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AUCTIONS AND MARKET GAMES

by

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### 1. INTRODUCTION

Some simple market games were briefly treated in their monumental work by John von Neumann and Oskar Morgenstern [10]. These remained as intriguing special cases until Shapley [12] showed that an interesting class of market games could also be solved as assignment problems, which constitute a special kind of linear program. Later, Shapley and Shubik [13] characterized the core of the assignment game.

From an entirely different point of view the ideas of auctions and bidding began in economic theory with the famous horse auction of Bohn-Baverk [3], which Shapley and Shubik later [13] identified as an assignment market game. Vickery [21], [22] discussed the idea of price setting in a sealed bid auction by the bid of the second highest bidder, which can also be called a Dutch auction, and showed that it was pareto optimal for the buyers. More recently other auction results were given in [1, 8, 11, 14, 17]. Barr and Shaftel [2] showed that Vickery's work could be extended to several buyers and sellers and different kinds of goods by using an assignment model. The author [19] remarked that the Barr-Shaftel model was really an assignment market game being solved for a special core point, the buyer surplus point. In [18] the author extended assignment market games to transportation market games and gave a constructive method for computing all the extreme points of their cores. He also discussed in [19] ways that market game theory could be applied to the solution of practical auctioning problems by using the recently obtained immense improvements in speed of solving assignment and transportation problems [4, 7, 15].

In [20] the author used the economic ideas of dutch auctions to devise a new and extremely efficient algorithm for solving assignment problems.

In the present paper a number of new results relating both to the cores of market games and to the corresponding auction applications. The first result has to do with the effect on the core of adding or dropping sellers and buyers. In [20] the author gave a recursive method for solving assignment problems which proceeds by starting with all the buyers and one dummy seller, and then adding sellers one by one. At all times a buyer surplus solution is maintained. In this process the sellers prices go down or stay the same and the buyers surpluses go up or stay the same. The same holds true if buyers are dropped and the reverse holds true if sellers are dropped or buyers are added. These results are of interest if one is considering dynamic auctions of indivisible goods.

The second result characterizes the maximum number of extreme points of the core as the binomial coefficient  $2n$  by  $n$ . This is a huge number for large  $n$  since it increases exponentially. Several other core results are also demonstrated such as: a dominant diagonal cost matrix has the maximum size core; a game with positive bids always has a core with an even number of extreme points; and the core of a primal nondegenerate transportation game has a core with exactly two extreme points.

Finally, in the last section, the graphs of the skeletons of several market games are drawn to show how they vary in size and in numbers of extreme points. Some remarks on auctions and fair division problems are also given.

## 2. NOTATION FOR MARKET GAMES

We use the same notation as [18], but summarize it here for completeness.

We denote the Index set of the sellers by

$$I = \{1, 2, \dots, m\} \tag{1}$$

and denote the index set of the buyers by

$$J = \{1, 2, \dots, n\}. \tag{2}$$

We assume that seller  $i \in I$  has

$$a_i > 0 \tag{3}$$

units of a good to sell, and that buyer  $j \in J$  wants to buy

$$b_j > 0 \tag{4}$$

units of the good. We let

$$*_{ji} L^0 \tag{5}$$

be the bid of buyer  $j$  for one unit of seller  $i$ 's goods. The nonnegativity requirement in (5) means that seller  $i$  can dispose of his goods without charge in case no one bids a positive amount for it.

We make the same economic assumptions as do Shapley and Shubik [13] in their treatment of the assignment market game, namely:

- (a) Utility is identified with money
- (b) Side payments are permitted
- (c) The objects of trade are indivisible
- (d) Supply and demand functions are inflexible.

The remarks they make about these assumptions are pertinent here and will not be repeated.

As In the assignment game [13], the only profitable coalitions are those containing some buyers and some sellers. Also, because of assumption (5) and the side payment condition (b), the only important coalition is the all-player coalition  $S = I \cup J$ . We shall concentrate on evaluating  $v(S)$  for this coalition only, since it is the only important coalition in the game.

