Linear Quadratic Optimal Control System Design by Chebyshev-Based State Parameterization

by

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Abstract

A Chebyshev-based representation of the state vector is proposed for designing optimal control trajectories of unconstrained, linear, dynamic systems with quadratic performance indices. By approximating each state variable by a finite-term, shifted Chebyshev series, the linear quadratic (LQ) optimal control problem can be cast as a quadratic programming (QP) problem. In solving this QP problem, one approach is to treat the state initial conditions as constraints that are included in the performance index using a Lagrange multiplier technique. A computationally more efficient approach is to tailor the state representation such that the boundary conditions (which include the known initial conditions) are decoupled from the coefficients of the Chebyshev series. In both approaches the necessary condition of optimality is derived as a system of linear algebraic equations from which near optimal trajectories can be designed.

1. Introduction

The optimal control, and corresponding state, trajectories of linear, lumped parameter models of dynamic systems are often determined from the necessary condition of optimality. Using variational methods, this optimality condition can be represented as a two-point boundary-value problem (TPBVP). One of the most well-known solution approaches is the Hamilton-Jacobi approach which converts the TPBVP to a terminal value problem involving a matrix differential Riccati equation. Although the Hamilton-Jacobi approach casts the optimal solution in closed-loop form making it a preferred approach for physical implementation, it is computationally intensive and sometimes difficult to employ in solving high order systems.

For time-invariant systems, a more efficient solution method for optimal trajectory planning is the open-loop transition matrix approach (Speyer, 1986; described in Appendix C). Typically, the transition matrix approach converts the TPBVP into an initial value problem which can be solved numerically. The transition matrix approach can encounter a problem of numerical instability in determining the optimal control of high order systems (Yen and Nagurka, 1990). This problem has been attributed principally to the error associated with the computation of large dimension state transition matrices. An accurate and computationally streamlined approach for calculation of state transition matrices of high order systems remains a research challenge (Moler and Loan, 1978).

To circumvent these numerical difficulties, and in the interest of seeking alternative solution strategies, trajectory parameterization methods have been investigated. In general, these approaches approximate the control, state, and/or co-state trajectories by finite-term orthogonal functions whose unknown coefficient values are sought giving a near optimal (or sub-optimal) solution. For example, approaches employing functions such as Walsh (Chen and Hsiao, 1975), block-pulse (Hsu and Cheng, 1981), Laguerre (Shih, Kung and Chao, 1986), Chcbyshev (Paraskevopoulcs, 1983; Chou and Horng, 1985; Vlassenbroeck and Van Dooren, 1988), and Fourier (Chung, 1987) have been suggested. Like the state transition matrix approach, many of these approaches employ algorithms that convert the TPBVP into an initial value problem. The initial value problem is then integrated with respect to time with the state and co-state vectors approximated by truncated orthogonal series. This technique (described in Appendix D) reduces the initial value problem into a static optimization problem represented by algebraic equations. The truncation of the orthogonal series results in errors, which can be minimized by including more terms, but the transition matrix (needed to convert the TPBVP to an initial value problem) must still be evaluated which, as mentioned above, can cause instability problems in high order systems.

This research is part of a broader effort toward the development of computational tools for solving optimal control problems via state parameterization. An advantage of state parameterization is that boundary condition requirements on the state variables, such as initial conditions, can be satisfied directly. A second advantage is that the state equations can be treated as algebraic equations in determining the corresponding control trajectory. This assumes that there are no constraints on the control Structure preventing an arbitrary representation of the state trajectory from being achieved

Earlier work on parameterization of the state vector via Fourier-type series (Yen and Nagurka, 1988) has shown that the necessary condition of optimality for an unconstrained linear quadratic (LQ) problem can be formulated as a system of linear algebraic equations. To ensure an arbitrary representation of the state trajectory and hence overcome the potential difficulty of trajectory inadmissibility (in which an arbitrary state trajectory can not be achieved), artificial control variables were proposed. These physically non-existent variables are driven small by being heavily penalized in the performance index. Simulation results indicated that the approach is accurate, computationally efficient, and robust relative to standard methods.

Studies of parameterization methods for prediction for optimal control of linear time-invariant systems have demonstrated advantages of expansions in terms of Chebyshev functions in comparison to Walsh, block-pulse, Hermite, Laguerre, and Legendre functions (Paraskevopoulos, 1983, 1985). Chebyshev functions can nearly uniformly approximate a broad class of functions, making them computationally attractive (Vlassenbroeck and Van Dooren, 1988).

This report explores the use of finite-term Chebyshev-based representations of the state trajectory. In one approach, each state variable of a dynamic system is approximated by a shifted Chebyshev series. The LQ problem is then converted to an equality constrained quadratic programming (QP) problem that minimizes the performance index and satisfies state initial conditions via Lagrange multipliers. In an alternate formulation, the state initial conditions are satisfied directly by representing each state variable by the superposition of a shifted Chebyshev series and a special third order polynomial. In both cases the necessary condition of optimality can be written as a system of linear algebraic equations from which the unknown state parameters can be solved.

2. Methodology

2.1 Problem Statement

The behavior of a linear dynamic system is governed by the state-space model.

$$\dot{x}(t) = \mathbf{A(t)x(t)} + \mathbf{B(t)u(t)} \qquad (1)$$

with known initial condition x(O)=xo where x is an Nxl state vector, **u** is an Mxl control vector, A is an NxN system matrix, and B is an NxM control influence matrix.

The design goal is to find the control u(t) and the corresponding state x(t) in the time interval [0,T] that minimizes the quadratic performance index L,

$$L = Li + L_2 \qquad (2)$$

where

$$Li = x^r(T)Hx(T) + h^r x(T) \qquad (3)$$

$$L_2 = \int_0^T [xr(t)Q(t)x(t)+u\wedge R(t)u(t)+x^T(t)S(t)u(t)+q^T(t)x(t)+r^T(t)u(t)]dt \qquad (4)$$

It is assumed that H and Q are real NxN symmetric and positive-semidefinite matrices, R is a MxM symmetric and positive definite matrix, S is a NxM weighting matrix, h, q are Nxl vectors and **r** is an Mxl vector. For now, it is assumed that the lengths of the state and control vectors are the same *(i.e.*$_9$ M=N) and B is invertible. These assumptions will be relaxed later.

2.2 Chebyshev Polynomials

Chebyshev polynomials are defined for the interval ^€[-1,1] and have the following analytical form:

$$cp\wedge) = \cos(k \cos^{-1}\wedge) \quad , \quad k = 0,1,2,\ldots \qquad (5)$$

or

$$\varphi_k(\xi) = \sum_{i=0} (-1)^i \frac{k!}{(2i)!(k-2i)!}(1-\xi)^i \xi^{-\cdots} \quad , \quad " = 1\ 2\ ,\ldots \qquad (6)$$

where the notation [k/2] means the greatest integer smaller than k/2. From Equation (6), the first few Chebyshev polynomials are

$$\begin{aligned}\varphi_0(\xi) &= 1\\ \varphi_1(\xi) &= \xi\\ \varphi_2(\xi) &= 2\xi^2-1\\ \varphi_3(\xi) &= 4\xi^3-3\xi\\ \varphi_4(\xi) &= 8\xi^4-8\xi^2+1\\ \varphi_5(\xi) &= 16\xi^5-20\xi^3+5\xi\end{aligned} \qquad \text{(7a-0}$$

The Chebyshev polynomials have several properties, such as satisfying *(i)* the recurrence relations

$$\begin{aligned}&\varphi_{k+1}(\xi) - 2\xi\varphi_k(\xi) + \varphi_{k-1}(\xi) = 0 \quad , \quad k\overline{\mp}, 2, \ldots\\ &\left(1-\xi^2\right)\dot{\varphi}_k(\xi) = -k\xi\varphi_k(\xi) + k\varphi_{k-1}(\xi) \quad , \quad k = 1, 2, \ldots\end{aligned} \qquad \text{(8a-b)}$$

where the dot indicates differentiation with respect to time, *(ii)* the initial, final and midpoint values

<Pk(D = 1

$\varphi_k(-1) = (-1)^k$

$\varphi_{2k}(0) = \quad {}^k$ **(9a-d)**

<P2k+i(0) = 0

and *(Hi)* the product relations

$$\begin{aligned}&\varphi_i(\xi)\varphi_j(\xi) = \frac{1}{2}[\varphi_{i+j}(\xi) + \varphi_{i-j}(\xi)] \quad , \quad i \geq j\\ &\varphi_k^2(\xi) = \frac{1}{2}[1 + \varphi_{2k}(\xi)]\end{aligned} \qquad \text{(lOa-b)}$$

The domain of the Chebyshev polynomials can be transformed to values between 0 and T by letting

$$\xi = 2\frac{t}{T} - 1 \qquad (11)$$

giving the shifted Chebyshev polynomial \|/$_k$(t) expressed as

$$\psi_k(t) = \varphi_k(\xi) = \varphi_k\left(2\frac{t}{T} - 1\right) \qquad (12)$$

From Equation (12) the first few shifted Chebyshev polynomials are

$$\psi_0(t) = 1$$
$$\psi_1(t) = 2\tau - 1$$
$$\psi_2(t) = 8\tau^2 - 8\tau + 1 \qquad (13a\text{-}e)$$
$$V_3(t) - 32t^3 - 48t^2 + 18t - l$$
$$\psi_4(t) = 128t^4 - 256t^3 + 16Ot^2 - 32t + l$$

where nondimensional time t = t / T. From Equations (5) and (12), the initial and final values of the shifted Chebyshev polynomial and its first time derivative can be obtained as

$$Vk(0) = (-D^k$$
$$\dot{V}k(0) = (-l)^{k+1}(2k^2/T)$$
$$\psi_k(T) = 1 \qquad (14a\text{-}d)$$
$$\dot{¥}k(T) = 2k^2/T$$

2.3 State Parameterization

2.3.1 Chebyshev State Parameterization A direct approach for state parameterization is to approximate each of the N state variables $x_n(t)$ by a K term shifted Chebyshev series.

$$X_n(t) = \sum_{k=1}^{K} c_k(t)\, y_{nk} \qquad (15)$$

for n=l,2,...,N where

$$c_K(t) = \psi_{k-1}(t) \qquad (16)$$

In Equation (15) ynk is the k-th unknown coefficient of the Chebyshev polynomial for the n-th state variable. Alternatively, equation (15) can be written as

$$x_n(t) = c\bar{r}(t) y_n \qquad (17)$$

where

$$c^r(t) = [c!(t)\ c_2(t) \dots\ CK(0] \qquad (18)$$

$$y_n = [y_{n1}\ y_{n2}\ \cdots\ y_{nK}]^T \qquad (19)$$

where y_n is the state parameter vector (containing the unknown coefficients) for the n-th state variable. The state vector containing the N state variables can be written in terms of a full state parameter vector y, i.e.,

$$\mathbf{x}(t) = \mathbf{C}(t)\mathbf{y} \tag{20}$$

where

$$C(t) = \begin{bmatrix} c^r(t) & & & 0 \\ & c^r(t) & & \\ & & \ddots & \\ 0 & & & c^r(t) \end{bmatrix}_{N\times NK} \tag{21}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} [y_{11} & y_{12} & \cdots & y_{1K}]^r \\ [y_{21} & y_{22} & \cdots & y_{2K}]^r \\ & \vdots & & \\ [y_{N1} & y_{N2} & \cdots & y_{NK}]^r \end{bmatrix}_{NK\times 1} \tag{22}$$

Similarly, the state rate vector can be written as

$$\dot{x}(t) = D(t)y \tag{23}$$

where

$$D(t) = \dot{C}(t) = \begin{bmatrix} d^r(t) & & & 0 \\ & d^r(t) & & \\ & & \ddots & \\ 0 & & & d^r(t) \end{bmatrix}_{N\times NK} \tag{24}$$

$$d^r(t) = [\dot{c}_1(t)\ \dot{c}_2(t) \ldots\ \dot{c}_K(t)] \tag{25}$$

The control vector u(t) can also be expressed as a function of y. From Equation (1)

$$\mathbf{u}(t) = \mathbf{B}^{-1}(t)\dot{\mathbf{x}}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{x}(t) \tag{26}$$

From Equations (20) and (23),

$$u(t) = \left[\mathbf{B}^{-1}(t)\mathbf{D}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{C}(t)\right]\mathbf{y} \tag{27}$$

Thus, the application of Chebyshev state parameterization allows the state vector, state rate vector, and control vector to be represented as functions of the state parameter vector.

