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OPTIMAL STRATEGIES FOR GENERAL
PRICE-ADVERTISING MODELS

by

J.-T. Teng and G.L. Thompson

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Jinn-Tsair Teng
University of Bridgeport
Bridgeport, Connecticut
U.S.A.

Gerald L. Thompson
Graduate School of Industrial Administration
Carnegie-Mellon University
Pittsburgh, Pennsylvania
U.S.A.

A general dynamic price-advertising model is developed in which price and advertising variables are control variables whose optimal values are to be determined over a finite planning period of length T . Theoretical results are obtained for the general model. Additional theoretical results which are true in special cases of the model are also derived. It is hoped that this analysis will help to understand past models and to aid the development and understanding of new ones.

INTRODUCTION

Over the past 30 years a number of marketing models have been devised in which price and/or advertising are control variables whose optimal values must be determined over a given planning period of length T . Such models are especially relevant during the introductory phase of a new product when price or advertising changes can affect more or less strongly the "new adopters" and "imitators" among the pool of potential buyers. It is also during this period that the most dramatic effects of the "learning curve" on production costs appear, which also must be taken into account when determining optimal price-advertising strategies.

References to papers containing such models are: Bass [1], Bass and Bultez [2], Doland and Jeuland [3], Dorfman and Steiner [4], Erickson [5], Gensch and Whelam [6,7], Horsky and Simon [8], Kalish [9], Kalish and Lilien [10], Kotowitz and Mathewson [11], Robinson and Lakhani [13], Sethi [15], and Teng and Thompson [16,18]. A survey of such models through 1977 appears in Sethi [14].

Each of these papers contains a theoretical treatment of its particular model, and, as might be expected, there is considerable duplication and overlap in these treatments. A lengthy survey paper would be needed to go into detailed comparisons of these models. However, we do not propose to do that here.

The purpose of this paper is to provide a general framework for and theoretical treatment of price-advertising models which contain as special cases all of the models mentioned in the previous paragraph as well as others yet to be devised.

In Section 2 we present the general model, derive the necessary conditions for its optimal solution, and obtain some theoretical results. In Section 3 we discuss some special cases for which we are able to obtain even stronger theoretical results. Finally in Section 4 we consider the important case in which advertising costs are concave. An Appendix contains the technical details of some of the proofs.

2. MODEL FORMULATION

In this section we formulate the general price-advertising model, and specify assumptions which are appropriate for such a model. Then we characterize the optimal price and advertising strategies for the model.

For many marketing situations, advertising is the most important strategic element in determining the market share achieved by a firm. Price is another important element in a marketing strategy which influences both the demand for and the profit obtained from a product. In addition, the demand of a product is also influenced by the cumulative number of adopters. One such influence is the word-of-mouth effect. Past buyers spread information about the product, thus reduce the uncertainty associated with, and increase the demand for the product. Therefore, we shall consider the growth of the demand for a product as a dynamic function of price, advertising, and cumulative sales volume.

The learning curve phenomenon, which states the labor costs decrease with cumulative output, has been used for many years to project labor costs, especially for new products. More recently, the Boston Consulting Group has observed that in many industries this learning curve phenomenon can be generalized to include total unit cost [17]. They have noted that the total unit cost (in constant dollars) declines by a factor of 10% to 50% each time the accumulated production volume doubles. So, we assume that the marginal cost of production is a non-increasing function of cumulative production volume, which contains as special cases the general learning curve phenomenon. For simplicity, we also assume that the product is a successful one, and thus sales are assumed to be equal to production.