Let  $x_{ij}$  be the number of units  $i$  sells to  $j$ . The value  $v(I \cup J)$  is obtained by solving the linear program:

$$\begin{aligned}
 & \text{Maximize } \sum_{i \in I} \sum_{j \in J} x_{ij} c_{ij} \\
 & \text{Subject to} \\
 & \sum_{j \in J} x_{ij} \leq a_i \quad \text{for } i \in I \\
 & \sum_{i \in I} x_{ij} \leq b_j \quad \text{for } j \in J \\
 & x_{ij} \geq 0
 \end{aligned} \tag{6}$$

The nonnegativity requirement on  $x_{ij}$  means that the exchange of property is from seller  $i$  to buyer  $j$ . The maximization objective in (6) means that we seek a set of transactions which maximizes the total gain of the coalition  $I \cup J$  of all sellers and buyers (see Shapley and Shubik [13]).

If  $b_j \geq 1$  for  $j \in J$  the problem is called a semi-assignment market game. If, in addition,  $a_i \geq 1$  for  $i \in I$  the problem is called an assignment market game.

The dual linear programming problem to (6) is easily written as

$$\begin{aligned}
 & \text{Minimize } \sum_{i \in I} u_i + \sum_{j \in J} v_j \\
 & \text{Subject to} \\
 & u_i + v_j \geq c_{ij} \quad \text{for } i \in I, j \in J \\
 & u_i \geq 0 \quad \text{for } i \in I \\
 & v_j \geq 0 \quad \text{for } j \in J
 \end{aligned} \tag{7}$$

where  $u_i$  and  $v_j$  are the dual variables associated with the first and second constraints in (6), respectively.

The core of the market game (i.e., the set of undominated imputations) is the set of all solutions to the dual problem (7). This was argued in [13] for the assignment case, and the same result holds here. Because of the non-negativity conditions (3), (4), (5) and well-known linear programming results, the core is a bounded convex polyhedral set.

We can turn (6) into the classical transportation problem of linear programming by adding a dummy seller  $\{nrfl\}$  and dummy buyer  $\{xri-1\}$  giving extended seller and buyer index sets

$$i' = I \cup \{Bri-1\} \tag{8}$$

$$J' = J \cup \{xrt-1\} . \tag{9}$$

We define the bids of these dummy players to be

$$c_{j'} = 0 \text{ for } j \in J \tag{10}$$

$$c_{i',n+1} = 0 \text{ for } i \in I \tag{11}$$

and note that (10) can be interpreted as a "free gift" option for the buyers and (11) can be interpreted as a "free disposal"<sup>1</sup> option for the sellers. To determine the amount sold by the dummy seller and the amount bought by the dummy buyer, we first define

$$S = \sum_{lei} a_i \tag{12}$$

$$T = \sum_{jeJ} b_j \tag{13}$$

and then define

$$a_{m+1} = \frac{1}{2}[|T - S| + (T - S)] \quad (14)$$

$$b_{n+x} = \frac{-h}{L}[|S - T| + (S - T)] \quad (15)$$

as the amount sold by the dummy seller and the amount purchased by dummy purchaser, respectively. It is easy to see that at least one (and possibly both) of  $a_{ffi'rl}$  and  $b_{nrrjL}$  is 0. In any case we retain both dummy players in the transportation problem for reasons that will become clear later.

We now use the above definitions to state a transportation problem from which the solution to (6) can be obtained.

$$\begin{aligned} & \text{Maximize} && \sum_{i \in I'} \sum_{j \in J'} x_{ij} c_{ij} \\ & \text{Subject to} && \sum_{j \in J'} x_{ij} = a_i \quad \text{for } i \in I' \\ & && \sum_{i \in I'} x_{ij} = b_j \quad \text{for } j \in J' \\ & && x_{ij} \geq 0 \end{aligned} \quad (16)$$

The dual problem to (16) is

$$\begin{aligned} & \text{Minimize} && \sum_{i \in I'} a_i u_i + \sum_{j \in J'} b_j v_j \\ & \text{Subject to} && u_i + v_j \geq c_{ij} \quad \text{for } i \in I', j \in J' \end{aligned} \quad (17)$$

Clearly the only difference between (7) and (17) is the nonnegativity requirements on the dual variables which are present in (7) but missing in (17).

In [18] it was shown that the set of nonnegative solutions to (17) is non-empty and bounded. Hence we impose the nonnegativity constraints



$$u_i \geq 0 \text{ for } i \in I, \quad \wedge \quad v_j \geq 0 \text{ for } j \in J' \quad (18)$$

on the solutions to (17), and now (17) and (18) together are identical to (7).