3.32 Conversion Process The first step in solving the state parameterized LQ problem is to convert it to a quadratic programming (QP) problem by rewriting the performance index as a function of the state parameter vector y. From Equation (20); the terminal state vector x(T) can be expressed as

$$x(T) = C(T)y \tag{28}$$

By substituting Equation (28) into (3), the cost Lj is

$$Li = y^{r}C(T)^{r}HC(T)y + h^{r}C(T)y \tag{29}$$

Similarly, by substituting Equation (26) into the integrand of Equation (4),

$$x^{r}Qx+u^{r}Ru+x^{r}Su+q^{r}x+r^{r}u = x^{r}Fix+x^{r}F_{2}x+x^{r}F3X+x^{r}fi+x^{r}f_{2} \tag{30}$$

where Fi, F2, and F3 are NxN matrices and fi and f_2 are Nxl vectors given by

$$Vi = Q+G^{r}RG+SG$$

$$F_{2} = B^{-r}RB^{"1} \tag{31a-c}$$

$$F_{3} = 2B^{T}RG+B^{r}S$$

$$fi = q+G^{r}r$$

$$f_{2} = B^{r}r \tag{32a-b}$$

where

$$G = -B\text{^}A \tag{33}$$

and superscript *-T* denotes inverse transpose. By substituting Equations (20) and (23) into (30), the integrand of Equation (4) can be expressed as a function of parameter vector y, *i.e.,*

$$x^{r}Qx+u^{r}Ru+x^{r}Su+q^{r}x+r^{r}u = y^{r}Py+y^{r}p \tag{34}$$

where

$$P\ m\ Fi®cc^{r}+F_{2}®dd^{r}+F3®dc^{r}$$

$$p = \text{ïf} \ \text{≰+f} \wedge d \tag{35a-b}$$

In Equation (35a-b), P is an NKxNK matrix, p is an NKxl matrix, and ® is a Kronecker product sign (Brewer, 1978), *e.g.*,

$$V \otimes W = \begin{bmatrix} V_{11}W & \cdots & V_{1n}W \\ V_{21}W & & \vdots \\ \vdots & & \vdots \\ V_{n1}W & \cdots & V_{nn}W \end{bmatrix} \quad (36)$$

where V is an nxn matrix and W is an arbitrary matrix. Thus, from Equation (34), the integral part of the performance index can be expressed as

$$L_2 = \int_0^T (y^T P y + y^T p)dt = y^T P^* y + y^T p^* \quad (37)$$

where

$$P^* = \int_0^T P dt$$

$$p^* = \int_0^T p dt \quad (38a\text{-}b)$$

Substituting Equations (29) and (37) into (2) gives the performance index L as a quadratic function of parameter vector y, *i.e.*.

$$L = y^T \Omega y + y^T \omega \quad (39)$$

where

$$\Omega = C(T)^T H C(T) + P^*$$

$$\omega = C(T)^T h + p^* \quad (40a\text{-}b)$$

For time-invariant problems, F_1, F_2, F_3, f_1 and f_2 are constants and can be removed from the integrals, enabling the remaining integral parts of P^* and p^* to be evaluated analytically. That is, Equation (38a-b) can be rewritten as

$$P^* = F_1 \otimes \left[\int_0 \quad \mathrm{lt} \right] + F_2 \otimes \left[\int (dd^r)dt \right] + F_3 \otimes \left[\int_0^T (dc^r)dt \right]$$

$$p^* = f_1 \otimes \left[\int_0 c\,dt \right] + fa \circledR \left[\int_0 d\,dt \right] \tag{41a-b}$$

The solutions of the integral parts of P* and p* (i.e., the terms in the brackets) have been derived and are summarized as recurrence formulas in Appendix A.

The initial conditions of the state variables can be expressed as

$$x_o = C_o y \tag{42}$$

where

$$x_0 = x(0)$$

$$C_0 = C(0) \tag{43a-b}$$

Hence, the problem is to minimize Equation (39) such that Equation (42) is satisfied.

<u>2.3.3 Lagrange Multiplier Solution Procedure</u> The above equality constrained QP problem can be reduced to an unconstrained problem by including the initial conditions as constraints in the performance index via a Lagrange multiplier vector X:

$$L(y,X) = y^T\Omega y + y^T co + \lambda^T[C_o y - x_o] \tag{44}$$

The necessary conditions of optimality are given by

$$\nabla_y L(y,\lambda) = (\Omega+\Omega^T)y + \omega + C_0^T\lambda = 0 \tag{45}$$

$$\nabla_\lambda L(y,\lambda) = C_0 y - x_0 = 0 \tag{46}$$

representing a system of linear algebraic equations in terms of the elements of y and *X*. Equations (45) and (46) can be written as

$$\begin{bmatrix} y \\ \lambda \end{bmatrix} = \begin{bmatrix} \Omega+\Omega^T & C_0^T \\ C_o & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\omega \\ x_0 \end{bmatrix} \tag{47}$$

from which the state parameter vector y can be solved.

2,3-4 Chebyshcv-Bascd State Parameterization. An alternate state representation is to approximate **each** of the N state variables $x_n(t)$ by the superposition of a third-order auxiliary polynomial and a (K-4) term shifted Chebyshev series. A motivation for this representation, subsequently called Chebyshev-Based State Parameterization (in contrast to just Chebyshev Parameterization above), is that the boundary values of the state variables can be decoupled from the unknown state parameters enabling the state initial conditions to be satisfied directly.

Mathematically, the state variable $x_n(t)$ for n=l,2,...,N is written as

$$x_n(t) = bno + bnix + b\wedge t^2 + b_{n3}t^3 + \overset{K-l}{£}\ a_{nk}y_k(t) \qquad (48)$$

Then the derivative of $x_n(t)$ is

$$\dot{x}_n(0 = \overset{1}{\underset{1}{\wedge}}(bm + 2bn2X + 3bn_3T^2) + \sum_{k=4}^{K-l} a_{nk}\backslash j/_k(t) \qquad \textbf{(49)}$$

The constants b's can be determined by substituting the initial and final values of time (0 and T) into Equations (48) and (49), using Equations (14a-d), and manipulating algebraically.

$$\mathbf{b_{n0} = x_{n0} - (-1)^k \sum_{k=4}^{K-l} a_{nk}}$$

$$\mathbf{b_{n1} = T\dot{x}_{n0} + (-1)^k \sum_{k=4}^{K-l} 2k^2 a_{nk}} \qquad (50a\text{-}d)$$

$$b_n2 - -3x_{nO}-2T\dot{x}_no+3x_{nT}-T\dot{x}nT+\sum_{k=4}^{K-l} H-3+2k^2)+(.l)^k(3-4k^2)]a_{nk}$$

$$\mathbf{b_{n3} = 2x_{nO}+T\dot{x}_no-2x_n\dot{T}+T\dot{x}_{nT}+\sum_{k=4}^{K-l} [(2-2k^2)-(-l)^k(2-2k^2)]a_{nk}}$$

where XnO, $\dot{x}_n o$, $*_nT$ *md* $\wedge$nT are the values of the state variable x_n and its derivative $\dot{x}_n$ at the boundaries of the time segment [0,T], *i.e.*,

$$x_{n0} = x_n(0)$$
$$\dot{x}_{n0} = \dot{x}_n(0)$$
$$x_{nT} = x_n(T) \qquad (51a\text{-}d)$$
$$\dot{x}_{nT} = \dot{x}_n(T)$$

By substituting Equations (50a-d) into (48), die state variable $x_n(t)$ can be rearranged as

$$x_n(t) = \sum_{k=1}^{K} c_k(t) y_{nk} \qquad (52)$$

where

$$c_1 = 1-3x^2+2x^3$$
$$c_2 = T(x-2x^2+x^3)$$
$$c_3 = 3x^2-2x^3 \qquad (53a\text{-}d)$$
$$c_4 = T(-x^2+x^3)$$

$$c_k=(-1)^k-2(-1)k(k-1)2t+[2k2-4k-1+(-1)k(4k2-8k+1)]x2+2k(2-k)[1+(-1)]x^3$$
$$+ V_{k-1}(t) \quad (k=5,6,\ldots,K) \qquad (54)$$

and where

$$y_{n1} = x_{n0}$$
$$y_{n2} = \dot{x}_{n0}$$
$$y_{n3} = x_{nT} \qquad (55a\text{-}e)$$
$$y_{n4} = \dot{x}_{nT}$$
$$y_{nk} = a_{n(k-4)} \quad (k=5,6,\cdots,K)$$

Then Equation (52) can be written as Equation (17) with different definitions of ynk's and Ck*s.

Using Equations (55a-e) the elements of the NxNK matrix C(T) in Equation (28) can be redefined as

$$c_{ij}(T) = \begin{cases} 1 & j=(i-1)K+3 \quad i=1,2,\ldots,N \\ 0 & \text{otherwise} \end{cases} \qquad (56)$$

The conversion process is similar to the previous approach. Because the Ck's in Equations (53a-d) and (54) are redefined, the integral parts of P*and p* change. For time-invariant problems new closed-form expressions for the integral parts have been derived. They are summarized in Appendix B.

<u>2.3.5 Direct Substitution Solution Procedure</u> The optimal control problem now can be viewed as the search for the unknown coefficients of the state parameter vector y that minimize Equation (39) subject to the equality constraints of Equation (55a). To isolate the known initial condition, a new state parameter vector z is introduced as

$$z = EIL \qquad (57)$$

where

$$z_2 = xo \qquad (58a\text{-}b)$$

with

$$XI = [MO \; {}^x20 \cdots x_{N0}]$$

$$*0^T = [*10 \;\; *20 \cdots *_{N0}]$$

$$x£ = [XIT \; X_2T \cdots XNT] \qquad (59a\text{-}e)$$

$$\dot{X}_T^T = [\dot{X}IT \;\; \dot{X}2T \cdots \dot{x}_{NT}]$$

$$a^T = [aw \;\; \&]s \ldots a_{1(k-1)} \; a_{24} \;\; a_{25} \ldots a_{2(k-1)} \ldots a_{N4} \ldots a_{N(k-1)}]$$

$$= [yi5 \;\; yie. \cdots yik \;\; yis \;\; y26.\text{--}y2k\text{--}.yN5 \cdots y_{Nk}]$$

Vector Z2 contains the known initial values of the state vector and vector *z* is the remaining subset of the parameter vector y. The two vectors z and y are related via a linear transformation:

$$y = Oz \qquad (60)$$

where O is an NKxNK matrix with elements

$\phi_{ij} = 1 \quad i = (n-1)K + k; \quad n = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, K$

$$j = \begin{cases} NK - N + n & k = 1 \\ NK - 4N + n & k = 2 \\ NK - 2N + n & k = 3 \\ NK - 3N + n & k = 4 \\ (n-1)(K-4) + (k-4) & k = 5, 6, \ldots, k \end{cases} \tag{61}$$

$\phi_{ij} = 0$ otherwise

The perfonnance index L in Equation (39) can thus be rewritten as a function of z

$$L = z^T Q^* z + z^T \omega^* \tag{62}$$

where

$$\Omega^* = \Phi^T \Omega \Phi$$
$$\omega^* = \Phi^T \omega \tag{63a-b}$$

By expanding Equation (62), the performance index can be expressed as

$$L = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix} \begin{bmatrix} \Omega_{11}^* & \Omega_{12}^* \\ \Omega_{21}^* & \Omega_{22}^* \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} z_1^T & z_2^T \end{bmatrix} \begin{bmatrix} \omega_1^* \\ \omega_2^* \end{bmatrix} \tag{64}$$

or, equivalently,

$$L = z_1^T \Omega_{11}^* z_1 + z_1^T \left(\Omega_{12}^* + \Omega_{21}^{*T} \right) z_2 + z_2^T \Omega_{22}^* z_2 + z_1^T \omega_1^* + z_2^T \omega_2^* \tag{65}$$

For an unconstrained LQ problem, the necessary condition of optimality can be obtained by differentiating the performance index with respect to the unknown state parameter vector z_1. This leads to

$$(\Omega_{11}^* + \Omega_{11}^{*T}) z_1 = -(\Omega_{12}^* + \Omega_{21}^{*T}) z_2 - \omega_1^* \tag{66}$$

which represents a system of linear algebraic equations from which the unknown vector z_1 can be solved.

