All ANALYSIS OF MIXED FINITE ELEMENT APPROXIMATIONS FOR PERIODIC ACOUSTIC WAVE PROPAGATION

by

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An Analysis of Mixed Finite Element Approximations for Periodic Acoustic Wave Propagation

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§1 Introduction. Let P^^P denote the fluid pressure, velocity, and density. Our starting point is the Eulerian equations of motion

au
~
it ~ ~ 1 p

•g— + (u.grad)u + — grad p = 0 (1.1)

\+ ~

f^ (a-grad)p + YP divu = 0 (1.2)

pp"Y = constant f (1.3)

where y is a ratio of specific heats. The particular case of interest in this paper is the acoustic disturbance about a mean flow $p_{Qi}\tilde{u}_o$^β, where the pressure has the form

$$P = P_o + e^{irt}P]L \quad , \quad |P_1/P_0I«1- \qquad (1.4)$$

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Assuming for example the mean flow is uniform and neglecting quadratic terms we obtain the following:

$$i\tau \underset{\sim}{u} + \frac{1}{\rho_0} \operatorname{grad} p = 0 \qquad (1.5)$$

$$\gamma p_0 \operatorname{div} \underset{\sim}{u} + i\tau p = 0 \qquad (1.6)$$

These equations are required to hold in the flow region Ω with linear combinations of the pressure and normal velocity being specified on the boundary Γ; say

$$\alpha \underset{\sim}{u} \cdot \underset{\sim}{\nu} + \beta p = \sigma \quad \text{on} \quad \Gamma, \qquad (1.7)$$

where $\underset{\sim}{\nu}$ denotes the outer normal.

The goal of this paper is to analyze Galerkin or mixed variational approximations to the first order system (1.5)-(1.6). At first glance such an approach may seem to be inferior to a discretization of (1.5)-(1.7) based on a least squares principle [1]. The primary reason for this is that in the least squares approach one can obtain second order accuracy in L_2 for both $\underset{\sim}{u}$ and p by using appropriate piecewise linear spaces for each. Such a combination, however, will be unstable in the Galerkin formulation ([2]-[4]). In the latter one typically uses an appropriate piecewise linear space for $\underset{\sim}{u}$ and a piecewise constant space for p. The degrees of freedom in the latter are virtually the same as in the least squares formulation for a comparable grid, yet yield only first order L_2 accuracy in p.

Two somewhat surprising results of the analysis in this paper offer hope, however, for the utility of the Galerkin approach. First we show that under suitable conditions one has superconvergence in the approximation to the pressure. For example, if $\hat{p}_h$ denotes the best L_2 approximation to p in a suitable space of piecewise constant functions, then we show that the L_2 error $\|p_h - \hat{p}_h\|_0$ in the Galerkin approximation p_h is actually second order if appropriate linear elements are used for the velocity u.

Second and of equal importance, the errors in the Galerkin approach do not deteriorate as rapidly when the frequency r increases. This is of particular importance in underwater acoustics where

$$r >> 1$$

is quite common [5].

This second property is also shared by the standard finite element formulation where (1.5) is differentiated and combined with (1.6) to give the Helmholtz equation

$$\Delta p + \omega^2 p = 0, \tag{1.8}$$

and the Galerkin method is applied directly to (1.8). So long as the coefficients Po'Ho'^o from the mean flow are smooth this approach is possibly preferable; however in many applications these coefficients come from measurements and are not smooth enough to be differentiated [1]. In such cases one must deal directly with a first order system like (1.5)-(1.6).

