equipment at each stage, the total number of states in terms of the number of available pieces of equipment in each stage is given by

$$TS=\prod_{j=1}^{M} (N_j + 1) \tag{15}$$

Furthermore, let us define condition Nj+1 as the one that corresponds to the case when no unit is available in stage j; since at this condition there is infeasible operation, the stochastic flexibility is zero. Therefore, total number of feasible states with $n_j \ge 1$ j=1,...,M is given by

$$TFS=\prod_{j=1}^{M} (N_j) \tag{16}$$

So for example if we have a plant with 3 stages (M=3), three units in stage 1, two units in stage 2 and one unit in stage 3, TS=24 states and TFS=6 states. Also note that the number of feasible states TFS, is considerably smaller that the total number of actual states 2^6=64.

The probability of a piece of equipment being active, pj, can be generated from failure rate and repair rate data. The determination of Pj in the context of availability is as follows. First assume that the equipment is in its useful life, where failures occur by chance, rather than by the initial break-in or wearout. In this case it can be assumed that the failure rate λ is constant. It will also be assumed the repair rate μ is constant. The reliability of this component is defined as follows:

$$R(t) = \exp(-\lambda t) \tag{17}$$

This expression can be used to define the mean time to failure (MTTF) of the components:

$$MTTF = \int_0^{\infty} \exp(-\lambda t)\, dt = 1/\lambda \tag{18}$$

Similarly the mean time to repair is given by

$$MTTR = \int_0^\infty \exp(-\mu t)\, dt = 1/\mu \qquad (19)$$

Using Markov Chain **theory** (Billinton and Allan 1983) it can be shown that the probability of being in an active state as $t \to \infty$ is

$$p_j = \frac{\mu}{\lambda+\mu} = \frac{MTTF}{MTTF+MTTR} \qquad (20)$$

Note that this differs from the reliability of the component which represents the probability of staying in the active state as a function of time. More complicated models are discussed by and Billinton and Allan (1983).

In order to compute the probability Pk of each state k, k=1,...TFS, in terms of the probabilities Pj, we use the following expression:

$$p_k = \prod_{j=1}^{M} \frac{(N_j)!}{(n_j^k)!\,(N_j - n_j^k)!}\, p_j^{\,n_j^k}\,(1-p_j)^{(N_j - n_j^k)} \qquad k=1,\dots,TFS \qquad (21)$$

To briefly justify this expression, the first portion simply represents the number of combinations of n_j^k items from Nj total items. The second part is the probability of that particular combination of active and inactive equipment. With (21) the expected stochastic flexibility E(SF) can be determined from

$$E(SF) = \sum_{k=1}^{TFS} SF_k * P_k \qquad (22)$$

Bounding Procedure

An efficient bounding scheme for the E(SF) has been developed by Straub and Grossmann (1989) to avoid analyzing all possible states. The basis for the scheme is that the feasible region gets smaller as the number of active components decreases (assuming that the volumes remain constant). That is $SF(S_a) \le SF(S_b)$ for $S_b \subset S_a$.

They show that a valid lower bound, LB, can be obtained by taking a partial summation of (22). This summation would **only** include terms whose SF has been evaluated. A valid upper bound, UB, can be obtained by adding to **the** lower bound the remaining terms with BSF(Sk) substituted for the SF(Sk). The BSF is an upper bound on the state SF.

The manner in which the BSF are obtained is best illustrated with an example. Consider a 2 stage batch plant in which $N_1=3$ and $N_2=2$. In this case TS=(3+1)(2+1)=12 and TFS=3*2=6 states. A network representation of the states is shown in Figure 4. In this network the lines connect a state to its substate. For example $S_5=\{2,1\}$ is a substate of both $S_2=\{3,1\}$ and $S_3=\{2,2\}$. Since $S_5 \subset S_2$ then $SF(S_5) \le SF(S_2)$; similarly $SF(S_5) \le SF(S_3)$. Thus, a valid upper bound of $SF(S_5)$ is given by $BSF(S_5)=\min[SF(S_2), SF(S_3)]$. Note that if $SF(S_2)$ and $SF(S_3)$ have not been evaluated yet, then their corresponding bounds, $BSF(S_2)$ and $BSF(S_3)$, can be used to compute the bound for S_5. That is, $BSF(S_5)=\min[BSF(S_2), BSF(S_3)]$.

To illustrate the bounding procedure more clearly, consider Figure 5 in which the probabilities for each discrete state are shown. These result from assuming p-^0.9 and $p_2=0.8$. For the network of Figure 5 only $SF(S_1)=0.95$ has been evaluated. Because all the remaining states are substates of S_1 they are all bounded by $SF(S_1)$. In this case the following bounds would be calculated as follows:

$$LB=SF(Si)*P(Si) \tag{23}$$

$$UB=SF(Si)*P(Si)+\sum_{i=2}^{TFS} BSF(Sj)*P(Sj) \tag{24}$$

Substituting the corresponding values in the equations above leads to LB=0.4433 and UB=0.9112. It is important to keep in mind that the summation is over TFS not TS, thus any state with nj=O is assigned SF=0.

Now assume the $SF(S_2)$ is evaluated next since it has, from the remaining states, the largest potential contribution to the E(SF); $BSF(S_2)*P(S_2)=0.2216$. Assume that $SF(S_2)$ is found to be 0.7, with which the changes to the BSPs are shown in Figure 6. In this case the BSF for states S_5 and S_8 change to 0.7 since they are substates of S_2.

The new bounds are then:

$$LB=SF(Si)*P(Si)+SF(S2)*P(S2) \tag{25}$$

$$UB=SF(Si)*P(Si)+SF(S2)*P(S2)+\sum_{i-3}^{TFS} BSF(S_i)*P(Si) \tag{26}$$

Substituting the corresponding values in the equations above leads to LB=0.6066 and UB=0.8312.

In this way, if we successively evaluate that state in the network with the largest BSF*P, and update the corresponding bounds in the substates, fast convergence for the lower and upper bounds can be obtained within a specified finite tolerance. This then avoids the problem of evaluating the SF for each state which is the major bottleneck in the computations.

The bounding procedure can be stated in general as follows. First we define the index sets E and U:

E={i|SF(Sj) is evaluated}
U={i|SF(Sj) is not evaluated} where EuU=TFS

The steps are then as follows for a specified tolerance e in the bounds:

1) Evaluate the stochastic flexibility SF(S-|) of the state S-jwith all components active. Set E={1}, U={2,3,...TFS}, and $BSF(S_i)=SF(S_1)$, iel).

2) Determine the bounds:

$$LB=\sum_{i\in E} SF(Si)*P(Si) \tag{27}$$

$$UB = \sum_{i \in E} SF(S_i)*P(S_i) + \sum_{i \in U} BSF(S_i)*P(S_i) \quad (28)$$

3) If UB-LB<e stop. Otherwise go to step 4

4) Let j be the index of the largest value of BSF(Sj)*P(Sj), ie U.
Set E=Eu{j}, U=U\{j}

5) Evaluate SF(Sj) update the BSF(Sj) for ie U and go to step 2.

Example 2

Consider the system presented in Example 1 for the design in Table 1(a). The only additional design parameters are the probabilities of the equipment being active. Initially we will let all Pj=0.9, j=1,2,3. Given N=(2,2,1) we can determine TS=18 and TFS=4 with state probabilities shown in Table 3. For the design presented in Table 2(a), the bounds converge to E(SF)=0.294. It was necessary to evaluate 3 states in order for the bounds to converge. This data is shown in Table 4. The result can be interpreted as follows, of all possible combinations of demands and system states, 29.4% of these combinations result in feasible operation.

There are some interesting aspects of the E(SF). First, if we increase the horizon time to 10,000 hours which effectively assigns SF=1 to all states that have a feasible path then the resulting E(SF) is equal to the systems reliability. For the given pj and the system configuration the resulting reliability is 0.882. When we evaluate the E(SF) the bounds also converge to 0.882. It is also important to note here that a feasible state for reliability is one in which the production is nonzero. One might ask how to calculate the reliability when constraints on product demand must be considered. This is easy to calculate with the present formulation. Suppose we define a feasible state as one producing a minimum of 180,000 units of Q^ and 40,000 units of Q2; then the reliability is equal to the E(SF) when we set the mean of Qj to 180,000 and 40,000 and set the standard deviations to a small positive number (e.g. 10). Doing so gives us a E(SF)=0.7217 which is then the reliability of the system for these demands.

Another aspect of the E(SF), which is more obvious, is that it converges to the SF of the state with all components active as Pj -> 1.00 (recall Example 1). This is demonstrated in Table 5.

Example 3

To demonstrate the effectiveness of the bounding procedure consider the following problem with 6 stages and 5 products. The data for the problem is given in Table 6. The horizon time for the problem is 6000 hours. Given the maximum available number of units in Table 6(d) we can calculate TS=864 and TFS=72. When the bounding scheme was applied to this problem the following bounds were obtained: $0.7210 \le E(SF) \le 0.7239$. The bounding scheme required examination of 7 states, see Table 7. Thus only -10% of the feasible states and <1% of the total states were required to evaluate the E(SF). The progression of the bounds is shown in Figure 7.

Optimization of SF

Having considered the evaluation of expected stochastic flexibility for a fixed design, the next step would be to consider the selection of volumes and number of units to maximize the expected stochastic flexibility and to minimize the investment cost. Since this leads to a bicriterion optimization problem, we will consider that the optimization problem is formulated as maximizing flexibility subject to a constraint for maximum investment. This problem can then be easily extended to generate trade-off curves that relate flexibility with cost.

