MINLP Model for Multiproduct Scheduling on Continuous Parallel Lines

by

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MINLP MODEL FOR MULTIPRODUCT SCHEDULING
ON CONTINUOUS PARALLEL LINES

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ABSTRACT

This paper addresses the problem of long range, multiproduct scheduling on continuous parallel production lines. This plant configuration is typically used in the manufacturing of specialty chemicals. The problem is modeled as a large-scale mixed integer nonlinear program (MINLP) and a reformulation technique is applied to linearize it in the space of the integer variables. The Kuhn-Tucker optimality conditions are exploited in the resulting model in order to effectively apply Generalized Benders Decomposition. This avoids the explicit solution of extremely large nonlinear subproblems. At the same time, a computational scheme is proposed to strengthen the bounds of the master problem and therefore achieve fast convergence. The proposed technique was applied to a real world problem for a polymer production plant. The corresponding MINLP contained 780 binary variables, 23,000 continuous variables and 3,200 constraints.

programming model that can accommodate resource constraints but where they assume that transitions are sequence independent. Their model was again applied to a real world problem with 5 lines and 10 products for the American Olean Company. Along almost the same lines, Boctor (1988) developed a model for the single line static demand problem; this is another mixed integer linear programming model which still has limited computational feasibility.

There are two major limitations of previous works addressing the short term scheduling problem. First, by *discretizing* time and demand, only very short term scheduling can be performed. For a longer scheduling horizon, the number of time slots and production lots needed in order to describe accurately the problem becomes very large, and it is impossible to solve the resulting models due to their size. Secondly, the assumption of *sequence independent transitions* may be satisfactory when one only needs to clean up a machine, but it is a severe limitation in the case of a continuously operating reactor whose operating conditions change whenever there is a changeover in production.

Clearly, the two types of problems that have been addressed in the literature are important. A solution approach to the first type of problem (long range horizon) would allow the examination of considerably longer time horizons, and would therefore be especially useful when one wishes to make strategic decisions, e.g. selection of cycle times, evaluation of proposed changes in equipment structure, selection of new products for production, and so forth. On the other hand, a solution approach to the second type of problem (short range horizon) would be more useful when on-line scheduling is required and orders come in with different due dates.

Both versions of the problem are difficult, however, mainly because they involve a combinatorial part (sequencing) and a continuous part (production times) at the same time. As a result of this difficulty, all previous works either *decompose* the two parts or solve *simplified versions* of the problem (e.g. sequence independence of set-ups), or else they are *computationally very limited.* The scope of this work is to develop a rigorous optimization model where transitions will be treated as being *sequence dependent,* processing times will not be discretized and nonlinear costs and inventory considerations will be accounted for. The next paragraph describes the problem.

3. PROBLEM STATEMENT

There are several products to be scheduled on a number of continuous production lines (plants, processors, reactors) during a long range horizon. Constant demand rates are

specified for a subset of products, while the production for the remaining ones is to be determined.

The lines operate in parallel in the sense that the *only* interrelationship between them is that they have to jointly satisfy the product demands. For each product the production rate and cost, and the set-up (changeover, transition) costs and times are different for different lines. Restrictions may be placed on the inventory levels to accommodate demand fluctuations and equipment failure.

The problem is to find when, on which line, and for what length of time each product must be processed so as to minimize the total cost (or maximize the total benefit) during a cyclic schedule. Trade-offs arise from the fact that - due to changeovers - production cost has to be increased if inventory holding cost is to be reduced and *vice versa.*

4. NATURE OF THE PROBLEM

Consider the following example. Two products, A and B, are to be produced in one continuous line. Two feasible cyclic schedules are shown in Fig. 1. In the first case (a), we start by producing A   Then production is changed to B, in which case a transition time is incurred. Later, production will be again changed to product A and another transition time will be incurred. This ▷. heme (A-transition-B-transition) will be repeated in time (cyclical operation). In case (b), the amounts of products produced are the same as in case (a). The only difference is that here transitions between A and B occur more frequently.

In answering the question which schedule is better, one might be tempted to say that, since in case (b) we have to pay a penalty for the frequent transitions, case (a) should be the recommended schedule. However, transition cost is only one facet of the problem. The other consideration is the inventory cost.

Since the inventory changes with time, the inventory cost will be proportional to the integral of the inventory function along time:

$$\text{Inventory Cost} = h \int_0^H I(t)\, dt$$

where   h = unit inventory holding cost

H = processing time

I = inventory.

Two different situations illustrating the inventory cost calculation are shown in Fig. 2, and they correspond to cases (a) and (b) of Fig. 1 above. In the first case, production of A begins at time 0 and proceeds until time t. During this time the inventory of A increases. At time t, production of A stops and (since the amount of A in stock is being used to satisfy demand) the inventory starts decreasing until it reaches 0 at time H. The integral of the inventory function along time is in this case equal to the area of the triangle. In case (b), the same amount of A is produced, only now in two identical cycles. The integral of the inventory is now equal to the sum of the areas of the two lower triangles, and there are savings in the inventory cost with respect to case (a). These savings are proportional to the shaded area.

We can clearly see from this example that, when production of A is repeated more frequently, the inventory cost is reduced. On the other hand, this implies more frequent transitions between products, and therefore increased transition cost. The conclusion is that there exists an optimum cycle time which is largely determined by the inventory and transition costs.

Another important issue stems from the fact that existing chemical plants are usually overdesigned. This means that using the existing capacity one may be able to find a schedule that not only produces the required products, but also creates some idle time for the processing lines. This time can then be used either to produce some additional new products, or to absorb demand fluctuations (and in this way increase the flexibility of the production system). In order to create this idle time, one should try to minimize transitions. This, on the other hand, would imply a larger cycle time and therefore increased inventory costs. So there is a trade-off between increasing the idle time of the machines and decreasing the inventory cost.

Finally, the selection of new products to be manufactured and the assignment of products to lines is another major decision that involves trade-offs. Different products may have different production rates and costs in different lines. We then need to decide (in an optimal fashion):
(i)   which (if any) new additional products to produce;
(ii)   which products to produce in each line;
(iii)   what is the sequence of production in each line, since this will affect transition times which are sequence dependent.

In the case of 25 products and only one line, the sequencing of products can be done in more than 10^{30} different ways. This is a large combinatorial problem and requires an efficient technique for its solution.

5. Model Development

5.1 Representation and Model

The following are the major assumptions in the model:

1. Demand rates are constant with time. They must be satisfied for the set of existing products and should not be exceeded for the set of selected new products.
2. Cyclical operation is desired, *Le.* a production schedule that will be repeated in time.
3. The lines operate in parallel and they do not share any common resources. Therefore, inventory cost calculations may be performed independently for each line.

The second assumption meets the wishes of production managers who prefer easily implementable schedules: a repetitive schedule clearly simplifies the allocation of labor and material requirements. The last assumption simplifies the inventory holding cost calculations and is especially applicable to the case where the lines are distinctly located plants or to the case where a product may not be assigned to more than one line.

The general idea of the model is shown in Fig. 3. For each line we postulate a potential number of time slots for each cycle. However, the length of each time slot is a continuous variable that is to be determined. The sum of the lengths of these time slots for each line is equal to the line's cycle time, which therefore becomes a variable as well. The assignment of products to lines, the production sequence for each line, the production times and the production levels are then to be determined.

In the following, the indices i and) will be used to denote products, / will be used for lines and k for time slots.

Product assignment to lines is modeled through binary variables defined as follows:

$$Y_{ilk} = \begin{cases} 1 & \text{if product } / \text{ is assigned to line } / \text{ during time slot } k \\ 0 & \text{otherwise} \end{cases}$$

A transition from product i to product) occurs if and only if product) is currently being produced and product / was previously being produced. In terms of the assignment variables this condition can be expressed as: " a transition from / toy' occurs in line / and in the beginning of time period k if and only if: $X_{j,k-1} \cdot X_{jik} = 1$ ", where " - 1 " denotes the previous time period in the cycle. We can therefore use binary products of the assignment variables to model the transitions.

One possible formulation of the problem is a mixed integer nonlinear program (MINLP) with many nonlinear terms in both the objective and the constraints:

$$x^* \quad t \quad \vee W \qquad T_{lk}X_{ilk} \quad , \quad x \cdot x \cdot W \cdot W \cdot \; \ldots \quad X_{il,k-l}X_{jlk}$$
$$/ \; / \; \bar{k} \qquad "I \qquad i \; j \; I \; k \; _{Hjl} \qquad _H 1$$
$$+ \sum_i \sum_{Ik} \sum m_{il} W_{ilk} \quad - \quad \sum_{i \in I_2} \sum_l \sum_k b_{il} \frac{W_{ilk}}{H_l} \qquad (1)$$

s.t.

$$\sum_i X'_{ilk} = 1 \qquad\qquad \vee /, \quad \vee /\!\text{fe} \qquad (2)$$

$$H/-lT_{/ik} \qquad\qquad \vee / \qquad (3)$$
$$k$$

$$^W itt = \;^r\!//1 \; ^T ik \; - \; I \; v \; x \, y \, a\text{-}i \;] \; ^x ak \qquad vi, \quad v/, \quad v^* \qquad (4)$$

$$\sum_l \sum_k \frac{W_{ilk}}{H_l} \geq d_i \qquad\qquad \forall i \in I_1 \qquad \mathbf{(5)}$$

$$\sum_l \sum_k \frac{W_{ilk}}{H_l} \leq d_i \qquad\qquad \forall i \in I_2 \qquad (6)$$

$$X/\!/ik = \{0,1\}, \quad Tik * 0, \quad Wuk\!\!>\!0, \quad H/ > 0 \qquad (7)$$

where the following are the parameters:

    $*c\#$ is the production cost of product i in line / per unit of time

    Sj_jy is the transition cost when production in line / is changed from product i to j

    m_{il} is the unit inventory holding cost for product J produced in line /

    b_{il} is the profit for each unit of a potentially new product / in line /

r_{il} is the production rate of product i in line /

τ_{ijl} is the transition time in line / when production is changed from product i toy

d;   is the demand rate of product /

NP   is the number of products $0 = 1, NP)$

NL   is the number of lines $(/ = 1, NL)$

NT   is the number of time slots postulated for each line $(k = 1, NT)$

and the following are the variables:

$X^{\wedge}$ is the binary variable to denote assignment of product i in line / and slot k

$T^{\wedge}$ is the length of time slot k in line /

$H^{\wedge}$ is the length of the cycle of line /

$W_{z}y\pounds$ is the amount of product / produced in line / during time slot k.

The next two subsections describe the model in detail.

5.2   Constraints

According to constraint (2), exactly one product must be assigned to each time slot. The total number of time slots will usually be larger than the number of products. Then the same product will be assigned to successive time slots, in which case there will be no transition times and costs for all but the first of these time slots.