2.4 Chebyshev-Based Approach for General Linear Systems

The approaches presented above are applicable to systems with square and invertiblc control influence matrices. For general linear systems, B is an NxM matrix where N is greater than M To apply the Chcbyshev and Ghebyshev-based approaches to this more common case, j.e., general linear systems which have fewer control variables than state variables, the state-space model of Equation (1) is modified to

$$\dot{x}(t) = A(t)x(t) + B'(t)u'(t) \tag{67}$$

where

$$B'(t) = B'_{N\times N} = \begin{bmatrix} I_{(N-M)\times(N-M)} & \\ 0_{M\times(N-M)} & B_{N\times M} \end{bmatrix} \tag{68}$$

$$u'(t) = u'_{N\times 1} = \begin{bmatrix} \hat{u}_{(N-M)\times 1} \\ u_{M\times 1} \end{bmatrix} \tag{69}$$

where $\hat{u}$ is an artificial (i.e., fictitious) control vector.

It can be guaranteed that B' is invertible if the last M rows of B are nonsingular. However, if the last M rows are singular, the first (N-M) columns of B' in Equation (68) can always be modified to make it invertible. In order to predict the optimal solution, the performance index is modified to

$$L' = L_1 + L_2' \tag{70}$$

where

$$L_2' = \int_0^T [x^T(t)Q(t)x(t)+u'^T(t)R'(t)u'(t)+x^T(t)S'(t)u'(t)+q^T(t)x(t)+r'^T(t)u'(t)]dt \tag{71}$$

$$R'(t) = R'_{N\times N} = \begin{bmatrix} \rho I_{(N-M)\times(N-M)} & 0_{(N-M)\times M} \\ 0_{M\times(N-M)} & R_{M\times M} \end{bmatrix}$$

$$S'(t) = S'_{N\times N} = \begin{bmatrix} \rho I_{(N-M)\times(N-M)} & \\ 0_{M\times(N-M)} & S_{N\times M} \end{bmatrix} \tag{72a-c}$$

$$r'(t) = r'_{N\times N} = [\rho \cdots \rho \quad r^T]^T$$

where p is a weighting constant chosen to be a large positive number. If S=0, q=0 and r=0, then Equation (71) simplifies to die more common penalty function:

$$L' = L + \rho \int_0^T [\hat{u}^T(t)\hat{u}(t)]dt$$ <73>

By penalizing the artificial control vector, the magnitude and influence of the artificial control variables can be made small and the solution of the modified optimal control problem can approximate the solution of the original LQ problem.

3. Simulation Study

To study the effectiveness of the approaches, the solutions of unconstrained, time-invariant LQ problems have been obtained by both the Chebyshev and Chebyshev-based state parameterization approaches and compared with the solutions from other numerical algorithms.

3.1 Example 1

This example considers an N input N-th order linear time-invariant dynamic expressed in canonical form.

$$\dot{x}(t) = Ax(t) + \mathbf{B}\mathbf{u}(t), \quad x^T(0) = [1 \; 2 \; \cdots \; N] \tag{74}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & -2 & 3 & \cdots & (-1)^{N+1}N \end{bmatrix}, \quad B = I_{N \times N} \tag{75a-d}$$

The problem is to determine the control **u** that minimizes the performance index

$$\mathbf{L} = x^T(1)Hx(1) + \int_0^1 (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt \tag{76}$$

where

$$H = 10\mathbf{I}_{N\times N}$$
$$Q = R = I_{N\times N} \tag{77a-b}$$

One of the most efficient methods commonly used for solving this unconstrained LQ problem is the transition matrix approach (described in Appendix C; more details can be found in (Speyer, 1986)). The approach converts an optimal control problem into a linear TPBVP (such as Equation (C.9)). By evaluating the transition matrix of this boundary value problem, the problem can be converted into an initial value problem which can be readily solved. In this study, the transition matrices were computed numerically using the algorithm presented in (Franklin and Powell, 1980).

An alternate approach is a Chebyshev approach adapted from (Paraskevopoulos, 1983). It also converts an optimal control problem into a linear initial value problem.

Then, the state and costate vectors in the linear homogeneous differential equations are expanded in Chebyshev series with unknown coefficients. By integrating the differential equations and introducing a "Chebyshev operational matrix", the unknown coefficients of the Chebyshev series may be determined The state and control vectors may then be obtained, as described in Appendix D. (In this study, die linear algebraic Equations (D.21) were solved by an LU-decomposition routine. In Appendix B of (Paraskevopoulos,1983), an algorithm which reduces the computational effort involved in solving Equation (D.21) was presented.) For comparison, this approach - henceforth referred to as the "previous Chebyshev" approach - was implemented.

In addition to the transition matrix approach and the previous Chebyshev approach, the Chebyshev and Chebyshev-based approaches described in Section 2.3 were used to solve this problem. The Gaussian elimination routine was used to solve the linear algebraic equations representing the conditions of optimality in Equations (47) and (66). For the Chebyshev approach, a six-term series is employed. For the Chebyshev-based approach, a two-term shifted Chebyshev series in conjunction with a third-order polynomial is used. The two-term series in the Chebyshev-based approach is equivalent to a six-term series in the Chebyshev approach since the third-order polynomial is essentially a reformulation of the first four terms of the shifted Chebyshev series.

Efforts were made to optimize the speeds of the computer codes, all of which were written in "C" and executed on a SUN-3/60 workstation. Simulation results for N=2,4,...,20 are summarized in Table 1. For the transition matrix, Chebyshev and Chebyshev-based approaches, the execution time includes the time to evaluate *(i)* the system response (control vector) at 100 equally-spaced points and *(ii)* the performance index. For the previous Chebyshev approach, the execution time includes only the time to evaluate the system response. (The table reports execution time for the transition matrix approach in seconds, and percent execution time relative to the time of the transition matrix approach for the previous Chebyshev, Chebyshev, and Chebyshev-based methods.)

The results show that the Chebyshev-based approach is the computationally most attractive approach with the relative error of the performance index less than one percent In comparison to the transition matrix approach, the Chebyshev-based approach is increasingly more efficient for N>2. For N=20, the Chebyshev-based results suggest greater than 70 percent savings in execution time. For N=2, the Chebyshev-based method is less efficient than the transition matrix approach since the time to evaluate the integrals in Equations (41a-b), a fixed time for any order system, is a significant fraction of the overall computation cost. For high order systems the principal computational cost is due to the solution of the linear algebraic equation (66), which is less intensive than the solution via the transition matrix method

The Chcbyshev approach offers less time savings than the Chcbyshcv-based approach for high order systems, but is still much faster than the transition matrix approach. The Chebyshev approach is more efficient than the Chcbyshev-based approach when N<4 since the integrals in Equations (41a-b) are easier to solve. Both the Chebyshev and Chebyshev-base approaches have the same values for the performance indices and control vectors because the terms of the series used to approximate the state variables are the same.

The previous Chebyshev approach is computationally more costly than the transition matrix approach. The advantage of this approach is that the relative error of the performance index does not grow significantly when the order of the system increases. The execution time is approximately twice the time of the transition matrix approach.

The time histories of the state and control variables for the case N=2 are plotted in Figures la and lb, respectively. The response curves from the transition matrix and Chebyshev/Chebyshev-based approaches drawn in these figures overlap for the scale shown. Hence, the Chebyshev and Chebyshev-based solutions achieve convergence on the trajectories of the state and control variables as well as on the value of the performance index.

3.2 Example 2

This example, adapted and modified from (Meirovitch, 1990, Example 6.3), considers a series arrangement of J masses and J springs. As shown in Figure 2, it represents a 2J order system with a single force input acting on the last mass, mj. The displacement of mass mj is denoted by qj. The mass and stiffness matrices are

$$M = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_J \end{bmatrix} \tag{78}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & 0 \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & & \ddots & & \\ & & -k_{J-1} & k_{J-1} + k_J & -k_J \\ 0 & & & & k_J \end{bmatrix} \tag{79}$$

The state equaticm of this system is

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{80}$$

where

$$x = [x_1 \; x_2 ... x_{2J}]^T = [q_1 \; q_2 ... q_J \; \dot{q}_1 \; \dot{q}_2 ... \dot{q}_J]^T \tag{81}$$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \tag{82}$$

$$B = [0 \; 0 ... 0 \; 1/m_J]^T \tag{83}$$

The initial conditions are

$$x(0) = [x_1(0) x_2(0) ... x_{2J}(0)]^T \tag{84}$$

where it is presumed

$$x_J(0) = 1$$

$$x_j(0) = 0 \qquad j = 1,2,...,J-1,J+1,...,2J \tag{85a-b}$$

implying that the last mass only has been displaced from rest

The problem is to find the optimal control history, u(t), that minimizes the performance index

$$L = \int_0^\infty (x^T Q x + uRu) dt \tag{86}$$

where

$$\mathbf{Q} = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}$$

$$R = l \qquad (87a\text{-}b)$$

The integrand term x^rQx with Q of Equation (87a) represents the sum of kinetic and potential energies of the system. The inclusion of the integrand term $u^rRu=u^2$ reflects the desire to minimize the force (as well as the total energy).

In this example, using the values mj=10[kg] and kj=l[N/m] (j=l,2,...,J) for two different systems, J=3 and J=5, the optimal solutions were determined using the transition matrix approach and the Chebyshev-based approach. To apply the Chebyshev-based approach, the artificial control variable technique of Section 2.4 was employed with $p=10^5$.

The resulting values of the performance index for the transition matrix and Chebyshev-based approaches are summarized in Table 2. This table shows that the performance index decreases as the number of terms of the Chebyshev-based series increases. The relative errors with four or more terms are less than one percent which indicates that convergence has been achieved In particular, when the number of terms is six or more, the Chebyshev-based solution is smaller than the solution obtained by the transition matrix approach. This is because the weighting constant p is used to penalize the artificial control vector. As shown in the table, for J=3 a 4-term Chebyshev-based approach offers a 20 percent savings in execution time compared with the transition matrix approach, while for J=5 it offers a 28 percent savings compared with the transition matrix approach. This suggests (as does Example 1) that the Chebyshev-based approach is more efficient in solving high order systems.