The first case we will present is the one in which the number of units is fixed and we seek the determine the optimal volumes. For the case of normal distributions this problem can be formulated as the NLP,

$$
\begin{aligned}
&\max_{V}\ SF=\int_0^H f(H_A)\,dH_A \\
&\text{s.t.}\ \ B_i=\min_j \{\frac{V_j}{S_{ij}}\} \quad i=1,\dots,NP \\
&\qquad \sum_j \alpha_j N_j V_j^{\beta_j} \le C \\
&\qquad \gamma_i = T_{L\,i}/B_i \quad i=1,\dots,NP \\
&\qquad \mu_{H_A}=\sum_i \gamma_i \mu_{Q_i} \\
&\qquad \sigma^2_{H_A}=\sum_i \gamma_i^2 \sigma^2_{Q_i} \\
&\qquad B_i^L \le B_i \le B_i^U \\
&\qquad \gamma_i \ge 0 \quad i=1,\dots,NP \\
&\qquad H,\ \mu_{H_A},\ \sigma^2_{H_A} \ge 0 \\
&\qquad V_j^L \le V_j \le V_j^U \quad j=1,\dots,M
\end{aligned}
\qquad (P1)
$$

where C is a limit on the capital investment, V_j^L and V_j^U are lower and upper bounds for the volumes, and B_i^L and B_i^U are lower and upper bounds on the batch size:

$$\min_j \frac{V_j^L}{S_{ij}} \le B_i \le \min_j \frac{V_j^U}{S_{ij}} \qquad (29)$$

Note that in this formulation the cycle time T_{Li} is a parameter, since its value is determined by the value of N_j which is fixed. Problem (P1) is a difficult problem to solve since the objective requires a numerical integration technique and the first set of constraints are non-differentiable. However, this formulation can be simplified a great deal. The first change would be to write the batch size constraint as the following system of inequalities.

$$B_i \le \{\frac{V_j}{S_{ij}}\} \quad i=1,\dots,NP \quad j=1,\dots,M \qquad (30)$$

The purpose of doing this is to avoid nondifferentiabilities in the NLP. Next, the following transformations can be used to convexify the problem (see Kocis and Grossmann, 1988):

$$
\begin{aligned}
&Bj=exp(bj)\\
&Vj-exp(vj)\\
&Nj=exp(tij)\\
&Ty=exp(tLi)\\
&\gamma_i=exp(\xi_i)
\end{aligned}
\qquad (31)
$$

Applying these transformation results in the following NLP:

$$
\begin{aligned}
\max_{V}\ & SF=\int_0^H f(H_A)dH_A\\
\text{s.t.}\ & b_i \wedge Vj - \log(S_{ij})\quad i=1,\ldots,NP\quad j=1,\ldots,M\\
& X \ll j\ exp(nj+Pj\ VJ) \le C\\
& \quad j\\
& \xi_i=t_{Li}-b_i\\
& \mu_{H_A}=\sum_i \exp(\xi_i)\,\mu_{Q_i}\\
& \sigma^2_{H_A}=\sum_i \exp(2\xi_i)\,\sigma^2_{Q_i}\\
& \ln(B_i^L)\le b_i \le \ln(B_i^U)\\
& -\infty \le \xi_i \le \infty\\
& H,\ \mu_{H_A},\ \sigma^2_{H_A}\ \ge 0\\
& \ln(V_j^L)\le v_j \le \ln(V_j^U)\quad j=1,\ldots,M
\end{aligned}
\qquad (P2)
$$

We can make one more simplification for the objective function. The SF is the cumulative probability of HA up to the value of H. Presently we need to use some type of numerical integration technique to evaluate the SF. But recall that $f(H_A)$ is assumed to be a normal distribution. Thus we can apply the following transformation to normalize the distribution.

$$ {}^{z}=({}^{H}"HHA)^{/a}HA \qquad (32) $$

The transformed variable is distributed normally with mean 0 and standard deviation 1. More important though, is the fact that the SF can now be written as

$$ \qquad (33) $$

For all practical purposes the lower bound is essentially $-\infty$, since the function is essentially constant between $-\infty$ and -4, and the lower bound is in general less than -4. Thus the SF is only a function of z. A plot of SF versus z is shown in Figure 8. Note that SF and z are monotonic, thus maximizing SF is equivalent to maximizing z. This fact allows us to rewrite the NLP as follows:

$$
\begin{aligned}
\max\ & z=(H-ti_{HA})/a_HA\\
\text{s.t.}\ & bi\le Vj - \log(Sij)\quad i=1,\ldots,NP\quad j=1,\ldots,M\\
& X\ \alpha_j \exp(ij+Pj\ VJ) \le C\\
& \quad j\\
& \xi_i=t_{Li}-b_i\\
& \mu_{H_A}=\sum_i \exp(\xi_i)\,\mu_{Q_i}
\end{aligned}
$$

$$\sigma^2_{HA}=\sum_i \exp(2\xi_i)\,\sigma^2_{Qi} \qquad \text{(P3)}$$
$$\ln(B_i^L)\le b_i \le \ln(B_i^U)$$
$$-\infty \le \xi_i \le \infty$$
$$H,\ \mu_{HA},\ \sigma^2_{HA} \ge 0$$
$$\ln(V_j^L)\le v_j \le \ln(V_j^U) \quad j=1,\dots,M$$

This problem reduces to a standard NLP problem that is much easier to solve than the original problem (P1).

Example 4

To demonstrate the optimization of the SF consider the system presented in Example 1, the 3 stage 2 product problem. Using the formulation in P3 we have determined the optimal SF for N=(2,2,1) and various values of the cost limit C using GAMS/MINOS on an HP 800. The trade off curve is shown in Figure 9. It should be noted that the alternative solution in Example 1 corresponds to a point in the tradeoff curve. Also note how much the optimal SF increases between $100,000 and $110,000 as compared from $110,000 to $120,000 where it becomes increasingly more expensive to increase the SF. It is interesting to compare the volumes at $100,000 where SF=0.023 and at $110,000 where SF=0.816. For the former V=(1076, 1614, 2152) while for the latter V=(1265, 1897, 2500). As can be seen a modest increase in the sizes leads to a large increase in the stochastic flexibility. Also in this case the overdesigns are of the order of 17%.

It is also of interest to optimize over Nj the number of units in each stage. This is a straightforward extension of problem P3. First we need to add the cycle time as a constraint in the formulation. For the same reasons we modified the batch size constraint, we will also formulate the cycle time constraint as a system of inequalities, applying the convexification transformation:

$$t_{Lj} \ge \log(t_{ij}) - r_{ij} \quad i=1\dots NP \quad j=1\dots M \qquad (34)$$

We also need the following constraints which restrict the values of Nj to be integer values. In terms of the transformed variables η_j this is given by:

$$\eta_j=\sum_r y_{jr}*\log(r) \qquad (35)$$

$$\sum_r y_{jr}=1 \qquad y_{jr}=0,1 \quad j=1\dots M \quad r=1\dots N_j^U$$

where N_j^u is an upper bound on the number of units in stage j. The formulation (P3) with the additional constraints in (34) and (35) corresponds to an MINLP problem which will be denoted as (P4).

Example 5

To demonstrate the optimization over V and N consider the 6 stage 5 product problem in Example 2. For this problem the tradeoff curve shown in Figure 10 has been generated by solving the corresponding MINLP problem with DICOPT++ (see Viswanathan and Grossmann, 1990). It is interesting to compare the solutions at $260,000 with SF=0.109 and $290,000 with SF=0.877. The first solution is N=*(222211) and V=(3000, 1849 1974, 2560, 2316, 2062). The second solution is N=*(223211) and V=(3000, 1984, 1974, 2748, 2442, 2213).

Optimization of E(SF)

Having developed a NLP to optimize the SF over V we will now extend the formulation to determine the volumes which maximize the E(SF) for a fixed number of units and a specified cost. The NLP to solve this problem is very similar to (P3).

$$
\begin{aligned}
\max\ & E(SF)=\sum_k P_k * SF_k(z_k) \\
\text{s.t. } & z_k=(H-\mu_{H_A}(k))/\sigma_{H_A}(k) \\
& b_i \le v_j-\log(S_{ij}) \quad i=1,\dots,NP \quad j=1,\dots,M \\
& \sum_j \alpha_j \exp(\eta_j^{TS}+\beta_j v_j) \le C \\
& \xi_i(k)=t_{L_i}(k)-b_i \\
& \mu_{H_A}(k)=\sum_i \exp(\xi_i(k))\,\mu_{Q_i} \\
& \sigma^2_{H_A}(k)=\sum_i \exp(2\xi_i(k))*\sigma^2_{Q_i} \\
& \ln(B_i^L)\le b_i \le \ln(B_i^U) \\
& -\infty \le \xi_i \le \infty \\
& H,\ \mu_{H_A},\ \sigma^2_{H_A} \ge 0 \\
& \ln(V_j^L)\le v_j \le \ln(V_j^U) \quad j=1,\dots,M
\end{aligned}
\qquad (P5)
$$

where η_j^{TS} in the cost constraint are the number of units in the top state (i.e. $\exp(\eta_j^{TS})= N_j$). For each state S_k ($k=1,\dots,TFS$) we will have a different cycle time $t_L(k)$, mean $\mu_{H_A}(k)$, standard deviation $\sigma_{H_A}(k)$ and SF(k). In this case we cannot simply maximize the summation with z_k substituted for SF_k. The reason for this is that E(SF) and E(z) do not have a direct relationship as opposed to SF and z which does. In order to avoid evaluating the SF by numerical integration we can employ the following nonlinear approximation of the curve shown in Figure 8.

$$
SF(z)=\frac{\exp(1.7009*z)}{1+\exp(1.7009*z)} \qquad (36)
$$

This approximation was determined by nonlinear curve fitting. Thus, problem (P5) can be simplified as the NLP problem:

$$
\begin{aligned}
\max\ & E(SF)=\sum P_k * SF_k(z_k) \\
\text{s.t.}\ & z_k=(H-\mu_{HA}(k))/\sigma_{HA}(k) \\
& SF(z_k)=\frac{\exp(1.7009 * z_k)}{1+\exp(1.7009 * z_k)} \\
& b_i \le v_j-\log(S_{ij}) \quad i=1,\ldots,NP \quad j=1,\ldots,M \\
& \sum_j \alpha_j \exp(n_j^{TS}+\beta_j v_j) \le C \\
& \zeta_i(k)=tL_i(k)-b_i \\
& \mu_{H_A}(k)=\sum_i \exp(\xi_i(k))\,\mu_{Q_i} \\
& \sigma^2_{H_A}(k)=\sum_i \exp(2\xi_i(k)) * \sigma^2_{Q_i} \qquad \text{(P6)} \\
& \ln(B_i^L)\le b_i \le \ln(B_i^U) \\
& -\infty \le \xi_i \le \infty \\
& H,\ \mu_{H_A},\ \sigma^2_{H_A} \ge 0 \\
& \ln(V_j^L)\le v_j \le \ln(V_j^U) \quad j=1,\ldots,M
\end{aligned}
$$