Equation (3) states the fact that the length of the cycle time (H/) of a line is equal to the summation of the lengths of all the time slots $(T/^{\wedge})$ in that line. In equation (4) we calculate the amounts of production $(W^{\wedge})$ which are proportional to the production time. The proportionality constant is the production rate $r^{\wedge}/$ (which is product and line dependent) and the production time is equal to the length of the time slot $(T^{\wedge})$ minus any transition times *XJU* (which are sequence and line dependent). In the right hand side we multiply by the assignment variable, since a product may be produced in a time slot only in the case that it is assigned to that time slot.

The constraint (5) states that demand must be satisfied for the subset (Ii) of products for which such a commitment has been made (existing products in the plant). On the other hand, constraint (6) applies to a set (I2) of potentially new products. For these products, we include a profit term in the objective function, while their production rate should not exceed the demand rate.

5.3 Objective Function

The objective function to be minimized is the total cost per unit time which consists of four components.

a) Production cost. This is represented by the first term of (1), and is proportional to the production time. The term is nonzero only when a product is assigned to a time slot. The reason for dividing by the cycle length H_l is because different lines may have different cycle lengths in which case we need to scale the terms down to the same time unit.

b) Transition cost. This is represented by a fixed charge term whenever a transition occurs. The fixed charge is adequate even when the transitions are sequence dependent and the transition cost depends on the duration of the transition. This is true since the transition times are known *a priori* and they can therefore be used to calculate the transition cost coefficients s_{ijl}.

c) Inventory holding cost

(c1) The case of one line

Consider first the case of one line where we are interested in deriving a common cycle solution. This means that, $NL = 1$ and that each product should be produced exactly once in the line. The cost coefficient m_{il} should include the combined effect of three different types of inventory holding cost:

(i) In the simplest case, one only needs to calculate the *working inventory* holding cost. The working inventory is an amount exactly enough to satisfy the constant demand rate. This is the inventory as discussed in the previous section and, by assuming linear holding cost (proportionality constant h_{il}), the corresponding term in the objective can be calculated by evaluating the triangle area. This is shown in Fig. 4.

(ii) Even though we have made the assumption of constant demand, there are usually uncertainties in the demand rate which may be changing with time. In this case, a *safety inventory* is carried. The cost of carrying safety inventory whose level depends on the length of the cycle can be included in the objective function. This is shown in Fig. 4 for the case where the safety inventory is equal to the demand for a period of time equal to the cycle's length. This, in effect, is a way of partially relaxing the assumption of constant demand rate.

(iii) When carrying inventory (working or safety) there is a possibility that a fraction of it will deteriorate or not be desirable at the end of a long range horizon. This part of the inventory will be called *dead inventory*. The amount of the dead inventory is usually a percentage (q)

of the safety inventory that has to be discarded per unit time and, when the safety inventory is equal to one cycle's production, the contribution of the dead inventory cost to the objective will be a term of the form

$$q \sum_i \sum_l \sum_t \hat{c}_{il} \, W_{ilt}$$

where $c\#$ is the cost associated with each lost unit of product l produced in line l.

It follows from the above discussion and Fig. 4 that the inventory cost coefficient in (1) should be

$$m_u = m_{fl}^{\text{working}} \overset{\&}{+} m_{il}^{\text{safety}} + m_u^{\text{dead}} = \frac{1}{2} jh_u + h_{fl} + q c_{fl}^{\sim} \qquad (8)$$

For this common cycle approach, even if we do not enforce the constraint that a product should not be produced more than once (in nonconsecutive time slots), the model has no incentive to do so. In fact, due to the way the inventory cost is being calculated, repeating more than once the production of a product in a line would result in the same inventory cost, but the same if not higher transition cost. The transition cost is higher in the case in which direct transitions between products are cheaper than indirect. By using a network flow algorithm, one can find the cheapest (direct or indirect) transitions between products (see Appendix A). These can then be used in the transition matrix and it can therefore be assumed without loss of generality that a triangle inequality property holds for the transition cost elements. Then for the above inventory cost model, the constraints enforcing the common cycle on all products need not be included and an upper bound for the number of the postulated time slots is the number of products.

Now, let us consider a more general case: one line where a product may be produced more than once during a cycle. For this "generalized common cycle" approach, the changes of the inventory level with time are as shown in Fig. 5 and it is obvious that the inventory holding cost calculations involve the evaluation of areas of trapezoids instead of triangles. The resulting cost formula for the working inventory is the following:

$$\text{Inventory Cost} = \sum_i \sum_l \sum_k 2 * L_{il} \frac{I_{il,k-1} + I_{ilk}}{2H_l} U_{lk} \qquad (9)$$

where

$$I_{ilk} = {}^I_{iUk-\backslash} + {}^w_{ilk} {}^\prime {}^d_i \ Hlk \qquad\qquad Vi, \ V/, \ V£ \qquad (10)$$

and *Itfo* $= I//^V7$ (NT = last time slot in the cycle) is a new degree of freedom for the problem (the inventory at the beginning of each cycle).

(c2) Multiple lines in parallel

The general case of multiple lines presents difficulties in the development of an inventory cost model that does not involve too many nonlinearities. This happens because in the model we do not use a common time measurement for all lines, but different time-variables for the intervals of each line. Fortunately, there are two reasonable alternate assumptions that simplify the calculations:

(Assumption-1): Each product may be produced in exactly one line. Lines with different technological characteristics may produce the same product but with slight quality differences. Customers, on the other hand, prefer to purchase uniform quality products (eg. paper). In order to enforce this restriction, one needs to define the variables:

$$Y_{il} = \begin{cases} 1 \text{ if product } / \text{ is assigned to line } / \\ 0 \text{ otherwise} \end{cases}$$

and include the following constraints in the model:

$$V/, \ V/, \ V* \qquad (11)$$

$$Vi \qquad (12)$$

$$(13)$$

(Assumption-2): Since the lines operate in parallel, we may assume that the inventory cost calculations are to be made independently for each line with the withdrawal rate for each product taken as equal to its production rate in the line. This is a reasonable assumption if the different lines are different plants which are distinctly located.

Both assumptions result in a situation where the inventory holding cost calculations may be performed independently for each line. Therefore, one can make use of the cost models^ for a single line. For the sake of simplicity of presentation only, we shall use the formulas developed for the common cycle case. The generalized common cycle case can be treated similarly.

d) Profit boost for excess of capacity. In addition to the cost terms, we must include in (1) a profit term for the production of products belonging to the set I2 of potentially new products. For each unit of such a product i which is produced in line /, we incur a profit of b^.

Even if there are no potentially new products, credit is to be given for creating idle time that will absorb demand fluctuations. This can be accounted for by including an additional dummy product.

6. EXACT LINEARIZATION TECHNIQUE

The model in the previous section is the most straightforward approach to the problem and includes many nonlinear terms in both the objective and the constraints. In order to avoid some of these nonlinearities, we will reformulate the model by including some additional variables and constraints.

First, we disaggregate the variables denoting the lengths of the time slots. The product (T/£ times $X;/^$) that appears in the first nonlinear term of the objective function (1) and the first nonlinear term of (4), will be replaced by a new variable t//£. This variable can be interpreted as the time in time slot k of line / which is devoted to the production of product /. Now, since a product cannot be produced unless it is assigned to the corresponding time slot:

$$Ulk *^{u} X_{ilk} \qquad\qquad Vi\backslash \quad V/, \quad V* \qquad (14)$$

where U is a large positive quantity (the selection of tight upper bounds is discussed in section 10).

Note that according to (2) exactly one product is assigned to each time slot and therefore, (3) can now be replaced by

$$H_l = \sum_{/} \sum_{k} t_{ilk} \qquad\qquad V/ \qquad (15)$$

Let us now turn our attention to some nonlinearities in the above model where we used binary products of the assignment variables as a way to model the transitions. This has led to nonlinearities in the transition cost in the objective function and in the evaluation of the transition times in equation (4). We can avoid both of these nonlinearities by defining the following transition variables:

$$Z_{ijlk} = \begin{cases} 1 & \text{if product / is followed by product) in line / at the beginning of time slot } k \\ 0 & \text{otherwise} \end{cases}$$

These variables have to be linked to the binary variables in such a way that $Z_{ijlk} = 1$ if and only if $X_{il,k-1} \cdot X_{jlk} = 1$. One way to enforce this condition is by including in the model the following constraint:

$$Z_{ijlk} \geq X_{il,k-1} + X_{jlk} - 1 \qquad\qquad \forall i, \ \forall j, \ \forall l, \ \forall k \qquad (16)$$

As a result of this constraint, Z_{ijlk} will become 1 if both $X_{il,k-1}$ and X_{jlk} are one. On the other hand, if at least one of them is zero, the constraint becomes redundant and, since transitions represent cost terms in the objective function, the optimization program will naturally set Z_{ijlk} to zero. This obviously also happens when the transition variables are relaxed in the interval $[0, 1]$.

The above reformulation can be derived under the guidelines of general reformulation strategies which transform a polynomial integer program into a linear integer program (Fortet, 1959; Watters, 1967; Glover and Woolsey, 1974). Another (this time problem specific) way of enforcing the same condition is to use the following set of constraints:

$$\sum_i Z_{ijlk} = X_{jlk} \qquad\qquad \forall j, \ \forall l, \ \forall k \qquad (17a)$$

$$\sum_j Z_{ijlk} = X_{il,k-1} \qquad\qquad \forall i, \ \forall l, \ \forall k \qquad (17b)$$

According to (17a), exactly one transition *to* product j occurs in the beginning of any time period if and only if j is being produced during that time period. According to (17b), exactly one transition *from* product i occurs in the beginning of any time period if and only if i was being produced during the previous time period. The transition variables once again need not be declared as binaries, but they may be relaxed in the interval $[0, 1]$ as shown below.

Theorem: Constraints (17), in conjunction with (2) and the integrality of the assignment variables X_{ilk}, have the following unique solution:

$$Z_{ijlk} = X_{il,k-1} \cdot X_{jlk} \qquad\qquad \forall i, \ \forall l, \ \forall k \qquad (18)$$

when the transition variables Z_{ijlk} are treated as nonnegative continuous variables.

