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**New Design Paradigms for MIMO
Control System Synthesis**

by

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New Design Paradigms for MIMO Control System Synthesis

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This report examines the use of Gain Plots (GPs), a new graphical representation and perspective on the Evans root locus, for analysis and design of multivariable feedback control systems. The development is based on the adjustment of a scalar, forward loop, proportional control gain cascaded with a square multi-input, multi-output (MIMO) plant employed in an output feedback configuration. By tracking the closed-loop eigenvalues as an explicit function of gain, it is possible to visualize the MIMO root loci in a set of plots, the GPs, depicting the polar coordinates of each eigenvalue in the complex plane. The GPs consist of two graphs: (i) magnitude of system eigenvalues vs. gain, and (ii) argument (angle) of system eigenvalues vs. gain. The concept of GPs is developed in detail in a companion report focusing on single-input, single-output systems (Kurfess and Nagurka, 1991a).

By identifying closed-loop eigenvalue trajectories, the GPs impart significant insight for determining the values of scalar gain that render a MIMO closed-loop system either stable or unstable. Furthermore, by exposing the correspondence of gain values to specific eigenvalue angles and magnitudes, the GPs are useful for evaluating the migration of closed-loop eigenvalues toward finite and infinite transmission zeros. The GPs reveal MIMO eigenvalue information unambiguously in a new and precise manner, that is not available in a standard MIMO root locus plot. Thus, GPs significantly enhance the control engineer's multivariable systems toolbox.

Introduction

Since their introduction, classical controls tools have been popular for analysis and design of single-input, single-output (SISO) systems. These methods may be viewed as specialized versions of more general tools that are applicable to multi-input, multi-output (MIMO) systems. Although modern "state-space" control methods (relying on dynamic models of internal structure) have been promoted as the predominant tools for multivariable system analysis, the classical control extensions offer several advantages, including requiring only an input-output map and providing direct insight into stability, performance, and robustness of MIMO systems. Furthermore, multivariable frequency-domain techniques extend the control engineer's MIMO tool set, complementing the tools of modern state-space control. The promise of intuitive, graphically-based methods for the analysis and design of MIMO systems has been a prime motivator of this work in post-modern (neo-classical) controls.

An early graphical method for investigating the stability of linear, time-invariant (LTI) SISO systems was developed by Nyquist (1932) and is based on a polar plot of the loop transmission transfer function. The MIMO analog of the Nyquist diagram is the multivariable Nyquist diagram which is used in conjunction with the corresponding multivariable Nyquist criterion (Rosenbrock, 1974; Lehtomaki, *et al*, 1981; Friedland, 1986). This criterion is complicated because it is expressed in terms of the determinant of the return difference transfer function matrix ($[I + G(s)]$ where $G(s)$ is the plant transfer function matrix, rather than just $1 + g(s)$ for the SISO case where $g(s)$ is the plant transfer function). Despite the complication, significant research has supported the MIMO Nyquist extension for assessment of multivariable system stability and robustness (MacFarlane and Postlethwaite, 1977).

It can be rationalized that the Bode plots (Bode, 1940) recast the information of the Nyquist diagram, with frequency extracted as an explicit parameter, and follow from a logical progression of frequency-domain tools (Kurfess and Nagurka, 1991a). The MIMO analog or extension of the classical Bode plots is the singular value Bode-type plot that shows maximum and minimum singular values of transfer function matrices as a function of frequency (Doyle and Stein, 1981). These generalized magnitude *vs.* frequency plots are useful for analysis, providing tremendous insight into performance in terms of command following, disturbance rejection, and sensor noise sensitivity, as well as for design, in terms of frequency shaping (Doyle and Stein, 1981; Safanov, *et al.*, 1981; Athans, 1982; Maciejowski, 1989).

Although promoted as an SISO tool, Evans root locus method (Evans, 1954) is applicable to both SISO and MIMO systems, since it plots the trajectories of closed-loop eigenvalues (of either SISO or MIMO systems) in a complex plane. However, the generalization to the multivariable root loci has not met with the success of the MIMO versions of the Nyquist diagram and Bode plot. The MIMO root locus plot does not, in general, follow the straight-forward sketching rules applicable to SISO systems and does not provide insight into stability, performance, or robustness. Part of the problem lies in the fact that "multivariable root loci live on a Riemann surface ... as compared with the single-input, single-output case where the root loci lie on a simple complex plane (a trivial, i.e., one sheeted, Riemann surface)" (Postlethwaite and MacFarlane, 1979). As a result, multivariable root loci tend to have strange looking patterns when drawn in a single complex plane and are generally not useful for compensator design insight. The possibility of loci being multi-valued functions of gain makes the MIMO root locus plot confusing and hence avoided.

Just as frequency-domain tools such as the Nyquist diagram and Bode plots have been extended to multivariable Nyquist diagrams and singular-value Bode-type plots, respectively, gain-domain tools such as Evans root locus plot can be enhanced by a set of Gain Plots (GPs) that are applicable to MIMO systems. A distinct advantage of the GPs is that they provide, in a unique fashion, significant insight into the stability, performance, and robustness of LTI MIMO systems. The conceptual framework of the GPs and their applicability to SISO systems has been developed in a companion report (Kurfess and Nagurka, 1991a).

This report presents an overview of the GPs, introduces basic definitions and concepts of MIMO systems, and presents a sequence of examples of increasingly more complicated multivariable systems that demonstrate the usefulness of the GPs. In addition to being an important analysis and design tool, the GPs promise to be a highly effective research tool for MIMO system synthesis.

Gain Plots

SISO System Overview

Gain Plots (Kurfess and Nagurka, 1991a) are an alternate graphical representation of the root locus plot that depict the behavior of system eigenvalues as an explicit function of gain. This section summarizes the concept of GPs. A SISO system is analyzed since it

provides significant intuition on GPs. A subsequent section develops GPs for multivariable systems.