The response histories for the state variables x_3 and x_6 (the displacement and velocity of the last mass, respectively) and the control variable u for J=3 with four and six term Chebyshev-based series are compared with the state and control variables of the transition matrix approach in Figures 3a and 3b. Both the four-term and six-term solutions are close to the transition matrix solutions. To verify that the artificial control variable technique is successful, the time histories of the artificial control variable $\hat{u}_i$ for four and six-term Chebyshev-based series arc plotted in Figure 3c. As shown in the Figure, the artificial control variable based on a six-term Chebyshev-based series is smaller in magnitude (closer to zero) than the artificial control variable based on a four-term series. However, the magnitudes are small for both cases and hence the influence of the artificial control variables on the system dynamics is negligible.

In summary, this example demonstrates the applicability of the Chebyshev-based approach to general linear systems (with fewer control variables than state variables).

3.3 Example 3

This problem, adapted from (Huntley, 1979), considers the distributed parameter problem

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial y^2} + u(y,t) \qquad 0 \le t \le T, \ 0 \le y \le Y \tag{88}$$

with boundary conditions

$$\frac{\partial x}{\partial y}(0,t) = \frac{\partial x}{\partial y}(Y,t) = 0 \tag{89}$$

and initial condition

$$x(y,0) = 1+y \tag{90}$$

The performance index to be minimized is

$$L = \frac{1}{2}\int_0 \int_0 [x^2(y,t)+u^2(y,t)]\,dy\,dt \tag{91}$$

Using a finite difference approximation, this distributed parameter system can be approximated by the following lumped parameter N+1 order system

$$\dot{x} = Ax + Bu \tag{92}$$

where

$$A = \frac{1}{(\Delta y)^2}\begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix}$$

$B * I_{(N+1)x(N+1)}$

$$\Delta y = \frac{Y}{N} \quad (93a\text{-}e)$$

$x = [x_0\ x_1\ ...\ x_N]^T \qquad x_n = x(n\Delta y),\ \ n=0,1,...,N$

$u = [u_0\ u_1\ ...\ u_N]^T \qquad u_n = u(n\Delta y),\ \ n=0,1,...,N$

with initial conditions

$$x_n(0) = 1 + n\Delta y \quad , n=0,1,...,N \quad (94)$$

The performance index is approximated by

$$L = \frac{\Delta y}{2}\int_0^T (x^TQx+u^TRu)dt \quad (95)$$

where

$$Q = R = \mathrm{diag}\left[\frac{1}{2}, 1, ..., 1, \frac{1}{2}\right]_{(N+1)x(N+1)} \quad (96)$$

In this example, the optimal value of the performance index and the optimal trajectories of the state and control vectors at 101 equally-spaced points were solved by the Chebyshev-based approach, a Riccati equation solver (Speyer, 1986), a transition matrix approach (Speyer, 1986), and a Fourier-based state parameterization approach (Yen and Nagurka, 1988).

The simulation results for T=1, L=4 and N=4, 5, 8, 10, 16, 20 and 32 are summarized in Table 3. Although the Riccati equation solver provides accurate solutions in all cases, it is time-consuming for high order systems. The transition matrix approach is computationally more efficient than the Riccati equation solver but it encounters numerical difficulties and fails to provide reasonable solutions when N is equal to or larger than 16. This instability is caused by the error in computing the transition matrix for the Hamiltonian system. State parameterization approaches such as the Fourier-based approach and the Chebyshev-based approach provide solutions with satisfactory accuracy and arc computationally more efficient than transition matrix approach for all cases. As shown in Table 3, the 3-term Chebyshev-based approach is more accurate and computationally more efficient than the 2-term Fourier-based approach. A K-term Chebyshev-based approach involves N(K+3) linear algebraic equations representing the conditions of optimality. In comparison, a K-term Fourier approach involves N(2K+3) linear algebraic equations (see Yen, Nagurka, 1988). As N grows, the Fourier approach needs to spend relatively greater time on solving the sytem of linear algebraic equations. For example, with N=32, the 3-term Chebyshev-based results suggest greater than 35 percent savings in execution time when compared with the 2-term Fourier-based approach. However, for N>10 in both approaches, the performance indices increase slightly as the order of the system grows, while the solutions from the Riccati equation solver indicate that the performance index should decrease. Adding terms to the series improves the accuracy of the solutions. For example, the performance index of a 4-term Chebyshev-type series does not increase as the order of the system increases when N is equal to or smaller than 16.

The time histories of the state and control variables for the case N=4 obtained via a transition matrix approach and a 3-term Chebyshev-based approach are plotted in Figures 4a and 4b, respectively. The overlap of the state and control trajectories from both approaches indicate that convergence has been achieved.

In summary, this example compares the Chebyshev-based approach with a Riccati equation approach, a transition matrix approach and a Fourier-based approach. The simulation results indicate that the Chebyshev-based approach is computationally more efficient than other approaches, especially in solving high order system.

4. Discussion

4.1 Chebyshev-based State Representation

Two state representations are presented in Sections 23.1 and 2.3.4. These two representations can be viewed as equivalent, differing only in the way the QP problem is solved. The approach described in Section 2.3.3 solves the unconstrained QP problem via a Lagrange multiplier technique. The second approach solves the same problem by direct substitution. To illustrate the difference between these two approaches, consider the following problem.

$$\textit{Minimize} \quad x^2+y^2 \quad \textit{subject to} \quad x+y=l$$

This problem can be readily solved by reformulation as

$$\textit{Minimize} \quad x^2+y^2+\lambda(x+y-1)$$

where X is a Lagrange multiplier. The above expression can be differentiated with respect to x,y and X to obtain three linear equations from which the optimal values of x,y and *X* can be computed. Alternately, this problem can be solved by direct substitution. The equality constraint can first be rewritten as

$$y=l-x$$

and then substituted into the original objective function. The problem is then converted into the unconstrained problem:

$$\textit{Minimize} \quad x^2+(1-x)^2$$

By differentiating the above expression with respect to x, one can obtain a single linear equation. Direct substitution, the method used in Section 2.3.5, is computationally more efficient than the Lagrange multiplier method, the method of Section 2.3.3. This argument is also supported by the simulation results summarized in Table 1.

4.2 Motivation of Auxiliary Polynomial

The inclusion of the third order polynomial in the Chebyshev-based state representation decouples the state parameters from the initial conditions. This enables the direct substitution technique to be used in solving the converted QP problem. Furthermore, the elements of C(T) in Equations (40a-b) can be obtained by Equation (56) instead of via direct substitution of Equation (16). The latter is computationally much more expensive as the order of the system increases.

Because the auxiliary polynomial of Equation (48) in Section 2.3.4 is essentially a reformulation of the first four terms of the Ghebyshev-type series of Equation (IS), the Chebyshev and the Chebyshev-based approaches result in the same solutions. Further study of the simulation results has shown that the evaluation of the integrals involving shifted Chebyshev terms is computationally more expensive than the evaluation of the integrals involving only polynomial terms. However, the advantage of using the integrals involving shifted Chebyshev terms is that the system of linear algebraic equations, Equation (66), becomes better conditioned as more Chebyshev terms are used. That is, the condition number of the coefficient matrix $Cl*_n + Q\backslash\backslash$ becomes larger as K grows, as shown in Table 4 indicating a better conditioned system. (Here, the condition number is taken as (Come, 1965).) Hence, roundoff errors should not present a problem as more terms are included.

4.3 Selection of the Terms of the Chebyshev-based Series

The example problems demonstrate that solutions with high accuracies can be achieved using two or three term Chebyshev-based series *{e.g.,* see Example 1 and Example 3). However, more terms are needed in Example 2 to achieve the required accuracy. Increasing the number of terms of the Chebyshev-based series improves the accuracy of the solution while sacrificing computational time. A recommended procedure for selecting the "optimum" number of the terms is to solve the problem using a K term series and a K+I term series (where "I" is an integer increment), respectively, and check whether the relative error of the performance index is within the required tolerance. If the relative error is within the required tolerance, the K term series is acceptable.

For example, considering the performance index for J=3 in Table 2, the percent relative error is 61 percent comparing K=2 and K=4 and is 0.92 percent comparing K=4 and K=6. (Here the percent relative error is defined as the magnitude of percent relative error of a (K+2) term performance index with respect to a K term performance index.) With a tolerance of one percent, the results show that a four term series leads to an accurate approximation.

5. Conclusions

This report presents a state parameterization method based on a finite-term Chebyshev representation of the state trajectory. Such a representation is used for predicting the optimal state and control trajectories of unconstrained linear time-invariant dynamic systems with quadratic performance indices. In one method, the time history of each state variable is approximated by a shifted Chebyshev series. The unconstrained LQ problem is then converted to an equality constrained QP problem that minimizes the performance index and satisfies the state initial conditions via Lagrange multipliers. In a second method, the time history of each state variable is represented by the superposition of a shifted Chebyshev series and a third order polynomial. The inclusion of the auxiliary polynomial improves the speed of evaluation of the integral parts in Equations (41a-b) in comparison to a standard Chebyshev series. In both methods, the necessary condition of optimality gives a system of linear algebraic equations from which the unknown state parameters can be solved The results of simulation studies demonstrate computational advantages of the Chebyshev-based method relative to the Chebyshev method, a previous Chebyshev method and a standard state transition matrix approach.

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Appendix A Integrals for Chebyshev Approach

A.1 Integrals of Chebyshev Polynomials

First, the integrals of $\xi^i \varphi_j$, $\xi^i \dot{\varphi}_j$, $\varphi_i \varphi_j$, $\dot{\varphi}_i \varphi_j$ and $\dot{\varphi}_i \dot{\varphi}_j$ are defined as

$$\alpha_{ij} \equiv \int_{-1}^{1} \xi^i \varphi_j(\xi)\, d\xi \tag{A.1}$$

$$\beta_{ij} \equiv \int_{-1}^{1} \xi^i \dot{\varphi}_j(\xi)\, d\xi \tag{A.2}$$

$$\gamma_{ij} \equiv \int_{-1}^{1} \varphi_i(\xi)\, \varphi_j(\xi)\, d\xi \tag{A.3}$$

$$\delta_{ij} \equiv \int_{-1}^{1} \dot{\varphi}_i(\xi)\, \varphi_j(\xi)\, d\xi \tag{A.4}$$

$$\varepsilon_{ij} \equiv \int_{-1}^{1} \dot{\varphi}_i(\xi)\, \dot{\varphi}_j(\xi)\, d\xi \tag{A.5}$$

Applying Equations (7a-d), (8a-b), (9a-b) and (10a), the integrals of $\xi^i \varphi_j$, $\xi^i \dot{\varphi}_j$, $\varphi_i \varphi_j$, $\dot{\varphi}_i \varphi_j$ and $\dot{\varphi}_i \dot{\varphi}_j$ can be derived by "integration by parts". In this section, the recurrence fonnulas for these integrals are summarized as follows.

For i=0,1,2,...; j=0,1,2,...

$$\alpha_{ij} = 0 \quad \text{for } j - i = 1$$

$$\alpha_{ij} = \frac{-1}{(j+1)(j-i-1)}\left[1 + (-1)^{i+j} + ij\,\alpha_{(i-1)(j+1)}\right] \quad \text{for } j \neq i+1 \tag{A.6a-b}$$

$$\beta_{ij} = 1 - (-1)^{i+j} - i\,\alpha_{(i-1)j} \tag{A.7}$$

$$\gamma_{ij} = \frac{1}{2}\left[\alpha_{0(i+j)} + \alpha_{0|i-j|}\right] \tag{A.8}$$

$\delta_{0j} = 0$

810 = Poi for i * 0

811=pn fori*0

812 = 2P2i - Poi for i * 0 (A.9a-0

5i3=4p$_{3i}$-3pii for i *0

$\delta_{ij} = 2\delta_{i(j-2)} - \delta_{i(j-4)} + 2i\left[\gamma_{(i+1)(j-2)} - \gamma_{(i-1)(j-2)}\right]$ for $i \neq 0$, j=4,5,...