Previous work on Galerkin approximations has centered on the Poisson equation

$$\underset{\sim}{u} - \text{grad}\,\varphi = 0 \tag{1.9}$$

$$\text{div}\,\underset{\sim}{u} = f. \tag{1.10}$$

Let $\mathcal{V}^h$ denote the finite dimensional space of velocities and $\mathcal{S}^h$ the finite dimensional space of pressures. Brezzi [3] showed that optimal convergence in the norms

$$\|\psi\|_{\mathcal{S}} = \|\psi\|_o \qquad \|\underset{\sim}{v}\|_{\mathcal{V}} = \{\|\underset{\sim}{v}\|_o^2 + \|\text{div}\,\underset{\sim}{v}\|_o^2\}^{1/2\,(*)} \tag{1.11}$$

will occur provided

$$\sup\left\{\frac{\int \psi\,\text{div}\,\underset{\sim}{v}}{\|v\|_{\mathcal{V}}}\right\} \geq \alpha\|\psi\|_{\mathcal{S}} \quad \text{all} \quad \psi\in\mathcal{S}^h \tag{1.12}$$

holds for a fixed $\alpha, 0<\alpha<\infty$, independent of h, where the sup is taken over all $\underset{\sim}{v}$ in $\mathcal{V}^h$. There are a variety of spaces satisfying this condition including piecewise linear functions on a suitable grid for the velocities and suitable piecewise constants for the pressure [6]. This combination gives first order accuracy in the mesh spacing h in the norms (1.11). These results are generalized in Section 3 for the acoustic equations.

In subsequent work ([2],[4]) on the Poisson equation it was shown that optimal accuracy in the norms

$$\|\psi\|_o \ , \ \|\underset{\sim}{v}\|_o \tag{1.13}$$

can be obtained under appropriate conditions. In particular, one needs the <u>inclusion property</u>

(*) Throughout this paper $\|\cdot\|_r$ denotes the norm on the Sobolev space $H^r(\Omega)$.

$$S^h = \mathrm{div}[V^h] \tag{1.14}$$

as well as a <u>decomposj-tion</u> <u>property</u>. The latter states there is an a; <Ka<oo, independent of h such that each Xfc^lr can be written

Xh ^Xh + ^i' (1_15)

where JSUJSU *** ^ satisfy

$$\mathrm{div}\, z_h = 0 \ , \ \alpha\|w_h\|_o \le \|\mathrm{div}\, v_h\|_{-1} \, . \tag{1.16}$$

In many mixed finite element formulations the analogs of (1.14)-(1.16) and (1.12) are equivalent ([7]). in this setting they are not. *In* fact, one can shew that in this case (1.12) is equivalent to *an* inclusion and decomposition property but with different norms in (1.16) [7]. Nevertheless the finite element spaces which are known to satisfy (1.14)-(1.16) also satisfy (1.12) and conversely ([4]). In Section 4 we generalize the error estimates using (1.14)-(1.16) to cover the acoustic equations. The most important aspect of this analysis is the superconvergence in the pressure p.

<u>§2 The Galerkin formulation.</u> For simplicity we consider the boundary value problem

$$\underset{\sim}{u} - \text{grad}\ \varphi = \underset{\sim}{f} \quad \text{in} \quad \Omega \tag{2.1}$$

$$\text{div}\ \underset{\sim}{u} + \omega\varphi = g \quad \text{in} \quad \Omega \tag{2.2}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma \tag{2.3}$$

The mixed variational formulation of this problem is based on Galerkin's method and takes the following form. Given $\underset{\sim}{f}_0 \in L_2(\Omega)$, $g_0 \in L_2(\Omega)$ find

$$\underset{\sim}{u}_0 \in V = H(\text{div};\Omega)^*, \quad \varphi_0 \in \mathcal{S} = L^2(\Omega) \tag{2.4}$$

such that

$$a(\underset{\sim}{u}_0,\underset{\sim}{v}) + b(\underset{\sim}{v},\varphi_0) = (\underset{\sim}{f}_0,\underset{\sim}{v}) \tag{2.5}$$

$$b(u_0,\psi) + c(\varphi_0,\psi) = \langle g_0,\psi\rangle \tag{2.6}$$

holds for all $v \in V$, $\psi \in \mathcal{S}$. The forms are defined as follows:

$$a(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega \underset{\sim}{u}\cdot\underset{\sim}{v}, \quad b(\underset{\sim}{v},\psi) = \int_\Omega \psi\,\text{div}\ \underset{\sim}{v} \tag{2.7}$$

$$c(\varphi,\psi) = \int_\Omega \omega\varphi\psi \qquad \langle\varphi,\psi\rangle = \int_\Omega \varphi\psi \tag{2.8}$$

(*) $H(\text{div};\Omega)$ consists of $\underset{\sim}{v} \in \vec{L}_2(\Omega)$ such that $\text{div}\ \underset{\sim}{v} \in L_2(\Omega)$.