Given a plant transfer function, $g(s)$, embedded in a standard closed-loop negative feedback system, the closed-loop stability can be determined by examining the real parts of the eigenvalues. Assuming a static compensator, given by scalar k in the forward loop, the eigenvalues are the solutions of the closed-loop characteristic equation

$$kg(s) = -1 \quad (1)$$

The root locus is the solution set of equation (1) as the gain k varies from zero to infinity. Equation (1) is equivalent to two conditions: the angle criterion,

$$\sum \angle kg(s) = \pm 180^\circ (2m + 1), \quad m = 0, 1, 2, \dots \quad (2)$$

and the magnitude criterion,

$$|kg(s)| = 1 \quad (3)$$

The shape of the root locus plot is determined entirely by the angle criterion. Then, for any eigenvalue, s , on the root locus, the magnitude criterion is invoked to solve for the corresponding value of k .

For example, Figure 1 is the root locus plot of the open-loop transfer function, $g(s)$,

$$G(s) = \frac{(s+3)}{s(s+1)(s+2)} \quad \langle 4 \rangle$$

Each branch of the root locus starts at $k=0$, corresponding to a system open-loop pole ($s=-1, -2$), and asymptotically approaches either a finite ($s=-3$) or an infinite ($s \rightarrow -\infty$) transmission zero. The transmission zeros represent complex frequencies at which system transmission paths vanish.

An alternative visualization of the root locus plot can be obtained by explicitly graphing the eigenvalue magnitude vs. gain in a Magnitude Gain Plot (MGP) and the eigenvalue angle vs. gain in an Angle Gain Plot (AGP). Figure 2a,b is the set of GPs for the system of equation (4). The MGP is presented using a log-log scale and the AGP using a semi-log scale (Kurfess and Nagurka, 1991a).

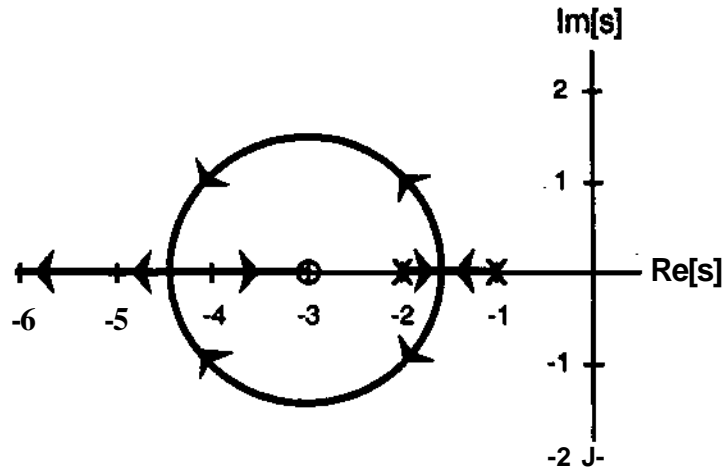


Figure 1. Evans Root Locus Plot of Equation (4).

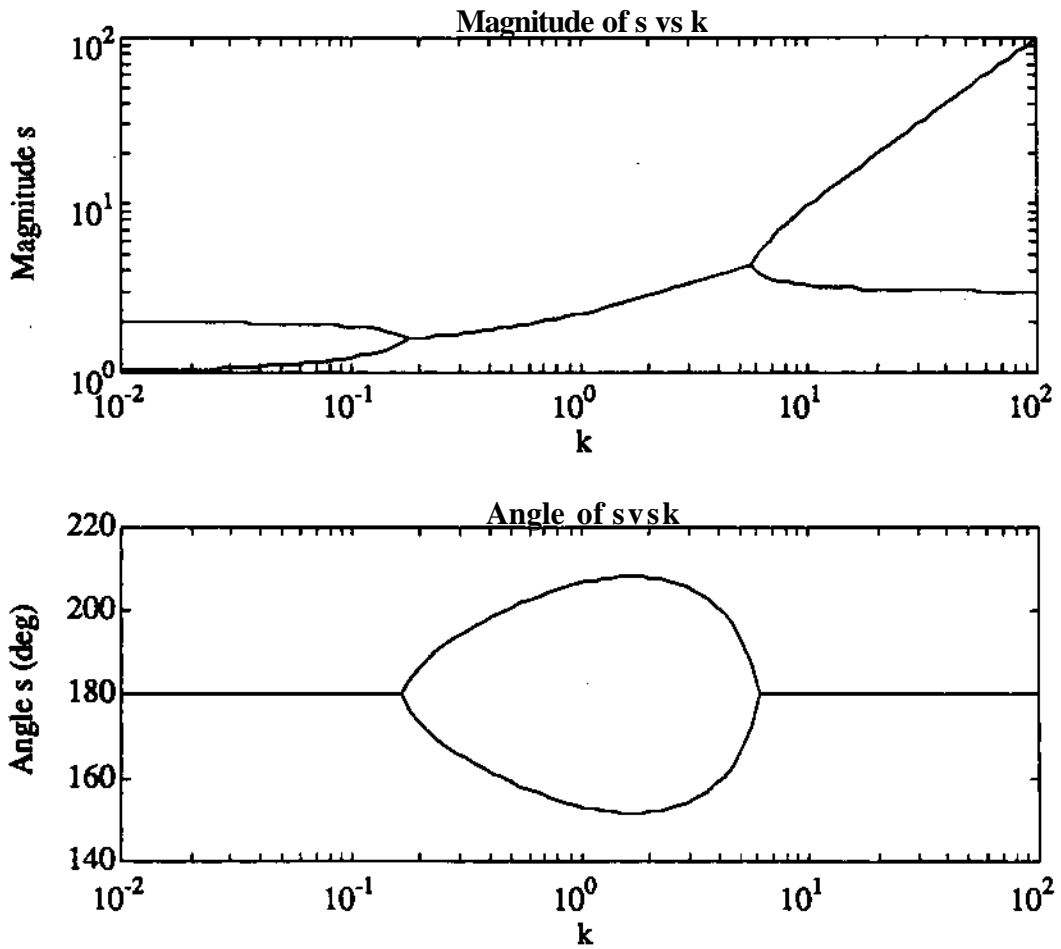


Figure 2a,b. Magnitude and Angle Gain Plots of Equation (4).