When the market game is an auction the interpretation of the  $u_i$ 's are the selling prices received by the sellers for their goods and the  $v_j$ 's are the buyer surpluses attained by the buyers. For elaboration of these interpretations, see [12], [13], and [19].

### 3. PREVIOUS RESULTS

This section will summarize, without giving proofs, results previously obtained concerning the structure of the core; see [18, 19, 20].

For purposes of simplicity let us assume that the solution to the primal problem (16) is unique. This solution gives a way of exchanging the goods that maximizes the total value of all goods exchanged. As noted above the core of the game is the set of all nonnegative dual solutions to (17) and (18). To give economic interpretations to the core solutions we make the following definitions.

DEFINITION 1. The core of the market game (6) is the set of all non-negative solutions to its dual problem (7); i.e., the core is the set of all solutions to (17) and (18). We denote the core by  $C \gg (C(U), C(V))$  where  $C(U)$  is the set of row dual solutions  $U$  which we call the seller core, and  $C(V)$  is the set of column dual solutions  $V$  which we call the buyer core.

REMARK 1. It can easily be shown that the core is non-empty. From standard linear programming theory we know the core is a bounded convex polyhedral set having a finite number of extreme points. Later (Theorem 4) we will characterize the maximum number of such extreme points.

DEFINITION 2. Given a market game the ~~maximum seller surplus~~  $u_i$  for seller  $i$  is given by

$$U_i^* = \text{Maximum}_{U \in C(U)} u_i \quad (19)$$

The ~~minimum~~ seller surplus  $u_i$  is defined by replacing the word Maximum by the word Minimum in (19). The vectors  $u$  and  $u^*$  with components  $u_i$  and  $u_i^*$  are the ~~minimum~~ and ~~maximum~~ seller surplus vectors\*

DEFINITION 3. Given a market game the ~~maximum buyer surplus~~  $v_j$  for buyer  $j$  is given by

$$v_j^* = \text{Maximum}_{V \in COO} v_j \quad (20)$$

The ~~minimum~~ buyer surplus  $v_j$  is defined by replacing the Maximum by the word Minimum in (20). The vectors  $v$  and  $v^*$  with components  $v_j$  and  $v_j^*$  are the maximum and ~~minimum~~ buyer surplus vectors.

THEOREM 1. Given a market game (6), the vector pairs  $(u, v^*)$  and

- (a) are in the core;
- (b) are the furthest distance apart of any two vectors in the core;
- (c) individually and collectively maximize, or minimize, buyer or seller surpluses.

For a proof see [13] or [19].

Theorem 1 characterizes the two "end points" of the core. The next theorem shows how these two distinguished core solutions can be calculated by making use of well-known perturbation techniques for the transportation problem\*

We define two kinds of perturbation (F1) and (F2), by the following transformations, where the arrow " $\rightarrow$ " means "is replaced by":

$$(PI) \begin{cases} a_i \rightarrow a_i & \text{for } i \in I, \quad \forall i \\ b_j \rightarrow b_j + e & \text{for } j \in J, \quad b_{n+1} \rightarrow b_{n+1} \\ \text{where } 0 < e < \frac{1}{2(n+1)} \end{cases}$$

$$(P2) \begin{cases} a_i \rightarrow a_i + e & \text{for } i \in I, \quad a_{m+1} \rightarrow a_{m+1} \\ b_j \rightarrow b_j & \text{for } j \in J, \quad b_{n+1} \rightarrow b_{n+1} + me \\ \text{where } 0 < e < \frac{1}{2(m-1)} \end{cases}$$

As shown in Dantzig [6] either of these perturbations, when applied to a transportation problem, gives a primal non-degenerate problem. Srinivasan and Thompson [16] showed that, given integer rim data, the primal solution  $X(e)$  to the perturbed problem, when scientifically rounded, yields an optimal integer primal solution  $T(X(e))$  to the original problem.

THEOREM 2. Let (6) be a market game with integer rim data and let (16) and (17) be the corresponding transportation problem; we assume (for convenience) the latter is dual non-degenerate.

(A) The dual solutions to (16), (17) after applying perturbation (PI) give the core vector pair  $(u, v^{\wedge})$ .

(B) The dual solutions to (16), (17) after applying perturbation (P2) give the core vector pair  $(u^{\wedge}, v^*)$ .

For a proof see [18].