To approxima'te we introduce finite dimensional spaces

$$»^h £ \% , \quad V^h c V \qquad (2.9)$$

and seek u^V^{11}, ^ $S^{1 1}$ such that (2.5)-(2.6) holds (with U_O replaced with u^ and $<p_Q$ replaced with *tp^)* for all $v c V^h$, $\phi \in S^h$.

It may happen that (2.1)-(2.3) does not have a unique solution, a case which arises for example if ID is am eigenvalue of the homogeneous problem. We explicitly rule this out by assuming that the adjoint equation (which in this case is the same as (2.1)-(2*3)) is uniquely solvable. More precisely, we assume that for each pair $\hat{f}$eH(div,O), §€L_O(C1) there us a unique pair $\hat{V}$clr, $€& for which

$$a(w,£) + b(w,\$) = (|,w) \qquad (2.11)$$

$$b(\phi,\xi) + c(\xi,\hat{\phi}) = \langle\hat{g},\psi\rangle \qquad (2.12)$$

holds for all weV, ?eS. Moreover, we assume that the solution of (2.11)-(2.12) satisfies the standard apriori bound for the Helmholtz equation

$$A\$ + oj\$ - \hat{g} - \text{div } £.$$

Namely,

$$!\$!l_0 + !|J''_1 \wedge K^H\hat{g}ll^ + ll£»_0) \qquad (2.13)$$

The constant K_Q approaches infinity with $\sqrt{|u)|}$.

The effect of the frequency to in our analysis will also be seen in the constant $0 < K_1 < \infty$ satisfying

$$|c(q>,\$)| \wedge K_X |W||_0 lto!|_0 \quad all \quad *,*\in* \tag{2.14}$$

For the model problem (2.1)-(2.3) we can take K^ = ω. In the general acoustic equations it will be a more complicated function of CD but will still approach infinity linearly with a>.

As noted in the introduction the case 0) = 0 has received considerable attention. We shall use this work to define a mapping from VXS to ir^S*1 - called the Poisson projector P^h — as follows. We let

$$\{\hat{v}_h,\hat{\phi}_h\} = P^h\{\hat{v},\hat{\phi}\}$$

when

$$a(w^h,\hat{v}_h) + b(w^h,\phi_h) = a(\mathrm{W}\pounds,) + b(\$> \tag{2.15}$$

$$b(\hat{v}_h,?^h) - b(\hat{v},?^h) \tag{2.16}$$

holds for all $w^h \in V^h$, $?^h \in 5^h$. That is C^,^} is the mixed finite element approximation to the Poisson equation generated by $\{\hat{v},\hat{1}\}$.

If Brezzi's condition (1.12) holds, then there is a constant $0 < C_\alpha < \infty$, depending only on the number a in (1.12) such that

$$\|\hat{v} - \hat{v}_h\|_0 + \|\mathrm{div}(\hat{v} - \hat{v}_h)\|_0 + \hat{«*}_0 \|\hat{\phi}_h\|_0 \leq C_\alpha E_1(\hat{v},\hat{\phi}), \tag{2.17}$$

where

$$E_1(\hat{v},\hat{\phi}) = \inf \{\|\hat{v} - *\% + \|div(£ - J>)\| + \|i - \wedge\|_Q) \qquad (2.18)$$

and where the inf is taken over $w^\circ \in V$, $\S^h \in S^h$. (see [3])