To define a class of monopoly price-advertising models, we first state the following notation:

- T = terminal time
- p = discount rate
- $x(t)$ * cumulative sales at time t (state variable)
- $p(t)$ • prices at time t (control variable)
- $u(t)$ → advertising rate at time t (control variable)
- $C(x)$ • learning curve production cost; we assume $C'_x = dC/dx > 0$.
- $\dot{x}(t)$ - $dx/dt = f(p, u, x)$ (state equation)
 - sales rate at time t is a dynamic function of p , u , and x ; we assume

$$f_p = \partial f / \partial p < 0 \quad \text{and} \quad f_u = \partial f / \partial u > 0$$

$g(u) = ku =$ linear advertising cost

$\lambda(t) =$ current-value adjoint variable at time t which can be interpreted as the increase in future profit resulting from an increase in x at time t . (For a given t it behaves in much the same way as a dual variable in linear programming.)

The discounted profits of the firm at time t are

$$\pi(t) = e^{-\rho t} \{ [p(t) - c(x)] \dot{x}(t) - g(u) \}. \quad (1)$$

We assume that the firm wants to maximize its total discounted profits during the planning horizon which has length T . Hence, the objective function for the firm is:

$$\max_{u,p} V = \int_0^T \pi(t) dt, \quad (2)$$

subject to the constraint

$$\dot{x} = f(p,u,x), \quad x(0) = x_0. \quad (3)$$

The model is a general extension of the models given in Bass [1], Dolan and Jeuland [3], Kalish [9], and Robinson and Lakhani [13], in which the sales rate f is a function of price and cumulative sales only, and also in Horsky and Simon [8], Teng and Thompson [16] who considered the sales rate to be a function of advertising and cumulative sales only. The model is also applicable to the models of Erickson [5], Kalish and Lilien [10] and Thompson and Teng [18].

2.1 Optimal Solutions

We now apply the maximum principle to the model to characterize the optimal policy. To apply to Pontryagin's Maximum Principle [12], we formulate the current-value Hamiltonian using the adjoint variable as follows:

$$H = [p(t) - C(x) + \lambda(t)] f(p,u,x) - g(u) \quad (4)$$

where the current-value adjoint variable $\lambda(t)$ satisfies the following differential equation:

$$\begin{aligned} \dot{\lambda} &= \rho \lambda - \partial H / \partial x \\ &= \rho \lambda - (p-C+\lambda) f_x + C_x f, \quad \text{and} \quad \lambda(T) = 0. \end{aligned} \quad (5)$$

By differentiating H with respect to p , and setting the result to zero, we derive the optimal price rule:

$$\begin{aligned} H_p &= \partial H / \partial p = (p-C+\lambda) f_p + f = 0 \\ \Rightarrow p^* &= C - \lambda - f / f_p \end{aligned} \quad (6)$$

$$\text{or} \quad p^* = \eta(C-\lambda) / (\eta-1), \quad (7)$$

where n is the elasticity of demand with respect to price, i.e.

$$n = -f_p^* p / f.$$

Similarly, we derive the optimal advertising rate as follows:

$$H_u - 3H/3u - (p-C+X)f_u - g_u - 0. \quad (8)$$

The solution of Equation (8) for u gives the optimal advertising rate u^* .

2>2 Theoretical Results

The general formulation and characteristics of the model help in gaining some insight into the important factors influencing the optimal policies. However, with some specific assumptions about the sales rate (or demand rate), we can obtain stronger results. To prove the theoretical results, we assume from now on that the discount rate, p , is zero. The analysis is conducted in constant dollars, so that while the zero discount rate assumption is not accurate, it is not an unreasonable approximation, except for very high risk investments.

Equations (5) to (8) imply the following theoretical results.

Theorem, (a) The marginal revenue product of advertising over the marginal advertising cost is always equal to the price elasticity, I.e.,

$$(p \backslash) / g_u = n. \quad (9)$$

(b) If the price elasticity n is constant, then

$$\dot{p}^* - dp^*/dt \gg -n(n-1)r^1 f f_x / f_p \quad (10)$$

That is, price increases if $n-1$ and f_x have the same sign, and price decreases if $n-1$ and f_x have opposite signs.

