A BOUND ON THE MULTIPLICATION EFFICIENCY OF ITERATION

by

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ABSTRACT

For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, \ldots, x_{i-d+1})$, define the multiplication efficiency measure E to be $\frac{\log_2 p}{M}$, where p is the order of convergence, and M is the number of multiplications or divisions needed to compute φ. Then, if φ is any multivariate rational function, $E \leq 1$. Since $E = 1$ for the sequence $\{x_i\}$ generated by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit $-\frac{1}{2}$, the bound on E is sharp.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications. Then $P_M \leq 2^M$ for all positive integer M. Moreover this bound is sharp.

I. INTRODUCTION

For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, \ldots, x_{i-d+1})$, define the multiplication efficiency measure E to be $\frac{\log_2 p}{M}$, where p is the order of convergence, and M is the number of multiplications or divisions needed to compute φ. In [1] Paterson showed that if

(i) φ is a rational function,

(ii) $d = 1$,

(iii) $\lim_{i\to\infty} x_i$ is an algebraic number, and

(iv) φ has rational coefficients,

then $E \le 1$. In this note we show $E \le 1$ removing all these restrictions except (i). Since condition (i) is not a restriction for a computer algorithm, this is a very general result. In particular, we shall show that $E = 1$ for the sequence $\{x_i\}$ defined by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit $-\frac{1}{2}$. Hence our bound on E is sharp.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications. Since $E \le 1$, it follows that $P_M \le 2^M$ for all positive integer M. Moreover, we shall show that this bound is sharp.

Paterson used results from approximation by rational numbers to obtain his result, while we use a completely different approach here. With the technique we use here, the case $d = 1$ would be very easy to prove. We show that a rational iteration function which generates a p^{th} order convergent sequence must have degree (degree will be defined below) $\ge p$, and therefore must employ at least $\lceil \log_2 p \rceil$ multiplications or divisions (except by constants). Hence, $E = \frac{\log_2 p}{M} \le 1$.

The result belongs to analytic computational complexity which deals with optimality theory of analytic processes [2].

II. NOTATION

We work over the field of real numbers or the field of complex numbers. Let $\{x_i\}$ be any convergent sequence with limit α, and $x_i \neq \alpha$ for all i. Denote $e_i = |x_i - \alpha|$ for all i.

<u>Definition 1</u>: (<u>Order</u>) <u>The sequence $\{x_i\}$ has an order $p > 1$ (or $\{x_i\}$ is a p^{th} order sequence) iff $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^{p-\epsilon}} = 0$ and $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^{p+\epsilon}} \neq 0$ for any $\epsilon > 0$.</u>

From the above definition, it is easy to see that if $\{x_i\}$ has order p, then

(2.1) $p = \sup\{r \mid \lim_{i\to\infty} \frac{e_{i+1}}{e_i^r} = 0\}$, and

(2.2) for any fixed positive integer n, $\{x_{in}\}_{i=0}^{\infty}$ has order p^n.

It should be noted that in our proofs the only properties of order needed are (2.1) and (2.2), although (2.1) has been used as a definition of order by many people. Definition 1 is the weakest definition on order we have found which enjoys both properties (2.1) and (2.2).

For each number α, we define a class $F(\alpha)$ of convergent sequences with the same limit α as follows: $\{x_i\} \in F(\alpha)$ iff

(i) $x_i \neq \alpha$ for all but finitely many i

(ii) $\{x_i\}$ has an order $p > 1$

(iii) $x_{i+1} = \alpha(x_i, x_{i-1}, \ldots, x_{i-d+1})$ for all i, for some multivariate rational expression $\alpha(y_1, y_2, \ldots, y_d)$ of d variables, say, $\varphi(y_1,\ldots,y_d) = \dfrac{\varphi_1(y_1,y_2,\ldots,y_d)}{\varphi_2(y_1,y_2,\ldots,y_d)}$, where $\varphi_1(y_1,y_2,\ldots,y_d)$ and $\varphi_2(y_1,y_2,\ldots,y_d)$ are two relatively prime multivariate polynomials of d variables $y_1,y_2,\ldots,y_d$. We say that $\{x_i\}$ is generated by the rational iteration φ. For examples of these φ's, see [3].