Similarly if the inclusion and decomposition properties (1.14)-(1.16) hold, then

$$H\hat{Z} - l«0 * {}^c a^E 0<\hat{Z}> \qquad (2.19)$$

and

$$"* " Mo * {}^c a«V2\hat{0}> + {}^E 0<\wedge 0» \qquad (2.20)$$

Where

$$E\wedge) - \inf \|£ - w^h\|_0, \quad E_Q(\$) « \inf \|S - \S^h l\|_0. \qquad (2.21)$$

said now C_α depends on the constant a in (1.16) (See! [4]) •

Throughout this paper we shall assume that the spaces V^h, S^h have the standard approximation properties. More precisely, we assume that we can approximate in V^h to order k in the sense that given $w \in \vec{H}^k(\Omega)$

$$\inf \{\|w - w^h\|_0 + h\|w - w^h\|_1\} \leq C_A h^k \|w\|_k \qquad (2.22)$$

for a fixed constant $0 < C_A < \infty$ independent of w and h* Moreover, we assume (2.22) holds for k replaced with any smaller kj^,satisfying $1 < k\wedge \leq k$. Similarly, we assume that for any

$\xi \in H^{\ell}(\Omega)$

$$\inf \{\|\xi - \phi^h\|_0\} \leq C_A h^{\ell} \|\xi\|_{\ell} \qquad (2.23)$$

with this inequality holding for ℓ replaced with any ℓ_1 satisfying $0 < \ell_1 \leq \ell$. For most spaces satisfying the Brezzi condition (1.12) or the inclusion condition (1.14) we have $t = k - 1$,

Observe that if (2.22)-(2.23) hold, then

$$E_1(\underset{\sim}{w}, \xi) = O(h^{k-1}) + O(h^{\ell})$$

while

$$E_0(\underset{\sim}{w}) = O(h^k), \qquad E_0(\xi) = O(h^{\ell})$$

provided $\underset{\sim}{w} \in \vec{H}^k(\Omega)$, $\xi \in \vec{H}^{\ell}(\Omega)$.

§3 The first error estimates. In this section we assume that Brezzi's condition (1.12) is valid, and estimate the errors Bo ~ £h' 'o ~ *h# The proof that the discrete system (2.5)-(2.6) on $V^h \times \$^h$ has a unique solution JJ^^^u is similar in structure to the. error analysis so we shall give only the latter• Also we shall assume that the regularity (2.13) and approximability conditions^ (2.22)-(2.23) hold.

Theorem 7. Let (1.12) hold. Then there is a constant 0 < C < oo depending only on a in (1.12) and to such that if hC < 1, then

$$\text{-to-} \bullet 2_h\|_0 + \|\varphi_0 - \varphi_h\|_0 \leq \left(\frac{C}{1-Ch}\right) E_1(\underset{\sim}{u}_0, \varphi_0) \qquad (3.1)$$

Moreover, C approaches infinity linearly with $(\mathrm{u}\triangleright|^{3/2}$•

Proof. Let

$$B((\underset{\sim}{u}, <p), (\underset{\sim}{v}, \$)) \text{ » } a(\underset{\sim}{u}, \underset{\sim}{v}) + b(\underset{\sim}{v}, (p) + b(\underset{\sim}{u}, j(>) + c(_v, j/)) \,. \qquad (3.2)$$

Then the defining equations (2.5)-(2.6) give

$$B((\underset{\sim}{u}_h, \varphi_h), (\underset{\sim}{v}^h, \psi^h)) = B((\underset{\sim}{u}_0, \varphi_0), \wedge, \wedge)) \qquad (3.3)$$

for all $v^h \in V^h$, $\wedge^h \in S^h$. Let

$$(\hat{\underset{\sim}{u}}_h, \hat{\varphi}_h) \in \mathrm{U}^h \times S^h$$

be given and put

$$\underset{\sim}{e}_h = \underset{\sim}{u}_h - \hat{\underset{\sim}{u}}_h, \qquad \underset{\sim}{e} = \underset{\sim}{u}_0 - \hat{\underset{\sim}{u}}_h \tag{3.4}$$

$$\epsilon_h = \varphi_h - \hat{\varphi}_h, \qquad \epsilon = \varphi_0 - \hat{\varphi}_h \tag{3.5}$$