<u>Proof:</u>

First of all, the problem decomposes into NL x NT independent subproblems. Therefore we may consider any / and any *k*. From (2), exactly one product must be assigned to line / and time slot *k* and exactly one product to line / and time slot *k*-1. Assume $X_m/£-1 = 1$ and $X_niJc = 1$. Then (2) gives

$$\mathbf{X}_{jlk} = 0 \qquad\qquad V_,/*n \qquad\qquad (19a)$$

$$X,v*{\sim}i = 0 \qquad\qquad V/*m \qquad\qquad (19b)$$

Now (a) and (b) of (17) and (19) give respectively:

$$Zijlk = 0 \qquad\qquad Vi, \quad Vy * m \qquad\qquad (20a)$$

$$Zijlk = 0 \qquad\qquad Vi^{•} * n, \ V; \qquad\qquad (20b)$$

From the last two relations, it follows that *Zijlk* can be 1 only for /=m and;'=n. Furthermore, if (17) is to be satisfied, at least one *Zyik* must become one. Therefore

$$Zijlk = X;/,£{\sim}7 • Xjik = 0 \qquad\qquad V(/,j) * (m, n)$$

$$\mathbf{Zmnlk = Xml,k»l * *nlk = 1}$$

and the claim is true. •

The reformulation (17) should be used instead of the refonr.ulation (16) for two reasons. First, any feasible point of (17) is also feasible to (16) but the inverse is not true since (16) satisfies (18) only at the optimum solution. Therefore, (17) is a tighter formulation. At the same time, it requires the same number of variables but fewer number of constraints.

7. THE REFORMULATED MODEL

Using the above reformulation techniques, the model becomes an MINLP with nonlinear objective function and mostly linear constraints:

$$\text{Min}_. f = \sum_{i-l}\sum_{J-^}\sum_{tmd} c_{:i}\frac{t_{ilk}}{\underset{H_l}{\overset{O,}{U}}} + \sum_{4^}\sum_{J-i}\sum_{J-l}\sum_{t-l} s_{:n}_{jll}\frac{Z_{ijlk}}{\underset{H_l}{U,}}$$

$$+ \sum_i\sum_l\sum_k m_{il}W_{ilk} - \sum_{i\in I_2}\sum_l\sum_k \triangleright_{il}-Q_{\overline{r_{1}l}} \qquad (21)$$

s.t.

$$X^X_i <7* = 1 \qquad\qquad VZ, \; V* \qquad\qquad (2)$$

$$\sum_i Z_{ijlk}= \; {}^x jlk \qquad\qquad V\zeta, \; V/, \; \backslash\!/k \qquad\qquad \textbf{(17a)}$$

$$X^z_{ijlk}= \; {}^x il,k\text{-}l \qquad\qquad V\textbf{i}, \; V/, \; \textbf{V}\textbf{fc} \qquad\qquad \textbf{(17b)}$$

$$ty* \; {}^{\wedge\; u\; x}/\!/* \qquad\qquad V/, \; V/, \; Vk \qquad\qquad \textbf{(14)}$$

$$H_l = \mathbf{IS\text{'}itt}_{i\;k} \qquad\qquad \mathbf{V\!/} \qquad\qquad \mathbf{(15)}$$

$$W_{ilk} = {}^r il \; [\; Hlk\text{-}\; \underset{j\neq i}{\%{}^x jil^Z jilk}{}^\wedge \qquad\qquad V/, \; V/, \; V^\wedge \qquad\qquad \cdot \textbf{(22)}$$

$$II \quad {}_1{}^{\wedge\;\wedge}{}_/ \overset{W_{:n.}}{} \qquad\qquad \textbf{Vtel!} \qquad\qquad \textbf{(5)}$$

$$\sum_l \sum_k \frac{W_{ilk}}{H_l} \leq d_i \qquad\qquad \forall i \in I_2 \qquad\qquad (6)$$

$$\mathbf{X}_{ilk} = \{0, 1\}, \quad Z_{ijlk} * 0, \quad t/tt \succeq 0, \quad W,7* \succcurlyeq 0, \quad H/ > 0 \qquad (23)$$

The size of this model is an important issue. For instance, for the problem of scheduling 26 products in 3 lines and where 10 time slots are postulated for each cycle, the model contains: 780 binary variables, 23,000 continuous variables, 3,200 constraints. The large number of continuous variables and constraints is the effect of the reformulation. However, if the assignment variables are temporarily fixed, the large-scale MINLP reduces to a much smaller reduced nonlinear problem (rNLP). For example, the transition variables are simply binary products of the (fixed) assignment variables: $Z_{ijlk} = {}^\wedge il_y k\text{-}l * Xjlk$. In

addition, constraint (14) implies that the processing time and the production amount are 0 whenever the corresponding assignment variable is zero. All these variables (transition variables, zero processing times and zero production amounts) as well as the associated constraints ((17), (14) and (22)) need not be included in the NLP.

These observations suggest the use of the Generalized Benders Decomposition method (Benders, 1962; Geoffrion, 1972) as illustrated in Fig. 6. We start by fixing the complicating variables (the assignment variables). Then the problem reduces to a much smaller and easier nonlinear programming problem (rNLP) whose solution gives an upper bound for the minimization problem. In addition, the solution of the dual of this rNLP provides the necessary information to formulate the Master Problem that will predict new values for the complicating variables and a lower bound for the problem. We keep iterating between these two subproblems until the two bounds become equal, in which case we have found the optimum. Although we solve only the reduced (small) primal rNLP, we actually need the solution to the dual of the original (large) NLP. This is the NLP before the elimination of the transition variables, the zero processing times, and the zero production amounts with all the associated constraints. However, the dual solution of this problem can be found analytically by making use of the Kuhn-Tucker optimality conditions (Bazaraa and Shetty, 1979), once the solution to the dual rNLP is known. It can be shown (Sahinidis and Grossmann, 1989) that this analytical solution is easy to find for any MINLP where the primal NLP subproblem has some special structure that allows the solution of a smaller problem (rNLP). This principle is applied in Section 9.

8. THE SOLUTION PROCEDURE

As we have already explained in the previous section, the assignment variables (X//£) are the ones that complicate the problem. Let u be the vector of non-complicating variables for the problem, *i.e.*

$$u = [\,^{z}ijlk.\ Hlk > \,^{w}i7* > \,^{H}/\,]\bullet$$

The detailed *solution procedure* is then as follows:

Step 1. Select X^{λ} that satisfies (2); s e t $/^{u} = +<*>J^{L} = -^{\circ\circ}, R = 1.$

Step 2. a) Fix the variables X^{Λ} and solve the MINLP problem as an NLP to determine f^{R} and *vfi*. In particular use the procedure described in Section 9:

　　　i) Solve the reduced NLP problem.

ii) Having found the optimal primal and dual solution of the reduced problem, find the optimal dual solution of the original NLP by analytically solving its Kuhn-Tucker optimality conditions.

b) Update the upper bound by setting $f^U = \min \{ f^U, fR \}$.

Step 3.   To determine new values X_{ilk}^{R+1} for the 0-1 variables and a lower bound to f, solve the pseudo-integer master problem:

$$f^L = \min \; \eta \tag{24}$$

$$\text{s.t.} \quad \eta \geq L^r(X_{ilk}) \qquad r = 1, R \tag{25}$$

$$\sum_i X_{ilk} = 1 \qquad \forall l, \; \forall k \tag{2}$$

$$\eta \in \mathfrak{R}^1, \quad X_{ilk} = 0,1 \tag{26}$$

where the lagrangian

$$L^r(X_{ilk}) = f(u^r) + \sum_i \sum_l \sum_k \mu_{ilk}^{1,r} [\, t_{ilk}^r - U X_{ilk} \,] \tag{27}$$

$$+ \sum_i \sum_l \sum_k \mu_{ilk}^{2,r} [\, X_{il,k-1}^r - X_{il,k-1} \,] + \sum_j \sum_l \sum_k \mu_{jlk}^{3,r} [\, X_{jlk}^r - X_{jlk} \,]$$

and $f(u^r)$ is our objective function with all continuous variables u^r fixed and $\mu_{ilk}^{1,r}$, $\mu_{ilk}^{2,r}$, and $\mu_{jlk}^{3,r}$ are the Lagrange multipliers of constraints (14), (17a) and (17b), respectively, in the NLP solution of Step 2.

Step 4.   If $f^L = f^U$, stop. Otherwise set $R = R + 1$, and return to Step 2.

9. NLP SUBPROBLEMS

The solution to the original NLP in Step 2 of our solution procedure could be difficult due to the large number of continuous variables that it contains. However, as we have already mentioned, this NLP subproblem contains the constraint set (17) which has a unique solution once the right hand side is fixed according to (2). In addition, it follows from (2) that exactly

one product must be assigned to each time slot. As a result, exactly one of the corresponding processing times and exactly one of the corresponding production amounts may be nonzero. The rest will be forced to zero by (14) and (22). We are led to the conclusion that instead of solving the original NLP we can solve a reduced one. The original NLP, where we have eliminated the variables W//£ from the formulation using (22), is as follows:

Model NLP

$$
\text{Min} \quad \sum_i \sum_l \sum_k \cdots t_{ilk} \quad \cdots + ZZ2\text{-}Z^s y7\text{-}jJ\text{-} z_{ijlk} \tag{28}
$$
$$
+ \sum_i \sum_l \sum_k m_{il} \, r_{il} \, t \, U_{lk} \sim X \, t_{jil} \, z_{jilk} \, _n l \big]_{j \pm i}
$$

s.t.

$$
T_{il} \, I \, U_{lk} \sim S \, x_{fil} \, z_{jilk} \, J
$$
$$
\sum_l \sum_k \text{———}^\wedge\text{———}^\wedge \, d_f \qquad \forall i \in l! \tag{29}
$$

$$
\sum_l \sum_k \frac{r_{il} \, [\, t_{ilk} - \sum_{j \ne i} \tau_{jil} \, z_{jilk} \,]}{H_l} \le d_i \qquad \forall i \in 12 \tag{30}
$$

$$
H_l = \sum_l \sum_k t_{ilk} \qquad \forall / \tag{15}
$$

$$
H_{lk} \, ^s \, S \, ^T y// \, Z/zV\text{it} \qquad \forall /, \; \forall /, \; \forall A: \tag{31}
$$

$$
H_{lk} \, * \, ^U X_{ilk} \qquad \forall i, \; \forall /, \; \forall A: \tag{14}
$$

$$\%^{Z}ijlk = \quad {}^{x}jlk \qquad\qquad V/, \quad V/, \quad V* \qquad (17a)$$

$$X^{z}ijlk = \quad {}^{x}iljc\text{-}l \qquad\qquad Vi, \quad V/, \quad V* \qquad \textbf{(17b)}$$
$$j$$

$$\mathbf{Z}ijiic > 0, \quad H/ > 0$$

where (31) is used in place of the nonnegativity constraint for the eliminated variables Wj/£ and where:

$$\bar{c}_{il} = \left\{ \begin{array}{ll} c_{il} & if / G11 \\ c_{il} - b_{il}\, r_{il} & if \ i s l_2 \end{array} \right. \qquad 02)$$

$$'_{ijl} = \left\{ \begin{array}{ll} f^{s}//7 & if \,\acute{}e I_1 \\ sy/\acute{} + b// \ r// \ T;V/ & if \ i e I_2 \end{array} \right. \qquad \textbf{(33)}$$