Example 6

Again consider the 3 stage 2 product problem of Example 1. For this example let p_j=(0.93, 0.95, 0.89). The top state N_j is chosen to be (322). The network structure in Figure 11 shows all possible states S_k in the set TFS. The set TFS may contain a large number of states depending on the size of the system and the top state chosen. A simple way to reduce the size of the set TFS is to determine the states whose production capacity is limited by the number of units in each stage. These states can be determined by evaluating the SF for each state with the capacities equal to V_j^{max}. In this problem V_j^{max}=2500ℓ for all j. Evaluating the SF for each state as in Example 1 we obtain the results shown in Table 8. This table shows that 8 states do not have enough units to produce even the smallest value of the uncertain demand. Since these states cannot contribute to the E(SF) they can be removed from the set TFS, leaving 4 states. In order to formulate (P6) we need to determine the probability of each state and also the cycle time, using equations (21) and (1) respectively. This data is shown in Table 9. The results of solving problem (P6) for this example are shown in a trade off curve, Figure 12. It is interesting to compare the results at a cost of 135,000 and 150,000. The volumes are (995, 1493, 1990) and (1186, 1779, 2372) respectively. The E(SF) are 0.208 and 0.781 respectively. As expected, as the cost increases the E(SF) converges to 0.8790, the sum of the probabilities of the 4 states remaining in set TFS. That is, the SF of each of the states goes to 1.0.

Determination of the optimal number of units and the corresponding capacities to maximize the E(SF) is a much more complicated problem. The complexity is due to the change in the size of the set TFS, depending on the choice of the number of units in N_j, the top state. For example consider the network structure in Figure 13. Here the top state is (333) and there are a total of 27 states in the set TFS. But note that if the plant has 2 parallel units in stages 2 and 3 the top state is (322), as in the last example, we would only have 12 states in the set TFS. This dynamic change in the set size is very difficult to incorporate in an MINLP. An interesting side note is that the network structure for the top state (322) is embedded in the network structure for the top state (333).

In order to determine the optimal volumes and number of units to optimize the E(SF) for a fixed cost, an effective algorithm will be presented that involves evaluation of the SF and E(SF) and the optimization problems described earlier in the paper. The algorithm is best demonstrated with an example. A summary of the detailed steps will be presented after the example.

Example 7

Consider the 3 stage 2 product problem described in Example 6. In this case the maximum number of units N_j^u in any stage is 3. Therefore there are 27 different states as in Figure 13. One way to solve this problem is to solve problem (P6) for every possible combination of units. This is equivalent to solving (P6) for each of these states, i, and its corresponding substates, k(i). Although this is a valid method of solving the problem it can be computationally expensive to do so. An obvious simplification is to eliminate states from the set that needs to be evaluated. This will reduce the number of times (P6) needs to be applied. Note that eliminating a state from the set that needs to be evaluated doesn't imply that it is totally eliminated from the problem. It is still a substate and therefore may need to be included in the problem (P6) for other states that do need to be evaluated.

All 27 states are shown in Figure 13. This form is especially convenient since for any particular state its substates are easily identified. For example if N=(122) we can identify (112), (121), and (111) as substates to be included in (P6). Similarly if N=(333), (P6) would contain all the states shown in the figure.

The set of states for which we need to apply problem (P6) will be labeled PTS. States will be eliminated from this set if the upper bound on the E(SF) is less than the current lower bound on the E(SF). Initially all 27 states are included in this set. In addition a set SI contains the states which might need to be included in the program (P6) as substates. Initially all 27 states are included in this set also. Obviously not all states in the set SI are included in (P6), only those which are substates of the state being evaluated. Also states can be eliminated from the set SI if their SF=0.

The first step of the algorithm is to determine a lower bound on the optimal E(SF). This will be used in the next two steps to eliminate states from the set PTS. A good heuristic lower bound can be obtained by determining the optimal V and N that maximize the SF (i.e. solving problem P4) and using the results to evaluate the E(SF). In this example solving the MJNLP problem (P4) for a cost of $150,000 results in N*=(331) and V*=(1244, 1866,2488). The E(SF) was evaluated to be 0.865 (this number is actually the first three decimal places of the upper and lower bound from the bounding scheme), which gives us an initial lower bound on the E(SF).

The second step of the algorithm involves determining the absolute maximum SF that a state can have. This is done by evaluating the SF of each state at V^{max} using equation (14). The resulting SFj^{max} is an upper bound on the E(SF) for state i. Any state i whose $SFj^{max}<0.865$ can be eliminated from the set PTS. Also any state in which $SFj^{max}=O$ can be eliminated from the set SI since it cannot contribute to the summation. The results of the evaluation are shown in Table 10. Based on these results we can eliminate the following states from both PTS and SI since they have $SFj^{max}=O$, which is less than 0.865, (313), (123), (213), (312), (133), (122), (212), (311), (112), (121), (211), and (111). In addition states (133), (132), and (131) have $SFj^{max}=0.16$ and thus can be eliminated from the set PTS. Figure 14 is a revised version of Figure 13. In this figure the states eliminated from SI are also removed from the state network. The states that remain in the set PTS are circled. This step has eliminated 15 out of the 27 states in the set SI greatly reducing the number of states for which (P6) needs to be applied.

The third step in the algorithm is similar to the second step except that the SFj^{max} is determined using (P3). Since the cost constraint is being used to limit the volumes it is expected that the SFj^{max} in this step will be smaller than in the previous step. Given the new SFj^{max} states can be eliminated from the sets as in the last step. The results of solving (P3) are shown in Table 11. Based on the results state (133) can be eliminated from SI since $SFj^{max}=O$. Also states (333), (233), (323), (133), (223), (132), and (131) can be eliminated from PTS since SFj^{max}£0.856. At the end of this step SI contains 14 states and PTS contains 8 states. The resulting state network is shown in Figure 15. An interesting side note is that we could eliminate even more states at this point. As shown in Figure 15 (333), (233), (323), and (223) are not substates of any state in the set PTS and thus they could be eliminated from the set SI. Since this doesn't influence the computational efficiency of the algorithm this is not included as a step.

The fourth step involves calculating lower and upper bounds on the E(SF) for states in the set PTS. The lower bound is calculated in the following manner. For state i use the optimal V* from (P3) in the previous step to evaluate the E(SF) for state i. That is, for each substate k use V* from state i and N from the substate in equation (14) to get SFk(j)(V*) in equation (37). Depending on the number of substates, k(i), for i the bounding procedure described earlier can be used to obtain bounds on the lower bound, but in general the lower bound can be written:

$$E(SF)[^{X)}=Pi * SFf'' + \underset{k(i)}{X}\ {}^{p}k(i) * SF_{k(i)}(V^*) \qquad (37)$$

Actually, since this is a maximization problem, any V that satisfies the cost constraint will result in a lower bound. Intuitively though, it is expected that V* will give a very good good lower bound. An upper bound can be obtained by using the following expression:

$$E(SF \wedge P\ i * SF?^{ax} + \underset{k(i)}{X}\ Pk(i) * SFgg \qquad (38)$$

This is an upper bound since SF|<^{max} from (P3) in the previous step is an upper bound on SFj<. To clarify how these bounds are calculated consider state i=(231) in Figure 15 which has substates k=i (131) and k=2 (221) in the set SI. The optimal solution from (P3) for (231) is V*=(1618,2427,2500). This solution is used to evaluate the SF of (131) and (221), and results in SF(k=1)=0.1062 and SF(k=2)=0.9973. Using equation (21) to evaluate the probabilities the following result is obtained:

$$E(SF)^{LO} \quad «(0.6600)*(1.00)+(.0994)*(0.1062)+(0.1042)*(.9973)$$
$$=0.7745$$

The upper bound is obtained by using SF^{max} from (P3) in step 3 (see Table 11) for all states:

$$E(SF)^{UP} \quad =(0.6600)*(1.00)+(.0994)*(0.16)+(0.1042)*(1.00)$$
$$=0.7801$$

Repeating this for each state gives the results shown in Table 12. Having lower and upper bounds for each state in the set PTS the size of PTS can be reduced by eliminating states whose upper bound is smaller than the largest lower bound (i.e. 0.8635 for state 331). Doing this results in the elimination of the following states: (232), (222), (231), (321), and (221). The state network is shown in Figure 16. The size of PTS has been reduced to 3 states.

The fifth and final step is to use the NLP in (P6) to determine the optimal volumes that maximize E(SF) for each state in PTS. The optimal design is the one resulting in the largest E(SF). The results of this step are shown in Table 13. The optimal solution occurs with N=(331). In this instance the optimal V and N that maximize the SF also maximize the E(SF). This is not always the case but it demonstrates why the solution to the optimal SF problem is used in step 1. The circumstances under which the optimal solutions will be the same is not clearly defined. There are many trade offs that need to be considered. For example, when a stage has a very low probability it is appropriate to add an extra unit, but doing so reduces the capital that can be spent on the other stages. Thus, the E(SF) may actually decrease by adding the extra unit.

Table 14 shows the optimal number of units for various costs. This data is plotted in the tradeoff curve, Figure 17. Unlike the previous tradeoff curves this one is not as smooth due to the changing number of units as the capital investment increases.

The algorithm can be summarized as follows.

Step 0 Define the set PTS with the maximum number of units to be considered. Also setSUPTS.

Step 1 Determine the optimal V and N which maximize the SF by solving the MINLP P4. Then evaluate the E(SF) at these conditions; this value is the current lower bound on the E(SF), $E(SF)^{LB}$.

Step 2 Using (14) evaluate the upper bound SF^{max} for each state i in the set SI using V^{max} as the capacity (i.e. with no regard for the cost). Eliminate any state i for which $SF_i^{max} \leq E(SF)^{LB}$ from the set PTS. Also eliminate states i with $SF_i^{max}=0$ from the set SI.

