Inductive definitions over finite structures

Daniel Leivant
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School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

We give a simple proof of a theorem of Gurevich and Shelah, that the inductive closure of an inflationary operator is equivalent, over the class of finite structures, to the inductive closure (i.e. minimal fixpoint) of a positive operator. A variant of the same proof establishes a theorem of Immerman, that the class of inductive closures of positive first order operators is closed under complementation.

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Introduction

Inductive definitions have played a central role in the foundations of mathematics for over a century. They were used in the 1970's as the backbone of one major generalization of Recursive Functions Theory [Mos74,Acz77]. In recent years the relevance of inductive definitions (in particular over finite structures) to Database Theory, to Descriptive Computational Complexity, and to Logics of programs, has been recognized.

A seminal paper on inductive definitions in Database Theory is [CH82], where Chandra and Harel define a hierarchy of queries over finite structures, within which minor steps (successor ordinals) correspond to first-order quantifier alternations, and major steps (limit ordinals) correspond to uses of fixpoints. They left open the question of whether the hierarchy remains strict above the first major step (level ω). This problem was answered in the negative by Immerman [Imm86]. Since the collection of first order fixpoint queries over finite structures is closed under composition and first order operations other than negation [Mos74], the main component of Immerman's solution was

Theorem A [Imm86]. *The complement of a fixpoint query is equivalent, over finite structures, to a fixpoint query.*

A connection was also discovered between inductive definability and computational complexity. Particularly striking in its elegance and simplicity is

Theorem B [Imm86,Var82,Gur83]. *The polynomial time queries over ordered structures are precisely the inductive closures of systems of positive first order operators.*

This equivalence evidences the fundamental nature of polynomial time computability, and leads one to view inductive definability over finite structures as a generalization of PTime, from ordered structures to arbitrary (finite) structures.

In general, any operator Φ has an *inductive closure* obtained as the union of the increasing chain $\Phi_\alpha =_{Df} \cup_{\xi < \alpha} \Phi[\Phi_\xi]$. The classical theory of inductive definability has traditionally focused on monotone operators because their inductive closure is their minimal fixpoint, a crucial property for algebraic and model theoretic applications. (Technical notions are defined in §1 below.) The theory was developed primarily for *positive* operators mainly because, over the natural numbers, positive induction has an attractive theory that clarifies the recursion theoretic analogies between Σ_1^0 and Π_1^1 relations on ω (see [Acz77, Mos74]). Also, a first order operator which is monotone on *all* structures must be positive (Lyndon's Theorem, see e.g. [CK73]), perhaps naturally leading to the early focus on positive operators (even though on certain individual structures, e.g. ω^ω, monotone fixpoints are more general than positive fixpoints). However, a first order operator may be monotone over all finite structures while failing to be positive [AG88]. The restriction to positive operators is therefore less natural in the context of Computer Science [Liv83,Gur84]. A natural question is then whether Theorem A remains true for the broader class of inductive closures. A positive answer follows from Theorem A and the following result of Gurevich and Shelah. Let *FO* denote first-order logic, and let *FO + LFP* be first order logic enriched with simultaneous fixpoints of operators defined by positive formulas.

Theorem C [GS86]. *If an operator over finite structures is definable in FO + LFP, then its inductive closure is also definable in FO + LFP.*

[GS86] also derives Theorem A from Theorem C. Theorem C demonstrates the strong stability of the notion of inductive definability over finite structures. This stability, aside from being reassuring, is also somewhat surprising: in important respects Finite Model Theory is less well behaved than unrestricted Model Theory (for instance, the first order formulas valid in all finite models do not form a recursively enumerable set [Tra50]), whereas here we find the opposite. The importance of Theorem C has been manifested by the rapid discovery of its applications, in particular in relation to the use of negation in logic programs and database languages (see e.g. [KP88] and [AV88]).

We give an alternative proof of Theorem C, with several gains. The most obvious one is simplicity. The method of proof in [GS86] builds on Aczel's proof of Moschovakis's Stage Comparison Theorem [Mos69, Mos74, Acz77], and much of the effort there stems from the adaptation of an argument that works for all structures, not only for finite ones. Our construction uses the finiteness of structures from the outset, and is considerably simpler. Moreover, our construction yields directly both Theorem C and Theorem A.

Also, the proof in [GS86] yields, for an operator φ, a positive $\hat{\varphi}$ with the same inductive closure over finite structures, where $\hat{\varphi}$ is defined from φ using positive occurrences of both $\exists$ and $\forall$. Our construction makes do with $\exists$. (Note, however, that this does not imply the elimination of positive occurrences of $\forall$ already present in φ.)