Once the assignment variables are fixed:

$$Zijlk \ = \ Xujc\text{-}1 \ \bullet \ Xjlk$$

$$ink \ = \ 0 \ \ if \ X,7ik \ = \ 0$$

By eliminating these variables from the above program, we obtain the following reduced NLP:

Model rNLP:

$$\text{Min} \quad \underset{l}{2} - 2 - {}^{c}/*7 - iT - \quad + \quad {}_{l}\,\hat{}\,{}_{k}^{\cdot\prime}\quad \hat{}\,\overline{.l_l} \qquad (34)$$
$$+ \ \sum_{l}\sum_{*} {}^{m}/*/\,{}^{T}i*l \ t \ tj*\hat{}\, \sim \ {}^{x}(\eta\!\!:)*l\,]$$

s.t.

$$\sum_{(/*)^*} \frac{''''\_'\_''Z}{H_{/*}} \xrightarrow{7\,t\,\prime} \cdot \mathbf{d}; \qquad V/.I, \qquad (29)$$

$$\sum \frac{r_{il*}\,[\,t_{i(lk)*} - \tau_{\!*}\,*]}{j\cup} \;,\; \mathbf{d}, \qquad V/eI_2 \qquad (30)$$

$$^{H}/= \; \pounds</*/* \qquad\qquad \underset{\bullet}{V/} \qquad (15)$$

$$W^\wedge \qquad {}^T(;7)^*/ \qquad\qquad V/,\;\; VA \qquad (31)$$

$$H/ \;>\; 0$$

where we have used the notation:

for any time slot (l,k): $i^* = i$ such that $X//\pounds = 1$

$('^*/>*/^*= (ij)lk$ such that $X//^\wedge\text{-}/ \cdot Xjik = 1$

for any product (i): $I^* =$ any $/$ such that $X//\pounds = 1$ for any k

$k^* =$ any k such that $X//\# = 1$ for any $/$

$(l,k)^* =$ any (l,k) such that $Xj/jfc = 1.$

According to the suggestions of Sahinidis and Grossmann (1989), in order to find the dual solution of problem (NLP) given the primal and dual solution of problem (rNLP) we need to exploit the Kuhn-Tucker optimality conditions for problem (NLP). The Lagrangian for this problem is:

$$\mathcal{L} = \sum_i \sum_l \sum_k \bar{c}_{il} \frac{{}^{\dagger\cdots}}{H_l} \;+\; \sum_i \sum_j \sum_l \sum_k \bar{s}_{ijl} \frac{Z_{ijlk}}{H_l}$$

$$+ \mathbf{XZE}^m u \; r_{il}\,[\,t_{ilk} - \sum_{j \neq i} \tau_{jil}\, Z_{jilk}\,]$$

$$+ \sum_{i \in I_1} \lambda_i^1 \left[d_i - \sum_l \sum_k \frac{r_{il} \left[t_{ilk} - \sum_{j \neq i} \tau_{jil} z_{jilk} \right]}{H_l} \right]$$

$$+ \sum_{i \in I_2} \lambda_i^2 \left[\sum_l \sum_k \frac{r_{il} \left[t_{ilk} - \sum_{j \neq i} \tau_{jil} z_{jilk} \right]}{H_l} - d_i \right]$$

$$+ \sum_l \lambda_l^3 \left[H_l - \sum_i \sum_k t_{ilk} \right] + \sum_i \sum_l \sum_k \lambda_{ilk}^4 \left[\sum_{j \neq i} \tau_{jil} z_{jilk} - t_{ilk} \right]$$

$$+ \sum_i \sum_l \sum_k \mu_{ilk}^1 \left[t_{ilk} - U X_{ilk} \right] + \sum_i \sum_l \sum_k \mu_{ilk}^2 \left[\sum_j z_{ijlk} - X_{il,k-1} \right]$$

$$+ \sum_j \sum_l \sum_k \mu_{jlk}^3 \left[\sum_i z_{ijlk} - X_{jlk} \right] - \sum_i \sum_j \sum_l \sum_k v_{ijlk} z_{ijlk}$$

where λ^1 (≥ 0), λ^2 (≥ 0), λ^3, λ^4 (≥ 0), μ^1 (≥ 0), μ^2, and μ^3 are the dual variables for constraints (29), (30), (15), (31), (14), (17b) and (17a), respectively, and v (≥ 0) are the dual variables associated with the nonnegativity of the transition variables (Z). Therefore the Kuhn-Tucker condition $\nabla L = 0$ gives:

$$\frac{\partial L}{\partial t_{ilk}} = \frac{\bar{c}_{il}}{H_l} + m_{il} r_{il} - \lambda_i^1 \frac{r_{il}}{H_l} \bigg|_{i \in I_1} + \lambda_i^2 \frac{r_{il}}{H_l} \bigg|_{i \in I_2} - \lambda_l^3 - \lambda_{ilk}^4$$

$$+ \mu_{ilk}^1 = 0 \qquad\qquad (35)$$

$$\frac{\partial L}{\partial z_{ijlk}} = \frac{\bar{s}_{ijl}}{H_l} - m_{jl} r_{jl} \tau_{ijl} + \lambda_j^1 \frac{r_{jl} \tau_{ijl}}{H_l} \bigg|_{j \in I_1} - \lambda_j^2 \frac{r_{jl} \tau_{ijl}}{H_l} \bigg|_{j \in I_2} + \lambda_{jlk}^4 \tau_{ijl}$$

$$+ \mu_{ilk}^2 + \mu_{jlk}^3 - v_{ijlk} = 0 \qquad\qquad (36)$$

where $\dot{A}\big|_B$ means that the term A should be included only if condition B holds true.

We also have to satisfy the following complementarity condition:

$$z_{ijlk} \, v_{ijlk} = 0 \qquad\qquad (37)$$

Once problem (rNLP) is solved, the values of A^{1}, k^{2} and $X\$$ are known. The same is true for the primal solution (t, H) of (NLP). Then we can solve (35) to (37) for the dual variables μ_i, in two steps:

In the first step, we solve (35) for p^{1}:

$$vk \sim {}^*lik = -| - m\ m\ +\ {}^*/\ j\pounds\ Ifei!\ -\ tf\ {}^\wedge\ 1_{fe}i_2\ +\ {}^*/^{3} \qquad (35a)$$

Both $\backslash\backslash j$ and $A4$ must be nonnegative. Depending on whether the right hand side of (35a) is negative or positive, we will set p^* or $X^\wedge$ equal to zero, respectively. Satisfying complementarity for constraint (14) is not an issue since thqse constraints of (14) which were not included in (rNLP) are always active ($0 = 0$). In this way the dual variables (I^* for the variable $uppei^{4}$ bounding constraints can be found from (35a) by simple substitutions.

The second step is to compute μ^{2} and $JJ3$ from (36). Let

$$q_{ijlk} = \frac{\bar{s}_{ijl}}{H_l} - m_{jl}\ r_{jl}\ \tau_{ijl} + \lambda_j^1\ \frac{r_{jl}\ \tau_{ijl}}{H_l}\ \Big|_{j\in I_1} - \lambda_j^2\ \frac{r_{jl}\ \tau_{ijl}}{H_l}\ \Big|_{j\in I_2} + \lambda_{jlk}^4\ \tau_{ijl}$$

This is a known parameter after the solution of (rNLP) and the calculation . The problem of computing ($I^2{}_{and}\ ^\wedge 3{}_{now}$ becomes that of solving the following system:

$$Vilk^{+}\ {}^{2}{}^*jlk \sim {}^{3}{}^vijlk + {}^*ijlk = {}^\circ \qquad (38a)$$

$$Z_{ijlk}\ {}^vijlk = {}^{\circ}> \qquad {}^vijlk * {}^\circ \qquad (38b)$$

The system (38) is to be used to calculate the dual variables of constraints (17). Constraints (17) have special structure: the system decomposes to independent transportation problems for each time slot (/, k). In fact, the system (38) can also be decomposed to independent problems for each time slot. Each of these problems is the Kuhn-Tucker conditions for the following transportation problem:

$$\text{Min} \quad \sum_i \sum_J q_{ijkl}\ z_{i\ kl} \qquad (39)$$

s.t.

$$\sum_i Z_{ujlk} = \tilde{x}_{jlk} \qquad\qquad \forall y \qquad\qquad (40a)$$

$$\sum_j Z_{ijlk} = X_{il,k-1} \qquad\qquad \forall i \qquad\qquad (41b)$$

Since we know the primal solution to this problem, we only need to apply any of the efficient methods that solve transportation problems in order to find the dual solution. The application of the (uv) method (Bazaraa and Jarvis, 1977), gives the following procedure:

Do for each time slot (/, k):

1. Say $X_{m}i_{,}^{\wedge}{}_{-}.j = X_w j\pounds = 1$. Then $Z_{mn}i_k = \backslash$ is the only nonzero basic variable.

2. Set $\backslash i_{nlk} = 0$. Then $v_{mnlk}=0$ requires $\overset{3}{\wedge} = \overset{2}{-q_{mnlk}}$.

2. Set $\backslash i_{nlk} = 0$. Then $v_{m/1/Jk}=0$ requires $\overset{3}{\wedge} = \overset{2}{-q_{mn/}}^{*}$.

4. For all $j^* n$, we now need (for all /) $v_{\mathbf{y}}.^{\wedge} \geqslant 0$ which implies $|!_{\mathbf{r}}-^{\wedge} \geqslant - (\overset{3}{M}-/^{\wedge} {}^{+} \overset{2}{Qy}\!fc)\bullet$

   This can be satisfied by letting $(\mathbf{j} -^{\wedge} = \max_i [- ((\overset{3}{i}^{\wedge} + \overset{2}{q}^{\wedge}))]\bullet$

Enddo.

The problem has only one nonzero basic variable and, in Step 3, we make basic all the variables in the transportation tableau which correspond to the column with the nonzero entry. This is achieved by setting their reduced cost equal to zero. Then, in Step 4, we have to evaluate the remaining dual variables in such a way that all the reduced costs (v) are nonnegative. Notice that the role of i and j in the above procedure may be interchanged, thus giving rise to an alternative dual solution. In this way we were able to generate two Benders cuts during each iteration of the solution procedure.

The solution of the reduced NLP problem as described above never becomes a problem. The number of constraints and variables in model (rNLP) is respectively equal to the number of products plus the number of lines (NP+NL). Therefore, it is easy to solve. Some difficulties with infeasibilities were resolved by including slack variables in constraint (5) and incorporating these variables in large penalty terms in the objective function. Finally, as shown

in Appendix B, this NLP is convex for the case where the processing rates for all products in a line are the same and it is equivalent to a linear program for the case where there are no inventory holding costs.