Step 3 For each state in the set SI determine the actual SF_i^{max} and its optimal volumes V^* by solving the NLP in (P3) with the corresponding cost constraint. Eliminate any state i for which $SF_i^{max} \leq E(SF)^{LB}$ from the set PTS. Also eliminate states i with $SF_i^{max}=0$ from the set SI.

Step 4 Calculate lower and upper bounds on the E(SF) for each state i in the set PTS:

$$E(SF)_i^{LO}=P_i * SF_i^{max} + \sum_{k(i)} P_{k(i)} * SF_{k(i)}(V^*)$$

$$E(SF)_i^{UP}=P_i * SF_i^{max} + \sum_{k(i)} P_{k(i)} * SF_{k(i)}^{max}$$

Here k(i) are the substates of i in the set SI. Eliminate from the set PTS any state whose upper bound is smaller than the largest lower bound.

Step 5 Apply the formulation in (P6) to all states in the set PTS. The largest resulting E(SF) is the optimal solution.

The importance of this algorithm is that it provides a rigorous solution to the maximization of the E(SF) for choices of N_j and V_j without having to do an exhaustive evaluation of E(SF) for each discrete alternative.

Conclusions

The paper has presented a measure for the probability of feasible operation for batch plants with uncertainties in product demands and equipment availability. As has been shown efficient procedures for evaluation of both the SF and E(SF) can be developed for the case when the demands are described by normal distribution functions. Also efficient NLP's and MINLP and algorithms have been presented for the optimization of the flexibility measures by exploiting a number of properties. Finally a novel bounding procedure for the optimization of E(SF) with number of units and volumes as decision variables has been presented to greatly reduce the computational expense. As has been shown by the results the proposed methods can be used to assess the trade-offs involved in flexibility and investment cost.

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Appendix

Proof of Property 1

If XT and X_2 are normally distributed, then Y= *aX^* + bX_2 is also normally distributed with mean *aX,* + bX_2 and variance $a^2VAR(X_1)+b^2VAR(X_2)$

Mean:

$E(aXi + bXz) = J X(^{aXi} + bX_2)\ fi(Xi)f_2(X_2)\ dXi\ dX2$

$= (\text{"}flaXi\ fi(Xi)f2(X2)\ dXi\ dX_2 + J\text{"}J\text{"}bX2fi(Xi)f2(X2)dXi\ dX2$

$-a£Xifi(Xi)dXi\ +b\ \}\text{"}\ bX_2\ f_2(X_2)\ dX2$

$=a\ E(Xi)\ +b\ E(X2)$

Variance: first note that

$E((aXi + bX_2)^2) = J X^{(aXi + bX2)2\ fi}(^{Xi})^{f2}<X2)\ dXi\ dX_2$

$= \int\int (a^2X\wedge + 2abX_1X_2 + b^2x|)\ fi(Xi)f_2(X_2)\ dXi\ dX_2$

$= a^2ff\ Xtfii(Xi)f2(X2)\ dXi\ dX2$

$+2ab\int\int X_1X_2\ fi(Xi)f2(X2)\ dXi\ dX2$

$+b^2\int\int X_2^2\ fi(Xi)f2(X2)\ dXi\ dX2$

$= a^2j\text{"}\ X_1^2\ fi(Xi)\ dXi$

$+2ab\ (J\sim Xi\ fi(Xi)dXi\ )\ (£X2f2(X2)\ dX_2\ )$

$+b^2fXl\ f2(X2)\ dX2$

$= a^2E(X?)\ +2ab\ E(Xi)E(X_2)+b^2E(X|)$

$VAR(Y)=E(Y^2)-[E(Y)]^2$

$VAR(aXi + bX_2)=E((aXi + bX_2)^2)-[E(aXi + bX_2)]^2$

$= a^2E(X?)\ +2ab\ E(Xi)E(X2)+b^2E(X|)-[aE(Xi)\ +bE(X_2)]^2$

$= a^2E(X_1^2)\ +2ab\ E(Xi)E(X_2)+b^2E(Xi)-[a^2E(Xi)^2\ *2ab\ E(Xi)E(X_2)\ +b^2E(X_2)^2]$

$= a^2E(X?)\ -a^2E(Xi)^2\ +b^2E(X_2{}^2)-b^2E(X_2)^2$

$= a^2\{E(X?)\ -E(Xi)^2\}+b^2\ \{E(X|)-E(X_2)^2\}$

$= a^2\ VAR(Xi)+b^2\ VAR(X2)$

Table 1 Data for Example 1

(a) Size Factors (l/kg)

product	stage 1	stage 2	stage 3
1	2	3	4
2	4	6	3

(b) Processing Times (hr)

product	stage 1	stage 2	stage 3
1	8	20	8
2	16	4	4

(c) Mean and Standard Deviation for Q (kg)

product	μ	σ
1	200,000	10,000
2	100,000	10,000

(d) Cost Coefficients

stage j	α	β
1	250	0.6
2	250	0.6
3	250	0.6

Table 2 Alternative Designs

(a) Alternative 1

stage	N_j	V_j
1	2	1200
2	2	1800
3	1	2400

(b) **Alternative 2**

stage	N_j	V_j
1	2	1265
2	2	1900
3	1	2500

Table 3 State Probabilities for Example 2.

State Number	1	2	3	Probability
1	2	2	1	0.59049
2	2	2	0	0.06561
3	2	1	1	0.13122
4	2	1	0	0.01458
5	2	0	1	0.00729
6	2	0	0	0.00081
7	1	2	1	0.13122
8	1	2	0	0.01458
9	1	1	1	0.02916
10	1	1	0	0.00324
11	1	0	1	0.00162
12	1	0	0	0.00018
13	0	2	1	0.00729
14	0	2	0	0.00081
15	0	1	1	0.00162
16	0	1	0	0.00018
17	0	0	1	0.00009
18	0	0	0	0.00001

Table 4 Computation of bounds for Expected Stochastic Flexibility

Iteration	State evaluated	State SF	LB	UB
1	221	0.4986	0.2944	0.4398
2	121	0	0.2944	0.3598
3	211	0	0.2944	0.2944

Table 5 Convergence of E(SF) to SF as $p_j \rightarrow 1.0$.

p	E(SF)
0.9	0.2944
0.92	0.3286
0.95	0.3858
0.97	0.4282
0.99	0.4742
0.999	0.4962

Table 6 Data for Example 3

(a) Size Factors (l/kg)

Product	Stage 1	2	3	4	5	6
1	7.9	2.0	5.2	4.9	6.1	4.2
2	0.7	0.8	0.9	3.4	2.1	2.5
3	0.7	2.6	1.6	3.6	3.2	2.9
4	4.7	2.3	1.6	2.7	1.2	2.5
5	1.2	3.6	2.4	4.5	1.6	2.1

(b) Processing Times (hrs)

Product	Stage 1	2	3	4	5	6
1	6.4	4.7	8.3	3.9	2.1	1.2
2	6.8	6.4	6.5	4.4	2.3	3.2
3	1.0	6.3	5.4	11.9	5.7	6.2
4	3.2	3.0	3.5	3.3	2.8	3.4
5	2.1	2.5	4.2	3.6	3.7	2.2

(c) Mean and Standard Deviations for demands (kg)

product	μ	σ
1	250,000	10,000
2	150,000	8,000
3	180,000	9,000
4	160,000	6,000
5	120,000	3,000

(d) Cost, Probabilites and Number of units

Stage	α	β	p	N_j	V_j
1	250	0.6	0.96	3	3000
2	250	0.6	0.98	2	1900
3	250	0.6	0.97	3	2000
4	250	0.6	0.95	2	2600
5	250	0.6	0.93	1	2300
6	250	0.6	0.98	2	2100

Table 7 **States evaluated and** resulting bounds **for** Example 3

iteration	n	Sf	LB	UB
1	323212	0.9972	0.6234	0.9243
2	223212	0.9247	0.6957	0.9166
3	323112	0	0.6957	0.8293
4	322212	0	0.6957	0.7569
5	323211	0.9918	0.7210	0.7568
6	313212	0	0.7210	0.7271
7	123212	0	0.7210	0.7239

Table 8 SF **of** states in the set TFS at maximum capacity for Example 6

k	State	SF^{max}
1	322	1.0
2	222	1.0
3	312	0.0
4	321	1.0
5	122	0.0
6	212	0.0
7	221	1.0
8	311	0.0
9	112	0.0
10	121	0.0
11	211	0.0
12	111	0.0

Table 9 State probabilities and Cycle times for states in Example 6

k	State	Probability	T_{L1}	T_{L2}
1	322	0.5750	10	5.33
2	222	0.1298	10	8
4	321	0.1421	10	5.33
7	221	0.0321	10	8

Table 10 State SF from the second step in Example 7

State	SF^{max}
333	1.00
233	1.00
323	1.00
332	1.00
133	0.16
223	1.00
232	1.00
313	0.00
322	1.00
331	1.00
123	0.00
132	0.16
213	0.00
222	1.00
231	1.00
312	0.00
321	1.00
113	0.00
122	0.00
131	0.16
212	0.00
221	1.00
311	0.00
112	0.00
121	0.00
211	0.00
111	0.00

Table 11 State SF from the 3rd step in Example 7

State	SF^{max}
333	0.08
233	0.03
323	0.01
332	1.00
133	0.00
223	0.03
232	0.99
322	1.00
331	1.00
132	0.02
222	0.99
231	1.00
321	1.00
131	0.16
221	1.00

Table 12 Resulting Lower and Upper bounds from step 4 of Example 7

State	LB	UB
332	0.7035	0.9675
232	0.7159	0.8535
322	0.7821	0.8791
331	0.8635	0.8728
222	0.7692	0.7711
231	0.7745	0.7801
321	0.7899	0.7920
221	0.6947	0.6947

Table 13 Optimal E(SF) from 5th step of Example 7

State	E(SF)
332	0.7036
322	0.7820
331	0.8650

Table 14 Trade-off curve data for Example 7

cost	N	E(SF)
100000	221	0.0312615
102000	221	0.0923951
105000	221	0.20841
106000	221	0.284827
109000	221	0.5196356
112000	221	0.6425975
115000	221	0.6694
120000	221	0.6913
125000	321	0.728
135000	321	0.7866
145000	331	0.8504
150000	331	0.865
160000	332	0.8808
165000	332	0.9277
170000	332	0.9454
180000	332	0.9635
190000	332	0.9644
195000	332	0.9651
200000	332	0.9662
210000	333	0.9729

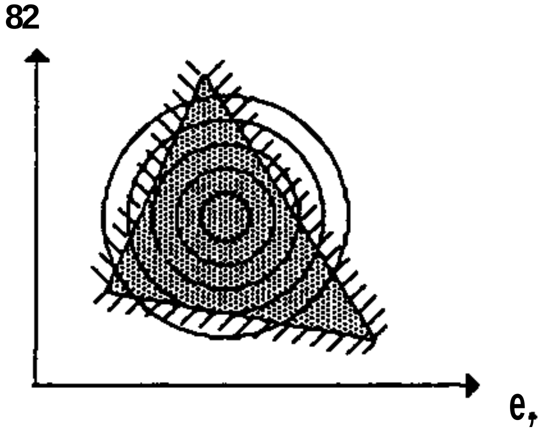

Figure 1 The stochastic flexibility equals the probability that the combination 91 82 lie within the feasible region.