1. Preliminaries

1.1. Inductive closure of global operators

Let U be a set, $F : \mathcal{P}^k(U) \to \mathcal{P}^k(U)$ an operator over the collection $\mathcal{P}^k(U)$ of k-ary relations over U. Let $F^{[0]} =_{Df} \emptyset$, $F^{[i+1]} =_{Df} F[F^{[i]}]$. F is *inductive* if $F^{[i+1]} \supseteq F^{[i]}$ for all i. If U has $n < \infty$ elements, and F is inductive, then $F^{[j+1]} = F^{[j]}$ for some $j \leq n^k$, and $F^\infty =_{Df} F^{[j]}$ is the *inductive closure* of F. We write $|F|$ for the first j for which $F^{[j]} = F^\infty$.

Examples of inductive operators are the monotone operators and the inflationary operators. F is *monotone* if $R \subseteq R'$ implies $F[R] \subseteq F[R']$. The inductive closure of a monotone operator F is its *least* fixpoint: if $F[R] = R$ then $R \supseteq F^{[i]}$ for all i (by induction on i), so $R \supseteq F^\infty$. F is *inflationary* if $F[R] \supseteq R$. For any operator F, the *cumulative closure* of F, defined by

$$F^{\text{cum}}[R] =_{Df} F[R] \cup R$$

is inflationary. Clearly, if F is inductive then $(F^{\text{cum}})^\infty = F^\infty$, and if F is inflationary then $F^{\text{cum}} = F$.

The inductive closure of an *arbitrary* operator F can be defined as the union of the increasing chain $F^{(i)} =_{Df} \bigcup_{j<i} F[F^{(j)}]$. If F is inductive, then $F^{[i]} = F^{(i)}$ (by induction on i), so this definition agrees with the previous one. Note that $F^{(i)} = (F^{\text{cum}})^{[i]}$; hence $F^\infty = (F^{\text{cum}})^\infty$ for every F, i.e. every inductive closure is the inductive closure of an inflationary operator.

Let C be a collection of structures. A *global (k-ary) relation* over C is a mapping Φ that assigns to each structure $S \in C$ a (k-ary) relation Φ^S over the universe $|S|$ of S [Tar52,BG86,Gur87]. A *global (k-ary) operator* over C is a mapping Φ that assigns to each structure $S \in C$ an operator over k-ary relations over $|S|$. If P is a property of relations, we say that Φ is P if Φ^S is P for each $S \in C$. The *inductive closure* Φ^∞ of Φ is the global relation over C determined by $(\Phi^\infty)^S =_{Df} (\Phi^S)^\infty$.

Let L be a language extending propositional logic. An occurrence of an identifier in an L-formula φ is *negative* if it is in the scope of an odd number of negations (where each implication $\alpha \to \beta$ is read as $\neg\alpha \vee \beta$). A relational identifier R is *positive* in φ if φ has no negative occurrences of R. We call a language L *monotone* if

1. L is closed under first order operations; and

2. for every L-formula φ and relational identifier R positive in φ, the formula
   $\forall \bar{z}.(P(\bar{z}) \to Q(\bar{z})) \to \varphi[P/R] \to \varphi[Q/R]$ is valid in every structure for L. (P and Q are relational variables not free in φ, with $arity(P) = arity(Q) = arity(R)$.)

For example, first order logic *FO*, second order logic, ω-order Logic, and *FO + LFP* (as defined in the introduction are monotone, whereas *FO+* the quantifier "there are finitely many" is not

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monotone. An operator defined by an L-formula is an *L-operator*. **For the rest of the paper L will be a monotone language.**

If C consists only of structures of some vocabulary σ, and φ is an L-formula over σ, all of whose free variables are among $x_1 \ldots x_k$ (abbreviated as $\hat{x}$), then $\lambda \hat{x}.\varphi$ is a global k-ary relation over C. Namely, for each $S \in C$,

$$(\lambda \hat{x}.\varphi)^S = \{ \hat{x} \in |S|^k : S, [\hat{x}/\hat{x}] \models \varphi \}.$$

If $R \not\subseteq \sigma$ is a k-ary relational identifier, and φ is an L-formula over the vocabulary $\sigma \cup \{R\}$, all of whose free variables are among $\hat{x}$, then $\lambda \hat{x}.\varphi$ is a global k-ary operator over C: for $S \in C$ and $\mathbf{R} \subseteq |S|^k$

$$(\lambda \hat{x}.\varphi)^S[\mathbf{R}] = \{ \hat{x} : \langle S, \mathbf{R} \rangle, [\hat{x}/\hat{x}] \models \varphi \}.$$

(Here $\langle S, \mathbf{R} \rangle$ is the expansion of S to the vocabulary $\sigma \cup \{R\}$, with R interpreted as $\mathbf{R}$.) If R is positive in φ, we say that the global operator $\lambda \hat{x}.\varphi$ is *positive*. Every positive L-operator is monotone, because L is assumed to be a monotone language.