Consider a sequence in $F(\alpha)$ generated by φ. For the purpose of this note, we assume the cost in generating the sequence to be the number of multiplications or divisions needed to compute φ at each stage. Then it is natural to give the following definition about the measure of efficiency.

Definition 2: (Multiplication Efficiency) The multiplication efficiency E of a sequence in $F(\alpha)$ generated by φ is defined to be $\frac{\log_2 p}{M}$ where p is the order of the sequence and M is the number of multiplications or divisions needed to compute φ, after doing any preconditioning of coefficients (i.e., preconditioning is not counted).

Definition 3: (Optimality) A sequence in $F(\alpha)$ is called optimal if it has the largest multiplication efficiency among all sequences in $F(\alpha)$.

From (2.2) we can check that a very desirable property holds, namely, for any fixed positive integer n, $\{x_i\}$ and $\{x_{in}\}_{i=0}^{\infty}$ have the same multiplication efficiency. In fact, this invariance under composition property implies that any efficiency measure must be a strictly monotonic function of E [4]. Therefore, as far as optimality is concerned, it makes no difference if E or any other possible efficiency measure is used. For instance, the efficiency measure $p^{\frac{1}{M}}$ will give the same answer in optimality problems as E will, since it is a strictly monotonic function of E.

<u>Definition 4</u>: (<u>Degree</u>) <u>Let $\varphi(y_1,y_2,\ldots,y_d) = \dfrac{\varphi_1(y_1,y_2,\ldots,y_d)}{\varphi_2(y_1,y_2,\ldots,y_d)}$ be a multivariate rational expression, where $\varphi_1(y_1,y_2,\ldots,y_d)$ and $\varphi_2(y_1,y_2,\ldots,y_d)$ are two relatively prime multivariate polynomials. If $D(\varphi_i)$ is the degree of $\varphi_i(y_1,y_2,\ldots,y_d)$ for $i = 1,2$, then the degree $D(\varphi)$ of $\varphi(y_1,y_2,\ldots,y_d)$ is defined to be $\max(D(\varphi_1), D(\varphi_2))$.</u>

III. PRELIMINARY LEMMA

For each positive integer d, we define an order (>) on the set $I_d = \{(j_1,j_2,\ldots,j_d) | j_i$ is a non-negative integer for $i = 1,2,\ldots,d\}$ as follows: for $(j_1,j_2,\ldots,j_d)$, $(\ell_1,\ell_2,\ldots,\ell_d) \in I_d$, $(j_1,j_2,\ldots,j_d) > (\ell_1,\ell_2,\ldots,\ell_d)$ iff there exists $k \in \{1,2,\ldots,d\}$ such that $j_k > \ell_k$ and $j_i = \ell_i$ for $i < k$.

Lemma 1: For any number α, let $\{x_i\}$ be any p^{th} order sequence in $F(\alpha)$ generated by φ, and let $e_i = |x_i - \alpha|$ for all i. Suppose that φ has d variables. Then we have the following:

(i) if $(j_1,j_2,\ldots,j_d) \in I_d$ with $\sum_{i=1}^{d} j_i < p$,

then $\lim_{i\to\infty} \dfrac{e_i^{p-\varepsilon}}{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}} = 0$, for $\varepsilon > 0$ and

sufficiently small, and

(ii) if $(j_1,j_2,\ldots,j_d)$, $(\ell_1,\ell_2,\ldots,\ell_d) \in I_d$

with $(j_1,j_2,\ldots,j_d) > (\ell_1,\ell_2,\ldots,\ell_d)$

and $\sum_{i=1}^{d} \ell_i < p$, then

$$\lim_{i\to\infty} \frac{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}}{e_i^{\ell_1} e_{i-1}^{\ell_2} \cdots e_{i-d+1}^{\ell_d}} = 0.$$

Proof:

(i) Choose ε such that $0 < \varepsilon < p - \sum_{i=1}^{d} j_i$

and $0 < \varepsilon < p - 1$. Then

$$\lim_{i\to\infty} \frac{e_i}{e_{i-1}} = \lim_{i\to\infty} \frac{e_i}{e_{i-1}^{p-\varepsilon}} \cdot e_{i-1}^{p-\varepsilon-1} = 0, \text{ and then}$$

$$\lim_{i\to\infty} \frac{e_i}{e_{i-2}} = \lim_{i\to\infty} \frac{e_i}{e_{i-1}} \cdot \frac{e_{i-1}}{e_{i-2}} = 0.$$

In general, $\lim_{i\to\infty} \frac{e_i}{e_{i-k}} = 0$ for any positive integer k. Hence,

$$0 \le \lim_{i\to\infty} \frac{e_i^{p-\varepsilon}}{e_i^{j_1} \cdots e_{i-d+1}^{j_d}} \le \lim_{i\to\infty} \frac{e_i^{\sum_{i=1}^{d} j_i}}{e_i^{j_1} \cdots e_{i-d+1}^{j_d}} = \lim_{i\to\infty} \left(\frac{e_i}{e_i}\right)^{j_1} \cdots \left(\frac{e_i}{e_{i-d+1}}\right)^{j_d} = 0.$$

(ii) Choose ε such that $0 < \varepsilon < p - \sum_{i=1}^{d} \ell_i$. Let $Q_i = \dfrac{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}}{e_i^{\ell_1} e_{i-1}^{\ell_2} \cdots e_{i-d+1}^{\ell_d}}$.

Suppose that $j_k > \ell_k$ and $j_i = \ell_i$ for $i < k$. Then when i is so large that $e_i < 1$, we have

$$Q_i = e_{i-k+1}^{j_k - \ell_k} \cdot \frac{e_{i-k}^{j_{k+1}} \cdots e_{i-d+1}^{j_d}}{e_{i-k}^{\ell_{k+1}} \cdots e_{i-d+1}^{\ell_k}}$$

$$\le e_{i-k+1} \cdot \frac{e_{i-k}^{j_{k+1}} \cdots e_{i-d+1}^{j_d}}{e_{i-k}^{\ell_{k+1}} \cdots e_{i-d+1}^{\ell_k}}$$

$$= \frac{e_{i-k+1}}{e_{i-k}^{p-\varepsilon}} \cdot e_{i-k}^{(p-\varepsilon+j_{k+1}-\ell_{k+1})} \cdot \frac{e_{i-k-1}^{j_{k+2}} \cdots e_{i-d+1}^{j_d}}{e_{i-k-1}^{\ell_{k+2}} \cdots e_{i-d+1}^{\ell_d}}.$$

Case 1, $p - \varepsilon + j_{k+i} - \ell_{k+i} \geq 1$ for $k+i = k+1,\ldots,d$. Repeating the above procedure, we get

$$Q_i \leq \frac{e_{i-k+1}}{e_{i-k}^{p-\varepsilon}} \cdot e_{i-k} \cdot \frac{e_{i-k-1}^{j_{k+2}} \cdots e_{i-d+1}^{j_d}}{e_{i-k-1}^{\ell_{k+2}} \cdots e_{i-d+1}^{\ell_d}}$$

$$= \frac{e_{i-k+1}}{e_{i-k}^{p-\varepsilon}} \cdot \frac{e_{i-k}}{e_{i-k-1}^{p-\varepsilon}} \cdot e_{i-k-1}^{(p-\varepsilon+j_{k+2}-\ell_{k+2})}$$

$$\cdot \frac{e_{i-k-2}^{j_{k+3}} \cdots e_{i-d+1}^{j_d}}{e_{i-k-2}^{\ell_{k+3}} \cdots e_{i-d+1}^{\ell_d}}$$

$$\leq \ldots \leq \frac{e_{i-k+1}}{e_{i-k}^{p-\varepsilon}} \cdot \frac{e_{i-k}}{e_{i-k-1}^{p-\varepsilon}} \cdot \ldots \cdot \frac{e_{i-d+2}}{e_{i-d+1}^{p-\varepsilon}} .$$