The angle criterion dictates that the eigenvalues must lie on the real axis or be complex conjugates. Thus, a pair of complex conjugate eigenvalues is shown as a single curve in the **MGP** with corresponding angles symmetrically configured about the 180° line shown in the **AGP**. As the gain is adjusted, complex conjugate eigenvalues may become distinct real eigenvalues, causing their angles to become equal (at a multiple of 180°) and permitting their magnitudes to differ.

The **MGP** shows the presence of two open-loop poles with magnitudes 1 and 2 at $k = 0$. As $k \rightarrow \infty$ it shows a single finite transmission zero with magnitude 3 and an asymptote tending toward an infinite transmission zero. The **AGP** indicates that the two open-loop poles and finite and infinite transmission zeros are located in the left-half plane on the real axis, since they all have angles of 180°. High gain asymptotic behavior of the closed-loop eigenvalues is discussed in detail in (Kurfess and Nagurka, 1991b).

From the **GPs**, break points (corresponding to points where branches leave or enter the real axis of the root locus) can be observed to occur at $k \ll 0.17$ and at $k \approx 5.83$. Between these break points the **AGP** indicates that the loci of the two branch points are not on the real axis and the corresponding single curve of the **MGP** confirms that the trajectories are those of a complex conjugate pair.

Basic MIMO Definitions and Concepts

A LTI MIMO system can be represented in the standard state-space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (6)$$

where \mathbf{x} is the state vector of length n , \mathbf{u} is the plant command or control input vector of length m , and \mathbf{y} is the plant output vector of length, m . Matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are the system matrix, the control influence matrix, the output matrix, and the feed-forward matrix, respectively, with appropriate dimensions. The input-output dynamics are governed by a square transfer function matrix, $\mathbf{G}(s)$,

$$\mathbf{G}(s) = \mathbf{C}[\mathbf{sI} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \quad (7)$$

The system is embedded in the closed-loop configuration, shown in Figure 3, where the controller is a static compensator, k_1 , implying that each input channel is scaled by the same constant gain k . (Note that the plant transfer function matrix and any dynamic

compensation may be combined in the transfer function matrix $G(s)$.) The control law is given by

$$u(t) = ke(t) \quad (8)$$

where

$$e(t) = r(t) - y(t) \quad (9)$$

is the error and $r(t)$ is the reference (command) signal vector of length m that $y(t)$ must track. The closed-loop transfer function matrix is

$$G_{CL}(s) = [I + kG(s)]^{-1}kG(s) \quad (10)$$

To develop the MIMO root locus plot, the migration of the eigenvalues of $G_{CL}(s)$ in the complex plane is graphed as scalar k varies in the range $0 \leq k < \infty$. The eigenvalues of the closed-loop system, $s = X_i$ ($i=1,2,\dots,n$), are the roots of $\phi_{CL}(s)$, the closed-loop characteristic polynomial,

$$\phi_{CL}(s) = \phi_{OL}(s) \det[I + kG(s)] \quad (11)$$

where $\phi_{OL}(s)$ is the open-loop characteristic polynomial,

$$\phi_{OL}(s) = \det[sI - A] \quad (12)$$

The roots, or solutions of equation (12), are the open-loop poles. By equating the determinant in equation (11) to zero, the MIMO generalization of the SISO characteristic equation ($1 + kg(s) = 0$) is obtained. The presence of the determinant is the major challenge in generalizing the SISO root locus sketching rules to MIMO systems and complicates the root locus plot. For example, the root locus branches "move" between several copies (Riemann sheets) in the s -plane that are connected at singularity points known as branch points (Yagle, 1981; Athans, 1982).

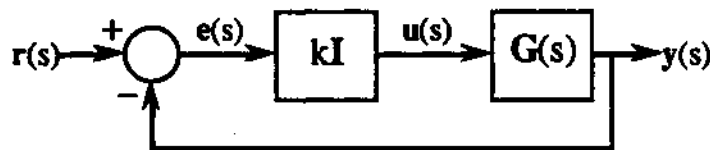


Figure 3. MIMO Closed-Loop Negative Feedback Configuration.

Although it is not possible to sketch MIMO root loci by inspection, the closed-loop system eigenvalues may be computed numerically from equations (5) - (9) as

$$\lambda_i = \text{eig}[A - B(kI)C] \quad , \quad i = 1, 2, \dots, n \quad (13)$$

In the examples, the loci of the eigenvalues are calculated from equation (13) as k is monotonically increased from zero.

As the gain increases from zero to infinity, the closed-loop eigenvalues trace out "root loci" in the complex plane. At zero gain, the poles of the closed-loop system are the open-loop eigenvalues. At infinite gain some of these eigenvalues approach finite transmission zeros, defined to be those values of s that satisfy the generalized eigenvalue problem

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(0) \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (14)$$

where $\begin{bmatrix} x(0) & u \end{bmatrix}^T$ is the right generalized eigenvector corresponding to the generalized eigenvalue, *i.e.*, transmission zero, with $x(0)$ representing the initial state and u being a vector representing input direction in the multi-input case. In the absence of pole/zero cancellation, the finite transmission zeros are the roots of the determinant of $G(s)$. Algorithms have been developed for efficient and accurate computation of transmission zeros (Davison and Wang, 1974; Laub and Moore, 1978; Westreich, 1991).