1.2. Simultaneous Fixpoints

One may consider operators over

$$\mathcal{P}^{k,l}(U) =_{Df} \mathcal{P}^k(U) \times \mathcal{P}^l(U),$$

rather than over $\mathcal{P}^k(U)$, as above. Such operators are pairs $\langle F_1, F_2 \rangle$, where

$$F_1 : \mathcal{P}^{k,l}(U) \to \mathcal{P}^k(U) \text{ and } F_2 : \mathcal{P}^{k,l}(U) \to \mathcal{P}^l(U).$$

We call these *bi-operators*. Bi-operators are the paradigm of the more general *multi-operators*, over cartesian products $\prod_i \mathcal{P}^{k_i}(U)$. Everything we say about bi-operators generalizes straightforwardly to multi-operators.

For F as above, one defines the sequence of pairs of relations $F^{[0]} \equiv \langle F_1^{[0]}, F_2^{[0]} \rangle =_{Df} \langle \emptyset, \emptyset \rangle$, $F^{[i+1]} \equiv \langle F_1^{[i+1]}, F_2^{[i+1]} \rangle =_{Df} F[F^{[i]}]$. F is *inductive* if this sequence is increasing: $F_1^{[i+1]} \supseteq F_1^{[i]}$ and $F_2^{[i+1]} \supseteq F_2^{[i]}$, for all i. If U has $n < \infty$ elements, and F is inductive, then $F^{[j+1]} = F^{[j]}$ for some $j \leq n^{k+l}$, and $F^\infty \equiv \langle F_1^\infty, F_2^\infty \rangle =_{Df} F^{[j]}$ is a fixpoint of F, which we call the *inductive closure* of F. We say that F_1^∞ and F_2^∞ are defined by *simultaneous induction (on F)*. Just as for simple operators, we define *monotone* bi-operators and *inflationary* bi-operators. Again, both monotone and inflationary bi-operators are inductive, and the inductive closure of a monotone bi-operator is its *least* fixpoint.

A *global ((k, l)-ary) bi-operator* over a class of structures C is a mapping $\Phi = \langle \Phi_1, \Phi_2 \rangle$ that assigns to each $S \in C$ a bi-operator $\Phi^S = \langle \Phi_1^S, \Phi_2^S \rangle$ over $\mathcal{P}^{k,l}(|S|)$. If P is a property of bi-operators, then we say that Φ is P if Φ^S is P for each $S \in C$.

For example, suppose that C consists of σ-structures, φ_1 and φ_2 first order formulas in the vocabulary $\sigma \cup \{R, Q\}$, with $\mathbf{R}$ and $\mathbf{Q}$ positive in φ_1 and φ_2, with the variables free in φ_1 among $\hat{y} \equiv (y_1, \ldots, y_k)$ and the variables free in φ_2 among $\hat{z} \equiv (z_1, \ldots, z_l)$. Then the bi-operator $\langle \mathbf{R}, \mathbf{Q} \rangle \longmapsto \langle \lambda \hat{y}.\varphi_1[\mathbf{R}/R, \mathbf{Q}/Q], \lambda \hat{z}.\varphi_2[\mathbf{R}/R, \mathbf{Q}/Q] \rangle$ is monotone. Again, we dub such global operators *positive*.

Operators defined by simultaneous induction are well known to be definable already as inductive closures of simple operators, provided some means are available for "gluing together" relations (see e.g. [Mos74]). For simplicity, assume that our bi-operators all have arity $\langle k, l \rangle$ with $l = k$. This is no loss of generality, since the general case reduces to this one by padding and projection; alternatively, the discussion below needs only minor modifications to apply to the general case. Call a class C of finite structures *discriminating (via constant identifiers c, d)* if the structures therein are over a vocabulary with distinct constant identifiers c, d, and $c^S \neq d^S$ for every $S \in C$.

Lemma 1 (Simultaneous Induction With Constants) *Let C be a discriminating class of structures. Let $\Phi = \langle \Phi_1, \Phi_2 \rangle$ be a $\langle k, k \rangle$-ary inductive global bi-operator over C. There is a $(k+1)$-ary inductive global operator Ψ, defined from Φ and $=$ using only disjunction and conjunction, and such that $\Phi_1^\infty(\hat{u}) \equiv \Psi^\infty(c, \hat{u})$.*

In particular, if Φ is defined by a positive formula then so is Ψ.