Case 2, $p-\varepsilon+j_{k+n}-\ell_{k+n} < 1$ and $p-\varepsilon+j_{k+i}-\ell_{k+i} \geq 1$ for $k+i = k+1,\ldots,k+n-1$ for some n with $k+n-1 < d$. Since $p-\varepsilon-\ell_{k+n} > 0$, $j_{k+n} < \frac{p-\varepsilon+j_{k+n}-\ell_{k+n}}{d} < 1$. Hence we must have $j_{k+n} = 0$. Consequently, $1 > p-\varepsilon-\ell_{k+n} > \sum_{i=1}^{d} \ell_i - \ell_{k+n}$. This implies that $\ell_i = 0$ for all i except $i = k+n$. Then

$$Q_i \leq \frac{e_{i-k+1}}{e_{i-k}^{p-\varepsilon}} \cdot \ldots \cdot \frac{e_{i-k-n+2}}{e_{i-k-n+1}^{p-\varepsilon}} \cdot e_{i-k-n+1}^{p-\varepsilon+j_{k+n}-\ell_{k+n}} \cdot e_{i-k-n}^{j_{k+n+1}} \cdot \ldots \cdot e_{i-d+1}^{j_d} .$$

Note that $p-\varepsilon+j_{k+n} - \ell_{k+n} > 0$. Therefore, in both cases, $\lim_{i\to\infty} Q_i = 0$. ■

IV. MAIN RESULT

Theorem 1: For any number α, let $\{x_i\}$ be any p^{th} order sequence generated by φ. Then $D(\varphi) \geq p$.

Proof: Write

$$(4.1) \quad \varphi_1(y_1,y_2,\ldots,y_d) - \alpha\varphi_2(y_1,y_2,\ldots,y_d)$$

$$= \sum_{(j_1,\ldots,j_d)\in I_d} C(j_1,\ldots,j_d)(y_1-\alpha)^{j_1}\ldots(y_d-\alpha)^{j_d}$$

for constants $C(j_1,\ldots,j_d)$. Suppose that $D(\varphi) < p$. Then $C(j_1,\ldots,j_d) = 0$ for all $(j_1,\ldots,j_d) \in I_d$ with $\sum_{i=1}^{d} j_i \geq p$. Moreover, we shall use induction to show that $C(j_1,\ldots,j_d) = 0$ for all $(j_1,\ldots,j_d)$ with $\sum_{i=1}^{d} j_i < p$. Note that for $\epsilon > 0$,

$$0 = \lim_{i\to\infty} \frac{|x_{i+1} - \alpha|}{|x_i - \alpha|^{p-\epsilon}} = \lim_{i\to\infty} \frac{|\varphi(x_i,x_{i-1},\ldots,x_{i-d+1})-\alpha|}{|x_i - \alpha|^{p-\epsilon}}$$

Then, by (4.1), we have

$$(4.2) \quad \lim_{i\to\infty} \frac{\left|\sum_{j_1+j_2+\ldots+j_d<p} C(j_1,\ldots,j_d)(x_i-\alpha)^{j_1}\ldots(x_{i-d+1}-\alpha)^{j_d}\right|}{e_i^{p-\epsilon}} = 0 .$$

Since $\lim_{i\to\infty} e_k = 0$ for $k=i,\ldots,i-d+1$, from (4.2) it follows that $C(0,\ldots,0) = 0$. Suppose that $C(j_1,\ldots,j_d) = 0$ whenever $(j_1,\ldots,j_d) < (\ell_1,\ldots,\ell_d)$ for some $(\ell_1,\ldots,\ell_d) \in I_d$ with $\sum_{i=1}^{d} \ell_i < p$. (4.2) may be written as

$$\lim_{i\to\infty} \frac{\left| \sum_{(j_1,\ldots,j_d) \geq (\ell_1,\ldots,\ell_d)} C(j_1,\ldots,j_d) \frac{(x_i-\alpha)^{j_1}\cdots(x_{i-d+1}-\alpha)^{j_d}}{e_i^{\ell_1}\cdots e_{i-d+1}^{\ell_d}} \right|}{\dfrac{e_i^{p-\varepsilon}}{e_i^{\ell_1}\cdots e_{i-d+1}^{\ell_d}}} = 0.$$

Using Lemma 1 for sufficiently small ε, we must have $C(\ell_1,\ldots,\ell_d) = 0$. This completes the induction proof.