(c) If the demand rate is a function of price and advertising only, then the optimal price is constant and the price elasticity is greater than 1. In fact,

$$P^*(t) = n(n-1)^{-1} C(x(T)) \quad \text{and} \quad n > 1. \quad (11)$$

(d) If the price elasticity is constant and the demand function is separable, i.e., $f(p,u,x) \gg P(p)U(u)X(x)$, then

$$\dot{p}^* = dp^*/dt = -\frac{-nP^2 UX'}{(n-1)P'} \quad (12)$$

$$\dot{u}^* \ll du^*/dt = 0 \quad (13)$$

$$j\dot{s}^* - d^2 p^*/dt^2 = \frac{-nP^2}{(n-1)^2 P'^2} t(x')^2 - X^* X \quad (14)$$

where $x' = dX/dx$, $P' = dP/dp$, and $X^* = d^2 X/dx^2$.

Proof. From Equations (6) and (8), we can easily get

$$\frac{p^* C+X}{v} = \frac{g}{s_u} \frac{It}{u} - \frac{-f}{f_p} \quad (15)$$

or

$$p^* f_u / g_u = n,$$

which is (9). Using Equation (5) and (7), we have

$$\dot{p}^* = n(n-1)^{-1}[\dot{C}-\dot{\lambda}] = -n(n-1)^{-1}ff_x/f_p,$$

if n is constant, which is (10). If the demand rate is a function of price and advertising only, then $f_x = 0$, $i = C^*f$, and $X(T) = 0$. This implies that $X(t) = C(x(t)) = C(x(T))$ and $p^*(t) = nCn-D^*CfrCT) = C(x(T)) = f/f_p$ for all $t \in [0, T]$; therefore

$$n \gg 1 - fC(x(T))/f_p > 1.$$

It is clear that $P(p) = mp^{-n}(t)$ if $f(p, u, x) = P(p)U(u)X(x)$ and the price elasticity n is constant, where m is a constant. The proof of Equation (12) is immediately obtained by substituting $f = PUX$, $f_x = PUf'_i$ and $f_p = P'UX$ into Equation (10). The detailed proofs of Equations (13) and (14) are very complicated, and are presented in the Appendix.

Remark 1» Equation (9) is a general extension of the results in Dorfman and Steiner [4], and Erickson [5]; in these references $g_u = 1$.

Remark 2. The result in (c) which says that if demand is a function of price and advertising only, then the optimal price is the constant given in (11), is very surprising* It provides some justification for the constant price assumption made by Gensch and Welam in their market segment model [6, 7].

Remark 3. The result in equation (13) is also very surprising and deserves addition discussion. If $U(u) = ku$ i.e., linear, then the optimal advertising policy is bang-bang (see Teng-Thompson [16]); equation (13) still holds except for the jump points. If $U(u)$ is nonlinear, then (13) implies $u^* = \text{constant}$.

Remark 4. In general, $X^* \wedge 0$ (e.g. $X(x) = x^a$, $0 < a \wedge 1$ for nondurable goods, or $X(x) = Y_1(M-x) + Y_2(M-x)x$ for durable goods, where M is the maximum potential sales (for details see Examples 3.2 and 3.3) and $n > 1$. Then we know that $\dot{p}^* \wedge 0$ by (14), so the optimal price is a concave function of t .

3. SPECIAL TYPES OF MODELS

In this section, we apply the above theoretical results to the investigation of several important models, and obtain stronger theoretical results. First, we consider the case where the demand function is separable but not a function of cumulative sales. This situation is characterized by the fact that the rate of price (or advertising) increase is zero if the price elasticity is constant. Second, we study the case in which price is the only control variable and cumulative sales have some effects (word-of-mouth and saturation) on current demand. We shall give theoretical justification for the computational result due to Robinson and Lakhani [13] that the optimal price increases at introduction time for a new product, then decreases if the planning horizon T is long enough. Third, we survey the situation in which the demand rate is a function of advertising and cumulative sales only. Finally, we investigate several models that do not fall into the previous classifications.