Proof. Assume C is discriminating via constant identifiers c, d. Let

$$\Psi[R](z, \hat{u}) \quad =_{Df} \quad z = c \,\wedge\, \Phi_1[\lambda \hat{v}.R(c, \hat{v}), \lambda \hat{v}.R(d, \hat{v})](\hat{u})$$
$$\vee\, z = d \,\wedge\, \Phi_2[\lambda \hat{v}.R(c, \hat{v}), \lambda \hat{v}.R(d, \hat{v})](\hat{u}).$$

Then, by induction on i, $\Psi^{[i]}(c, \hat{u}) \leftrightarrow \Phi_1^{[i]}(\hat{u})$, and $\Psi^{[i]}(d, \hat{u}) \leftrightarrow \Phi_2^{[i]}(\hat{u})$. So $\Phi_1^\infty(\hat{u}) \equiv \Psi^\infty(c, \hat{u})$. ⊣

Lemma 2 (Simultaneous Induction Without Constants) *Let C be a class of structures, in each one of which there are at least two elements. Let $\Phi = \langle \Phi_1, \Phi_2 \rangle$ be a $\langle k, k \rangle$-ary inductive global bi-operator over C. There is a $(k+3)$-ary inductive global operator Ψ, defined from Φ, $=$, and $\neq$ using only disjunction and conjunction, and such that $\Phi_1^\infty(\hat{u}) \equiv \exists x, z \, \Psi^\infty(x, x, z, \hat{u})$.*

In particular, if Φ is defined by a positive formula then so is Ψ.

Proof. Let

$$\Psi[R](x, y, z, \hat{u}) \quad =_{Df} \quad x = y \neq z \,\wedge\, \Phi_1[\lambda \hat{v}.R(x, y, z, \hat{v}), \lambda \hat{v}.R(x, z, y, \hat{v})](\hat{u})$$
$$\vee\; x = z \neq y \,\wedge\, \Phi_2[\lambda \hat{v}.R(x, z, y, \hat{v}), \lambda \hat{v}.R(x, y, z, \hat{v})](\hat{u}).$$

The proof is concluded as for Lemma 1. ⊣

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Moschovakis's exposition [Mos74] is free of distinctions as in the lemmas above, because the operators referred to there allow structure elements as parameters. Immerman ([Imm86], Lemma 4.1) proves Lemma 2 using only two extra variables, but with quantified formulas as relational substitutions.

1.3. Composition

The method used above for combining simultaneous fixpoints into single fixpoints can be used to compose fixpoints. Suppose given $\Phi[R, P](\hat{u})$, with $arity(R) = arity(P) = arity(\hat{u})$. Given any structure and interpretation $\mathbf{P}$ for P therein, $\Phi[R, \mathbf{P}](\hat{u})$ has an inductive closure $\Phi^\infty[\mathbf{P}]$. In particular, if $\mathbf{P} = \lambda\hat{u}.\Psi^\infty(\hat{u})$ for some operator $\Psi[Q](\hat{u})$, we have the composition $\Phi^\infty[\Psi^\infty]$.

Lemma 3 (Composition [Mos74]) *Let C be a class of structures of size ≥ 2. Let Φ and Ψ be as above. Then there is an operator $\Xi[S](\hat{x})$, defined from Φ and Ψ using only operator application, equality, $\wedge$ and $\vee$, such that $\Phi^\infty[\Psi^\infty]$ is equivalent, over C, to an instance of Ξ^∞ In particular, if Φ and Ψ are positive first order then so is Ξ.*

Proof. Suppose C is a discriminating class of structures. Define

$$\Xi[S](z, \hat{v}) =_{Df} \quad z = c \ \wedge \ \Phi[\lambda\hat{u}.S(c, \hat{u})](\hat{v})$$
$$\vee \ z = d \ \wedge \ \Psi[\lambda\hat{u}.S(d, \hat{u}), \lambda\hat{u}.S(c, \hat{u})](\hat{v}).$$

We claim that $\Xi^\infty(d, \hat{v}) \equiv \Psi^\infty[\Phi^\infty](\hat{v})$.

By induction on m, $\Xi^m(c, \hat{v}) \equiv \Phi^m(\hat{v})$. Since $\Phi^m \subseteq \Phi^\infty$, and Φ and Ψ are positive, this implies (again by induction on m) $\Xi^m(d, \hat{v}) \rightarrow \Psi^m[\Phi^\infty](\hat{v})$. But $\Psi^m[R] \subset \Psi^\infty[R]$ for all R, so $\Xi^m(d, \hat{v}) \rightarrow \Psi^\infty[\Phi^\infty](\hat{v})$, for all m, whence $\Xi^\infty(d, \hat{v}) \rightarrow \Psi^\infty[\Phi^\infty](\hat{v})$, proving the forward direction of the claim.

For the converse, assume $\Psi^\infty[\Phi^\infty](\hat{v})$, that is $\Psi^\infty[\Phi^m](\hat{v})$, where $m = |\Phi|$. Then, for some n, $\Psi^n[\Phi^m](\hat{v})$. By induction on n, this implies $\Xi^{m+n}(d, \hat{v})$, and so $\Xi^\infty(d, \hat{v})$.