The computational importance of Theorem 2 is immediately obvious. For by solving just two transportation problems it is possible to find the two distinguished extreme points of the core  $(u^*, v^{\wedge})$  and  $(u^{\wedge}, v^*)$ . By using one of the current transportation codes [1, 5, 9] this computation can be made

in a few seconds or minutes, even for problems having hundreds of buyers and sellers. Since the core tends to be long and thin with the other points in the core usually lying quite close to the line segment between these two extreme points, finding them already gives a very good idea of what the core is like.

In [18] the author gave an algorithm for finding all the (finite number of) extreme points of the core of a transportation market game. The algorithm consisted of shifting cells that ship zero in the optimal basis according to certain rules which were described there.

In the case of assignment problems every basic optimal solution to (16) and (17) has exactly  $2n+1$  basis cells of which  $n$  ship 1, and  $n+1$  ship 0. One of these cells that ship 0 is always the cell  $(n+1, n+1)$ . Hence, there are always 11 (other) ~~cells that ship zero and which are distributed among the  $n+1$  rows of the assignment tableau.~~ We will make use of this fact in the proof of Theorem 4.

#### 4. NEW RESULTS FOR CORES OF ASSIGNMENT MARKET GAMES

We present here some new results concerning the structure of cores of assignment market games- Only sketches of the proofs of these results will be given because many of the technical details are contained in other papers by the author and are too long to be included here.

The first theorem compares the solutions of two assignment market games both solved at either the buyer surplus or seller surplus points and which have different numbers of sellers or buyers. We assume supplies and demands have been changed according to (12)-(15) to make solutions possible.

THEOREM 3. (A) If we maintain ~~the~~ buyer surplus solutions while adding sellers or dropping buyers then seller prices are non-increasing and buyer surplus are non-decreasing.

(B) If we maintain maximum seller surplus solutions while adding buyers or dropping sellers then seller prices are non-decreasing and buyer surpluses are non-decreasing.

Proof. (A) Reference [20] gives an algorithm for assignment problems which starts with all the buyers and only one dummy seller and proceeds by adding the real sellers one by one while maintaining a maximum buyer surplus solution. It was shown there that seller prices are non-increasing and buyer surpluses are non-decreasing. The proof for the case of dropping buyers requires the development of a similar algorithm, and will not be given here. The proofs of the statements in (B) are similar.

The results given in Theorem 3 agree with our economic intuition that having more buyers is better for the sellers and having more sellers is better for the buyers.

THEOREM 4. The maximum number extreme points in the core of an assignment market game is  $\binom{2n}{n} J$ .

Proof. As remarked at the end of the preceding section, every basic optimal solution assigns  $n$  cells that ship zero to the  $n+1$  rows of the assignment tableau. This is a special case of the Ehrenfest model for 1-dimensional distribution of gas molecules, and is also a special kind of the combinatorial problem of counting the number of ways of putting  $r$  balls into  $n$  boxes discussed on p. 121 of [9]. According to the result given there, the number of ways of putting  $n$  balls in  $n+1$  boxes is  $\binom{2n}{n}$ .

It is easy to show that

$$2 \cdot 3^{n-1} \leq \binom{2n}{n} \leq 2 \cdot 4^{n-1}$$

so that the ~~number~~ number of core points for an  $n \times n$  assignment market game increases exponentially with  $n$ . As remarked in [13], it is unlikely that all

all the core points for general assignment games will be computed except for small values of  $n$ .

We next show that for every  $n$  there is an assignment game having a maximal size core. For this purpose we define a matrix  $A$  with entries

$$a_{ij} = \frac{1}{a^{i+j-2}} \tag{21}$$

where  $a$  is a positive integer. Then the assignment game with cost matrix

$$C = a^I + A \tag{22}$$

where  $I$  is the  $n \times n$  identity matrix, will be called a dominant diagonal matrix

As an example let  $a \gg 2$  and  $n \gg 3$ ; then

$$C = \begin{pmatrix} 32 & 8 & 4 \\ 8 & 20 & 2 \\ 4 & 2 & 17 \end{pmatrix} \tag{23}$$

Notice that  $C$  is constructed so that the entries on the main diagonal are large. By applying the algorithm given in [18] it is possible to show that the following theorem holds.

**THEOREM 5.** The  $n \times n$  assignment matrix game whose bid matrix is the dominant diagonal matrix of (22) has a core with the myfinti number of extreme points given in Theorem 4\*

**THEOREM 6.** Let  $C$  be the bid matrix of an assignment game: if  $C > 0$ , i.e.,  $c_{ij} > 0$  for all  $i$  and  $j$ , then the core of the game has an even number of extreme points.