Subtracting $B((\underset{\sim}{u}_h,\varphi_h),(\underset{\sim}{v}^h,\psi^h))$ from both sides of (3.3) gives

$$B((\underset{\sim}{e}_h,\epsilon_h),(\underset{\sim}{v}^h,\psi^h)) = B((\underset{\sim}{e},\epsilon),(\underset{\sim}{v}^h,\psi^h)). \tag{3.5}$$

Let us first estimate the left hand side of (3.4), and in the process make a definite choice for $\underset{\sim}{v}^h$ and ψ^h. The idea is to choose these functions so that the left hand side becomes essentially $\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2$. To do this we first solve the adjoint problem (2.11)-(2.12) with data $\underset{\sim}{f} = \underset{\sim}{e}_h, g = \epsilon_h$. Letting $\underset{\sim}{w} = \underset{\sim}{e}_h$, and $\xi = \epsilon_h$ in (2.11)-(2.12) gives

$$B((\underset{\sim}{e}_h,\epsilon_h),(\hat{\underset{\sim}{v}},\hat{\psi})) = \|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2. \tag{3.6}$$

Since $B(\cdot,\cdot)$ is linear in each variable it follows from (3.5) that

$$\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2 = B((\underset{\sim}{e}_h,\epsilon_h),(\hat{\underset{\sim}{v}} - \underset{\sim}{v}^h,\hat{\psi} - \psi^h)) + B((\underset{\sim}{e},\epsilon),(\underset{\sim}{v}^h,\psi^h)) \tag{3.7}$$

We now let $\{\underset{\sim}{v}^h,\psi^h\} = \{\hat{\underset{\sim}{v}}_h,\hat{\psi}_h\}$ be the Poisson projection of $\{\hat{\underset{\sim}{v}}^h,\hat{\psi}\})$, i.e., (2.15)-(2.16) holds. Thus putting $\underset{\sim}{w}^h = \underset{\sim}{e}_h$, $\xi^h = \epsilon_h$ in (2.15)-(2.16) we get

$$a((\underset{\sim}{e}_h,\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h) + b(\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h,\epsilon_h) + b(\underset{\sim}{e}_h,\hat{\psi} - \hat{\psi}_h) = 0. \tag{3.8}$$

Thus

$$B((\underset{\sim}{e}_h,\epsilon_h),(\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h,\hat{\psi} - \hat{\psi}_h)) = c(\epsilon_h,\hat{\psi} - \hat{\psi}_h), \tag{3.9}$$

and so

$$\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2 = c(\epsilon_h, \hat{\phi} - \hat{\phi}_h) + B((\underset{\sim}{e},\epsilon),(\hat{v}_h,\hat{\phi}_h)). \quad (3.10)$$

We treat the second term on the right hand side of (3.10) in a similar way. In particular, we let $\{\hat{u}_h, \hat{\varphi}_h\}$ be the Poisson projection of $\{u_0, \varphi_0\}$. This gives

$$a(\underset{\sim}{e}, \hat{\underset{\sim}{v}}_h) + b(\underset{\sim}{e}, \hat{\phi}_h) + b(\hat{\underset{\sim}{v}}_h, \epsilon) = 0, \quad (3.11)$$

and so

$$B((\underset{\sim}{e},\epsilon),(\hat{\underset{\sim}{v}}_h,\hat{\phi}_h)) = c(\epsilon,\hat{\phi}_h). \quad (3.12)$$

Combining <u>this</u> with (3.10) we obtain

$$\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2 = c(\epsilon_h, \hat{\phi} - \hat{\phi}_h) + c(\epsilon,\hat{\phi}_h) \quad (3.13)$$