If C is not discriminating, the definition of Ξ is modified as in the proof of Lemma 2. ⊣

A corollary of the Composition Lemma is

Lemma 4 *The class of positive fixpoints is closed under conjunction, disjunction, and quantification.*

Proof. The operations listed are trivially defined as positive fixpoints. For instance, $\lambda\hat{u}. \ \exists x \ Q(x, \hat{u}) \equiv \Phi^1 \equiv \Phi^\infty$, where $\Phi[R] \equiv \exists x Q(x, \hat{u})$ (here Φ has a constant value: there is no occurrence of R). Hence the Composition Lemma applies. ⊣

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2. Positive inductive definability of stage comparison relations

In this section we prove the following refinement of Theorem C.

Theorem I *Let C be a class of structures. Suppose Φ is an inductive global L-operator over C, whose inductive closure is reached after finitely many stages in every structure in C. Then Φ^∞ is definable as the diagonal of a relation defined by simultaneous induction on a positive global L-multi-operator Ψ.*

Moreover, Ψ is defined from Φ using only operator-application, boolean operations, and positive occurrences of $\exists$. In particular, if Φ is defined by a first order formula with no positive occurrence of $\exists$ nor negative occurrence of $\forall$ (so the prenex form of φ is purely existential), then Ψ is defined by a first order purely existential positive formula.

This implies, by Lemmas 1 and 2,

Corollary 5 *Let C be a class of finite structures of size ≥ 2. If Φ is an inductive global L-operator over C, then there is a* positive *global L-operator Ψ such that Φ^∞ is defined from Ψ^∞ by projection (and $\exists$, if C is not discriminating), and Ψ is defined from Φ as in Theorem 1.*

Let S be a finite structure, F an inductive k-ary operator over $|S|$. F determines a function

$$|\hat{x}| \equiv |\hat{x}|_F =_{Df} \begin{cases} \min_i[\hat{x} \in F^{[i]}] & \text{if } \hat{x} \in F^\infty, \\ |F| + 1 & \text{otherwise.} \end{cases}$$

In infinite structures a fixpoint of a monotone operator F may fail to be reached after finite iterations, and one needs to define $F^{[\lambda]} =_{Df} \cup_{\alpha<\lambda} F^{[\alpha]}$ for limit ordinals λ so as to reach a fixpoint. However, when each element of $|S|$ has a finite stage in the build-up of F^∞, as in finite structures, one can define properties of a stage from the properties of the predecessor stage, which always exists.

Let $\Phi = \lambda\hat{x}.\varphi[R]$ be an inductive global operator over a class C of structures, which closes after finitely many stages over every structure in C. Write φ as $\varphi[R, \neg R]$, where the exhibited form $\varphi[R, Q]$ is positive in its two arguments. To simplify notations, assume that φ defines a unary operator (i.e. R is unary and $\hat{x}$ is a 1-tuple).

A formula $\varphi[R, S]$ with one free variable, x, is *regular* if $\lambda x.\varphi[\emptyset, \emptyset] = \emptyset$ and $\lambda x.\varphi[U, U] = U$ (where $\emptyset \equiv$ **false** $\equiv \lambda u.(u \neq u)$, and $U \equiv$ **true** $\equiv \lambda u.(u = u)$).

Lemma 6 *For every formula $\varphi[R, S]$ as above there exists a regular formula $\varphi'[R, S]$ such that $\varphi[R, \neg R] \equiv \varphi'[R, \neg R]$.*

Proof. Let $\varphi'[R, S](x) =_{Df} (\varphi[R, S](x) \wedge (R(x) \vee S(x))) \vee (R(x) \wedge S(x))$. ⊣

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Let $\Phi = \lambda u.\varphi$ be an inductive global L-operator over a class of structures $\mathcal{C}$, which closes after finitely many stages in every structure in $\mathcal{C}$. Consider the following global relations over $\mathcal{C}$.

$$
\begin{aligned}
(x \dot{\prec}^\varphi y)^S &=_{Df} \quad |x| < |y| \leq |\Phi| \quad \text{where } |\Phi| = |\Phi^S|,\ |z| = |z|_{\Phi^S} \\
(x \dot{\preceq}^\varphi y)^S &=_{Df} \quad |x| \leq |y| \leq |\Phi| \\
(x \dot{\prec}_1^\varphi y)^S &=_{Df} \quad |x| + 1 = |y| \leq |\Phi| \\
(x \dot{\not\prec}^\varphi y)^S &=_{Df} \quad |x| \not< |y| \leq |\Phi| \\
(x \dot{\not\preceq}^\varphi y)^S &=_{Df} \quad |x| \not\leq |y| \leq |\Phi|
\end{aligned}
$$

Note that $|x| \leq |\Phi|$ is not required in the last two relations. The notations use dots to differentiate between these relations and the common stage comparison relations, see §3 below.