Proof. Let  $(u,v)$  be an extreme point of the core of the assignment game. Then either there is an  $i$  such that  $u_i = 0$  or there is a  $j$  such that

$v_j \neq 0$  (or else it is easy to show that  $(u,v)$  is not an extreme point of the core.) Suppose the former, that there is an  $i$  with  $u_i = 0$ . Then, since

$$v_j = u_i + v_j \geq c_{ij} > 0$$

for all  $i$  and  $j$  it follows that  $v_j > 0$  for all  $i$ . Let  $w = \min v_j$ . Then the pair  $(u', v')$  where

$$u'_i = u_i + w \text{ and } v'_j = v_j - w$$

is also an extreme core point. Since  $(u,v)$  and  $(u',v')$  are distinct extreme core points which are uniquely related, it follows that the core extreme points can be paired together so that there are an even number of them.

The result in Theorem 6 can be used to improve the algorithm in [1].

THEOREM 7. (A) The core of the  $n \times n$  assignment game with bid matrix  $C - I$ , where  $I$  is the  $n \times n$  identity matrix, has  $2^n$  extreme points.

(B) The core of the game  $C * I + aE$ , where  $E$  has entries  $e_{ij} = 1$  for all  $i$  and  $j$ , has  $2^n - 2$  extreme points.

Proof. (A) Let  $u$  be any binary vector, that is, a vector whose components are 0 or 1, (except  $u_i = 0$ ), and let  $v$  be the complementary vector that is, with the 0's and 1's interchanged (and  $v_i = 0$ ). Then it is easy to check that  $(u,v)$  is an extreme core point of the market game.

Clearly, there are  $2^n$  binary vectors  $u$ , hence that number of core points.

(B) Except for the maximum buyer surplus and the minimum seller surplus points, each extreme point  $(u,v)$  of the game  $I$  splits into two extreme points for the game  $I + aE$  when we either add  $a$  to all the components of the  $u$  vector or else add  $a$  to all the components of the  $v$  vector. Hence there are  $2(2^n - 2) + 2 * 2^{n-1} - 2$  extreme points in the core of the game  $I + aE$ .

By considering the structure of the basis of an optimal basic solution of an  $n \times n$  assignment game it is easy to prove the following theorem.

THEOREM 8. If  $C$  is the bid matrix of a positive  $n \times n$  assignment market game and either

- (i)  $c_{ij} = c_i$  for  $i, j = 1, \dots, n$ , that is, if the sellers goods are equally attractive to all buyers, or
- (ii)  $c_{ij} = c_j$  for  $k, j = 1, \dots, n$ , that is, if the buyers have identical tastes,

or both, then the core of the game has exactly two extreme points.

#### 5. NEW RESULTS FOR CORES OF TRANSPORTATION MARKET GAMES

We present some new results on the structure of the core of a transportation market game. We find that the size of the core varies directly with the amount of primal degeneracy of the problem, being smallest (2 points) for a non-degenerate transportation game and largest for a semi-assignment game.

DEFINITION 4. A proper submarket of a transportation market game consists of subsets of sellers,  $I_1 \subseteq I$ , and buyers,  $J_1 \subseteq J$ , with at least one of  $I_1$  or  $J_1$  being proper subsets (of  $I$  or  $J$ , respectively) such that the sellers in  $I_1$  can exactly satisfy the demands of the buyers in  $J_1$ . A primal, non-degenerate transportation market game is one which has no proper submarkets.

Note that an assignment market game is maximally primal degenerate, since any set of  $k$  sellers can satisfy any set of  $k$  buyers. Also, an  $m \times n$  semi-assignment problem has a proper submarkets for each seller, say seller  $i$ , who can exactly satisfy the needs of any subset having exactly  $a_i$  buyers.



THEOREM 9. A primal non-degenerate positive transportation market game has a core containing exactly two extreme points.

Proof. Since the game is positive, it has an even number of extreme points by Theorem 6, hence its core has at least two extreme points. The algorithm given in [18] showed that extreme points of the core were obtained from other extreme points by "shifting zeros." However, it is easy to see that a zero  $x_{ij}$  value implies that there is a proper submarket. Hence if the core of the game had more than two extreme points, the game must have at least one proper submarket, contrary to hypotheses\*

THEOREM 10. Suppose  $m < n$ , and  $C$  is an  $m \times n$  positive transportation market game; then the maximum number of extreme points in its core is  $\binom{2m}{m} J$ ; this certainly occurs in the case of a semi-assignment market game.