The high gain behavior of the root loci can be viewed another way (Friedland, 1986). The eigenvalues can be considered as always migrating from the open-loop poles to their matching transmission zeros. However, those eigenvalues that do not have matching zeros in the finite part of the s -plane are considered to have matching zeros at infinity. In the global SISO perspective, whenever the excess of poles over zeros is greater than two, the eigenvalues migrate towards a magnitude of ∞ in a Butterworth configuration and therefore as $k \rightarrow \infty$ the closed-loop eigenvalues must become unstable. However, this may not be the case with MIMO systems as is demonstrated in the following section.

MIMO Examples

Decoupled MIMO Example

This example demonstrates the use of the GPs for exploring the behavior of a decoupled multivariable system. The state space model of the system is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t) \quad (15)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) \quad (16)$$

corresponding to the diagonal transfer function matrix

$$G(S) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \quad (17)$$

It represents two decoupled first order SISO systems with eigenvalues at $s = \{-1, -2\}$. The open-loop system is assumed to be embedded in the feedback configuration of Figure 3. Since the system is decoupled, the multivariable root locus may be considered to be the superposition of two SISO root locus plots. That is, the MIMO root locus diagram, shown in Figure 4, depicts two eigenvalue trajectories, one beginning at $s = -1$ and the other beginning at $s = -2$. Both trajectories follow a straight line path along the negative real axis. The MIMO root locus does not follow the rules of the familiar SISO root locus (*e.g.*, the SISO rule for the portion of the root locus on the real axis is violated), and is not intuitive.



Figure 4. MIMO Root Locus for System Given by Equation (17).

Figure 5 is the MGP for this decoupled MIMO system. Although not shown, the AGP indicates that both eigenvalues have angles of 180° for all values of k . Thus, at low gain the GPs confirm that the two open-loop eigenvalues are at $s = \{-1, -2\}$. Furthermore, as k increases, both eigenvalues proceed deeper into the left half plane along the negative real axis at the same constant rate. From the MGP, there is no ambiguity as to the number or location of the eigenvalues. The MGP also provides the control engineer with other useful information. For example, from the MGP it can be seen that gains greater than 10 correspond to system time constants faster than approximately 0.1 sec. In summary, for this decoupled MIMO example, the GPs provide more insight into the behavior of the closed-loop system than does the MIMO root locus of Figure 4.

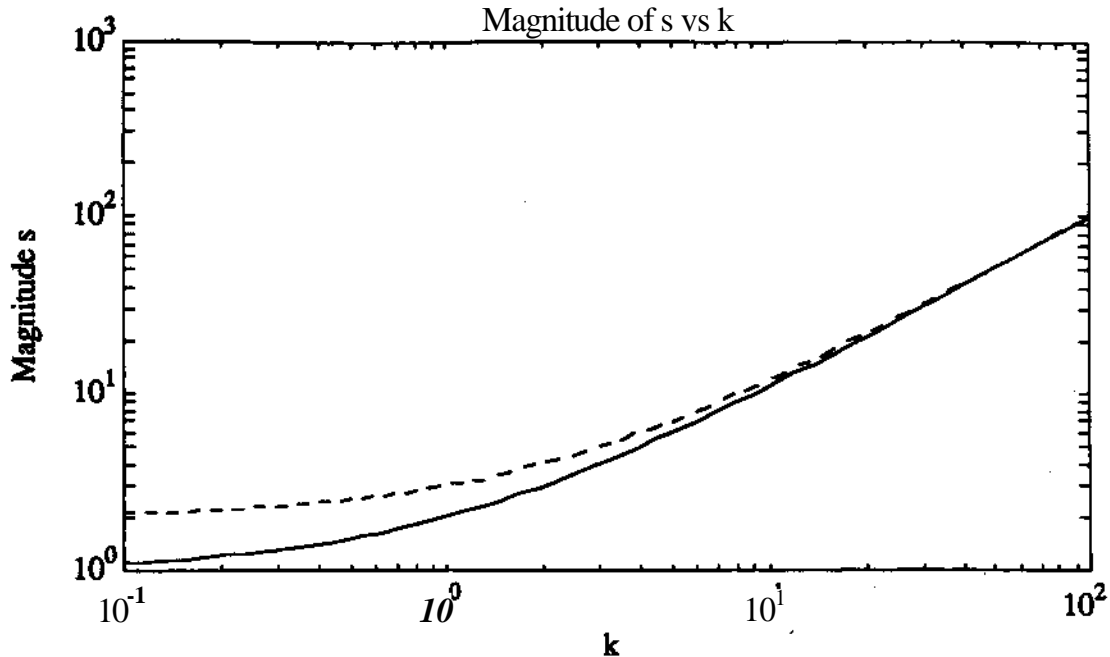


Figure 5- Magnitude Gain Plot for System Given by Equation (17).

Coupled MIMO Example

This example demonstrates the use of the GPs for understanding the closed-loop behavior of a coupled multivariable system. The open-loop plant dynamics of this system are given by the state space model

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \mathbf{u}(t) \quad (18)$$

$$\mathbf{y}(t) = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \mathbf{x}(t) \quad (19)$$

corresponding to the **transfer** function matrix

$$G(s) = \begin{bmatrix} \frac{(s-1)}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{-6}{(s+1)(s+2)} & \frac{(s-2)}{(s+1)(s+2)} \end{bmatrix} \quad (20)$$

(Equation (20) is used as an example by Postlethwaite and MacFarlane (1979) and later by Yagle (1981).) The presence of off-diagonal terms in $G(s)$ of equation (20) indicates a coupled **MIMO** system. Also, from equations (18) or (20), the system has open-loop eigenvalues at $s = \{-1, -2\}$.

Since the system is coupled, the multivariable root locus is more complicated than superimposed SISO root locus plots. The MEMO root locus diagram, shown in Figure 6, depicts two eigenvalue trajectories, one beginning at $s=-1$ and the other beginning at $s=-2$. As in the decoupled example, the eigenvalue at $s=-2$ follows along the negative real axis as the gain increases. The eigenvalue at $s=-1$ does not follow the same trajectory. It initially migrates to the right, proceeding to $s=1/24 * 0.042$, and then reverses. As the gain increases, it moves back to the left of the imaginary axis along the real axis. Figure 6 does not follow the rules of the familiar SISO root locus, and is extremely counter intuitive.