Theorem II *Let $\Phi = \lambda x.\varphi$ be an inductive global L-operator (or multi-operator) over a class $\mathcal{C}$ of structures, whose inductive closure is reached after finitely many stages in every structure in $\mathcal{C}$. There is a positive L-multi-operator Ψ, defined by positive existential quantification of propositional forms in ϕ, such that*

$$
\Psi^\infty = \langle \dot{\prec}^\varphi,\ \dot{\preceq}^\varphi,\ \dot{\prec}_1^\varphi,\ \dot{\not\prec}^\varphi,\ \dot{\not\preceq}^\varphi \rangle.
$$

Proof of Theorem 1. $\Phi^\infty = \{\, x \ :\ x \dot{\preceq}^\varphi x \,\}$, by Theorem 2, where $\dot{\preceq}^\varphi$ is as required. $\dashv$

Proof of Theorem 2. Let Ψ be the positive first-order 5-operator whose inductive closure, $\langle \dot{\prec},\ \dot{\preceq},\ \dot{\prec}_1,\ \dot{\not\prec},\ \dot{\not\preceq} \rangle$, satisfies the following fixpoint equivalences:

$$
\begin{aligned}
x \dot{\prec} y &\equiv \exists z\, (x \dot{\preceq} z \dot{\prec}_1 y) \\
x \dot{\preceq} y &\equiv \varphi[\lambda u \dot{\prec} y, \lambda u \dot{\not\prec} y](x) \\
x \dot{\prec}_1 y &\equiv \neg\varphi[\neg\, \lambda u \dot{\not\prec} x, \neg\, \lambda u \dot{\prec} x](y) \\
& \qquad \wedge\ \varphi[\lambda u \dot{\preceq} x, \lambda u \dot{\not\preceq} x](y) \\
x \dot{\not\prec} y &\equiv \exists z\, (x \dot{\not\preceq} z \dot{\prec}_1 y)\ \vee\ \varphi[\emptyset, U](y) \\
x \dot{\not\preceq} y &\equiv \neg\varphi[\neg\lambda u \dot{\not\prec} y, \neg\lambda u \dot{\prec} y](x)
\end{aligned}
$$

Note that the first and fourth equivalences depend on the stages being all finite. Suppose all stages in the build-up of Φ^∞ over S are finite. We show, for $R \in \{\dot{\prec}, \dot{\preceq}, \dot{\prec}_1, \dot{\not\prec}, \dot{\not\preceq}\}$, that, for all $y \in \Phi^\infty$ and all x,

$$
x R^\varphi y \quad \text{IFF} \quad x R y. \tag{*}
$$

The multi-operator Ψ is positive, so its inductive closure $\langle \dot{\prec}, \dot{\preceq}, \dot{\prec}_1, \dot{\not\prec}, \dot{\not\preceq} \rangle$ is its minimal fixpoint. The backward direction of (*) is therefore guaranteed once we verify that $\langle \dot{\prec}^\varphi, \dot{\preceq}^\varphi, \dot{\prec}_1^\varphi, \dot{\not\prec}^\varphi, \dot{\not\preceq}^\varphi \rangle$ satisfies the fixpoint equivalences above. We assume that φ is regular, which by Lemma 6 is no loss of generality. The equivalence for $\dot{\prec}$ is immediate. The

equivalence for $\precsim$ is immediate if $|y| \leq |\Phi|$, and the case $|y| = |\Phi| + 1$ holds (with both sides empty) by regularity. The equivalence for $\prec_1$ is immediate if $|x| < |\Phi|$, $|y| \leq |\Phi|$; if $|x| = |\Phi|$ the first conjunct fails if $|y| \leq |\Phi|$, and the second if $|y| = |\Phi| + 1$; if $|x| = |\Phi| + 1$ the first conjunct fails by regularity. The remaining two equivalences are similar.

The forward direction of (*) is proved by induction on $|y|$.

Induction Basis: $|y| = 1$.

$R \equiv \prec_1$: $x\prec_1^\varphi y$ holds for no x, so (*) is true vacuously.

$R \equiv \prec$: Similar.

$R \equiv \not\prec$: Since $|y| = 1$, $\varphi[\emptyset, U](y)$. Hence the second disjunct of $x\not\prec y$ is true for all x, so (*) holds.

$R \equiv \precsim$: If $x\precsim^\varphi y$ then $|x| = 1$, i.e. $\varphi[\emptyset, U](x)$. By the two cases above $\lambda u \prec y = \emptyset$ and $\lambda u \not\prec y = U$, so $x\precsim y$.

$R \equiv \precnsim$: If $x\precnsim^\varphi y$, then $|x| > 1$, i.e. $\neg\varphi[\emptyset, U](x)$. As in the last case above, this implies $x\precnsim y$.

Induction Step: Assume $1 < |y| = k + 1 \leq |\Phi|$, and that (*) holds for all x and all y with $|y| \leq k$.

$R \equiv \prec_1$: Assume $x\prec_1^\varphi y$, so $|x| = k$. We show both conjuncts in the clause for $x\prec_1 y$.