Hence $C(j_1,\ldots,j_d) = 0$ for all $(j_1,\ldots,j_d) \in I_d$.
From (4.1), $\varphi_1(y_1,\ldots,y_d) - \alpha\varphi_2(y_1,\ldots,y_d) \equiv 0$.
Hence $\varphi(y_1,\ldots,y_d) \equiv \alpha$. This is a contradiction.
Hence, $D(\varphi) \geq p$. ■

<u>Theorem 2</u>: <u>If $\varphi(y_1,\ldots,y_d)$ is a multivariate rational expression and $\bar{M}$ is the number of multiplications or divisions (except by constants) needed to compute $\varphi(y_1,\ldots,y_d)$, then $\bar{M} \geq \log_2 D(\varphi)$.</u>

<u>Proof</u>: Observe that we compute $\varphi(y_1,\ldots,y_d)$ through a sequence of arithmetic operations. Let $R_i(y_1,\ldots,y_d)$ be the result immediately following the i^{th} multiplication or division (except by constants) for $i=1,2,\ldots,\bar{M}$. Let $R_0(y_1,\ldots,y_d)$ be one of $y_1,\ldots,y_d$. Observe that we have either

$$(4.3)\quad R_{n+1}(y_1,\ldots,y_d) = \left(\sum_{i=0}^{n} M_{i,n+1}R_i(y_1,\ldots,y_d) + A_{n+1}\right)$$

$$\times \left(\sum_{i=1}^{n} N_{i,n+1}R_i(y_1,\ldots,y_d) + B_{n+1}\right), \text{or}$$

$$(4.4)\quad R_{n+1}(y_1,\ldots,y_d) = \left(\sum_{i=0}^{n} M_{i,n+1}R_i(y_1,\ldots,y_d) + A_{n+1}\right)$$

$$\div \left(\sum_{i=1}^{n} N_{i,n+1}R_i(y_1,\ldots,y_d) + B_{n+1}\right)$$

where $M_{i,n+1}$, $N_{i,n+1}$, A_{n+1}, B_{n+1} are many numbers, for $n=0,1,\ldots,\bar{M}-1$.

We claim that, for $n=1,2,\ldots,\bar{M}$, the following is true. For any numbers $k_0,\ldots,k_n$, C, we have

$$(4.5) \qquad \sum_{i=0}^{n} k_iR_i(y_1,\ldots,y_d) + C = \frac{P_n(y_1,\ldots,y_d;\ k_0,\ldots,k_n,\ C)}{Q_n(y_1,\ldots,y_d)}$$

where $P_n(y_1,\ldots,y_d;\ k_0,\ldots,k_n,C)$ is a multivariate polynomial depending on $k_0,k_1,\ldots,k_n$, C and $Q_n(y_1,y_2,\ldots,y_d)$ is a multivariate polynomial independent of $k_0,k_1,\ldots,k_n$, C; moreover, both polynomials have degrees $\le 2^n$. We prove it by induction. It is clear that (4.5) is true for $n = 1$. Suppose that (4.5) is true for all $n \le N$ for some $N < \bar{M}$. Suppose that (4.3) is true for $n = N$. Then by (4.5) for $n = N$, we have

$$\sum_{i=0}^{N+1} k_iR_i(y_1,\ldots,y_d) + C = k_{N+1}\ R_{N+1}(y_1,\ldots,y_d) + \sum_{i=0}^{N} k_iR_i(y_1,\ldots,y_d) + C$$

$$= k_{N+1}\left(\sum_{i=0}^{N} M_{i,N+1}\ R_i(y_1,\ldots,y_d) + A_{N+1}\right) \times \left(\sum_{i=1}^{N} N_{i,N+1}R_i(y_1,\ldots,y_d) + B_{N+1}\right)$$