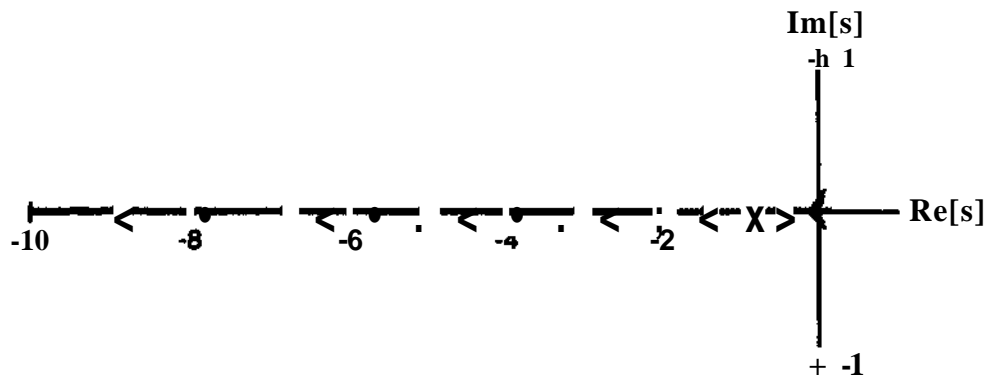


Figure 6. MIMO Root Locus for System Given by Equation (20).

Figure 7a,b presents the GPs for the coupled MIMO system. The MGP shows that the gain values for the unstable range are

$$1 \leq k < 2 \quad (2i)$$

and confirms the maximum magnitude of the eigenvalue at 0. An abrupt change in eigenvalue angle occurs when the closed-loop system becomes unstable. This is expected since there is a 180° jump in angle as the eigenvalue passes through the origin, highlighting the stable-unstable transition.

The MIMO root locus plot of this coupled MIMO example is confusing because of the collapse of the Riemann surface into a single complex plane. Since the plot is two dimensional, branch points that may be generated by more than one gain value may not be presented uniquely. The GPs, however, display eigenvalue magnitude and angle information in an unambiguous and concise manner.

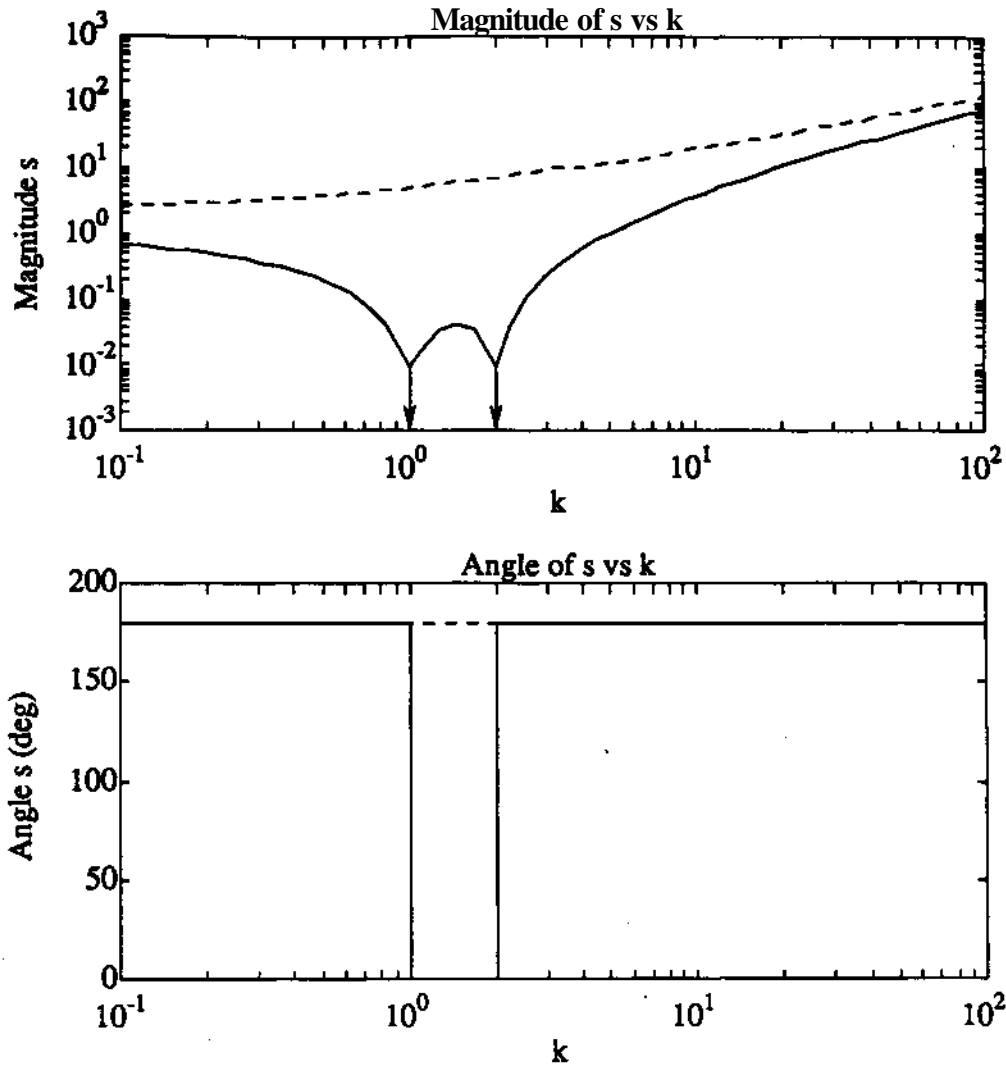


Figure 7a,b. Gain Plots for System Given by Equation (20).