$e_{i0} = 0$

Cii = Poi

e*2 = 4 P H (A.10a-e)

Ei3=12p2i-3poi

£ij = 2£i(j.2) - £j(j^> + 2j8i(j.i) + (8 - 2j)8i(j.3) for j=4,5,...

A.2 Integrals of Shifted Chebyshev Polynomials

In this section, the integrals of r>j, t'vj/j, ViV$_{jt}$ Vi\|/j and Wj are derived using the equations of the previous section. These integrals are needed to determine the integrals in Equations (41a-b) for the Chebyshev approach.

From Equations (11) and (12) the following equations are known:

$$\psi_k(t) = \varphi_k(\xi)$$

$$\dot{\psi}_k(t) = \frac{2}{T}\dot{\varphi}_k(\xi)$$

$$t = \frac{T}{2}(1 + \xi) \qquad \text{(A.I la-d)}$$

$$dt = \frac{T}{2}d\xi$$

Making use of Equations (A.I la-d), the following integrals are obtained:

$$\int^{T} (t)\,dt = \frac{T}{2}\alpha_{0i} \qquad (A.12)$$

$$\int^{T} (t)\,dt = \beta_{0i} \qquad (A.13)$$

$$\int \psi_i(t)\psi_j(t)\,dt = \frac{T}{2}\gamma_{ij} \qquad (A.14)$$

$$\int_0 (t)\,dt = \delta_{ij} \qquad (A.15)$$

$$\int^{T} (t)\,dt = \frac{T}{2}\varepsilon_{ij} \qquad (A.16)$$

$$\int \psi_i(t)\,dt = \frac{T}{4}(\alpha_{0i} + \alpha_{1i}) \qquad (A.17)$$

$$\int (\alpha_{0i} + 2\alpha_{1i} + \alpha_{2i}) \qquad (A.18)$$

$$\int_0 \psi_i(t)\,dt = \frac{T}{16}(\alpha_{0i} + 3\alpha_{1i} + 3\alpha_{2i} + \alpha_{3i}) \qquad (A.19)$$

$$\int \psi_i(t)\,dt = \frac{T}{2}(\beta_{0i} + \beta_{1i}) \qquad (A.20)$$

$$\int_0 \psi_i(t)\,dt = T(\beta_{0i} + 2\beta_{1i} + \beta_{2i}) \qquad (A.21)$$

$$\int t^3 \dot{\psi}_i(t)dt = \frac{T^3}{8}(\beta_{0i} + 3\beta_{1i} + 3\dot{\beta}_{2i} + \beta_{3i}) \tag{A.22}$$

for i=0,l,2,...; j=0,l,2,...

By introducing Ck = Vk-i(t) and dk = V̇k-i(t) into Equations (A. 12) - (A. 16) and applying Equations (18) and (25), the integrals in Equation (41a-b) for the Chebyshev approach can be summarized as:

$$\int_0^T c\,dt = \frac{T}{2}\alpha \tag{A.23}$$

$$\int_0^T d\,dt = P \tag{A.24}$$

$$(c\,c^T)dt = \frac{T}{2}\gamma \tag{A.25}$$

$$\int (dc^r)dt = 5 \tag{A.26}$$

$$\int (d\,d^T)dt = \frac{2}{T}\varepsilon \tag{A.27}$$

where

$$ot = [\alpha_{00} \quad \alpha_{01} \quad \alpha_{02} \quad \cdots \quad \alpha_{0(K-1)}]^T \tag{A.28}$$

$$P=[poo \quad Poi \quad Pra \quad \cdots \quad P<XK-1)]^T \tag{A.29}$$

$$\gamma \simeq \begin{bmatrix} \gamma_{00} & \cdots & \gamma_{0(K-1)} \\ \gamma_{10} & & \vdots \\ \vdots & & \\ \gamma_{(K-1)0} & \cdots & \gamma_{(K-1)(K-1)} \end{bmatrix} \tag{A.30}$$

$$\delta = \begin{bmatrix} \delta_{00} & \cdots & \delta_{0(K-1)} \\ \delta_{10} & & \vdots \\ \vdots & & \\ \delta_{(K-1)0} & \cdots & \delta_{(K-1)(K-1)} \end{bmatrix} \tag{A.31}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_{00} & \cdots & \varepsilon_{0(K-1)} \\ \varepsilon_{10} & & \vdots \\ \vdots & & \\ \varepsilon_{(K-1)0} & \cdots & \varepsilon_{(K-1)(K-1)} \end{bmatrix} \tag{A.32}$$

Appendix B Integrals for Chebyshev-based Approach

In order **to** formulate the integral parts of P* and p* for the Chebyshev-based approach, C_k ($k = 1,2,\ldots,K$) in Equations (53a-d) and (54) is redefined as

$$C_k = \mu_{0k} + \mu_{1k}\tau + \mu_{2k}\tau^2 + \mu_{3k}\tau^3 + \kappa_k\psi_{k-1}(t) \quad \text{(B.1)}$$

where $k = 1, 2, \ldots, K$, $\tau = t/T$,

$$\begin{bmatrix} \mu_{01} & \mu_{02} & \mu_{03} & \mu_{04} \\ \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ -3 & -2T & 3 & -T \\ 2 & T & -2 & T \end{bmatrix} \quad \text{(B.2)}$$

$$\begin{bmatrix} \mu_{0k} \\ \mu_{1k} \\ \mu_{2k} \\ \mu_{3k} \end{bmatrix} = \begin{bmatrix} (-1)^k \\ -2(-1)^k(k-1)^2 \\ 2k^2 - 4k - 1 + (-1)^k(4k^2 - 8k + 1) \\ 2k(2-k)[1 + (-1)^k] \end{bmatrix}, \quad k = 5, 6, \ldots, K \quad \text{(B.3)}$$

$$\kappa_k = \begin{cases} 1 & k = 5, 6, \ldots, K \\ 0 & k = 1, 2, 3, 4 \end{cases} \quad \text{(B.4)}$$

Then the elements of $\int_0^T c\,dt$, $\int_0^T d\,dt$, $\int_0^T (cc^T)dt$, $\int_0^T (dc^T)dt$ and $\int_0^T (dd^T)dt$ can be expressed as

$$\int_0^T c_i dt = T(\mu_{0i} + \mu_{1i}/2 + \mu_{2i}/3 + \mu_{3i}/4) + \kappa_i \int_0^T \psi_{i-1}(t)dt \quad \text{(B.5)}$$

$$\int_0^T \dot{d}_i dt = \mu_{1i} + \mu_{2i} + \mu_{3i} + \kappa_i \int_0^T \dot{\psi}_{i-1}(t)dt \quad \text{(B.6)}$$

$$\int_0^T c_i c_j dt = T[\mu_{0i}\mu_{0j} + (\mu_{1i}\mu_{0j} + \mu_{0i}\mu_{1j})/2 + (\mu_{2i}\mu_{0j} + \mu_{1i}\mu_{1j} + \mu_{0i}\mu_{2j})/3$$

$$+ (\mu_{3i}\mu_{0j} + \mu_{2i}\mu_{1j} + \mu_{1i}\mu_{2j} + \mu_{0i}\mu_{3j})/4 + (\mu_{3i}\mu_{1j} + \mu_{2i}\mu_{2j} + \mu_{1i}\mu_{3j})/5$$

$$+ (\mu_{2i}\mu_{3j} + \mu_{3i}\mu_{2j})/6 + \mu_{3i}\mu_{3j}/7]$$

$$+ \kappa_j\left[\mu_{0i}\int_0^T \psi_{j-1}(t)dt + \frac{\mu_{1i}}{T}\int_0^T t\psi_{j-1}(t)dt + \frac{\mu_{2i}}{T^2}\int_0^T t^2\psi_{j-1}(t)dt + \frac{\mu_{3i}}{T^3}\int_0^T t^3\psi_{j-1}(t)dt\right]$$

$$+ \kappa_i\left[\mu_{0j}\int_0^T \psi_{i-1}(t)dt + \frac{\mu_{1j}}{T}\int_0^T t\psi_{i-1}(t)dt + \frac{\mu_{2j}}{T^2}\int_0^T t^2\psi_{i-1}(t)dt + \frac{\mu_{3j}}{T^3}\int_0^T t^3\psi_{i-1}(t)dt\right]$$

$$+ \kappa_i\kappa_j\int_0^T \psi_{i-1}(t)\psi_{j-1}(t)dt \qquad \text{(B.7)}$$

$$\int_0^T \dot{c}_i c_j dt = \mu_{1i}\mu_{0j} + (\mu_{1i}\mu_{1j} + 2\mu_{2i}\mu_{2j})/2 + (\mu_{1i}\mu_{2j} + 2\mu_{2i}\mu_{1j} + 3\mu_{3i}\mu_{0j})/3$$

$$+ (\mu_{1i}\mu_{3j} + 2\mu_{2i}\mu_{2j} + 3\mu_{3i}\mu_{1j})/4 + (2\mu_{2i}\mu_{3j} + 3\mu_{3i}\mu_{2j})/5 + \mu_{3i}\mu_{2j}/2$$

$$+ \frac{\kappa_j}{T}\left[\mu_{1i}\int_0^T \psi_{j-1}(t)dt + \frac{2\mu_{2i}}{T}\int_0^T t\psi_{j-1}(t)dt + \frac{3\mu_{3i}}{T^2}\int_0^T t^2\psi_{j-1}(t)dt\right]$$

$$+ \kappa_j\left[\mu_{0j}\int_0^T \dot{\psi}_{i-1}(t)dt + \frac{\mu_{1j}}{T}\int_0^T t\dot{\psi}_{i-1}(t)dt + \frac{\mu_{2j}}{T^2}\int_0^T t^2\dot{\psi}_{j-1}(t)dt + \frac{\mu_{3j}}{T^3}\int_0^T t^3\dot{\psi}_{i-1}(t)dt\right]$$

$$+ \kappa_i\kappa_j\int_0^T \dot{\psi}_{i-1}(t)\psi_{j-1}(t)dt \qquad \text{(B.8)}$$

$$\int_0^T \phi_i \phi_j dt = \frac{1}{T}[\mu_{1i}\mu_{1j} + \mu_{2i}\mu_{1j} + \mu_{1i}\mu_{2j} + (3\mu_{3i}\mu_{1j} + 4\mu_{2i}\mu_{2j} + 3\mu_{1i}\mu_{3j})/3$$

$$+ (3\mu_{3i}\mu_{2j} + 3\mu_{2i}\mu_{3j})/2 + 9\mu_{3i}\mu_{3j}/5]$$

$$+ \frac{\kappa_j}{T}\left[\mu_{1i} \qquad \int_0 t\dot{\psi}_{j-1}(t)dt + \quad T\int_0^T t^2\dot{\psi}_{j-1}(t)dt\right]$$

$$+ \frac{\kappa_i}{T}\left[\int_0^T \dot{\psi}_{i-1}(t)dt + \frac{2\alpha_{2j}}{T}\int_0^T t\dot{\psi}_{i-1}(t)dt + \frac{3\alpha_{3j}}{T^2}\int_0^T t^2\dot{\psi}_{i-1}(t)dt\right]$$

$$+ \kappa_i\kappa_j \int \dot{\psi}_{i-1}(t)\,\dot{\psi}_{j-1}(t)dt \tag{B.9}$$

where the integrals on the right hand sides of Equations (B.6)-(B.10) come from Equations (A.12)-(A.22).