3.1 Demand Determined by Price and Advertising Only.

Here, we investigate the situation where both word-of-mouth and market saturation effects are negligible. Therefore, the demand of a product is a function of price and advertising only.

Example 3.1. Consider the following problem:

$$\max_{u,p} \int_0^T \{ [p(t) - C(x)] \dot{x}(t) - u(t) \} dt \quad (16)$$

subject to:

$$\dot{x}(t) = mp^{-n}(t)u^s(t), \quad x(0) = x_0 \quad (17)$$

and

$$\dot{y}(t) = u(t), \quad y(0) = 0, \quad \text{and } y(T) \leq Y \quad (18)$$

where y stands for cumulative advertising expenditure, Y is the maximum advertising budget available, m , n , and β are constants, $n > 1$, and $0 < \beta < 1$. The model is slightly different than Sethi's model [15], in which advertising is the only control variable.

From the parts (c) and (d) of the Theorem in Section 2.2, it immediately follows that both optimal price p^* and optimal advertising rate u^* are constants. Applying the maximum principle, and after integrating and making some algebraic manipulations, we get the following closed form solutions:

$$p^* = n(n-D)^{-1}C(x(T)) \quad (19)$$

$$u^* = \begin{cases} \frac{mg(n-1)^{n-1}}{n^n C^{n-1}(x(T))} & \text{if } u_0 \cdot T > Y \\ Y/T & \text{otherwise} \end{cases} \quad (20)$$

$$\dot{x} = \begin{cases} \frac{n^{n/(\beta-1)} (n-1)^{(\beta-n)/(\beta-1)}}{m^{1/(B-D)} \beta^{\beta/(\beta-1)}} [C(x(T))r^{(\beta-2)/(\beta-1)}] & \text{if } u_0 \cdot T > Y \\ \frac{(n-1)^n}{nV(x(T))} \left(\frac{Y}{\Delta T}\right)^\beta & \text{otherwise.} \end{cases} \quad (21)$$

Since

$$x(T) - x(0) = \int_0^T \dot{x} dt = \dot{x}T, \quad \text{and } \dot{x} \text{ is constant} \quad (22)$$

we can calculate $x(T)$ by using Equations (21) and (22). Then solve the problem.

3.2 Demand Determined by Price and Cumulative Sales Only.

Next we study the case in which price is the only control variable and cumulative sales have two effects on current demand (referred to as "word-of-mouth

effect" and "market-saturation effect"). Some authors (e.g. Dolan and Jeuland [3], Kalish [9], Bass [1], and Robinson and Lakhani [12]) have proposed several important models in the area. However, there is still lack of theoretical results concerning such models. Here, we shall propose two examples, one for nondurable goods and the other for durable goods.

Example 3.2. Consider the following nondurable goods model:

$$\max_P \int_0^T [p(t) - C(x)] \dot{x}(t) dt \quad (23)$$

subject to

$$\dot{x}(t) = mK(t) e^{-8p(t)} - x(t) \quad (24)$$

where m and g are positive constants. Then we obtain:

$$x(t) = C(x(t)) - C(x(T)) + \int_t^T \frac{m}{8} e^{-8p^*} dt, \text{ for all } t \quad (25)$$

$$p^*(t) = C(x(T)) + \frac{1}{8} - \int_t^T \frac{m}{g} e^{-gp^*} dt, \text{ for all } t \quad (26)$$

$$p^*(T) = C(x(T)) + 1/g \quad (27)$$

and

$$\dot{p}^*(t) = - \frac{m}{8} e^{-8p^*(t)} \quad (28)$$

The model shows that the optimal price increases for all t . From (25) to (28), we know that a higher value of $x(T)$ causes the optimal price $p^*(t)$ to be higher and $\dot{x}(t)$ to be lower for all $t \in [0, T]$. This implies that there exists a unique solution of $x(T)$ which satisfies (25) to (28). Therefore, we can use binary search to obtain the value of $x(T)$, then calculate the optimal prices backward by using (27) and (28).