Complex MIMO System

The previous two examples have addressed MIMO systems having eigenvalues that remain on the real axis. This example demonstrates the utility of the GPs for systems that possess complex conjugate root loci. The state space representation for this example (taken from Hung and MacFarlane (1982) and studied in detail by Maciejowski (1989)),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1.132 & 0 & -1 \\ 0 & -0.054 & -0.171 & 0 & 0.071 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.049 & 0 & -0.856 & -1.013 \\ 0 & -0.291 & 0 & 1.053 & -0.686 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.665 \\ 1.575 & 0 & -0.073 \end{bmatrix} \mathbf{u} \quad (22)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x} \quad (23)$$

represents a linearized model of the vertical plane dynamics of an aircraft with three inputs, three outputs and five state variables. The fifth order system has no finite transmission zeros and has open-loop eigenvalues of $\lambda = \{0, -0.7801 \pm 1.0296j, -0.0176 \pm 0.1826j\}$. The pole at the origin indicates that the open-loop system is marginally stable. To determine the stability of the closed-loop system, the MEMO root locus may be drawn; however, little insight is offered by this plot, shown in Figure 8, since it is not clear if all of the eigenvalues exist in the left-half plane for low values of gain. The MIMO root locus suggests that at high gain the system is unstable, but fails to indicate the range of gain for which instability occurs. In fact, the closed-loop system is never stable for positive gain values, although this is not evident from the root locus plot. Another interesting characteristic of the MIMO root locus is that it is completely counter intuitive. It should be noted, however, that poles with imaginary components are part of a complex conjugate pair as required since A is real.

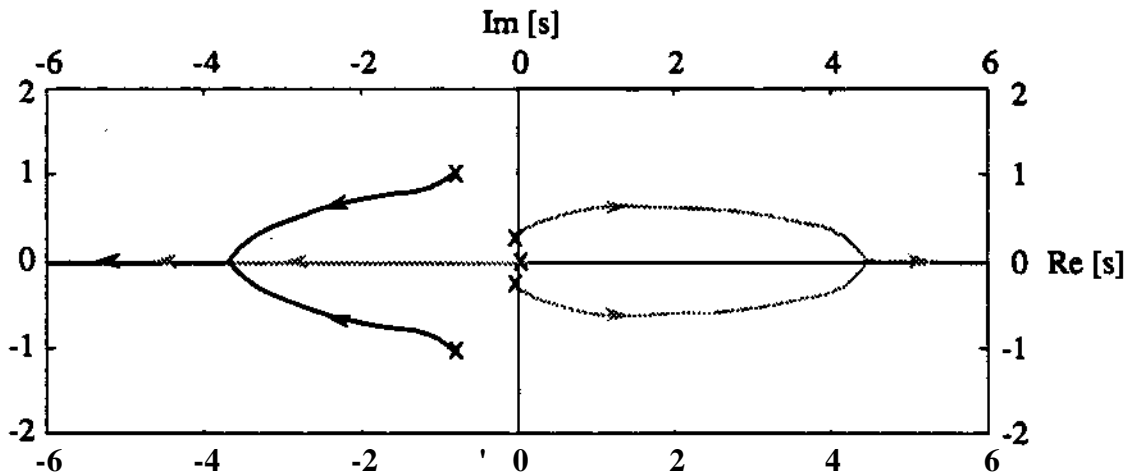


Figure 8- MIMO Root Locus for System Given by Equations (22) and (23)

In contrast to the MIMO root locus, the GPs for this *complex* system reveal a substantial amount of information about the closed-loop system stability. Figure 9a,b is the pair of GPs for the MIMO system given by equations (22) and (23). Several interesting phenomena occur as the gain increases. The first and most noticeable is that the eigenvalue at the origin initially migrates to the right-half plane, becoming unstable. Its maximum real value is $s \ll 0.010$ at a gain of $k \ll 0.018$, behavior which is obscured in the MIMO root

locus. This eigenvalue then becomes stable at a gain of $k \ll 0.043$, at which point the two other eigenvalues whose angles are symmetric about zero are already unstable. The AGP of Figure 9b shows this behavior clearly. Thus, the GPs provide an unambiguous means by which stability may be determined.