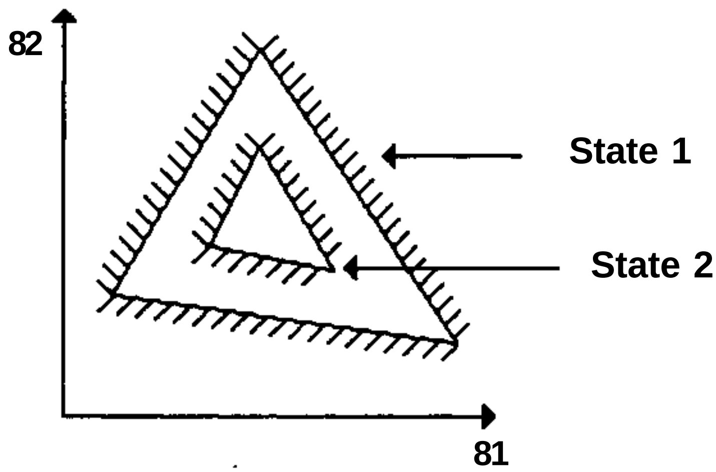

Figure 2 Showing the effect on the feasible region of equipment failure.

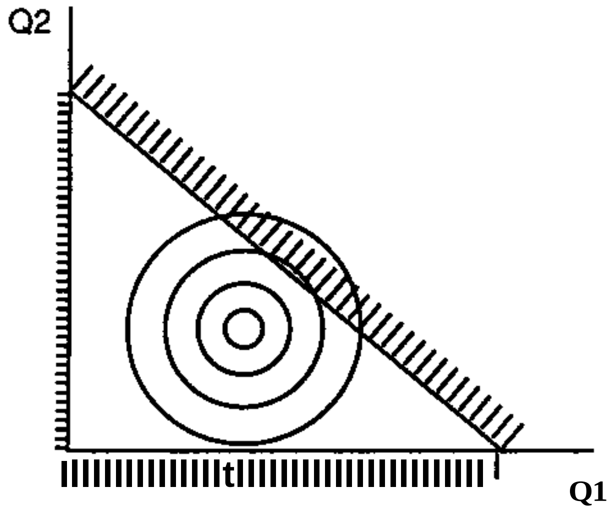

Figure 3 Linear feasible region in the space of the uncertain demands

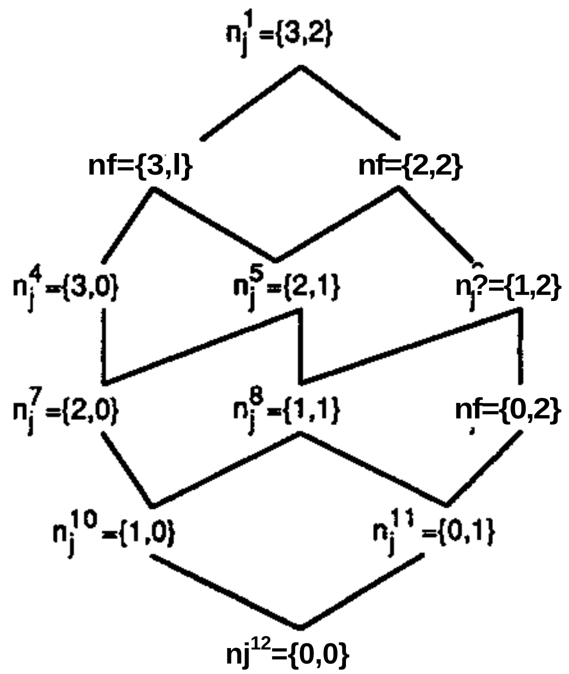

Figure 4 Network Representation of states

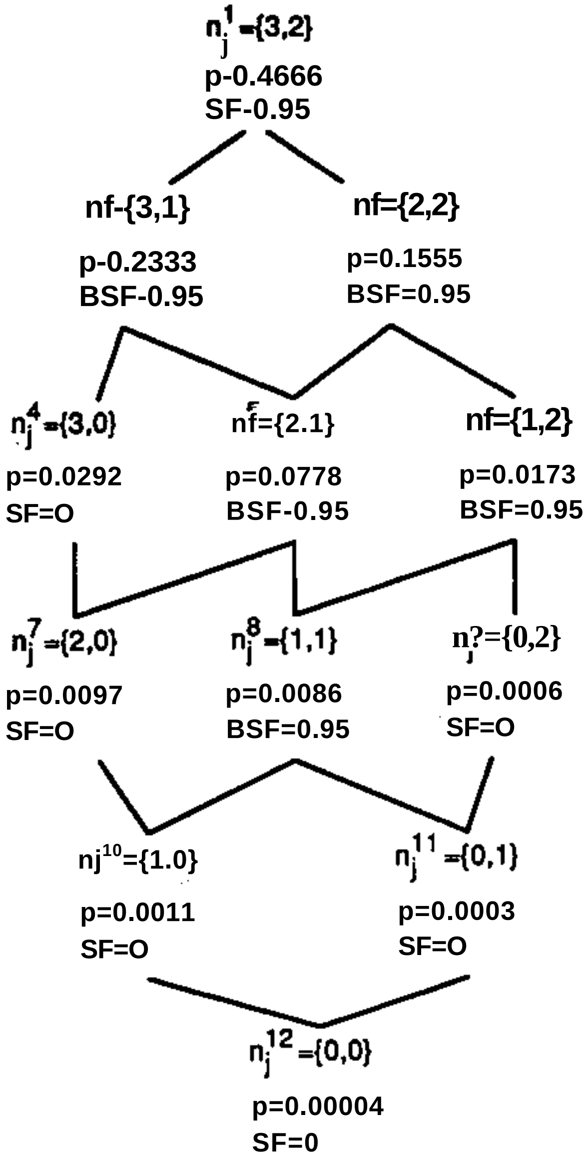

Figure 5 Network Representation shown with State Probabilities and BSF resulting from the evaluation of SF for State 1.

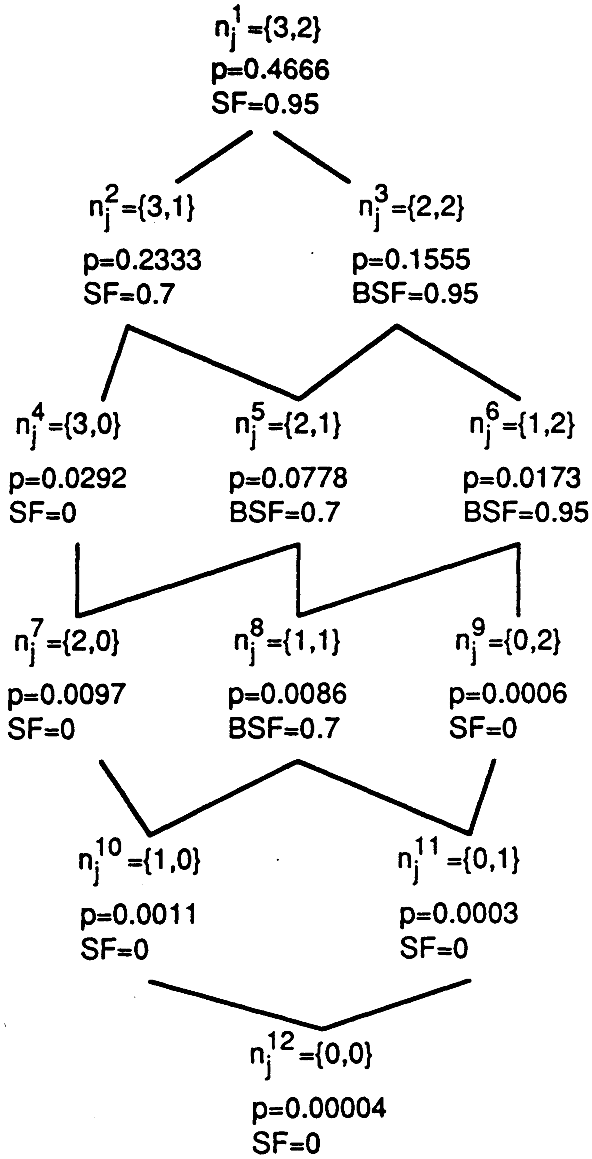

Figure 6 Network Representation shown with State Probabilities and BSF resulting from the evaluation of SF for State 1 and State 2.