10. MILP MASTER PROBLEMS

Contrary to the NLP subproblem, the MILP in Step 3 is much harder to solve. Although it has the desirable feature that the assignment variables $(X_i/\pounds)$ belong to Special Ordered Sets of type 1 (see Beale, 1979), applying special procedures while searching the branch and bound tree may still require a long time to solve it to optimality. For this reason, we have decided to use a variant of Benders decomposition proposed by Geoffrion and Graves (1974) and not to solve this problem to optimality, but only until we obtain a feasible solution which is lower than the current best upper bound. This is achieved by including an upper bound for $r\backslash$ in the master problem and by requiring the optimization package to terminate the search as soon as it finds a feasible integer solution. In this way, we take advantage of the fact that commercial MELP codes usually find the optimum in a short time, but they take an extremely long time to verify its global optimality. In this case, the procedure in Section 8 should be terminated as soon as the master problem becomes infeasible.

Another problem with the algorithm is that, although it has the advantage of involving an integer programming problem with only one continuous variable Tl in the master problem of Step 3, it has the disadvantage that this problem is often too relaxed. As a result, the algorithm has the tendency to initially yield very low values for the lower bound/$^\wedge$, and predict binary combinations which yield infeasible subproblems, and hence requires a large number of iterations. However, we can increase the computational effectiveness of the algorithm by directing the master problem to avoid binary combinations that will lead to infeasible NLP subproblems, or binary combinations that will lead to "equivalent" solutions.

In order to avoid some infeasible NLP subproblems, we include the following constraint in the master problem:

$$\sum_l \sum_k X_{ilk} \geq 1 \qquad\qquad \forall i \in I_j \qquad\qquad (42)$$

which will ensure that all products are assigned to at least one time slot. If that were not true, some of constraints (5) would be violated. In addition, we can include an outer-approximation of (5) into the master problem as follows:

$$r_{il} \, TF_{il} \geq d_{il} \qquad\qquad \forall i \in I_1 \qquad \forall l \qquad (43)$$

$$\sum_i TF_{il} \leq 1 \qquad\qquad \forall l \qquad\qquad (44)$$

$$TF_{il} \leq \frac{d_i}{r_{il}} \sum_k X_{ilk} \qquad\qquad \forall i \qquad \forall l \qquad (45)$$

$$\sum_l d_{il} = d_i \qquad\qquad \forall i \qquad\qquad (46)$$

where TA_{il} is a variable to approximate the fraction of the production time of line l which is devoted to production of product i, and d_{il} is a variable to approximate the amount of the demand rate which is satisfied by production in line l. By including (45) we express the fact that a product cannot be produced unless it is assigned. The upper bound in the right hand side of (45) is derived by assuming that all the demand is satisfied by production in one line. According to (44), the time assigned to all products cannot exceed the total available. Constraint (43) is then a relaxation of (5) where the changeover times have been neglected; it enforces the condition that demand must be satisfied. In equation (46) we are simply defining the variables d_{il} in terms of the total demand rate d_i.

There is some multiplicity in our model in the sense that some binary assignments give rise to exactly the same alternative optimal NLP solutions. Consider for example the case of a product being assigned in two successive time slots. If the processing times in this NLP solution are t1 and t2, it is clear that by allocating the total time t1+t2 in any feasible way to the two intervals, produces an alternative optimum. Therefore, we can avoid binary combinations that assign the same product to successive time periods. This can be done by including the following constraint in our master problem:

$$X_{il,k+1} \leq 1 - X_{il,k} + X_{il,NT} \qquad \forall i, \ \forall l, \ \forall k \qquad (47)$$

As a result a product cannot be assigned to two successive time slots unless it is assigned to the last (NT) time slot in the cycle.

Finally, a small note about the variable upper bounding constraints (14). Theoretically, it is sufficient to use any large value for U that does not limit t_{ilk} to become as large as it might be profitable. On the other hand, it should be obvious that the larger the U, the smaller the values of X_{ilk} in the relaxation of the model where the assignment variables are relaxed in the

interval [0, 1]. Therefore, in order to obtain a strong relaxation, the upper bounds should be made as small as possible. A valid upper bound can be derived by assuming that, when a product is assigned to a time slot, all its demand is going to be satisfied by production in that slot and by neglecting the transition times:

$$^T U\ ^l_{ilk}\ =\ ^d_i\ ^H/$$

and therefore

$$Hlk\ *\ \frac{d\cdot H}{-^{\pm}TiT} \tag{48}$$

where are upper bounds H^ for the lengths of the cycle times. The right hand side of (48) can be used for U in (14).

Our computational experience has indicated that using these bounds leads to stronger lower bounds in Step 3, while constraints (42) to (46) eliminate most of the infeasible subproblems which otherwise were outnumbering by far the feasible ones. On the other hand, constraints (47) create difficulties due to their large number; their use was abandoned after some computational experiments indicated this difficulty. By solving the master problem suboptimally, we were able to reduce the computational requirements for the solution of this problem by several orders of magnitude with respect to the case where master problems are optimally solved. At the same time the number of iterations required by decomposition procedure to converge did not change. Finally, another order of magnitude reduction in the solution of the master problem is achieved when special procedures are utilized for the branch and bound search based on the existence of the Special Ordered Sets (constraint (2)).

11. Computational Results

We have used the modeling system GAMS (Brooke *et aL,* 1988) in order to implement the solution procedure. The solvers used were MPSX (IBM, 1979) for the MILP master problems and MINOS (Murtagh and Saunders, 1986) for the NLP subproblems. The procedure was executed on an IBM-3090 at the Cornell Supercomputing Center.

11.1 Example

The first example considered involves the production of 3 products (A, B and C) in one line. The transition times are the ones used in the example problem of Appendix A. The production rates for the three products are 12, 15 and 10 Klb / day. The production costs are 0.76, 0.75 and 0.77 $ per lb. The transition costs are 10% of the production cost and that

includes the production cost during the transition time. The inventory holding cost is 0.0306 $ per Klb per hour and there is no profit boost. Finally, the product demands are 50%, 20% and 30% of the total demand of 10,000 Klb during a scheduling horizon of 8280 hrs   It is desired to determine a common cycle solution for the problem.

The resulting MINLP involved 9 binary variables, 46 continuous variables and 43 constraints. The solution to this MINLP was obtained in 2 iterations of the algorithm and indicates an optimum cost of 935 $ / hr when the cycle time is 102 hr. The optimum production sequence is A-(B)-C-B where the parenthesis indicates that B is not produced but used only as an intermediate product in order to induce a cheap transition between A and C.

11.2   A Real World Problem

The solution technique was also applied for the scheduling of 26 polymer products (25 existing, 1 dummy new) in 3 parallel plants that belong to a large chemical company. The scheduling horizon was 1 year. The resulting MINLP involved 780 binary variables, 23,000 continuous variables and 3,200 constraints.

The solution obtained in 20 CPU minutes on an IBM-3090 and 20 iterations is shown in Fig. 7. It indicates that the optimum cycle times are close to 2 months and the associated cost 44 m$/yr. The company has been until now operating with cycle times of 1 month. By constraining the cycle times in the model to be less than a month, the new solution indicated that the best possible schedule in this case has an increased cost of 52 m$/yr. Table 1 gives a summary of costs for the unconstrained and the constrained case. It is obvious that by allowing the cycle time to be larger than one month, the transition costs can be reduced. The inventory holding cost is increased but the fewer transitions provide extra capacity in order to produce a larger amount of new products. In fact, the profit boost from the production of the new products outbalances the increase in the inventory holding cost when the cycle time is increased.

We have also analyzed the possibility of making a capital investment of 3 m$ in order to increase the production rates of one of the lines. The optimum cost after these technological changes in the plant was found to be 38 m$/yr, indicating that the proposed investment plan has a payout time of 8 months. This shows that the model can be used as a planning tool in order to evaluate whether proposed equipment changes are worthy.

12.   Conclusions

The problem of multi-product scheduling on continuous parallel production lines has been discussed in this paper. It has been shown that this is a large combinatorial problem, with nonlinearities involved and where the major trade-off is between inventory and transition costs. A rigorous MINLP model has been developed for determining cyclic schedules for the case of constant demand rates for all products. Sequence dependent transitions are rigorously taken into account and production times are not discretized in this model. An exact linearization technique was applied to linearize this MINLP in the space of the integer variables. The Kuhn-Tucker optimality conditions were exploited in the resulting model in order to effectively apply Generalized Benders Decomposition. This avoids the explicit solution of extremely large nonlinear subproblems. At the same time, a computational scheme was proposed to strengthen the bounds of the master problem and therefore achieve fast convergence. Finally, the effectiveness of our solution technique has been demonstrated through the solution of a large-scale real world problem.

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Appendix A: Calculation of Shortest Transitions

Even in the common cycle case where a product is not allowed to be produced more than once in the cycle, it may be beneficial to assign it to more than one time slot only in order to induce cheaper - indirect - transitions. What becomes a major issue for the optimization problem is then the number of time slots postulated for each cycle. On the one hand, this number should be as small as possible in order to keep the number of integer variables small. On the other hand, it should be large enough so that schedules with many repetitions of products in the same cycle are not excluded if they are to provide a lower cost.

However, the problem of finding the shortest indirect transitions between products can be solved prior to the optimization step. This can be done by finding the shortest paths between all pairs of nodes in a graph whose vertices correspond to the products that can be produced in a line. The lengths of the arcs of this graph are the transition times for the direct transitions. The solution will give the shortest transitions between all pairs of products. Consider as an example the case where we have three products A, B and C. Let the transition times be the following:

	A	B	C
A		2	6
B	2		3
C	6	3	

The transitions graph is shown in Fig. A-1. In order to find the shortest transitions between all products we need to find the shortest paths between all pairs of nodes in this graph. The problem of finding the shortest paths between all pairs of nodes in a graph can be solved in polynomial time. Floyd's algorithm (1962) requires $O(N^{\wedge})$ operations where N is the number of vertices in the graph (= number of products). By letting $T=(x_z y)$ be the transition matrix, the algorithm is the following:

[11] For m from 1 to N, for all i and j from 1 to N, do: $1^{\wedge} = \min(x^{\wedge}, Tj_m + T_{w,j})$. M.

In this way, the table of shortest transition times for the three product example is found to be:

	A	B	C
A		2	5
B	2		3
C	5	3	

where now the transition between A and C is only 5 hrs and it requires going through B.

By using this procedure the transition matrices can be replaced by the shortest transition matrices where now the elements of these transition matrices satisfy the triangle inequality property. However, one such transition may now imply shifting the production through some intermediate products. Finally, we should mention that, in the case where the transition cost is not proportional to the transition time, a more appropriate choice for the lengths of the arcs of this graph would be the sum of the transition cost and the opportunity profit which is lost due to the transition time (profit boost times transition time).

Appendix B: On the convexity of the Nonlinear Program

The convexity of the NLP that appears as a subproblem in our decomposition approach is an important issue since it determines whether or not a globally optimal solution can be guaranteed for the problem at hand. We first show how to perform a variable transformation that leads to a problem with only one nonlinear term in the objective and linear constraints. Then we examine the convexity of the transformed problem.