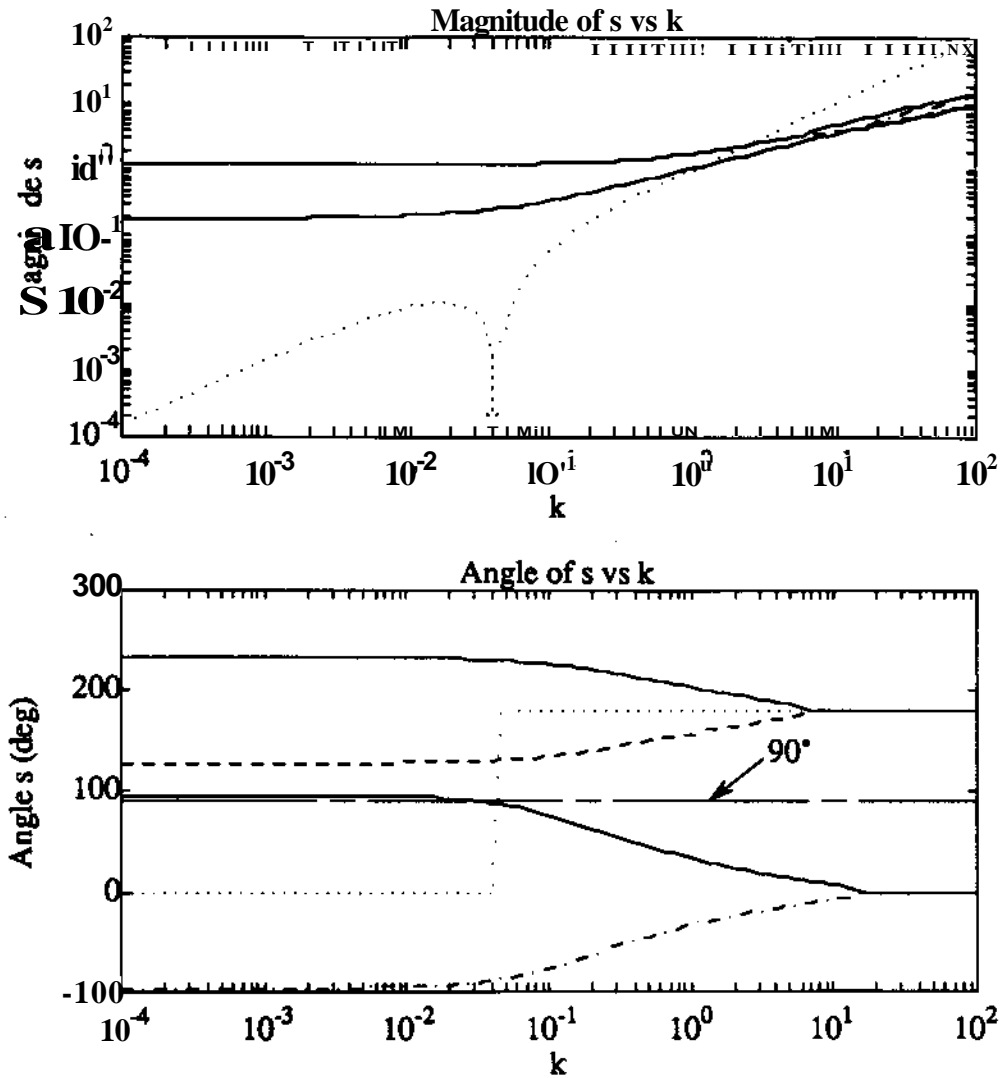


Figure 9a,b. Gain Plots for System Given by Equations (22) - (23).

The rate at which the eigenvalues migrate towards a magnitude of infinity is seen in the MGP. The single eigenvalue that begins at the origin proceeds towards infinity along the negative real axis at a rate proportional to k (the high gain MGP slope is unity). This slope is characteristic of a first order SISO system. The real eigenvalue possesses an MGP slope of unity for all gain values except when its magnitude is small (near zero). The two

complex conjugate pole pairs proceed toward infinity at a rate proportional to $k^{1/2}$ (shown as a high gain **MGP** slope of 1/2), indicative of a SISO second order system (Kurfess and Nagurka, 1991b).

Finally, unusual behavior is exhibited by the complex conjugate pole pairs as they break into the real axis and proceed to $\pm D^\circ$. Figure 10a,b presents the GPs shown in Figure 9a,b for gain values between 10^0 and 10^4 . Each complex conjugate pair demonstrates the typical break point behavior of the GPs. When the eigenvalues are complex, they are symmetric about either the 180° or 0° line with equal magnitudes. When they are purely real, they possess equal angles (180° or 0°) but different magnitudes. This striking behavior is seen even more dramatically in Figure 11, which is the MGP expanded vertically of Figure 10a.

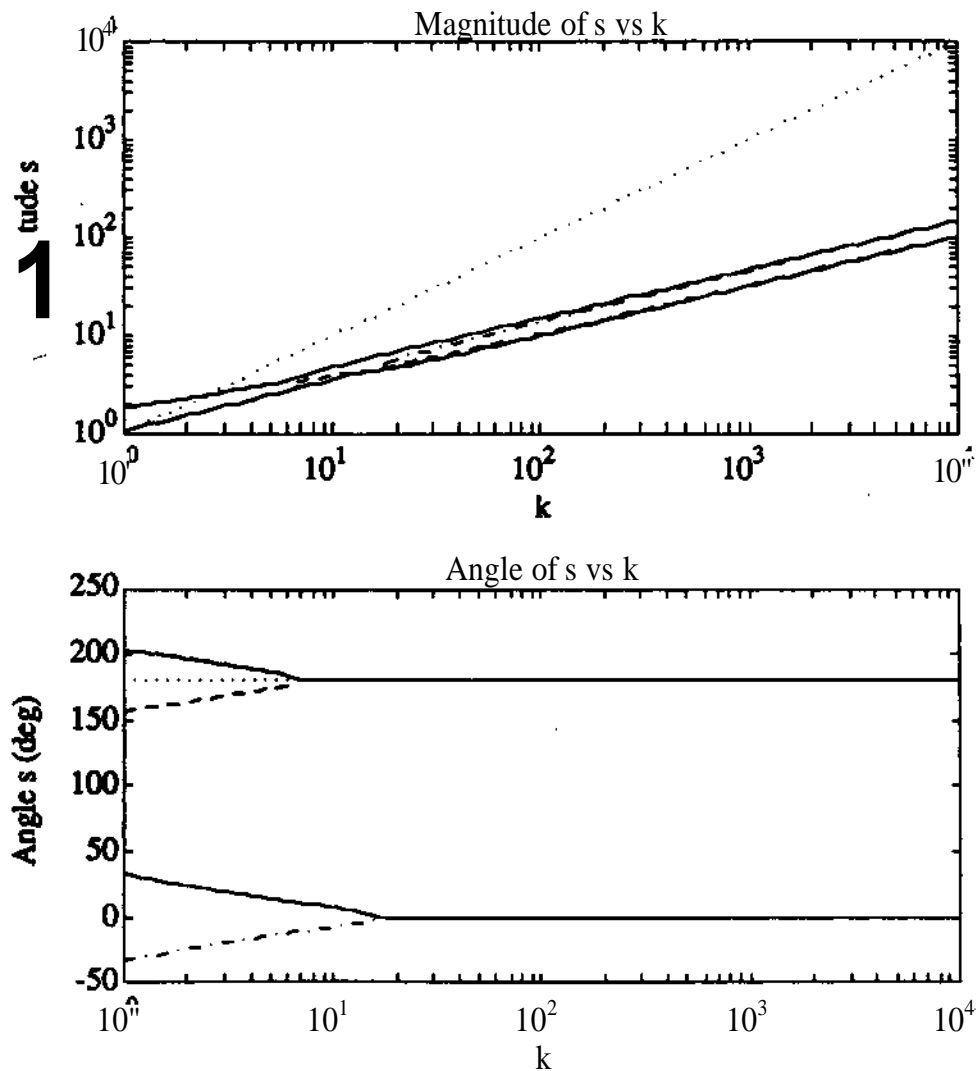


Figure 10a,b. Gain Plots for System Given by Equations (22) - (23).