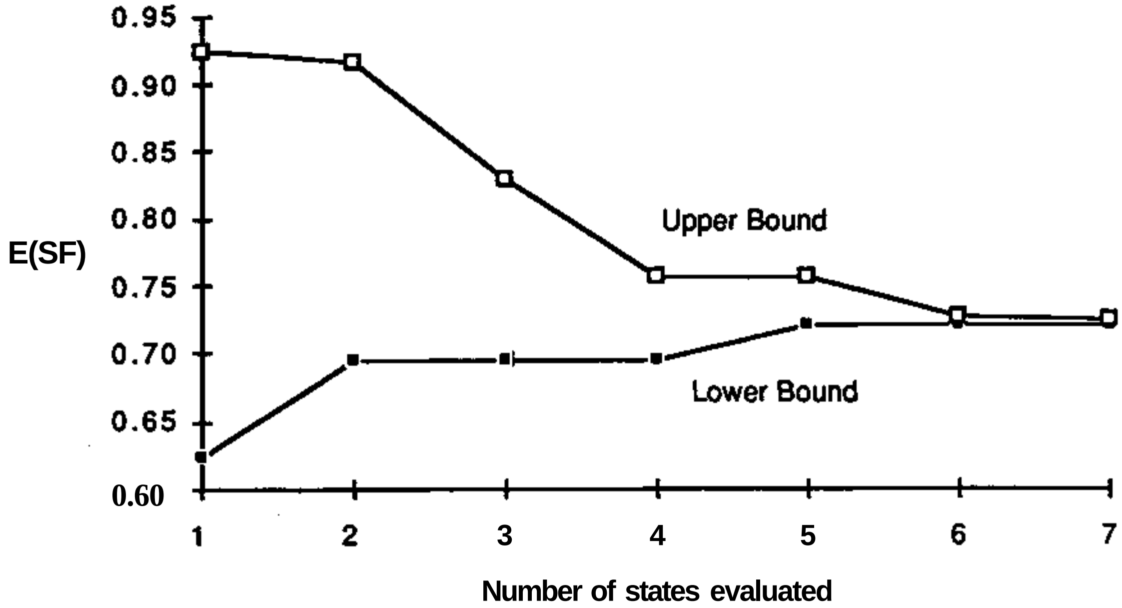

Figure 7 Convergence of bounds for Example 3

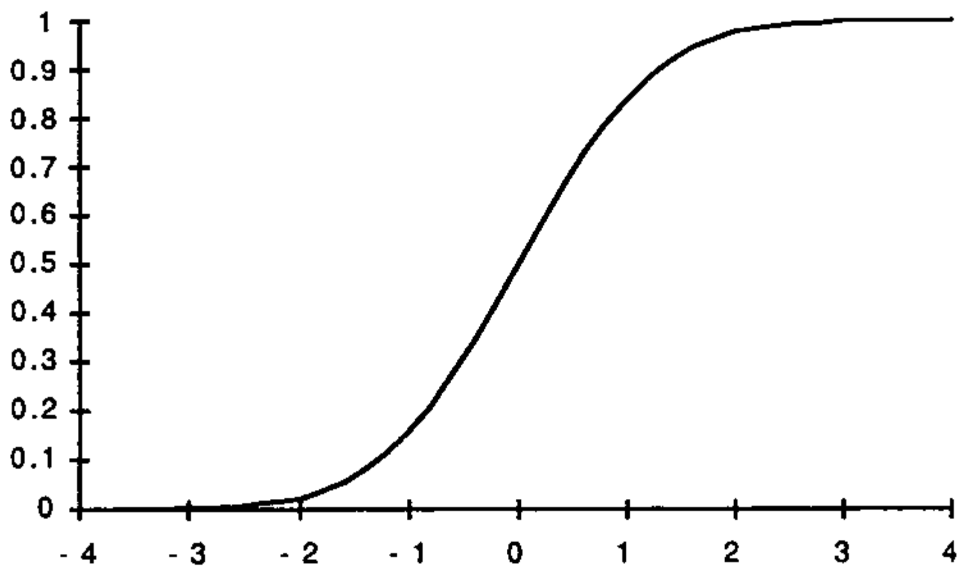

Figure 8 SF versus z

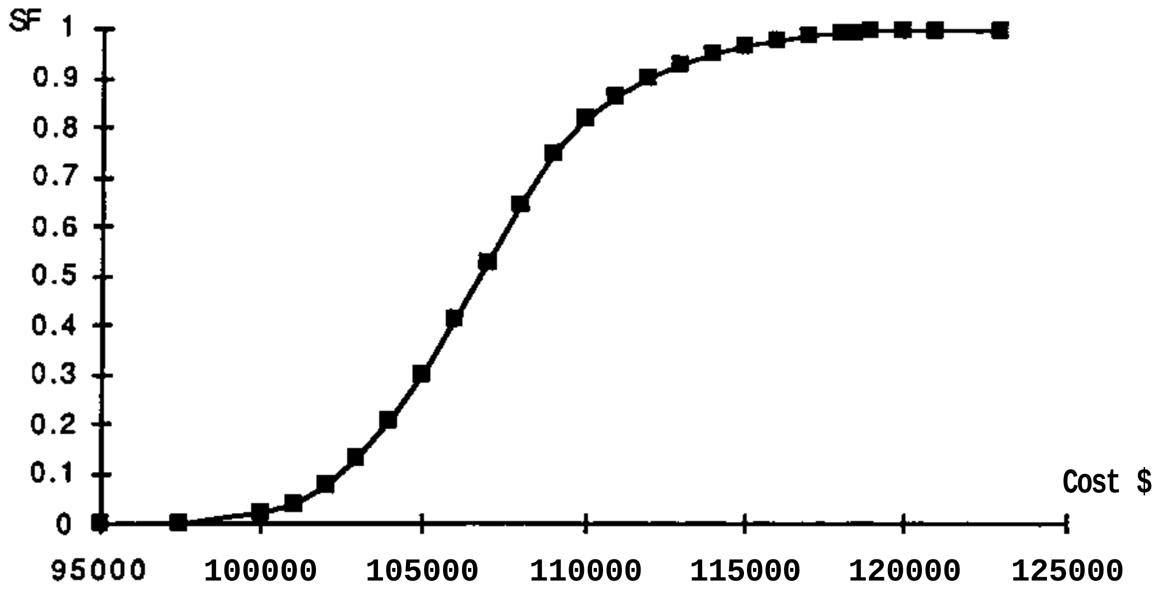

Figure 9 Trade-off curve for Example 4.