Appendix C Transition Matrix Approach

Consider **the** LQ problem that minimizes

$$L = x^r(T)Hx(T) + \int_{Jo}^{T} (x^r(t)Q(t)x(t)+u^r(t)R(t)u(t))dt \qquad \text{(C.I)}$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \ , \ \ x(0) = x_o \qquad \textbf{(C2)}$$

For simplicity, cross product and linear terms of the control and state vectors have been omitted from the performance index. The order of the system is assumed to be"N.

In the transition matrix approach (see, for example, Speyer, 1986), the Hamiltonian is first introduced as

$$H = \frac{1}{2}\mathbf{x}^T\mathbf{Qx} + \frac{1}{2}\mathbf{u}^T\mathbf{Ru} + \lambda^T\mathbf{Ax} + \lambda^T\mathbf{Bu} \qquad \textbf{(C.3)}$$

where *X* can be viewed as a Lagrangian multiplier vector whose elements are often called costate variables. It can be shown that the necessary conditions of optimality are

$$\dot{\mathbf{x}} = \frac{3//}{dX} = s\ Ax + \mathbf{Bu} \ , \ \ x(0) = x_0 \qquad \textbf{(C.4)}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\mathbf{Qx} - \mathbf{A}^T\lambda \ , \ \ \lambda(T) = \mathbf{Hx(T)} \qquad \text{(C.5)}$$

$$0 = \frac{\partial H}{\partial \mathbf{u}} = \mathbf{Ru} + \mathbf{B}^T\lambda \qquad \text{(C.6)}$$

Equation (C.6) can be solved for the optimal control **u** giving

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\lambda \qquad \textbf{(C.7)}$$

Substituting equation (C.7) into equation (C.4) yields

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BR}^{l}\mathbf{B}^{r}\mathbf{X} \qquad \textbf{(C.8)}$$

Combining equations (C.5) and (C.8) gives a TPBVP that consists of 2N linear homogeneous differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{X} \end{bmatrix} = \begin{bmatrix} A & -\mathbf{BR}^{-1}\mathbf{B}^T \\ -Q & -A^r \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} , \ \ \mathbf{x(0) = x_o} \ , \ \ \lambda(T) = \mathbf{Hx(T)} \qquad \text{(C.9)}$$

This system of equations is often called the Hamiltonian system. Its solution has the following forai

$$\begin{bmatrix} x(t_2) \\ \lambda(t_2) \end{bmatrix} = \begin{bmatrix} \\ \Lambda(t_1) \end{bmatrix} \qquad \text{(CIO)}$$

where ϕ is the transition matrix of the Hamiltonian system. By setting t2 = T and ti = 0, equation (CIO) gives

$$\begin{bmatrix} x(T) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} \phi_{11}(T,0) & <t>i_2(T,0) \\ \phi_{21}(T,0) & <|>22(T,0) \end{bmatrix} \begin{bmatrix} x(0) \\ \end{bmatrix} \qquad \text{(Cll)}$$

With the terminal condition X(T) = Hx(T) given by equation (C5), X(0) can be determined from equation (C. 11) as

$$*.(0)=K(T)x(0) \qquad \text{(C12)}$$

where

$$K(T) - [<|>22(T,0) - H\wedge I_2(T,0)]'^1[H\wedge I I(T,O) - <l>2i(T,0)] \qquad \text{(C. 13)}$$

The Hamiltonian system of equation (C.9) can thus be viewed as an initial value problem. Using equation (CIO), the solution of this initial value problem can be formulated as

$$\begin{bmatrix} x(t_p+At) \\ \lambda(t_p+\Delta t) \end{bmatrix} = \phi(t_p+\Delta t, t_p) \begin{bmatrix} x(t_p) \\ \lambda(t_p) \end{bmatrix} \quad \text{forp ss } l_f\text{...,}P \qquad \text{(C.14)}$$

where *P* is the number of equally-spaced points for which the solution is required and At = *TIP*. Note that for time-invariant problems, the transition matrix 4>(t/>+At, t_p) is independent of t_p and is only a function of At. A solution approach based on equation (C.14) is computationally much more efficient in general than solving equation (C.9) using numerical integration-based differential equation solvers such as Runge-Kutta methods. The corresponding optimal control u can be computed from equation (C7). Using this transition-matrix approach, it can also be shown that the corresponding performance index value is

$$L^* = x^r(0)K(T)x(0) \qquad \text{(C•15)}$$

Appendix D Paraskevopoulos' Approach for LQ Problems

This section summarizes the method of (Paraskevopoulos, 1985) for solving the unconstrained optimal LQ problem. Consider the LQ problem that minimizes

$$L = x^r(T)Hx(T) + \int_{J_0}^{F} (x^r(t)Q(t)x(t) + u^r(t)R(t)u(t))dt \qquad \text{(D.I)}$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad , \quad x(0) = x_0 \qquad \text{(D.2a-b)}$$

The order of the system is assumed to be N. The optimal control u*(t) is given by

$$u'(t) = -R\text{-}^{1}B^{r}X(t) \qquad \text{(D.3)}$$

where X(t) is a vector satisfying the canonical equation:

$$\begin{bmatrix} \mathbf{x(t)} \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ MO \end{bmatrix} = M \begin{bmatrix} x(t) \\ X(t) \end{bmatrix} \qquad \text{(D.4)}$$

with the boundary conditions specified as

$$x(0) = x_0$$
$$X(T) = \mathbf{Hx(t)} \qquad \text{(D.5a-b)}$$

Matrix M in Equation (D.4) is a transition matrix which can be written as a panitioned matrix

$$e^{Mt} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ .<I>2I(0 & 022(t) \end{bmatrix} \qquad \text{(D.6)}$$

Then the TPBVP (represented by Equations (D.4) and (D.5)) can be converted to an initial value problem with initial conditions

$$\mathbf{x(0) = x_0}$$
$$\mathbf{X(0) = K(T)x(0) = X_0} \qquad \text{(D.7a-b)}$$

where

$$K(t) = [<J>22(T) - HtOaCT)]'^{1} [H<J>n(T).\ \phi_{21}(T)] \qquad \text{(D.8)}$$

In the following, equation (D.4) will be solved using Chebyshev series. To this end, the transformation

$$\tau = 2\frac{t}{T} - 1 \tag{D.9}$$

is introduced such that when t=0 then t=-l and when t=T then t=l. Using Equation (D.9), Equation (D.4) may be written as

$$\begin{bmatrix} \dot{x}(\tau) \\ \dot{\lambda}(\tau) \end{bmatrix} = -N \begin{bmatrix} x(\tau) \\ \lambda(\tau) \end{bmatrix} \tag{D.10}$$

where

$$N = \frac{-T}{2} M \tag{D.ll}$$

Integration of Equation (D. 10) from -1 to t yields

$$\begin{bmatrix} x(\tau) \\ \lambda(\tau) \end{bmatrix} = -N \begin{bmatrix} \int_{-1}^{\tau} x(\sigma)d\sigma \\ \int_{-1}^{\tau} \lambda(\sigma)d\sigma \end{bmatrix} + \begin{bmatrix} x(\tau = -1) \\ \lambda(\tau = -1) \end{bmatrix} \tag{D.12}$$

where

$$x(\tau = -1) = x(t = 0) = x_0$$
$$\lambda(\tau = -1) = \lambda(t = 0) = \lambda_0 \tag{D.13a-b}$$

Vectors x(t) and *X(t)* are then expanded in a K-term Chebyshev series as follows

$$\begin{bmatrix} x(\tau) \\ \lambda(\tau) \end{bmatrix} = W\varphi(\tau) \tag{D.14}$$

where W is a 2NxK Chebyshev coefficient matrix (to be determined) and $\varphi(t)$ is a Chebyshev function vector.

$$W = [w_0 \quad w_1 \quad \cdots \quad w_{K-1}]_{2M}$$
$$\varphi(\tau) = [\varphi_0(\tau) \quad \varphi_1(\tau) \quad \cdots \quad \varphi_{K-1}(\tau)]^T \tag{D.15a-b}$$

Paraskevopoulos (1983) introduces a KxK Chebyshev operational matrix P which has the following property

$$\int_{-1}^{x} \gamma(\sigma)\,d\sigma \approx P\varphi(x) \qquad \text{(D.16)}$$

where

$$P = \begin{bmatrix} \alpha_0 & \beta_0 & \gamma_0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & \gamma_2 & 0 & \cdots & 0 & 0 \\ \alpha_3 & 0 & \beta_3 & 0 & \gamma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha_{k-3} & 0 & 0 & 0 & 0 & \cdots & \gamma_{k-3} & 0 \\ \alpha_{k-2} & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{k-2} \\ \alpha_{k-1} & 0 & 0 & 0 & 0 & \cdots & \beta_{k-1} & 0 \end{bmatrix} \qquad \text{(D.17)}$$

with

$$\alpha_n = \begin{cases} 1 & \text{for} \quad n = 0 \\ \dfrac{-1}{4} & \text{for} \quad n = 1 \\ \dfrac{(-1)^{n+1}}{n^2-1} & \text{for} \quad n = 2, 3, \ldots \end{cases}$$

$$\beta_n = \begin{cases} 0 & \text{for} \quad n = 0 \\ 1 & \text{for} \quad n = 1 \\ \dfrac{-1}{2(n-1)} & \text{for} \quad n = 2, 3, \ldots \end{cases} \qquad \text{(D.18a-c)}$$

$$\gamma_n = \begin{cases} 0 & \text{for} \quad n = 0 \\ \dfrac{1}{4} & \text{for} \quad n = 1 \\ \dfrac{1}{2(n+1)} & \text{for} \quad n = 2, 3, \ldots \end{cases}$$

By using Equations (D.14) and (D.16), Equation (D.12) may be written as

$$W\varphi(x) = -NWP\varphi(x) + S\varphi(x)$$

or

$$W + NWP = S \qquad \text{(D.19)}$$

where

$$\mathbf{S} = \begin{bmatrix} x(x - -1) & 0 & \cdots & 0 \\ X(x«-1) & 0 & \dots & 0 \end{bmatrix} =: [S_Q \quad Si \ \cdots \ \mathbf{S}_{K-1}]_{2N \times K} \tag{D.20}$$

Equation (D.19) can be written as

$$T\,w = S \tag{D.21}$$

where

$$\mathbf{T} = \mathbf{I} + \mathbf{P}^T \otimes \mathbf{N}$$
$$w = [wo \quad wi \quad \dots \quad w_{K}.i]^r \tag{D.22a-c}$$
$$S = [S_0 \quad S, \quad \dots \quad S_K.,f$$

and where the Kronecker product is defined as

$$\mathbf{P}^T \otimes \mathbf{N} = \begin{bmatrix} PnN & P_2iN & \cdots & \mathbf{P_{1K}N} \\ P_{12}N & P_{22}N & \cdots & P_{2K}N \\ \vdots & \vdots & & \vdots \\ \mathbf{P_{1K}N} & P_{2K}N & \cdots & P_{KK}N \end{bmatrix}_{2NKx2NK} \tag{D.23}$$

From Equation (D.21) w may be obtained. Once determined, the control vector can be calculated according to Equations (D.14) and (D.3).