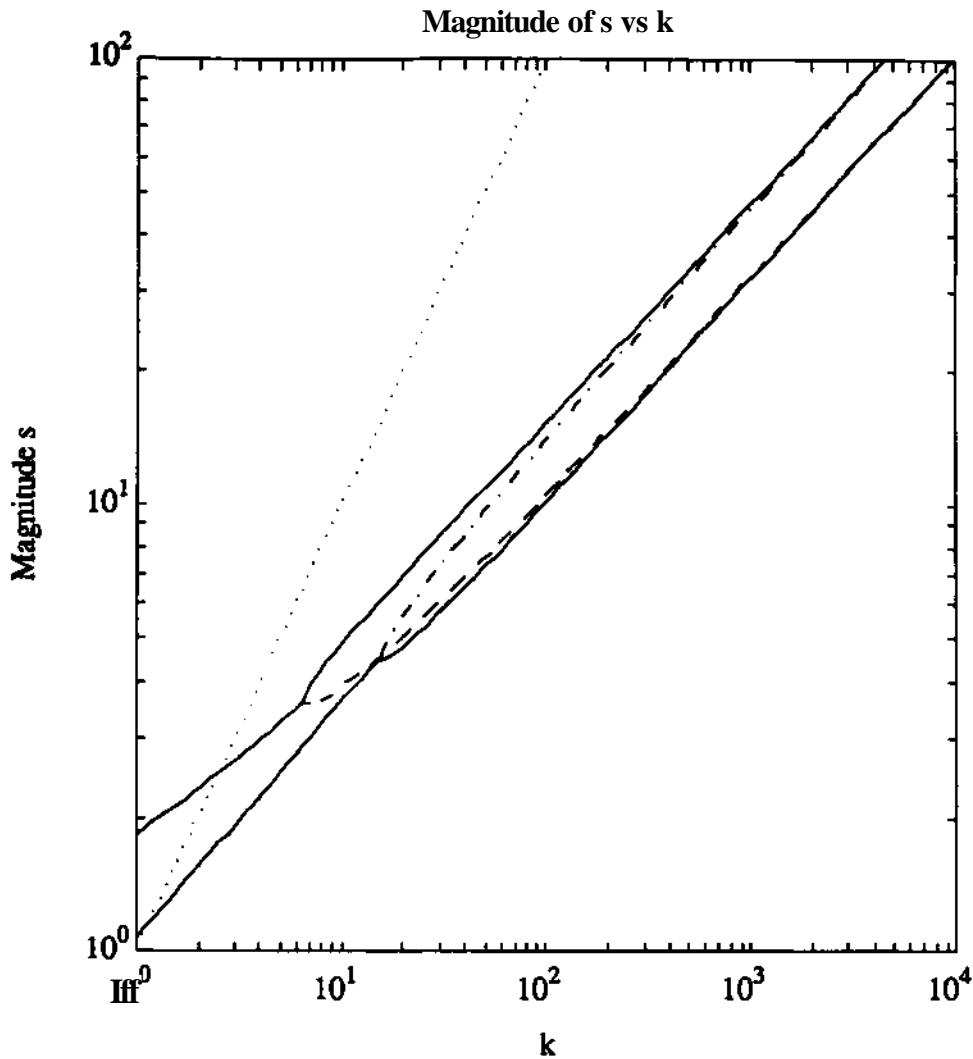


Figure 11. Magnitude Gain Plot for System Given by Equations (22) - (23).

From Figures 10 or 11, the MGP slopes of the two complex conjugate pole pairs at high gains have a value of $1/2$. As $k \rightarrow \infty$, there are four parallel lines with the same slopes. This group of four lines may be separated into two sets of identical lines within each set. The MGP in Figure 11 clearly depicts the two sets of two identical lines. An interesting phenomenon is that the two identical lines are comprised of an eigenvalue magnitude from each of the original complex conjugate pairs. It is as if the complex conjugate eigenvalues have *swapped partners*. This phenomenon is not visible from the MIMO root locus and does not appear to be reported in the controls literature.

Conclusions

GPs are promoted as offering significant advantages over standard root locus plots for MIMO systems. The major enhancement is the visualization of eigenvalue trajectories as an explicit function of gain (where the compensation has been assumed to be the same static gain applied to all control channels). This representation provides a unique description of the eigenvalues, and is contrasted to typical root locus plots that do not necessarily generate unique trajectories as some branches may overlap. This overlap, reducing the usefulness of the MIMO root locus, does not occur in the GPs.

Several interesting research topics may shed significant light on a more complete understanding of the GPs. In particular, work by MacFarlane and Postlethwaite (1977, 1979) and Hung and MacFarlane (1982) on relating characteristic frequency plots to characteristic gain plots may yield substantial insight into the relations between GP methods and singular value frequency methods.

In conclusion, GPs enhance the multivariable root locus in much the same way that singular value frequency plots are an alternate and extended presentation of the multivariable Nyquist diagram. A view of multivariable analysis and design tools that includes the GPs is shown in Figure 12. Thus, the GPs may be considered as filling a gap in the multivariable controls toolbox.

		Span Domain	
		Frequency, ω	Gain, k
ace	2-D	Multivariable Nyquist Diagram	Multivariable Evans Root Locus
	3-D	Multivariable Singular Value (Bode) Plots	Multivariable Gain Plots

Figure 12. The Multivariable Controls Toolbox.

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