Since $|x| < |y|$ we have $\neg\varphi[\lambda u \prec^\varphi x, \neg \lambda u \prec^\varphi x](y)$, i.e. (since $|x| < |\Phi|$) $\neg\varphi[\neg\lambda u \not\prec^\varphi x, \neg \lambda u \prec^\varphi x](y)$. By induction assumption $\lambda u \prec^\varphi x = \lambda u \prec x$, and $\lambda u \not\prec^\varphi x = \lambda u \not\prec x$, so $\neg\varphi[\neg\lambda u \not\prec x, \neg \lambda u \prec x](y)$.

Also, $|y| \leq |x|+1$. Since $y \in \Phi^\infty$, we have $\varphi[\lambda u \precsim^\varphi x, \neg\lambda u \precsim^\varphi x](y)$, i.e., $\varphi[\lambda u \precsim^\varphi x, \lambda u \precnsim^\varphi x](y)$. Hence, by induction assumption, $\varphi[\lambda u \precsim x, \lambda u \precnsim x](y)$. Since both conjuncts of the clause for $x\prec_1 y$ are implied by $x\prec_1^\varphi y$, we have (*).

$R \equiv \prec$: Since $|y| > 1$, we have $\exists z (x\precsim^\varphi z\prec_1^\varphi y)$. We have verified already that $z\prec_1 y$ iff $z\prec_1^\varphi y$. Also, by induction assumption, $x\precsim z$ iff $x\precsim^\varphi z$. So the clause for $x\prec y$ is true, and (*) holds.

$R \equiv \not\prec$: Similar to the above, with $\precnsim$ in place of $\precsim$.

$R \equiv \precsim$: If $x\precsim^\varphi y$, then $\varphi[\lambda u \prec^\varphi y, \lambda u \not\prec^\varphi y](x)$, since $|x| \leq |\Phi|$. We have just shown that $u\prec^\varphi y$ iff $u\prec y$, and $u\not\prec^\varphi y$ iff $u\not\prec y$. Thus $x\precsim y$, and so (*) holds.

$R \equiv \precnsim$: Similar.

$\dashv$

3. Positive inductive definability of strong stage comparison relations

Let $\Phi = \lambda x.\varphi$ be an inductive global L-operator over a class C of structures, whose inductive closure is reached after finitely many stages in every structure in C. Consider the following global relations over C.

$$
\begin{aligned}
(x \prec^\varphi y)^S \quad &=_{Df} \quad |x| < |y| \\
(x \preceq^\varphi y)^S \quad &=_{Df} \quad |x| \le |y| \ \wedge \ x \in \Phi^\infty \\
(x \prec_1^\varphi y)^S \quad &=_{Df} \quad |x| + 1 = |y| \\
(x \not\prec^\varphi y)^S \quad &=_{Df} \quad |x| \not< |y| \\
(x \not\preceq^\varphi y)^S \quad &=_{Df} \quad |x| \not\le |y| \ \vee \ x \notin \Phi^\infty
\end{aligned}
$$

These relations, familiar from the general theory of inductive definability [Mos74,Acz77], are analogous to the "dotted" relations defined above, except that the second argument, y, of each relation is not required to be in Φ^∞. The main result of this section is the following.

Theorem III *Let $\Phi = \lambda x.\varphi$ be an inductive global L-operator (or multi-operator) over a class C of structures, whose inductive closure is reached after finitely many stages in every structure in C. There is a positive L-multi-operator Ψ such that*

$$
\Psi^\infty \;=\; \langle \prec^\varphi, \ \preceq^\varphi, \ \prec_1^\varphi, \ \not\prec^\varphi, \ \not\preceq^\varphi \rangle.
$$

Compared to Theorem 2, we obtain here the inductive definability of more complex relations, but we shall need positive occurrences of universal quantification. We return to the proof of Theorem 3 momentarily.

If Ψ is a global k-ary operator over a class C of structures, then its *complement*, $-\Psi$ is the global operator over C defined by $(-\Psi)^S =_{Df} |S|^k - \Psi^S$.

An immediate corollary of Theorem 3 is

Theorem IV *Let Φ be an inductive global L-operator over a class C of structures, whose inductive closure is reached after finitely many stages in every structure in C. Then the complement $-\Phi^\infty$ of Φ is definable by simultaneous induction on a positive global L-multi-operator.*

Moreover, Ψ is defined from Φ using only operator-application and first order operations.