Proof. As remarked above any semi-assignment market game has many submarkets since seller  $i$  can satisfy the needs of any subset of  $a^i$  buyers. It follows that in any primal solution there are  $m$  zero values of  $x^i$ , hence by the same argument as given in Theorem 4 there are at most  $\binom{2m}{m} J$  extreme points of the core.

DEFINITION 5. By the fair division point of a transportation market game we shall mean the imputation

$$(u^f, v^f) \ll \frac{1}{2} [c^* + (u^*, v^*)]$$

where  $u^i, u^*, v^i,$  and  $v^*$  were characterized in Theorem 1.

The following theorem is obvious from Theorem 1 and Definition 5.

THEOREM U. The fair division Imputation  $(u^f, v^f)$  has the following property: the amount received by each buyer and each seller is exactly half way between the maximum amount and the minimum amount he can receive in any feasible imputation.

Because of the property given in Theorem 11, it is likely that the fair division imputation could be useful in solving practical allocation problems.

## 6. AN AUCTION APPLICATION

Use of transportation market games to solve auction models was suggested by the author in [19]. We explain it in terms of the house buying example of Shapley and Shubik [10]. In that example it is assumed that there are 3 sellers, each of whom has a house to sell, and three buyers, whose evaluations are shown in Figure 1.

Shapley and Shubik derived from these evaluations the assignment market game shown in Figure 2. The entries in the assignment game are

$$c_{ij} = \text{Max}[e_j - s_i, 0]$$

where  $e_j$  is the evaluation of buyer  $j$  and  $s_i$  is the evaluation of seller  $i$ . Note that the 0 in the lower right hand corner of Figure 2 came from the fact that  $e_3 = 17,000$  and  $s_3 = 19,000$  so that  $e_3 - s_3 = -2,000$  and so  $c_{ij} = 0$  from the above rule.

Figure lists all the extreme points of the core, a graph of which is shown in Figure 4. Note the fair division point, and observe that at  $(u^f, v^f)$  all buyers and sellers get positive surpluses, and also that the selling prices are exactly half way between their maximum and minimum values.

Because of the ease of computing the buyer and seller surplus points, and hence of the fair division point, the use of this method to solve large auction problems having hundreds of buyers and sellers is very practical.

## 7. QUALITATIVE REMARKS ON CORE SIZE; EXAMPLES.

The results of this paper show that the size of the core of a market game,

as measured by the number of extreme points, depends on two things: first, the degree of primal degeneracy of the problem, the core size being greatest for assignment problems and least for transportation problems having no submarkets; second, on the degree of divergence of tastes of the buyers, the core size being greatest when tastes are most divergent and least when they are identical. A complete explanation of this variation would require further investigation.

The reader may wish to get more of an idea as to what cores of market games look like. In Figure 5 all the qualitatively different cores for  $2 \times 2$  assignment market games are sketched. Note that the possible numbers of extreme points are 2, 3, 4, 5, and 6. Also several selected examples of cores for  $3 \times 3$  assignment games are sketched. In addition a  $3 \times 3$  core having the maximal size (20) is shown in Reference [18].

Houses	Evaluation	Buyer's Evaluations		
		Buyer 1	Buyer 2	Buyer 3
Seller 1	\$18,000	\$23,000	\$26,000	\$20,000
Seller 2	15,000	22,000	24,000	21,000
Seller 3	19,000	21,000	22,000	17,000

Figure 1. Shapley-Shubik House Auctioning Example.

5	8	2	1
7	9	6	1
2	1	0	1
1	1	1	

Figure 2. Derived Assignment Game.

Core Extreme Point	Buyer Surpluses			Selling Prices			Seller Surpluses		
	1	2	3	1	2	3	1	2	3
1*	1	3	0	23	21	20	5	6	1
2	2	3	0	23	21	19	5	6	0
3	1	4	0	22	21	20	4	6	1
4	2	5	0	21	21	19	3	6	0
5*	2	4	1	22	20	19	4	5	0
6*	2	5	1	21	20	19	3	5	0
Fair Division Point	$\frac{3}{2}$	4	$\frac{1}{2}$	22	$\frac{21}{2}$	$\frac{19}{2}$	4	$5\frac{1}{2}$	$\frac{1}{2}$

Figure 3. Extreme points of the core for the Shapley- Shubik example. Point 1 is the seller surplus point, and 6 is the buyer surplus point. The fair division point is the average of these two.