$$+ \sum_{i=0}^{N} k_iR_i(y_1,\ldots,y_d) + C = \frac{P_{N+1}(y_1,\ldots,y_d;\ k_0,\ldots,k_N,\ C)}{Q_{N+1}(y_1,\ldots,y_d)}$$

where $P_{N+1}(y_1,\ldots,y_d;\ k_0,\ldots,k_N,\ C) = k_{N+1}\ P_N(y_1,\ldots,y_d;\ M_{0,N+1},\ldots,M_{N,N+1},\ A_{N+1})$

$$\cdot\ P_N(y_1,\ldots,y_d;\ N_{0,N+1},\ldots,N_{N,N+1},\ B_{N+1}) + P_N(y_1,\ldots,y_d;\ k_0,\ldots,k_N,C)\ Q_N(y_1,\ldots,y_d),$$

and $Q_{N+1}(y_1,\ldots,y_d) = Q_N(y_1,\ldots,y_d)^2$. Then by the induction hypothesis, we have that $\sum_{i=0}^{N+1} k_iR_i(y_1,\ldots,y_d) + C$ has degree $\le 2^{N+1}$.

Similarly, from (4.4) we also have that $\sum_{i=0}^{N+1} k_i R_i(y_1,\ldots,y_d) + C$ has the form $\frac{P_{N+1}(y_1,\ldots,y_d;\ k_0,\ldots,k_N,C)}{Q_{N+1}(y_1,\ldots,y_d)}$ with degree $\le 2^{N+1}$ for some $P_{N+1}(y_1,\ldots,y_d;\ k_0,\ldots,k_N,C)$ and $Q_{N+1}(y_1,\ldots,y_d)$.

Hence, both cases imply that (4.5) is true for $n = N+1$. This completes the induction. Therefore, for any numbers $k_0,\ldots,k_n,C$, the degree of $\sum_{i=0}^{n} k_i R_i + C$ will not reach $D(\varphi)$ until $n \ge \log_2 D(\varphi)$. This implies that $\bar{M} \ge \log_2 D(\varphi)$. This completes the proof. ■

Note that $M \ge \bar{M}$, since preconditioning is only performed on constant coefficients. Thus, by Theorem 1, $M \ge \bar{M} \ge \log_2 D(\varphi) \ge \log_2 p$. Therefore, we have the following

<u>MAIN RESULT</u>: $E = \frac{\log_2 p}{M} \le 1.$

Now consider the sequence generated by $\psi(x) = x^2 + x - \frac{1}{4}$ with the limit $-1/2$. Since $\psi'(-1/2) = 0$ and $\psi''(-1/2) \ne 0$, we can easily show that this sequence has order 2. Obviously $M=1$ for this sequence. Thus $E = \frac{\log_2 2}{1} = 1$. Similarly, $E=1$ for the second order sequence generated by $\Gamma(x) = \frac{1}{x} + x - 1$ with the limit 1. Either example shows that our bound on E is sharp. Moreover, we have the following interesting result.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications. From our main result, we have the following

<u>Corollary</u>: <u>$P_M \le 2^M$ for all positive integer M. Moreover this bound is sharp.</u>

<u>Proof</u>: Let ψ_M be the composition of ψ with itself M times where $\psi(x) = x^2 + x - \frac{1}{4}$ as before. Then the sequence generated by ψ_M has order 2^M and ψ_M employs M

multiplications. Hence for each M the maximal order is achieved by the sequence generated by ψ_M. ∎

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13. ABSTRACT

For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, \ldots, x_{i-d+1})$, define the multiplication efficiency measure E to be $\frac{\log_2 p}{M}$, where p is the order of convergence, and M is the number of multiplications or divisions needed to compute φ. Then, if φ is any multivariate rational function, $E \le 1$. Since $E = 1$ for the sequence $\{x_i\}$ generated by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit -1/2, the bound on E is sharp.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications. Then $P_M \le 2^M$ for all positive integer M. Moreover this bound is sharp.

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