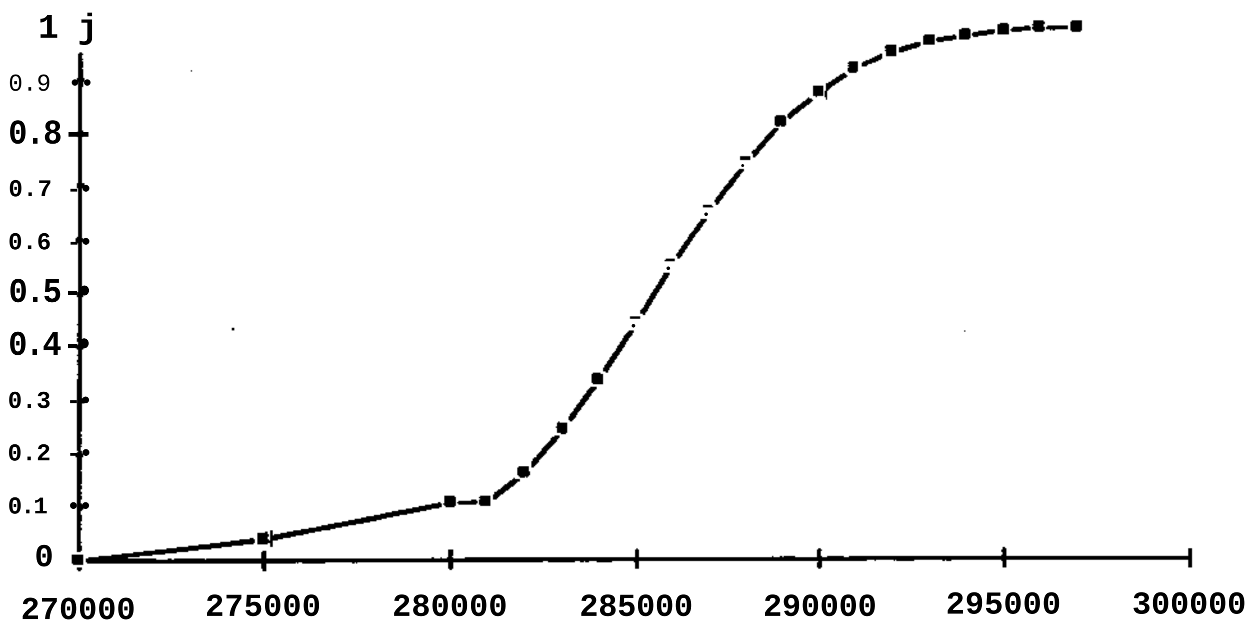

Figure 10 Trade off curve for Example 5

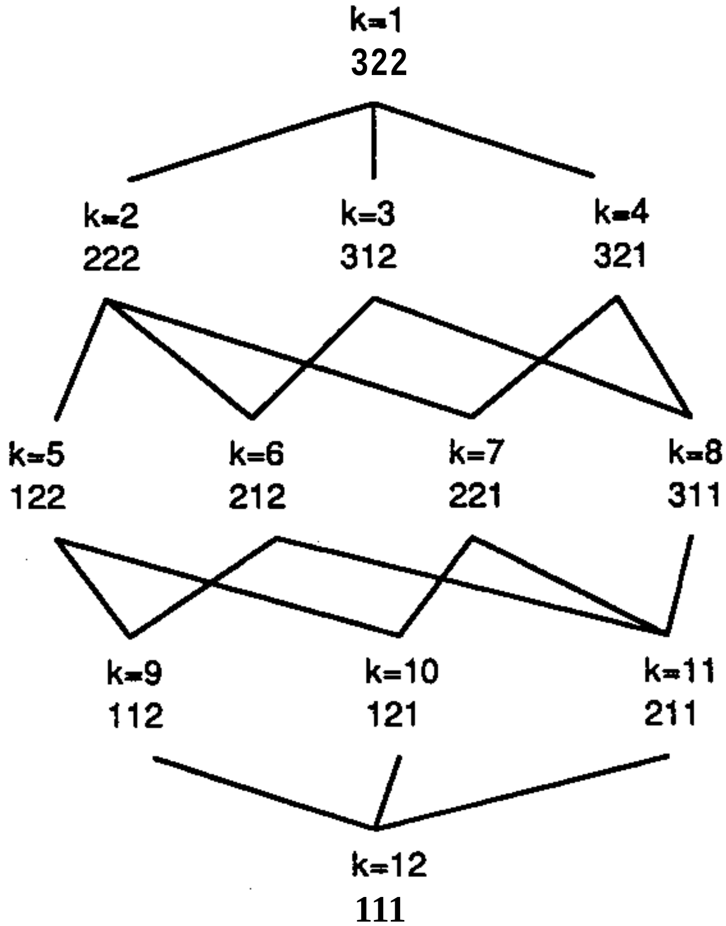

Figure 11 State Network for Example 6

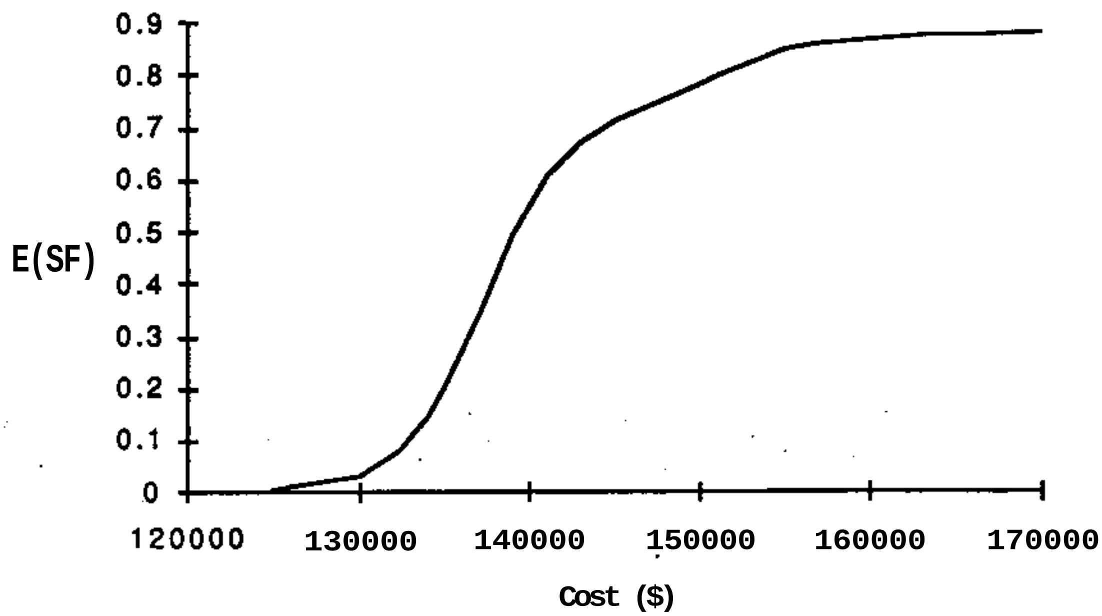

Figure 12 Trade off curve for example 6

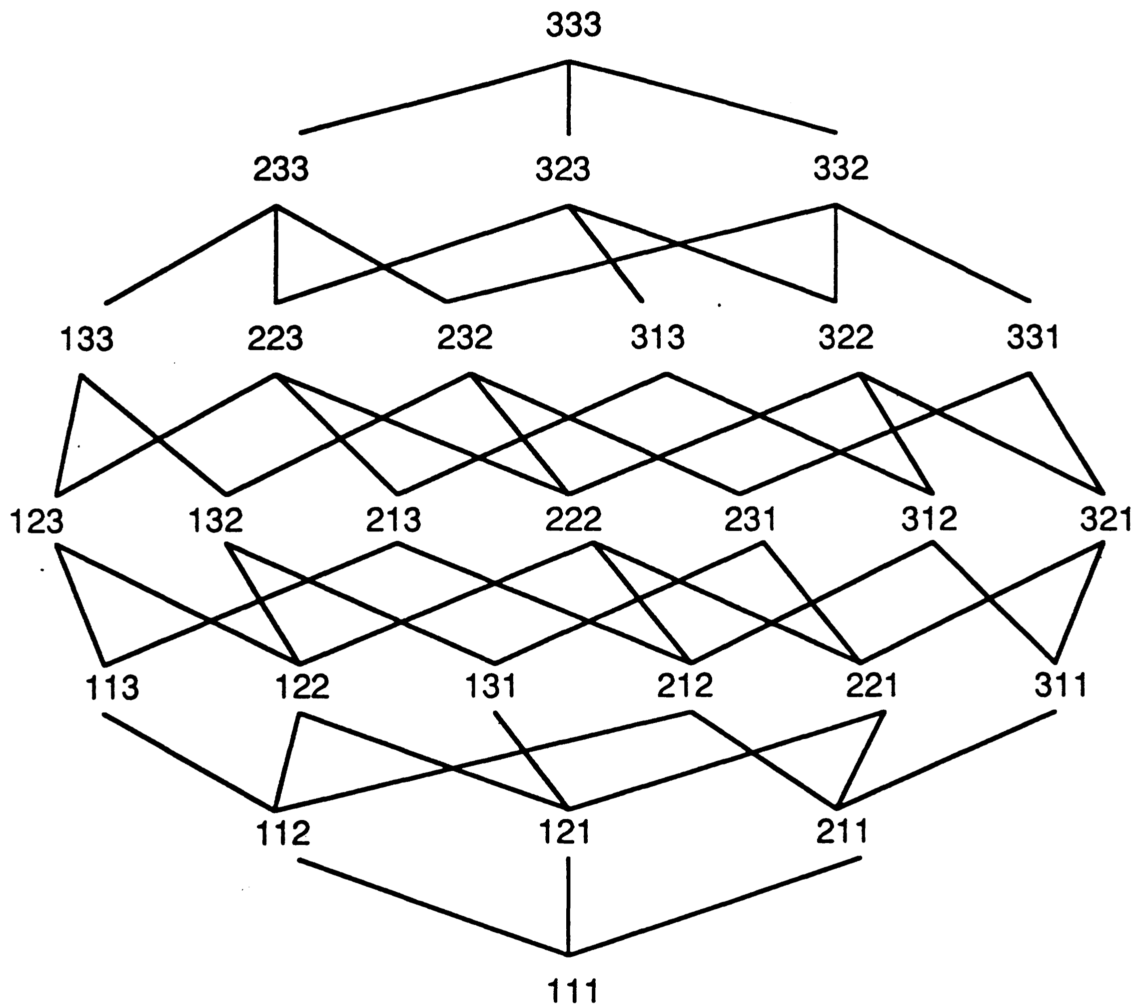

Figure 13 Network structure defined by N_j^U=3 for all j..

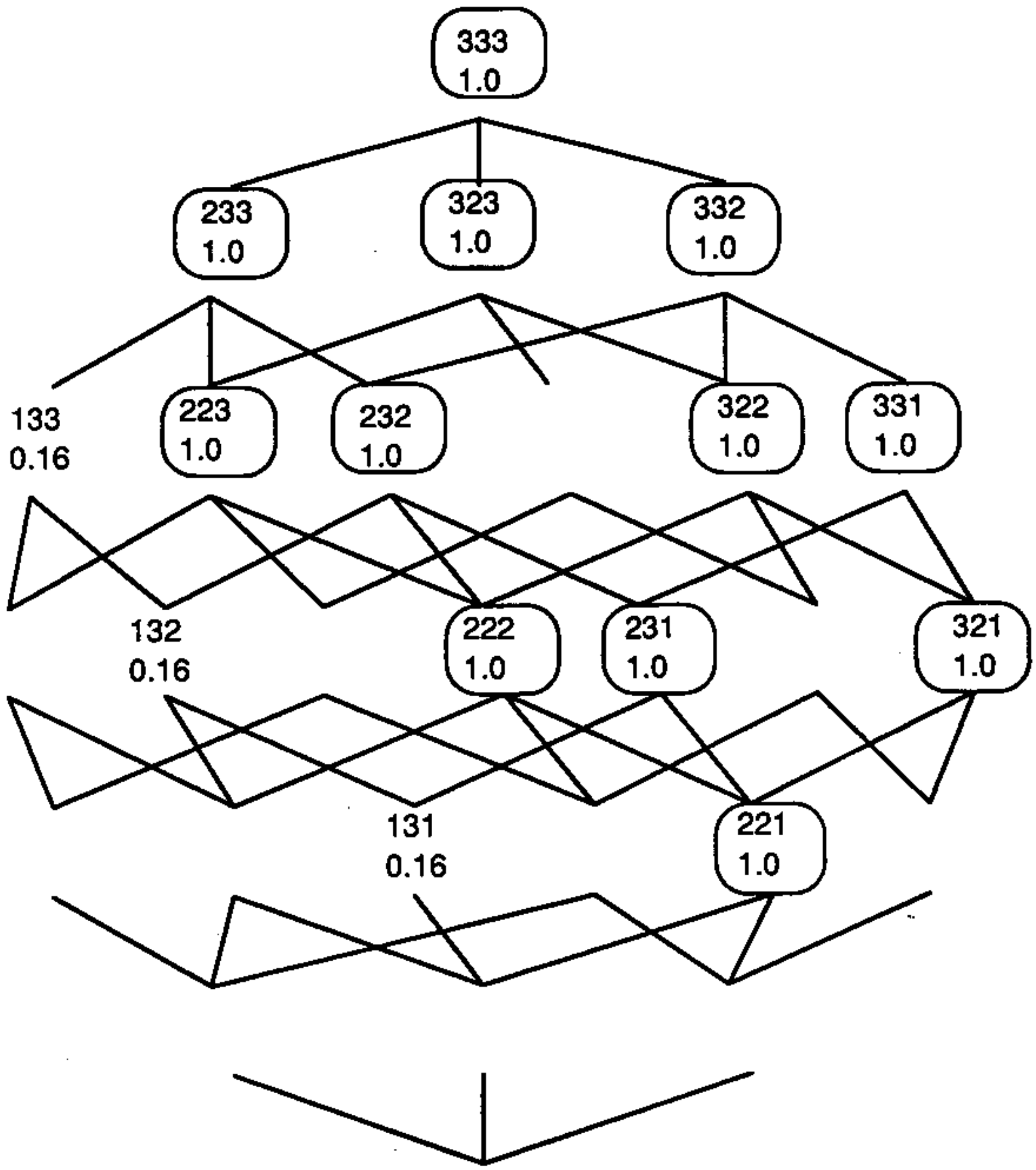

Figure 14 Network after limination of states in step 2.