Table 1. Comparison of Simulation Results for Example 1*

KI IN	Transition Matrix		Previous Chebyshev		Chebyshev		Chebyshev-based	
	Perf. Index	Time (sec)	%Er **	%Time ***	%Er **	%Time ***	%Er **	%Time ***
2	5.3591	0.50	1.29e-03	144.0	3.21e-05	120.0	3.21e-05	156.0
4	44.249	2.42	2.56e-03	171.9	7.67e-04	81.0	7.67e-04	67.8
6	153.75	7.06	3.67e-02	187.0	5.23e-03	73.4	5.23e-O3	48.4
8	373.02	15.86	1.56e-01	193.8	1.84c-02	69.4	1.84c-02	40.7
10	741.61	29.04	1.76e-01	202.8	4.41e-02	70.0	4.41c-02	38.2
12	1299.3	50.44	1.74e-01	204.0	8.32e-02	67.4	8.32e-02	35.1
14	2086.3	81.46	1.61e-01	198.3	1.34c-01	64.8	1.34e-01	33.2
16	3142.8	124.54	1.48c-01	197.0	1.94c-01	62.4	1.94e-01	30.8
18	4509.0	174.24	1.39e-01	199.6	2.61e-01	62.8	2.61e-01	30.4
20	6225.4	247.50	1.36e-01	191.8	3.31e-01	60.2	3.31c-01	28.7

* Six-term series for Chebyshev and Previous Chebyshev approaches. For Chebyshev-based approach, four terms are used in polynomial and two terms in Chebyshev series.

** Magnitude of percent relative error of Chebyshev performance index with respect to transition matrix performance index

*** Percent of execution time of Chebyshev approach relative to execution time of Transition Matrix approach

Table 2. Summary of Example 2 Results

J	Transition matrix		Chebyshev-based*			%Error**	%Time**»
	Performance Index	Time (sec)	Terms	Performance Index	Time (sec)		
3	7.62051	7.82	**2**	20.12276	3.06	164	39
			4	7.68190	6.26	0.81	80
			6	7.61147	11.74	0.12	150
5	7.62044	31.08	**2**	19.49309	10.02	156	32
			4	7.68581	22.44	0.86	72
			6	7.61124	43.20	0.13	139

* Four terms polynomial and two terms Chebyshev series are used.

** Magnitude of percent relative error of Chebyshev performance index with respect to transition matrix performance index

*** Percent of execution time of Chebyshev approach relative to execution time of Transition Matrix approach

Table 3. Comparison of Simulation Results for Example 3

N	Riccati* K1C.Call		Transition Matrix		Fourier 2 terms		Chebyshev 3 terms		Chebyshev 4 terms	
	Perf. Index	**Time (sec)**	**Perf. Index**	**Time** (sec)	**Perf. Index**	**Time (sec)**	**Perf. Index**	**Time (sec)**	**Perf. Index**	**Time (sec)**
4	15.180	9.08	15.180	4.36	15.180	3.42	**15.180**	2.94	15.180	4.06
5	15.112	17.08	15.112	7.06	15.112	5.18	15.112	4.30	15.112	5.90
8	15.042	73.36	15.042	21.56	15.043	13.76	15.043	10.74	15.043	15.22
10	15.027	154.36	15.027	38.66	15.031	23.38	15.030	17.72	15.030	25.58
16	15.011	820.08	Instability		15.042	78.00	15.042	55.98	15.027	80.90
20	15.007	4313.82	Instability		15.068	138.34	15.061	89.14	15.038	130.56
32	15.003	43351.62	Instability		15.170	510.18	15.165	328.94	15.112	496.26

* For N=4 to N=16, the Riccati equation is integrated backward in time by a fourth-order Runge-Kutta routine using a time step of 0.01 time unit. In order to ensure a numerically stable solution, a time step of 0.005 time unit is used for N=20 and a time step of 0.0025 time unit is used for N=32.

Table 4. Condition Numbers for Example 1,2 and 3

K	Example 1 (N=2)	Example 2 (J=3)	Example 3 (N=4)
4	1.6732e-1	**1.3490e83**	2.3709e8
6	1.3247e07	**5.1105e148**	1.3630e10
8	1.6116e17	-2.2184e226	2.0283e32