Example 3.3. Consider the following durable goods model:

$$\max_P \int_0^T [p(t) - C(x)] \dot{x}(t) dt \quad (29)$$

subject to

$$\dot{x} = [Y_1(M-x) + Y_2(M-x)x] p^n \quad (30)$$

innovators imitators

where y_1 and y_2 are positive constants of proportionality, M is the maximum potential sales, and $n > 0$. The model is similar to Robinson and Lakhani's [13], except that they assume that demand is an exponential function of the price. It is almost the same as Bass and Bultez's [2], which is a discrete time model.

After differentiation and making some algebraic manipulations, we obtain:

$$f_x = [\gamma_2 M - 2\gamma_2 x - \gamma_1] p^{-\eta} = \begin{cases} \text{positive} & \text{if } x < x^0 - \left| -\frac{\gamma_1}{2\gamma_2} \right| \\ \text{zero} & \text{if } x = x^0 \\ \text{negative} & \text{if } x > x^0 \end{cases} \quad (31)$$

and

$$\dot{p}^* = \begin{cases} \text{positive} & \text{if } (x < x^0 \text{ and } n > 1) \text{ or } (x > x^0 \text{ and } 0 < n < 1) \\ \text{negative} & \text{if } (x > x^0 \text{ and } n > 1) \text{ or } (x < x^0 \text{ and } 0 < n < 1). \end{cases} \quad (32)$$

This gives theoretical justification for the computational result due to Robinson and Lakhani [13] that we have penetration pricing at introduction time, then use skimming pricing. (In their case, $n > 1$).

3.3. Demand Determined by Advertising and Cumulative Sales Only

Many models have been introduced for the situation in which demand of a product is a function of advertising and cumulative sales only. For examples, Kotowitz and Mathewson [11], Horsky and Simon [8], Sethi [14, 15], and Teng and Thompson [16] have investigated this framework with diffusion models.

We have proved that there are only three quantitatively different kinds of optimal advertising policies in a linear advertising cost case as follows: (A) No advertising at all over the whole time horizon; (B) Advertising during an initial time, then no advertising when the market becomes saturated; (C) No advertising, then advertising finally no advertising again. For more details see Teng and Thompson [16].

3.4 Other Models.

Finally, we shall consider the case where the demand function is separable, and characterize the optimal price and advertising trajectories.

Example 3.4. Consider the following problem

$$\max_{u, p} \int_0^T \{ [p(t) - C(x)] \dot{x}(t) - u(t) \} dt \quad (33)$$

subject to

$$\dot{x}(t) = X(x) p^{-\eta}(t) u^{\beta}(t), \quad x(0) = x_0 \quad (34)$$

where $0 < \beta < 1$, $n > 0$. Applying the maximum principle again, we obtain the following results:

$$p^* = \eta(\eta-1)^{-1} (C-\lambda), \quad (35)$$

$$u^* = \left[\frac{e\lambda}{n(p^*)^{\eta-1}} \right]^{1/(1-\beta)} \quad \text{is positive constant, by theorem (d).} \quad (36)$$

$$\dot{p}^* = n^{-1} (p^*)^{-\eta+1} \left[\frac{BX}{n(p^*)^{\eta-1}} \right]^{0/(1-\beta)} \quad (37)$$

Note that Equation (37) shows that: (a) the optimal price is concave increasing all the time if $X(x) = mx^a(t)$, $0 < a < 1$, and $n > 1$, because from Equations (12) and (14), we have $p^* > 0$ and $p^* \wedge 0$; (b) the optimal price goes up first then down if $X(x) = m[Y_1(M-x) + Y_2(M-x)^2]$ and $*1 > 1$. The reason is that $p^* \wedge 0$ if $x < \frac{M}{Y} - \frac{1}{Z}$ and $p^* < 0$ elsewhere. This is a general case of Example 3.3.