Proof. $-\Phi^\infty = \{ x \ : \ x \not\preceq^\varphi x \}$, by Theorem 3, where $\not\preceq^\varphi$ is as required. $\dashv$

Theorem V (Immerman) *If a global relation over finite structures is definable in $FO + LFP$, then it is definable by a single fixpoint applied to a first order formula.*

Proof. By induction on the length of the $FO + LFP$ formula defining the global relation. The basis is trivial. The induction step uses Lemma 4 for $\wedge$, $\vee$, $\forall$, and $\exists$, Lemma 3 for the fixpoint operator, and Theorem 4 for negation. $\dashv$

Proof of Theorem 3. The proof is only a minor modification of the proof of Theorem 2. Let Ψ be the positive 5-operator whose inductive closure, $\langle \prec, \preceq, \prec_1, \not\prec, \not\preceq \rangle$, satisfies the following fixpoint equivalences:

$$x \prec y \quad\equiv\quad \exists z\,(x \preceq z \prec_1 y)$$

$$x \preceq y \quad\equiv\quad \varphi[\lambda u \prec y, \lambda u \not\prec y](x)$$

$$x \prec_1 y \quad\equiv\quad In(x)$$
$$\wedge \;\neg\varphi[\neg\,\lambda u \not\prec x, \neg\,\lambda u \prec x](y)$$
$$\wedge \;(\varphi[\lambda u \preceq x, \lambda u \not\preceq x](y) \;\vee\; Last(x))$$
$$\text{where}$$
$$In(x) \quad\equiv\quad \varphi[\lambda u \prec x, \lambda u \not\prec x](x),$$
$$Last(x) \quad\equiv\quad \forall z\,(\varphi[\lambda u \prec x, \lambda u \not\prec x](z) \;\vee\; \neg\varphi[\neg\,\lambda u \not\preceq x, \neg\,\lambda u \preceq x](z))$$

$$x \not\prec y \quad\equiv\quad \exists z\,(x \not\preceq z \prec_1 y) \;\vee\; \varphi[\emptyset, U](y) \;\vee\; Empty_\varphi$$
$$\text{where} \qquad Empty_\varphi \quad\equiv\quad \forall w\,\neg\varphi[\emptyset, U](w)$$

$$x \not\preceq y \quad\equiv\quad \neg\varphi[\neg\,\lambda u \not\prec y, \neg\,\lambda u \prec y](x)$$

We show, for $R \in \{\prec, \preceq, \prec_1, \not\prec, \not\preceq\}$, that, for all y and all x,

$$x R^\varphi y \quad\text{IFF}\quad x R y. \tag{*}$$

The proof of the backward direction is as for Theorem 2. The forward direction is proved, again, by induction on $|y|$. The induction basis is unchanged. The only difference for the induction step is the case for $\prec_1$. Assume $x \prec_1^\varphi y$ with $|y| = k + 1$, so $|x| = k$. We show each of the three conjuncts in the clause for $x \prec_1 y$.

Since $x \in \Phi^\infty$ (because $|x| \leq |\Phi|$), we have $\varphi[\lambda u \prec^\varphi x, \lambda u \not\prec^\varphi x](x)$. By induction assumption $u \prec^\varphi x$ iff $u \prec x$, and $u \not\prec^\varphi x$ iff $u \not\prec x$. So $\varphi[\lambda u \prec x, \lambda u \not\prec x](x)$, i.e. $In(x)$.

Since $|x| < |y|$ we have $\neg\varphi[\lambda u \prec^\varphi x, \neg\,\lambda u \prec^\varphi x](y)$, i.e., $\neg\varphi[\neg\,\lambda u \not\prec^\varphi x, \neg\,\lambda u \prec^\varphi x](y)$, which implies, by induction assumption as above, $\neg\varphi[\neg\,\lambda u \not\prec x, \neg\,\lambda u \prec x](y)$.

Finally, we know that $|y| \leq |x| + 1$. Consider two cases. First, suppose that $y \in \Phi^\infty$, i.e. $|y| \leq |\Phi|$. Then $\varphi[\lambda u \preceq^\varphi x, \neg\,\lambda u \preceq^\varphi x](y)$, i.e., $\varphi[\lambda u \preceq^\varphi x, \lambda u \not\preceq^\varphi x](y)$. So, by induction assumption, $\varphi[\lambda u \preceq x, \lambda u \not\preceq x](y)$. Otherwise, $|x| = |\Phi|$, so all elements generated at stage $|x| + 1$ are generated already at stage $|x|$:

$$\forall z\,(\varphi[\lambda u \prec^\varphi x, \neg\,\lambda u \prec^\varphi x](z) \;\vee\; \neg\varphi[\lambda u \preceq^\varphi x, \neg\,\lambda u \preceq^\varphi x](z)).$$

This implies, by induction assumption,

$$\forall z\,(\varphi[\lambda u \prec x, \neg\,\lambda u \prec x](z) \;\vee\; \neg\varphi[\neg\,\lambda u \not\preceq x, \neg\,\lambda u \preceq x](z)),$$

that is — $Last(x)$. In either case, the last conjunct of the clause for $x \prec_1 y$ holds.   ⊣

The property *Last* above is akin to a similar property used by Immerman [Imm86] in proving that inductive fixpoints are closed, over finite structures, under complementation.

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