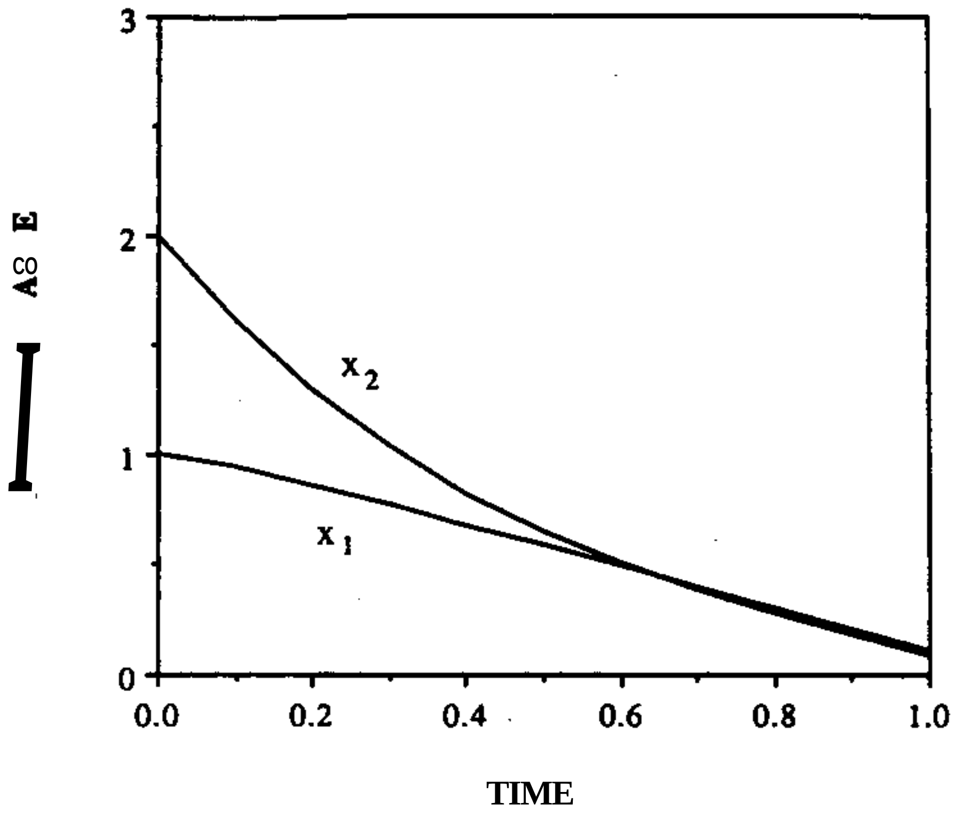

Figure la State Variable History for Example 1

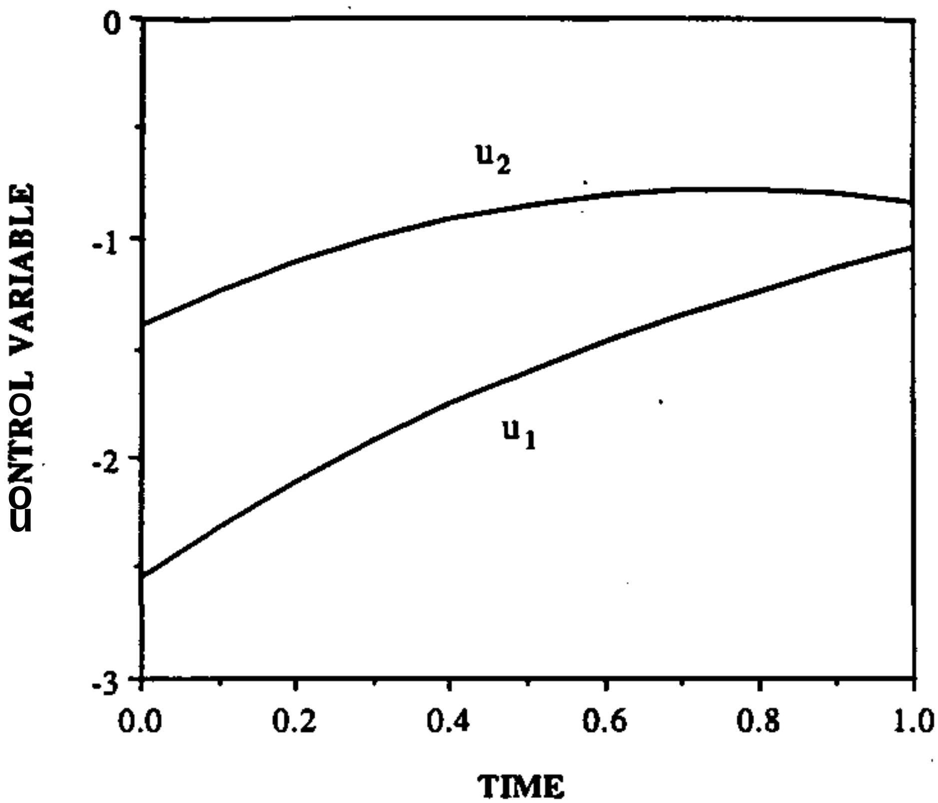

Figure lb Control Variable History for Example 1

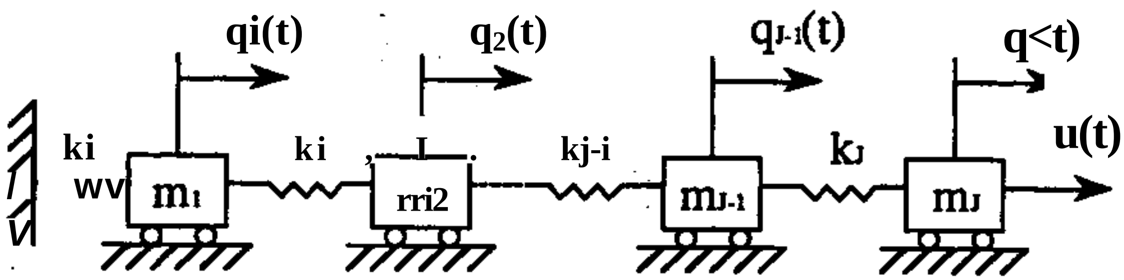

Figure 2 2J Order System for Example *2*

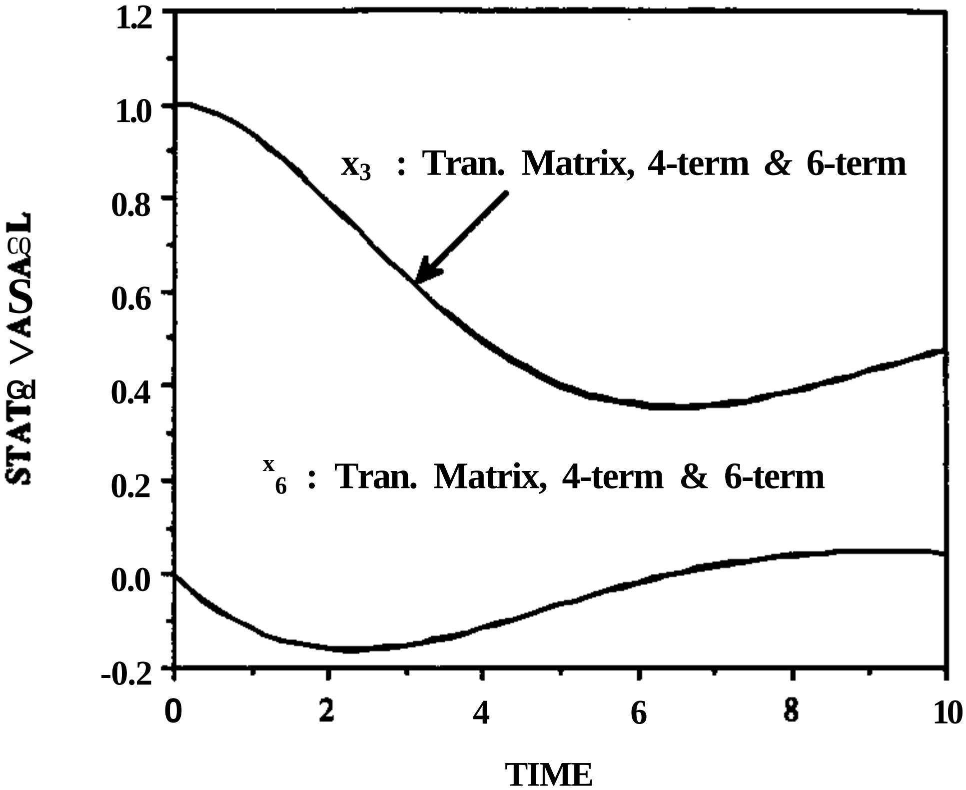

Figure 3a State Variable History for Example 2

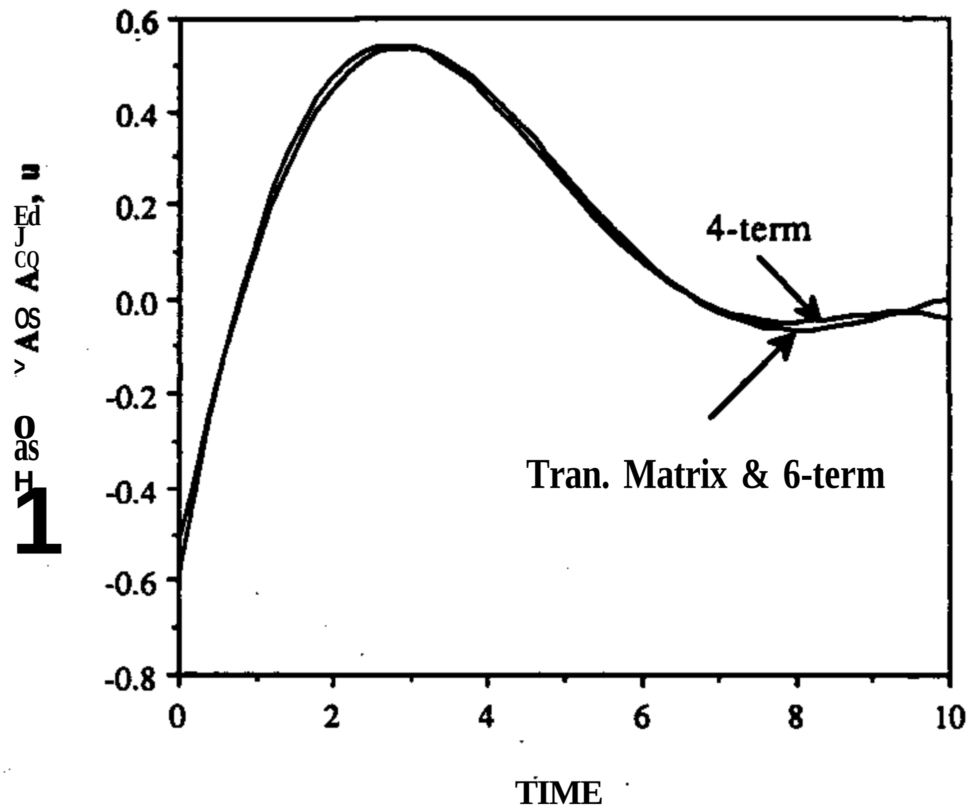

Figure 3b Control Variable History for Example 2

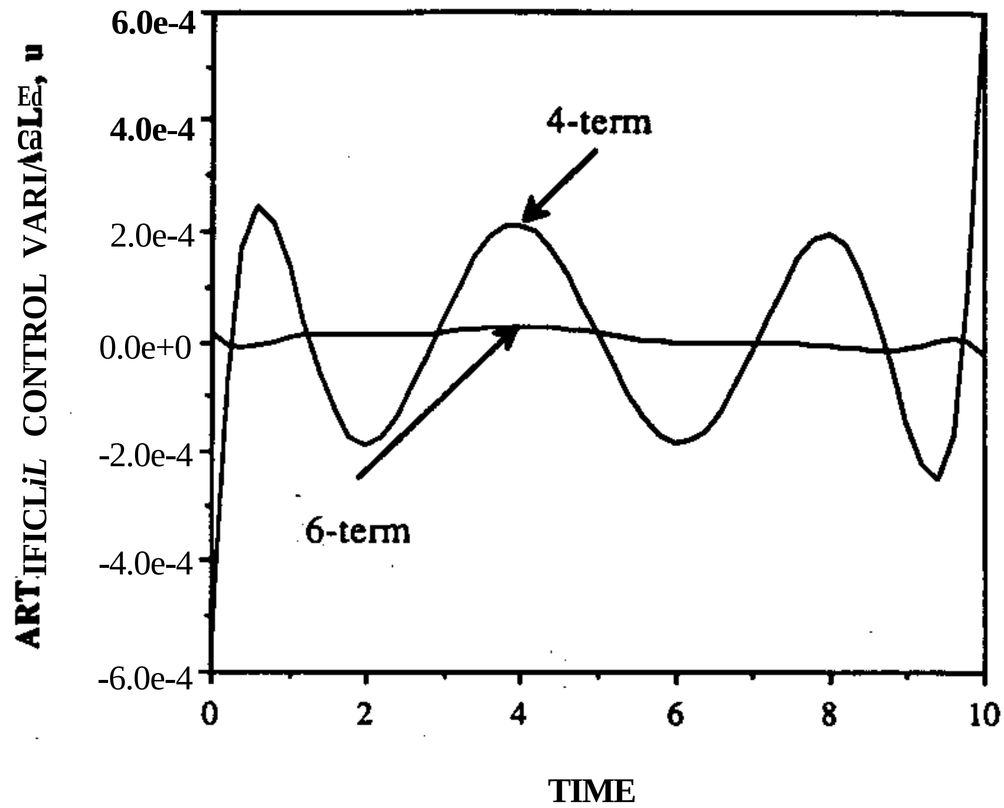

Figure 3c Artificial Control History for Example 2

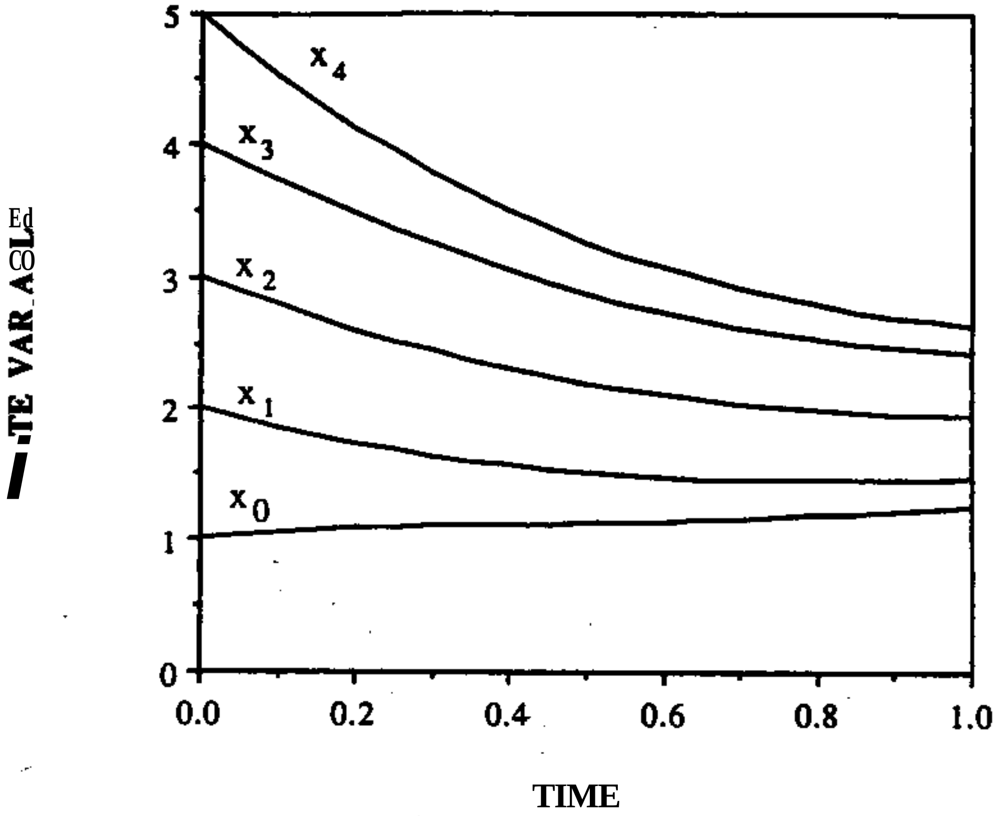

Figure 4a State Variable History for Example 3

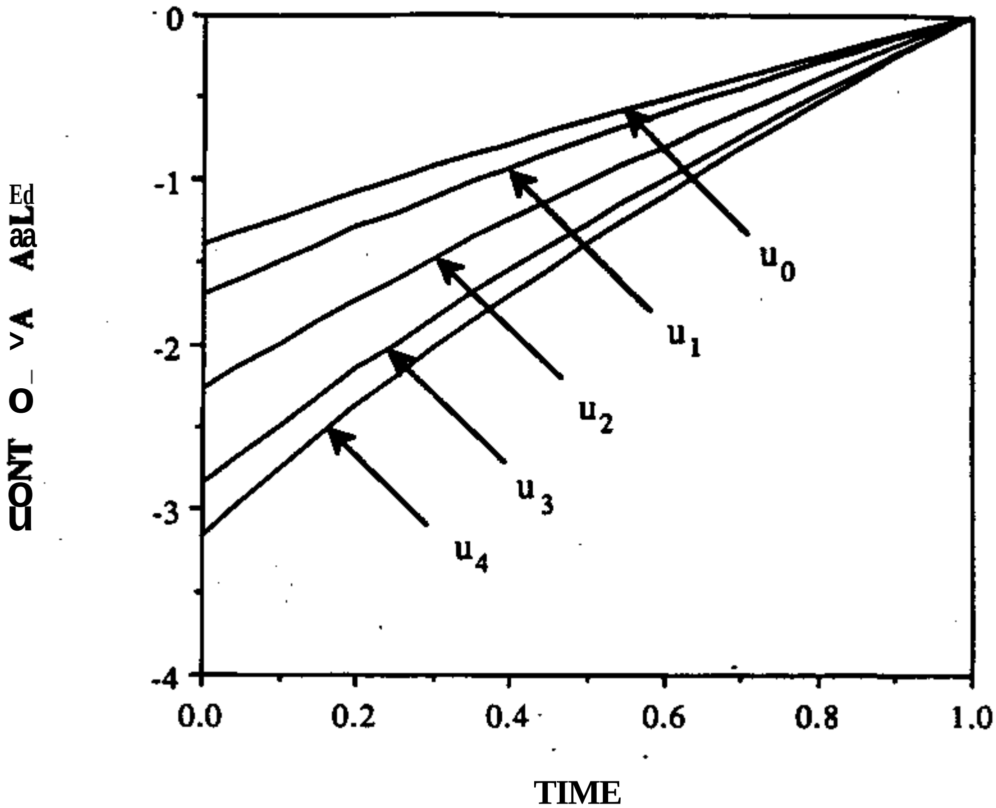

Figure 4b Control Variable History for Example 3