The NLP subproblem that needs to be solved at each iteration is the following:

$$\text{Min } f = \sum_i \sum_l \sum_k c_{il} \frac{t_{ilk}}{u} + \sum_i \sum_j \sum_I \sum_k s_{ijl} \frac{Z_{ijlk}}{H_l}$$

$$+ \sum_i \sum_l \sum_k {}^m_{il} {}^w_{i7*} \quad {}^{''} \quad \underset{i \in I_2}{I} \underset{l}{II} \underset{k}{} {}^b_{il} -\frac{W/ft}{TT}_l \tag{21}$$

S.t.

$$\underset{i}{*L^Z_{ijlk}} = {}^X_{ilk} \qquad V/, \qquad V/, \qquad V* \tag{17a}$$

$$\sum Z_{ijlk} = {}^x//,/t\text{--}1 \qquad\qquad V/, \ V/, \ V/k \tag{17b}$$

$$\wedge \ \wedge \ {}^u \ {}^x//yt \qquad\qquad Vi, \ V/, \ V* \tag{14}$$

$${}^H/ = XSt/Z/fe \qquad\qquad V/ \tag{15}$$

$$/ \ k$$

$$\cdot \ {}^w///t = {}^r//1 \ Uik \sim \underset{j \neq i}{I} V/^z ; «*] \qquad v/, \ v/, \ v \wedge \tag{22}$$

$$\sum_{l}\sum_{k} \frac{W_{ilk}}{11_{I}} \wedge d, \qquad\qquad V/eli \qquad\qquad (5)$$

$$\mathbf{II}_{l\ k} \quad \wedge\, \mathbf{d}\,; \qquad\qquad\qquad \mathbf{V/.I_2} \qquad\qquad \mathbf{(6)}$$

$$Zijlk \ * \ 0, \quad \not{t\!\!/}\!\!\not{k} \geqslant 0, \quad Wilk \geqslant 0, \quad H/ > 0 \qquad\qquad (C\text{-}l)$$

Constraints (17) have a unique solution $Z\!\!\!/M/\pounds = {}'X\text{-}Uk\!\!-\!l^{\wedge}\text{-}jlk$ $^{an\wedge}$ can therefore be used to eliminate the Z-variables from the formulation. Let us then eliminate constraints (17) and consider the Z's are given constants. Furthermore, let us define the following new variables:

$$\_\ Hlk \qquad\qquad \_\ 1 \qquad\qquad \_\ {}^{w}i7*$$

Then the NLP subproblem becomes as follows:

$$\text{Min} \quad \wedge = \sum_{i}\sum_{l}\sum_{k} c_{il}\, \chi_{ilk} \quad + \quad \sum_{i}\sum_{j}\sum_{I}\sum_{k} (\, s_{ijl}\, Z_{ijlk})\, \eta_{l}$$

$$+ \sum_{i}\sum_{l}\sum_{k} m_{il}\, \frac{0)_{ilx}}{T\backslash\!\!\lfloor} \qquad \sum_{i\,e}\sum_{i_2}\sum_{/}\sum_{k}{}^{u}ii\ ^{\wedge}ilfc \qquad\qquad \text{\footnotesize MM '''}$$

s.t.

$$\chi_{ilk} \leq U\ X_{\cdot\!:ilk} \qquad\qquad\qquad \mathbf{vi,\ v/,\ v^*} \qquad \mathbf{(C\text{-}3)}$$

$$\cdot\ 1\ =\ \sum_{i}\sum_{k} XUL_{\cdot\cdot ilk} \qquad\qquad V/ \qquad\qquad (C\text{-}4)$$

$$\omega_{ilk} = r_{il} [\chi_{ilk} - \sum_{j \neq i} (\tau_{jil} Z_{jilk}) \eta_l] \qquad \forall i, \ \forall l, \ \forall k \qquad \text{(C\_5)}$$

$$\sum_l \sum_{it} co_{j7/t} \geq d_f \qquad \text{Vfelx} \qquad \text{(C-6)}$$

$$\sum_l \sum_k \text{»/}_{\text{»}}\text{£<!;•} \qquad \text{v,'61}_2 \qquad \text{(C-7)}$$

$$X_{ilk} \geq 0, \quad co^\wedge \geq 0, \quad TI_Z > 0 \qquad ' \qquad \text{(C-8)}$$

This model involves only linear constraints and one nonlinear term in the objective function (the term corresponding to the inventory cost). Consider now the following cases:

Case I. There is no inventory holding cost (*i.e.* m// = 0). In this case, the nonlinear term in (C-2) vanishes and the NLP subproblem is exactly equivalent to a Linear Problem. Its global solution can therefore be guaranteed in this case.

Case II. By making use of (C-5), the nonlinear term in (C-2) can be written as:

$$\frac{r_{il} [\chi_{ilk} - \sum_{j \neq i} (\tau_{jil} Z_{jilk}) \eta_l]}{\eta_l}$$

$$i \ / \ k \qquad Ti/ \qquad i \ l \ k$$

$$= \ \underset{i \ l \ k}{v \ v \ v} \ , \ <^m U \ ^T{il} \wedge \frac{\chi,nk}{i\backslash i} \text{----constant}$$

$$= \ \sum_l \frac{1}{TI/} \ H \ (m_a \ r_u) \ xuk \ \sim \ \text{constant}$$

$$= \ \sum_l \frac{\alpha_l}{T\backslash l} \ 2X \ X/ifc \ \sim \ \text{constant} \qquad \text{if} \ m_{i7} \ r^\wedge = a^\wedge \ \text{for all}/$$

$$= \ \sum_l \frac{a_l}{r\backslash i} \text{----constant} \qquad \text{by (P-4)}$$

But since $a_{il} \geq 0$, the last term is convex and therefore the global optimal to our problem can be guaranteed in the case the product of the inventory holding cost and the

production rate is the same for all products in a line. This for example will happen if all products have the same production rates and the same inventory holding costs.

We haven't been able to theoretically characterize the convexity properties of the model in any other case. However, computational experiments have indicated that the objective function probably satisfies the convexity requirements for a global optimum since using different starting points has always yielded the same optimal solution.

Table 1. Cost distribution for example 2.

Item	No cycle time restriction	With cycle time restriction
Production	54,341,000	51,153,000
Transition	289,800	471,900
Inventory	6,864,000	4,214,000
Profit Boost	-16,990,000	-6,508,000
Total	$ 44,504,800 / yr	$ 49,330,900 / yr

Schedule (a)

A   B

|————————————————⟦////////////⟧————————————————| ▪ ▪ ▪

trans

Schedule (b)

A   B   A   B

|————i/^yyyyy.772————EZZZZZZZZ————————⟦////////////⟧———| ▪ ▪ ▪

trans   trans   trans

<u>Fia. 1</u>   Two feasible (cyclic) schedules

Fig. 2 Inventory Calculation for alternative schedules

Cycle time 1

Line 1

Cycle time 2

Line 2

TIME SLOT

Trans   Production

- Potential number of time slots postulated for each line
- Length of each slot is a variable
- Assignment of products to slots to be determined