4. CONCAVE ADVERTISING COSTS

In sections 2 and 3, we assume that the advertising cost $g(u)$ is linear, which makes $g_{uu} = 0$ and $H_{uu} \wedge 0$ so that we can guarantee u^* in (8) is optimal. In the real world the advertising costs may be concave increasing with respect to advertising rates, i.e., $g_{uu} < 0$, which implies have $H_{uu} \wedge 0$ may not be true and make the problem more difficult to solve, because u^* in (8) is not guaranteed to be an optimal solution if H_{uu} is not less than or equal to zero. However, $H_{uu} \wedge 0$ still holds in some concave advertising cost cases. Here, we shall present an example in which the advertising cost is concave increasing and $H_{uu} \wedge 0$ still holds.

Example 4.1. Consider the following concave advertising cost case

$$\int_0^T \{ [p(t) - CWI]^n - [a^2(t) + mM + c] \} dt \quad (38)$$

u, p

subject to

$$\dot{x}(t) = mp^n(t)u^6(t), \quad x(0) = X_0 \quad (39)$$

where $m, n > 6$, a, b , and c are constants, $a < 0$, $b > 0$, $n > 1$, and $0 < 6 < 1$. The model is similar to Example 3.1 which the advertising cost is linear.

From Theorems (a) and (c), and Equation (15), we know $p^* = n(n-1)u^1 C(x(T))$ is constant, and $g_u/f_u = -f/f_p$ which implies that

$$\frac{2au + b}{Bmp^{-n}u^{\beta-1}} = \frac{-mp^{-n}u^{\beta}}{-nmp^{-n-1}u^{\beta}} = \frac{p}{n}$$

and

$$Bmp^{-n+1}u^{\beta-1} = 2anu + bn \quad (40)$$

This implies u in (4) is constant, too. It immediately follows from (40) that

$$(\beta-1)Bmp^{-n+1}u^{\beta-2} = 2an + \frac{bn(g^{n-1})}{u} = 2an(2-6)$$

and

$$H_{uu} = (B-Demp)^{n-1}V^{\wedge}/n-Za - \frac{2a(2-6)}{u} \quad (41)$$

Clearly $H_{uu} < 0$ provided

$$b \geq |a| \frac{2(2-\beta)u^*}{1-\beta} . \quad (42)$$

Therefore, the optimal solutions for this problem are p^* in (11) and u^* in (40) if (42) holds, and (42) is true if b is sufficiently larger than the absolute value of a .

Example 4.2. Consider the same model as in Example 4.1, for which (42) is not satisfied. Also assume the advertising constraints

$$0 \leq L \leq u \leq U . \quad (43)$$

Here L is the lower bound, U is the upper bound on advertising. In this case, (41) and the fact that (42) does not hold imply that the solution of (40) does not give the optimal advertising strategy because the Hamiltonian is concave instead of convex. By substituting into the Hamiltonian we can determine whether $u = L$ or $u = U$ maximizes it. Therefore we conclude that the optimal advertising level is constant and that either

$$u = L \text{ or } u = U \quad (44)$$

is optimal when the Hamiltonian is concave.

5. CONCLUSIONS

During the past three decades many authors have published papers on dynamic price-advertising models. The present paper contains a general model which includes the previous models as special cases. Theoretical results were obtained for the general model and various special cases. The analysis given here should help the understanding of past models and should aid the development and understanding of new models.