To estimate the first term on the right hand side of (3.13) we note that (2.14) gives

$$|c(\epsilon_h, \hat{\phi} - \hat{\phi}_h)| \leqq K_1 \|\epsilon_h\|_0 \|\hat{\phi} - \hat{\phi}_h\|_0. \quad (3.14)$$

Our approximation assumption (2.23) gives

$$\|\hat{\phi} - \hat{\phi}_h\|_0 \leqq C_A h \|\hat{\phi}\|_1 \quad (3.15)$$

The regularity (2.13) of (2.11)-(2.12) can be used to bound $\hat{\phi}$ in terms of the data $\underset{\sim}{e}_h, \epsilon_h$ as follows:

$$\|\hat{\phi}\|_1 \leqq K_0 (\|\epsilon_h\|_0 + \|\underset{\sim}{e}_h\|_0) \quad (3.16)$$

Thus

$$|c(\epsilon_h, \hat{\phi} - \hat{\phi}_h)| \leq K_0 K_1 c_A h(\|\epsilon_h\|_0 + \|\underset{\sim}{e}_h\|_0)\|\epsilon_h\|_0 \qquad (3.17)$$

The second term on the right hand side of (3.13) is treated in a similar way. in particular,

$$|c(\epsilon, \phi_h)| \leq K_1 \|\epsilon\|_0 (\|\hat{\phi} - \hat{\phi}_h\|_0 + \|\hat{\phi}\|_0)$$

$$\leq K_1 \|\epsilon\|_0 \,{}^{\wedge}0 + c_A K_0 h)(\,|\epsilon_h\|_0 + \|\underset{\sim}{e}_h\|_0) \qquad (3.18)$$

Conibining (3«17)-(3.18) with (3.13) we obtain (3a).

Remark. The linear dependence on $|to|^{3/2}$ in the error estimates is an order of magnitude better than that obtained for the least squares approximation, where the dependence is quadratic*

The order of accuracy for a fixed u> is not best possible. For example, If the standard linear element - piecewise constant combination is used (i.e., k = 2 and I = 1), then we get only first order accuracy in ^ and tp_0 as in [3] and [6].

§4 Improved estimates. In this section we assume that the inclusion and decomposition properties (1.14)-(1.16) hold, and snow that the L_2 errors in $\hat{\chi}$ are best possible. Moreover, we show that a superconvergence result holds for the scalar φ_0. The starting point is to prove the latter result for the Poisson projection, and then using the approach of the previous section show that it also holds for to £ 0.

Lemma 1. Let $\{\hat{v}_h, \hat{\phi}_h\}$ be the Poisson projection of $(\hat{v}, \hat{\phi})$ defined by (2.15)-(2.16), and let $\bar{\phi}_h$ be the best L_2 approximation to $\hat{\phi}$ in S^h. Then

$$\|\hat{\phi}_h - \bar{\phi}_h\|_0 \leq (C_\alpha/\alpha) E_0(\hat{v}) \tag{4.1}$$

Remark, if S^h consists of piecewise constant functions, then the value of $\bar{\phi}_h$ is a given triangle T is equal to the average of $\hat{\phi}$ over T. The above estimate states that this func~ tion will differ from the Poisson projection by order $0(h^2)$ if linear elements are used to represent u_h or by order $0(h^3)$ if quadratic elements are used, to cite another popular combination.

Proof. Subtracting $\bar{\phi}^h$ in the right and left hand sides of (2.15) gives

$$b(w^h, \hat{\phi}_h - \bar{\phi}_h) = a(w^h, \hat{v} - \hat{v}_h) + b(w^h, \hat{\phi} - \bar{\phi}_h) \tag{4.2}$$

for all $w^h \in V^h$. We use the inclusion property (1.14) to write

$$\hat{\phi}_h - \bar{\phi}_h = \operatorname{div} \underset{\sim}{y}_h \tag{4.3}$$

for $\underset{\sim}{y}_h \in \underset{\sim}{V}^h$. In addition, we use the decomposition property (1.15)-(1.16) to write