Fig. 3   Model essentials

a) Working Inventory

$$\text{Area} = \frac{W * H}{2}$$

b) Safety Inventory

$$\text{Area} = W * H$$
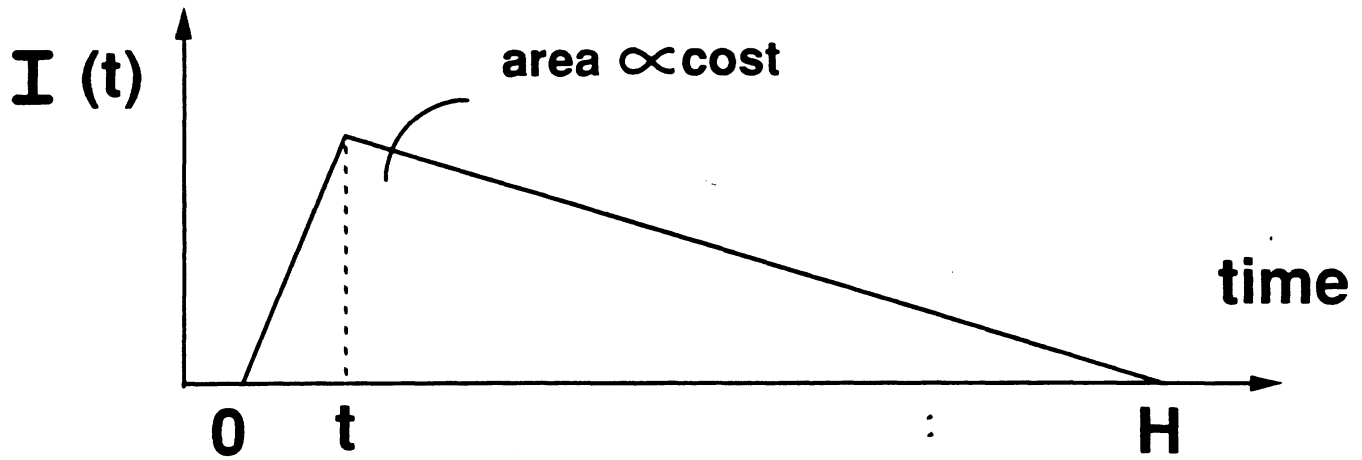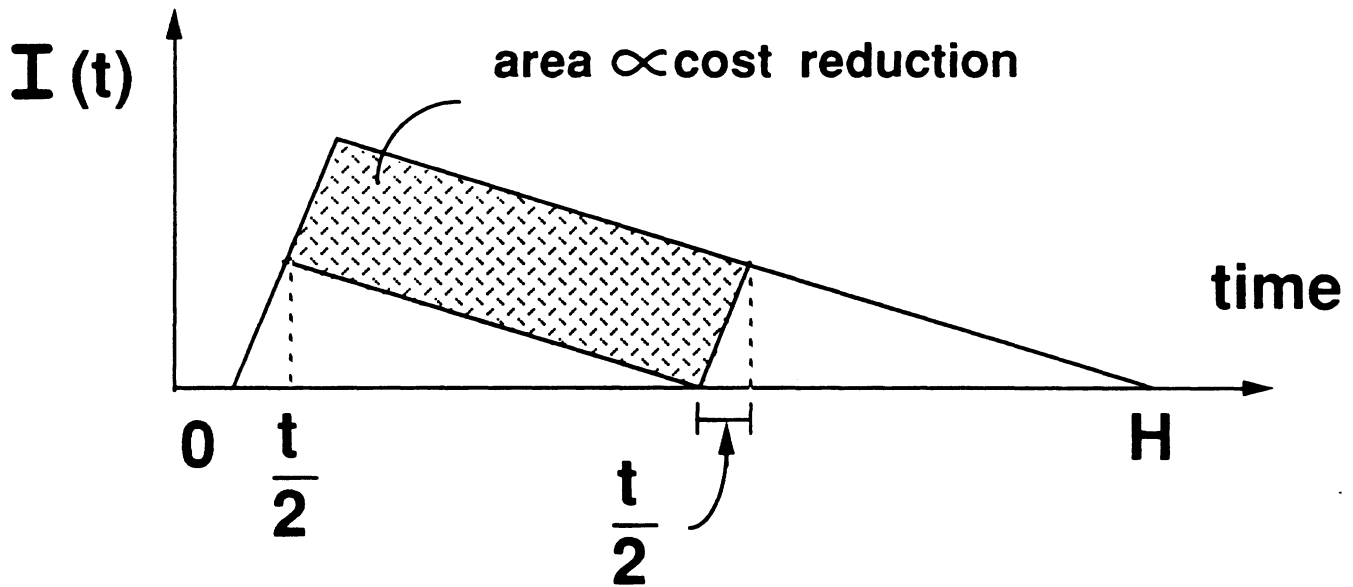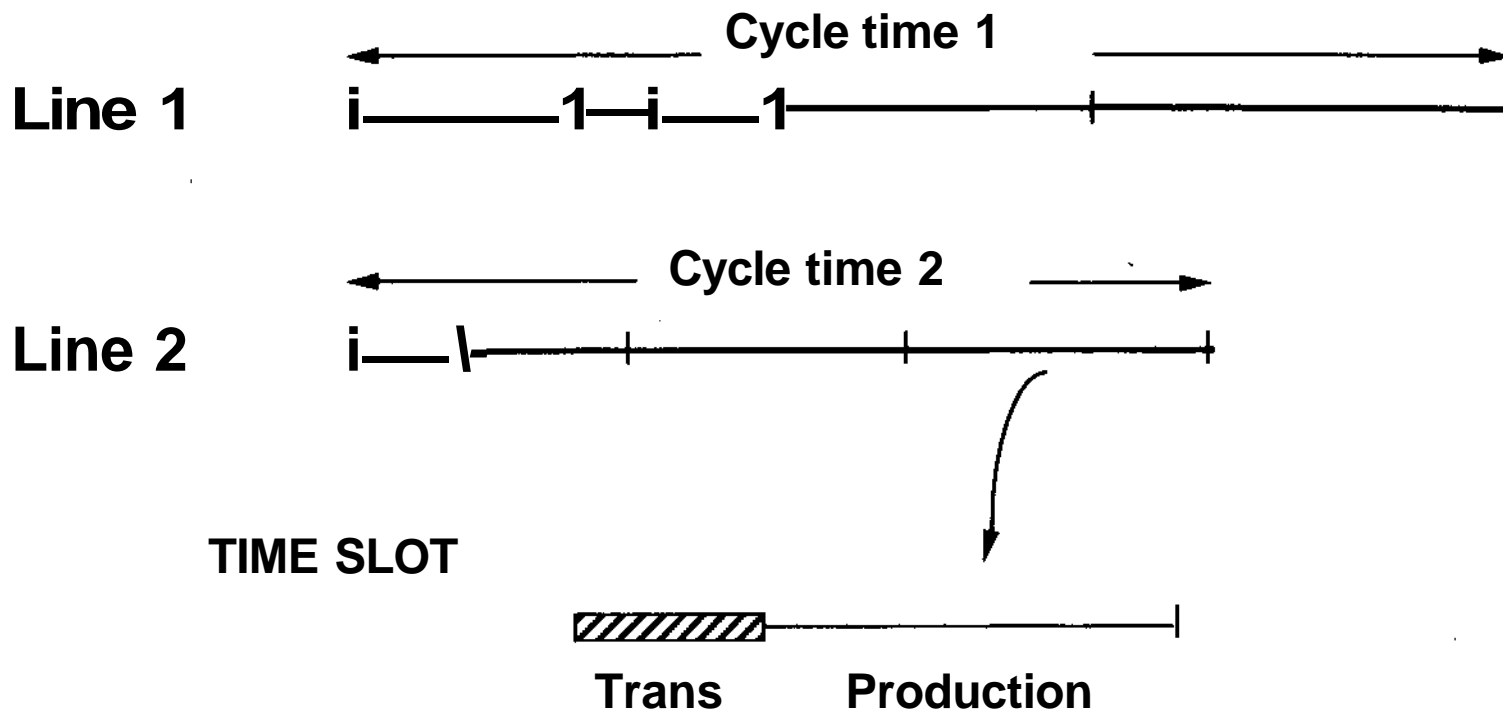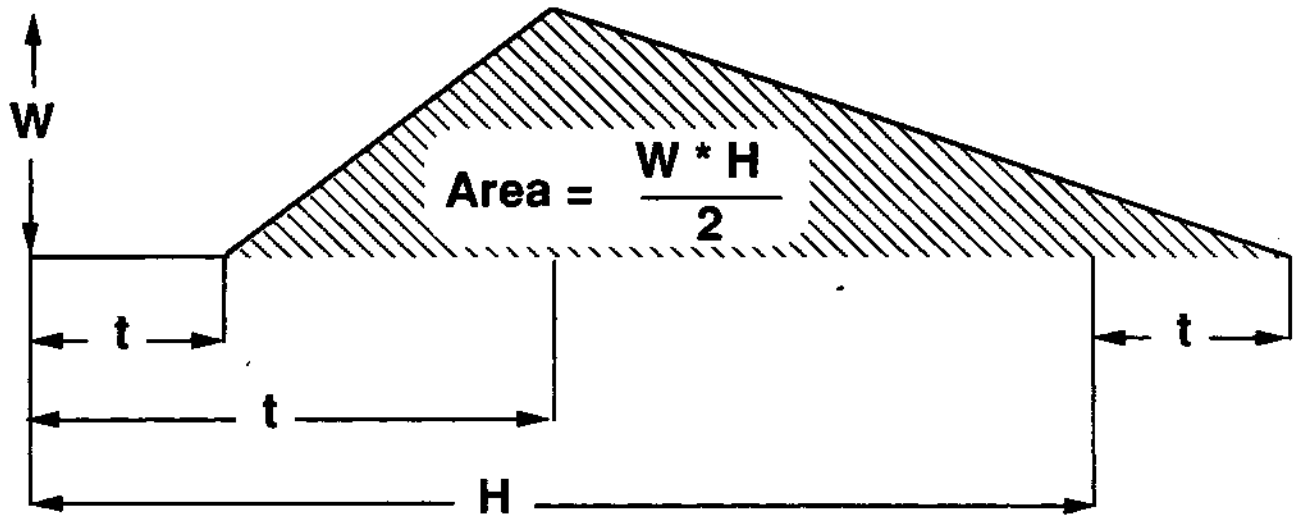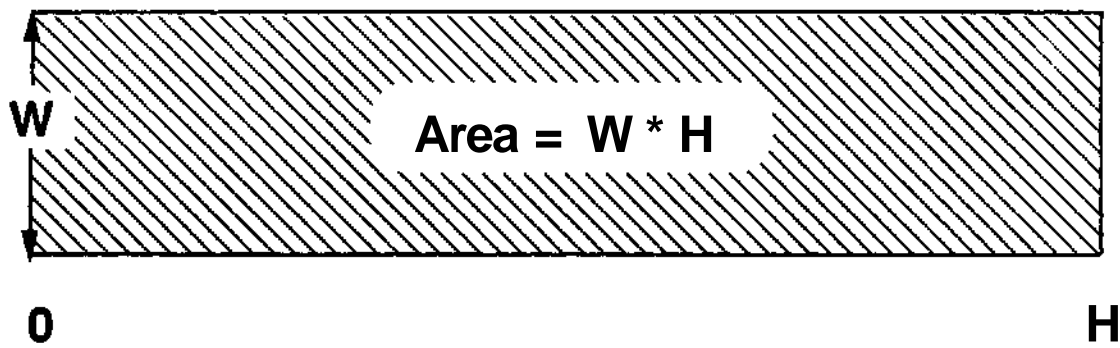
Fig.4   Inventory Cost Calculation
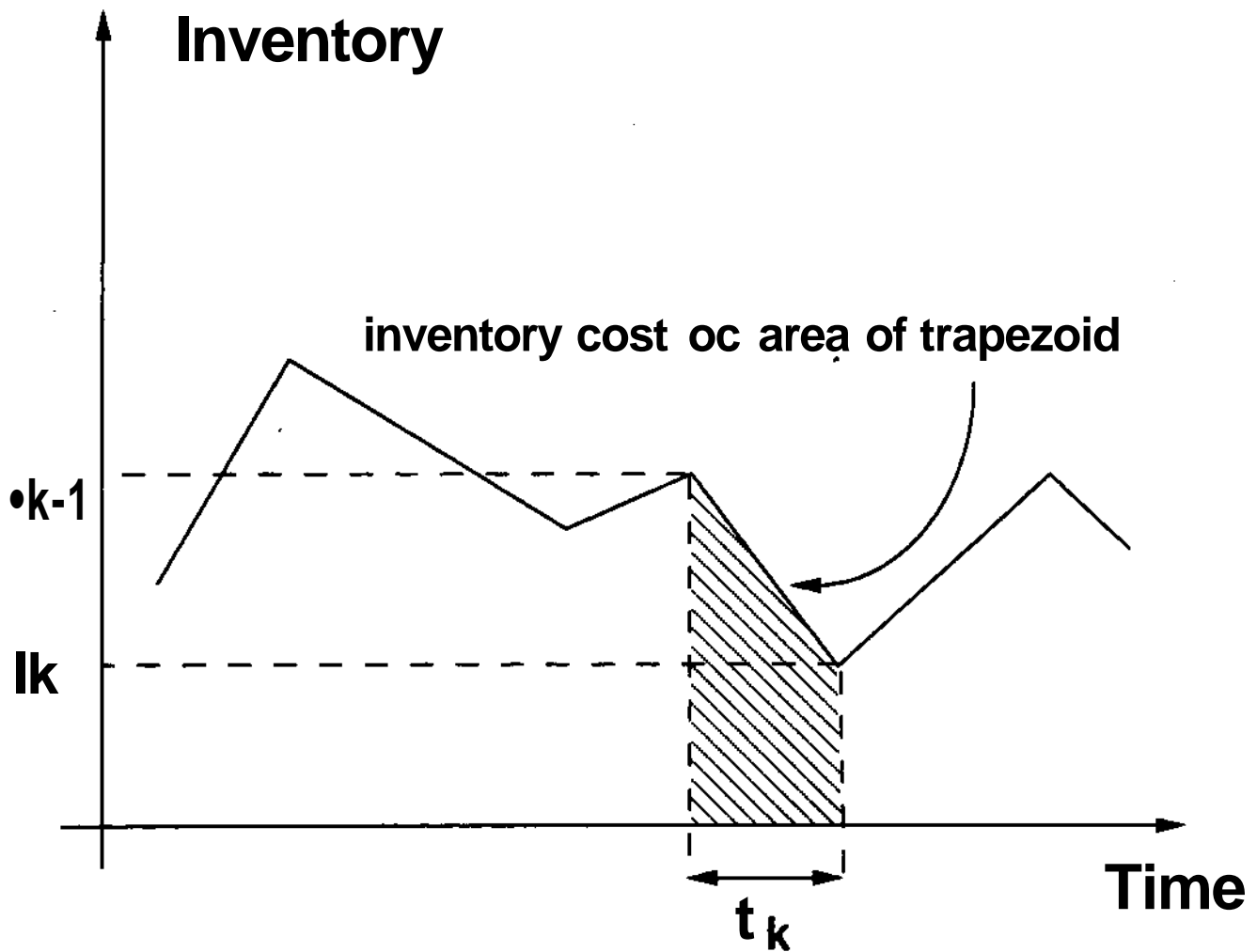
Fig. 5: Inventory changes for the generalized common cycle case
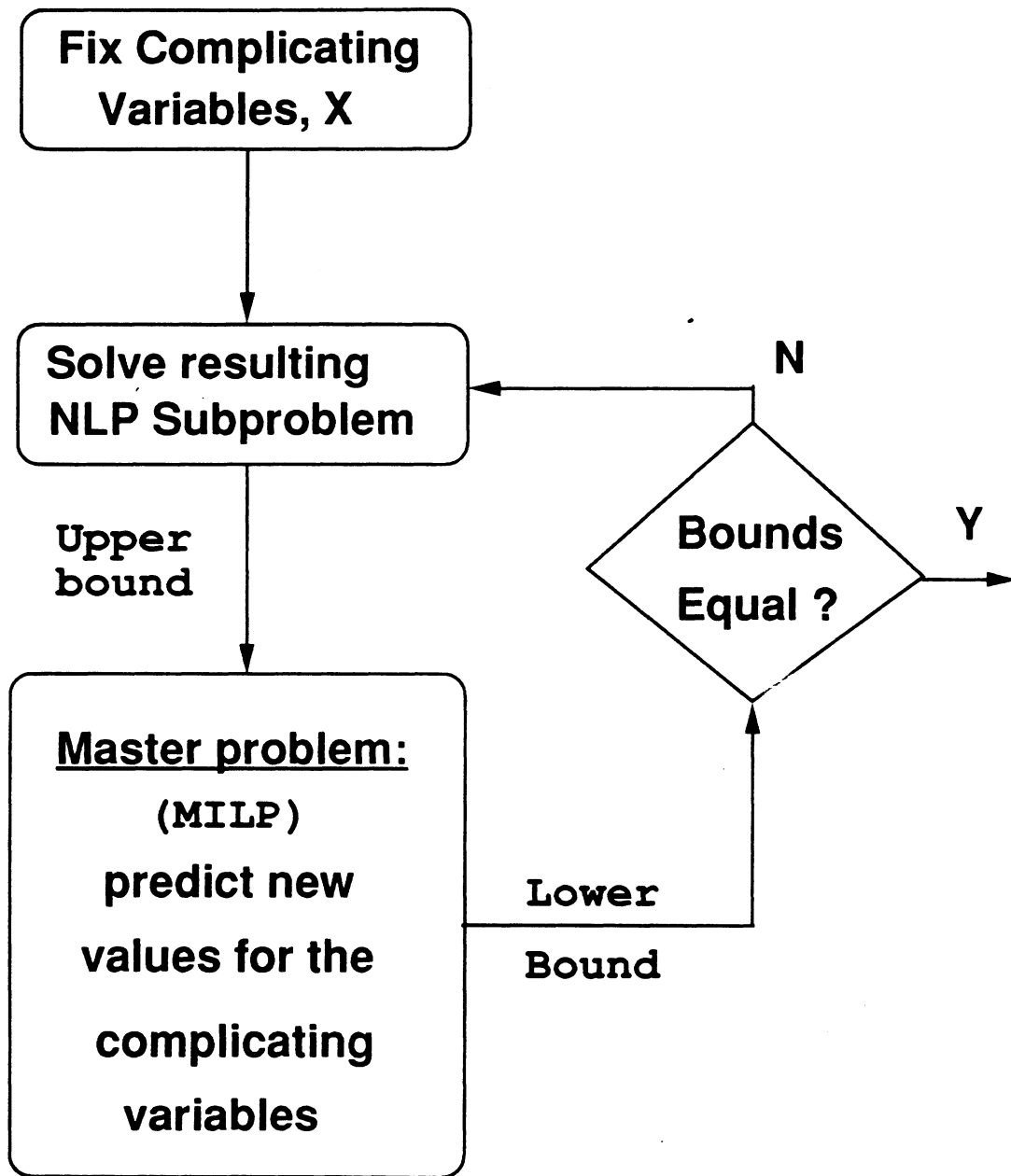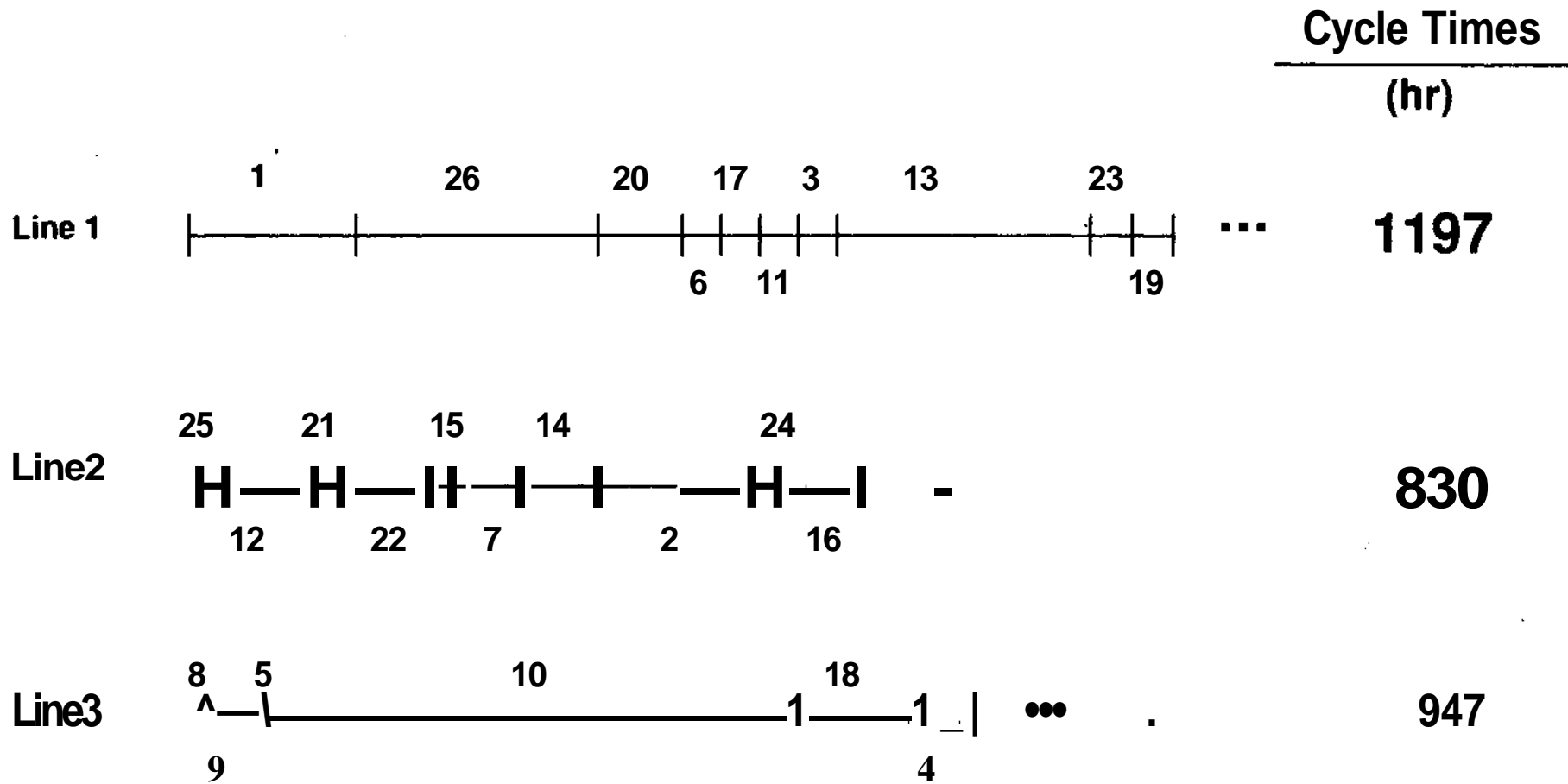
Fig.6 Solution Technique