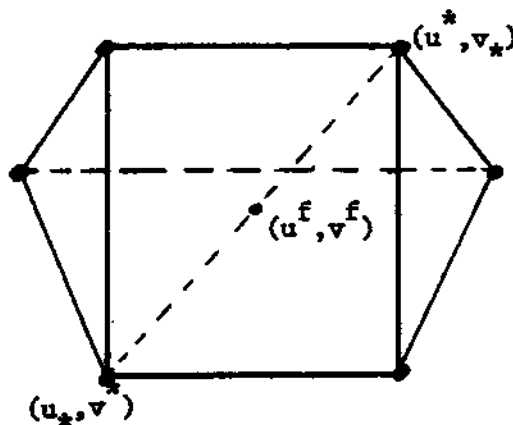


Figure 4. Core for House Auction.

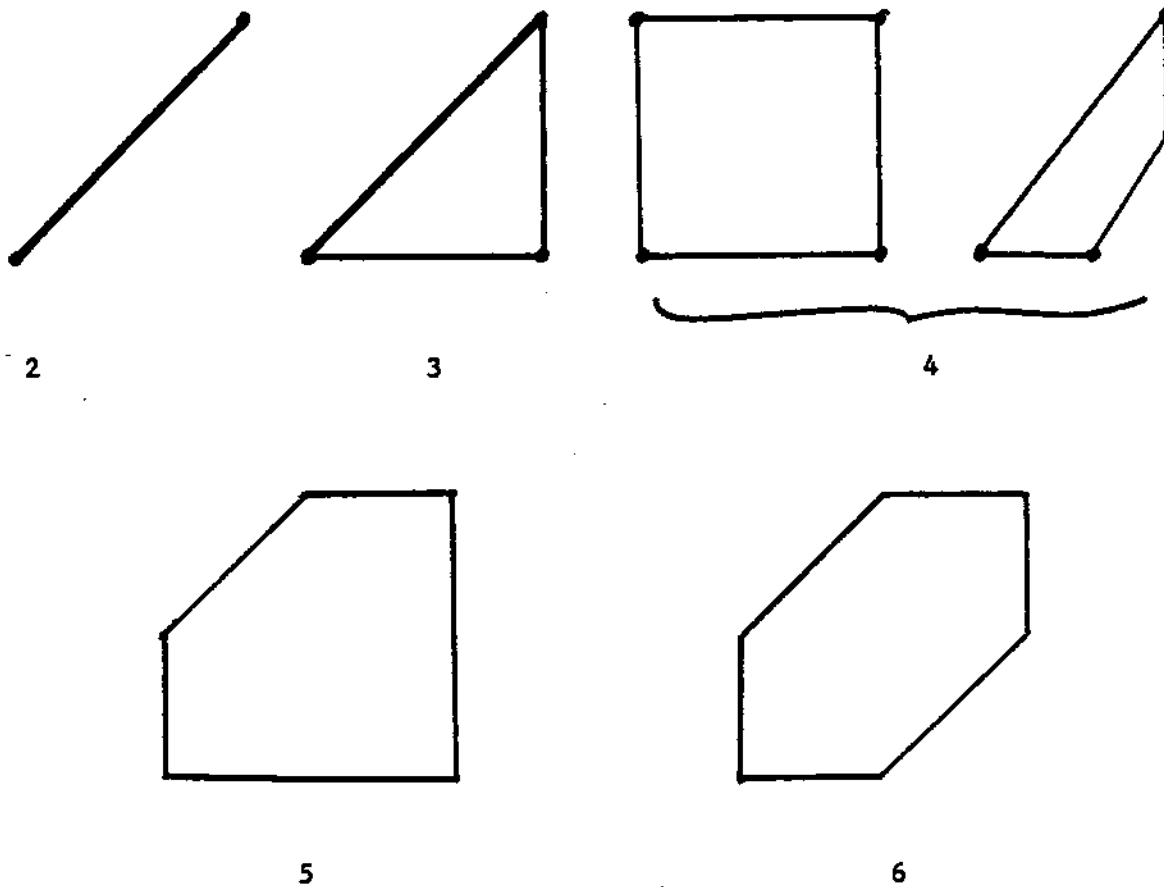


Figure 5. All the qualitatively different cores  
for a 2x2 assignment game.