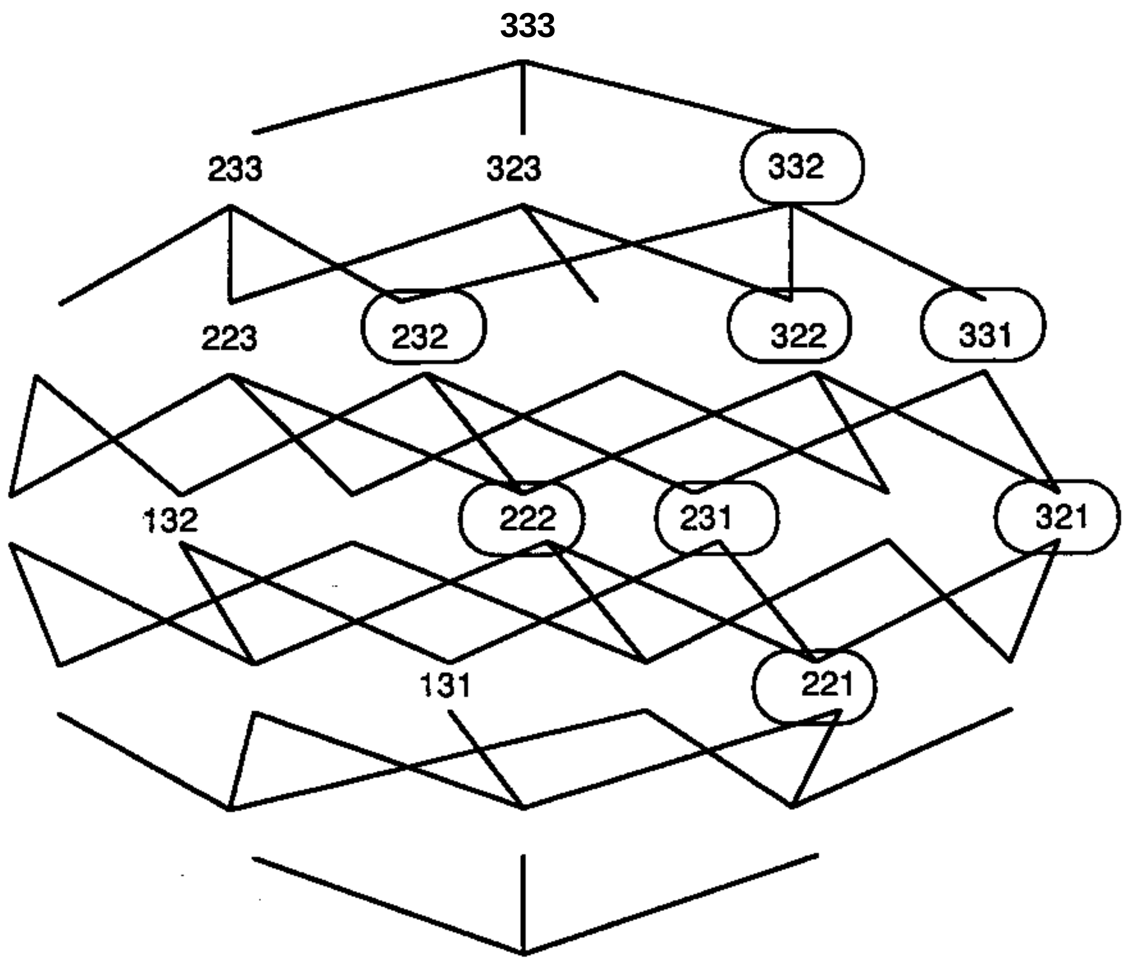

Figure 15 Network after elimination of states in step 3.

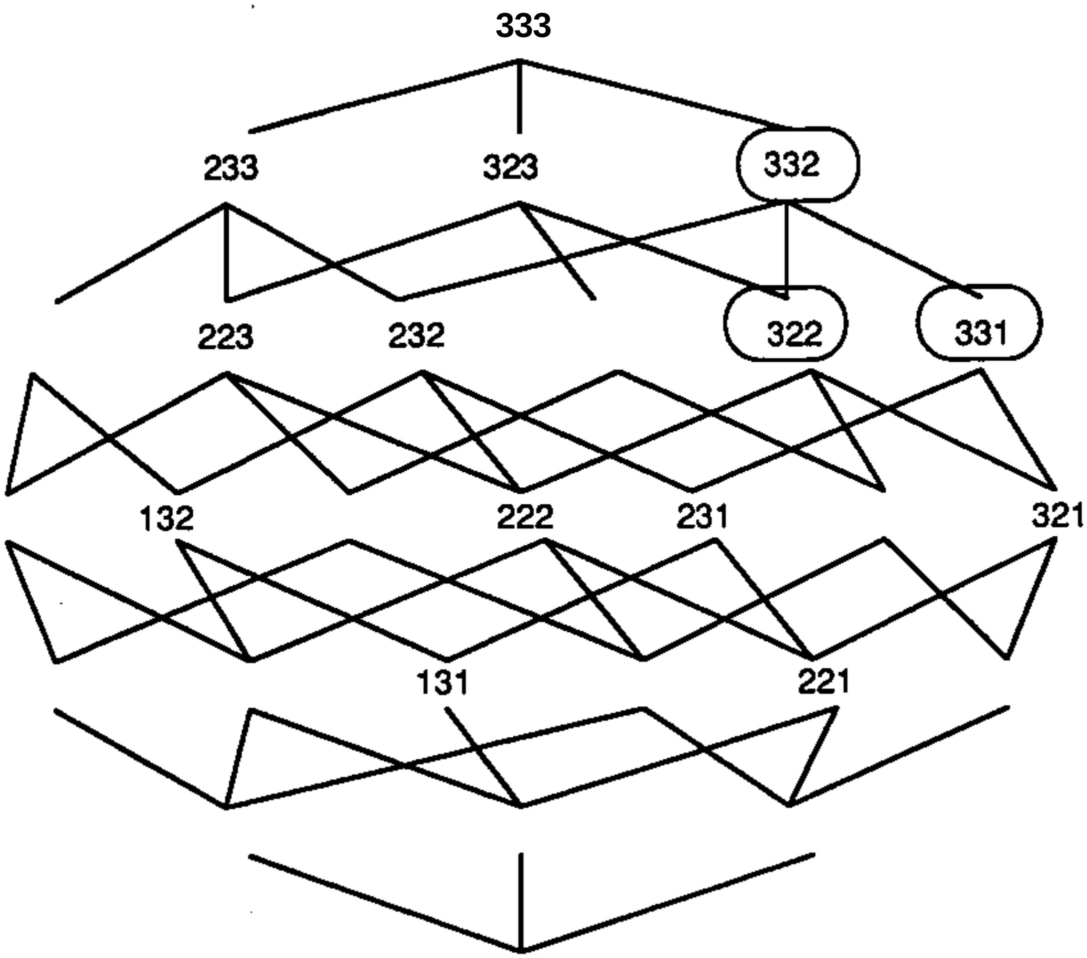

Figure 16 Network after elimination in step 4.

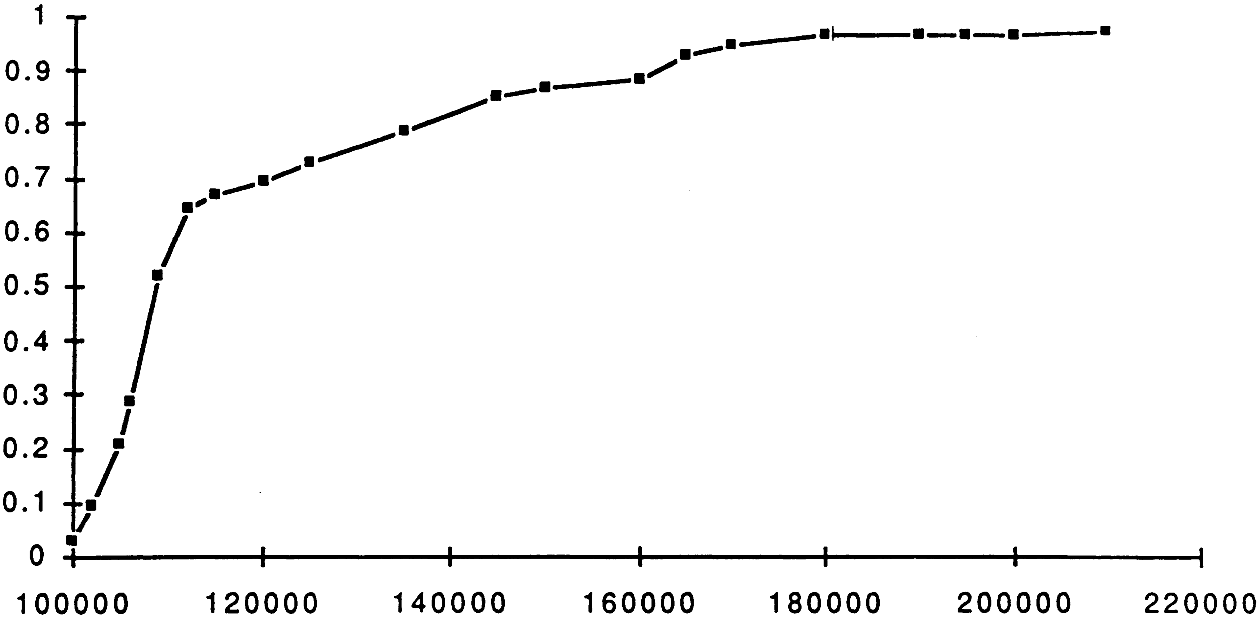

Figure 17 Trade off curve for the optimal E(SF) over V and N problem