Fig. 7

OPTIMAL SCHEDULE for the 26 products (1 ... 26)
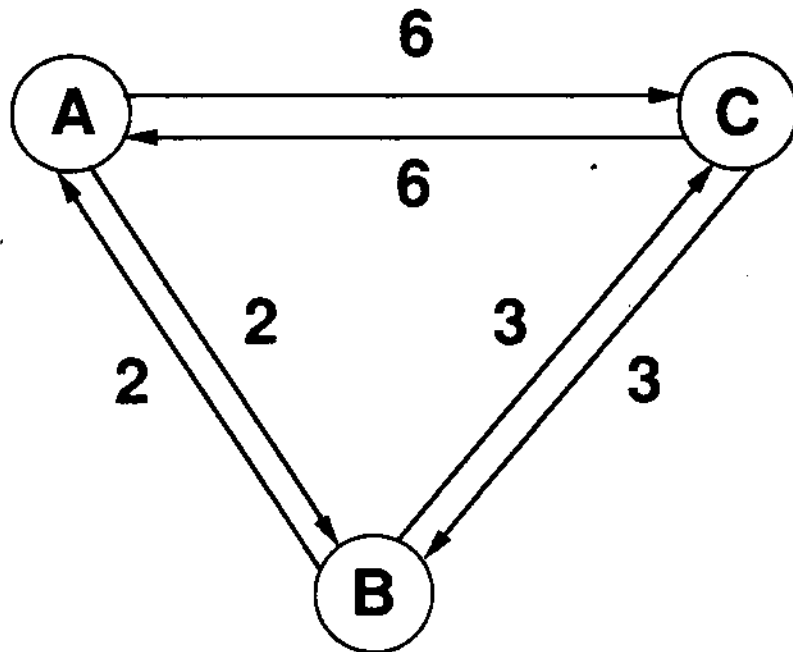
(one cycle shown; lengths of slots proportional to duration)

Fig. A-1: Product graph indicating transition times