APPENDIX

(A) Derivation of the optimal advertising rate with a separable demand function and a constant price elasticity.

$$Pf_u/g_u - n \tag{9}$$

$$pf_u = ng_u$$

$$\frac{d}{dt}[pf_u] = \frac{d}{dt}[ng_u]$$

$$p[f_{uu}\dot{u} + f_{ux}\dot{x} + f_{up}\dot{p}] + f_u\dot{p} - 0$$

$$p[f_{uu}\dot{u} + f_{ux}\dot{x} - f_{up}yf_x/f_p + yf_uif_x/f_p - 0$$

where $\dot{p} = yxf_x/f_p$ and $y \ll n(n-1)^{-1}$.

$$pf_{uu}\dot{u} - pf_{ux}\dot{x} + pf_{up}yf_x/f_p + yf_uif_x/f_p$$

or

$$pf_{pu}\dot{u}/x \gg -pf_{ux}\dot{x}/p + pf_{up}yf_x/f_p + yf_uif_x/f_p$$

Since $f(p,u,x) = P(p)U(u)X(x)$, we have

$$pf_{pu}\dot{u}/x = -px'n'xpup' + pxup' YX^{\wedge} P^{\wedge} YXO'PX'UP \\ - X'U/PXD(-pP' + p\gamma P' + \gamma P) \\ - XVPXU \left(\frac{1}{n-1} + \frac{1}{n-1} P \right)$$

Because f is separable and the price elasticity n is constant we obtain

$$P(p) \text{ is proportional to } p^{-n},$$

and $\frac{pP'}{p-1} + \frac{n}{p-1} P = 0$.

Therefore, we have $\dot{u} = 0$ if $f_{uu} > 0$. If $f_{uu} < 0$ then u^* is bang-bang and hence $\dot{u} = 0$ still holds except for the jump points.

(B) Calculation of the first and second derivatives of the optimal price with a separable demand function and a constant price elasticity.

$$\dot{p} \gg -Yff_x/f_p - -YP^{\wedge}'/P'$$

or

$$-\frac{1}{p}\dot{p} = ff_x/f = ff_x f^{-1} \\ - \frac{1}{p} \dot{p} = (f_u + f_x + f_p)ff_x^{-1} + (f_{ux} + f_{xx} + f_{xp}\dot{p})ff_x^{-1} \\ - f_x^{-2}(f_u + f_x + f_p)ff_x$$

Since $u = 0$, we can simplify the equation as follows:

$$\begin{aligned}
-\frac{f^2}{p} &= (f_{xx} \dot{x} + f_{xp} \dot{p}) f_x f_p + (f_{xx} \dot{x} + f_{xp} \dot{p}) f f_p - (f_{px} \dot{x} + f_{pp} \dot{p}) f f_x \\
&- (X'UPXUP + XUP'P)X'UPXUP' \\
&+ (X'UPXUP + X'DP'P)XUPXUP' \\
&- (X'UP'XUP + XUP'P)XUPX'UP \\
&= PXU^3 [(X'PXUP - YXX'UP^2)XV \\
&+ (X'PXUP - YX'^2UP)XP' \\
&- (X'P'XUP - YXP'^2S^*) X'P] \\
&= X^2 P^3 U^3 [X'u - YX'U)X'P' + (x'ux - YX'^2U)P \\
&\quad \cdot (x'P'u - YP' \frac{X'UP}{P}) X'] \\
&= X^2 P^3 U^4 [-2Y(X')^2 P' + XV + Y \frac{P'P(X')^2}{P}] .
\end{aligned}$$

Since $P(p) \cdot p^{-n}$, we obtain

$$\begin{aligned}
-\frac{f^2}{p} p' &= X^2 P^3 U^4 p^{-n-1} \left[\frac{2n^2}{n-1} (X')^2 - nX'X - \frac{n(n-1)}{n-1} (X')^2 \right] \\
&= nX^2 P^3 U^4 p^{-n-1} [(X')^2 - X'X] .
\end{aligned}$$

Hence

$$\begin{aligned}
ii. \quad \frac{-Y}{2} nX^2 P^3 U^4 p^{-n-1} [(X')^2 - X'X] \\
&= \frac{-U^2}{(n-1)} p^{-2n+1} [(X')^2 - X'X].
\end{aligned}$$

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