$$\underset{\sim}{y}_h = \underset{\sim}{w}_h + \underset{\sim}{z}_h, \quad \underset{\sim}{w}_h, \underset{\sim}{z}_h \in \underset{\sim}{V}^h, \tag{4.4}$$

where

$$\operatorname{div} \underset{\sim}{z}_h = 0, \quad \alpha\|\underset{\sim}{w}_h\|_0 \le \|\operatorname{div} \underset{\sim}{y}_h\|_0 = \|\hat{\phi}_h - \bar{\phi}_h\|_0 \tag{4.5}$$

Let $\underset{\sim}{w}^h = \underset{\sim}{w}_h$ in (4.2). Then

$$b(\underset{\sim}{w}_h, \hat{\phi}_h - \bar{\phi}_h) = \int_\Omega \operatorname{div} \underset{\sim}{w}_h (\hat{\phi}_h - \bar{\phi}_h) = \int_\Omega (\hat{\phi}_h - \bar{\phi}_h)^2 \tag{4.6}$$

Also, since $\bar{\phi}_h$ is the best L_2 approximation,

$$b(\underset{\sim}{z}_h, \hat{\phi} - \bar{\phi}_h) = \int_\Omega (\operatorname{div} \underset{\sim}{z}_h)(\hat{\phi} - \bar{\phi}_h) = 0 \tag{4.7}$$

(since div $\underset{\sim}{z}_h \in W^h$ by inclusion property). Thus, using (2.19) and (4.5), (4.2) becomes

$$\|\hat{\phi}_h - \bar{\phi}_h\|_0^2 = a(\underset{\sim}{w}_h, \hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h) \le \|\underset{\sim}{w}_h\|_0 \|\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h\|_0$$

$$\le \alpha^{-1}\|\hat{\phi}_h - \bar{\phi}_h\|_0 \, C_\alpha E_0(\hat{\underset{\sim}{v}}).$$

Cancellation of the common factor gives (4.1).

To apply this result to the case $m \ge 0$, we retain the approach of Section 3 except giving an alternate estimate for the term

$c(\epsilon, \hat{\phi}_h)$. To treat the latter we note that

$$c(\epsilon,\hat{\phi}_h) = \int_\Omega \varpi(\varphi_0 - \hat{\varphi}_h)\hat{\phi}_h , \qquad (4.8)$$

where $(\hat{u}_h, \hat{\varphi}_h)$ is the Poisson projector of $\{u_0, \varphi_0\}$. Let $\bar{\varphi}_h$ denote the best L_2 approximation to φ_0 in S^h. Then

$$c(\epsilon,\hat{\phi}_h) = \int_\Omega \varpi(\varphi_0 - \bar{\varphi}_h)\hat{\phi}_h + \int_\Omega \varpi(\bar{\varphi}_h - \hat{\varphi}_h)\hat{\phi}_h \qquad (4.9)$$

Assuming that UD is constant we have

$$\int_\Omega \varpi(\varphi_0 - \bar{\varphi}_h)\hat{\phi}_h = 0, \qquad (4.10)$$

since $\varphi_0 - \bar{\varphi}_h$ is orthogonal to S^h. Thus

$$|c(\epsilon,\hat{\phi}_h)| \le K_1 \|\bar{\varphi}_h - \hat{\varphi}_h\|_0 \|\hat{\phi}_h\|_0 \qquad (4.11)$$

Estimating the last term on the right of (4.11) as in (3*18) we obtain the following result.

<u>Theorem 2</u>. Let the inclusion and decomposition properties (1.14)-(1.16) and assume ID is a constant. There is a $0 < C < \infty$ depending only on a in (1.16) said w such that if $Ch < 1$, then

feo - 2h«o + K- »hllo " ≤ Tlh)E 0(Ho) • (4.12)

where $\bar{\varphi}_h$ is the best L_2 approximation to φ_0 in 3^h.

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