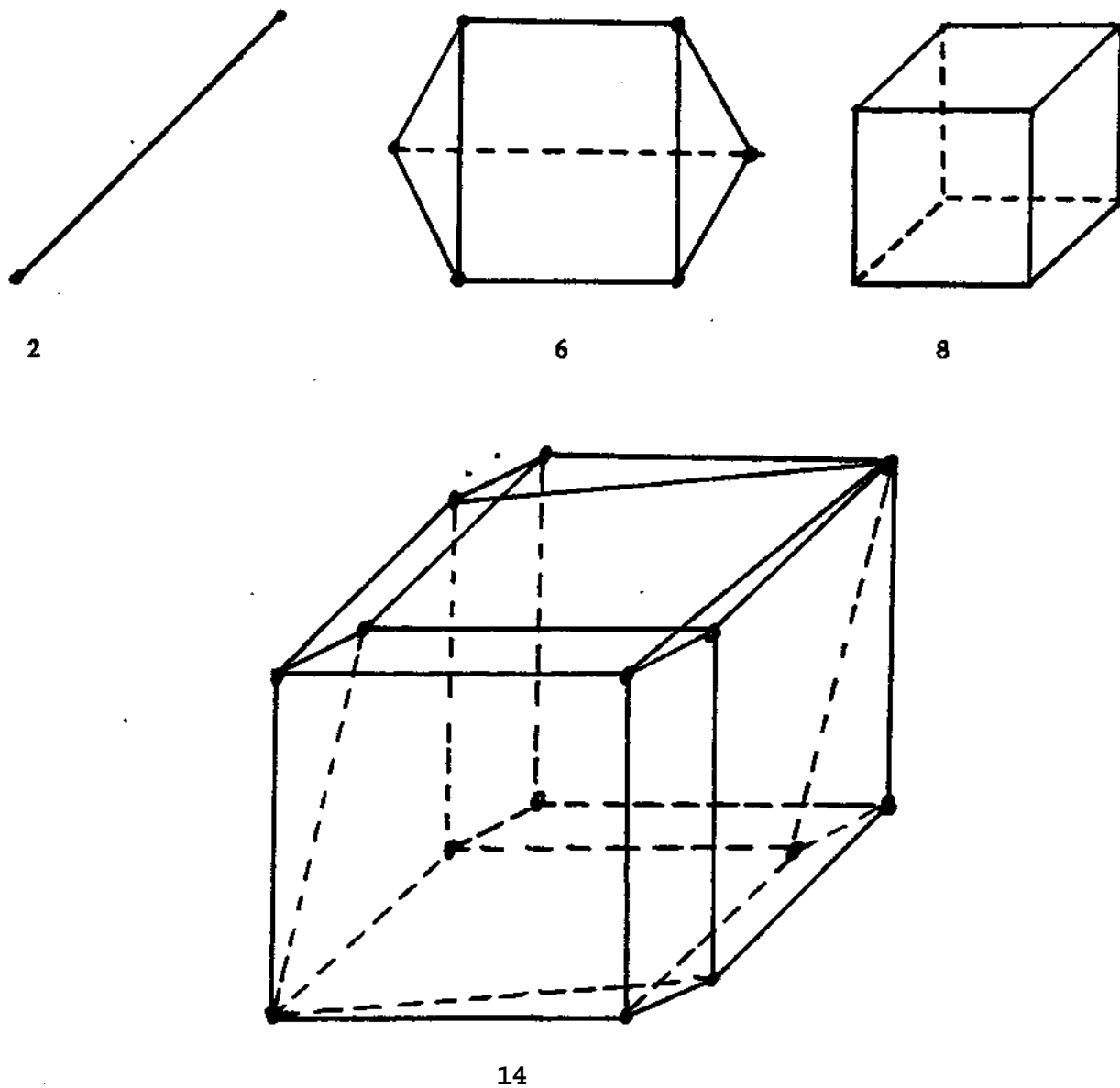


Figure 6. Selected Cores for 3x3 assignment games. A picture of a 3x3 core having 20 extreme points appears "in"reference [18].

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