An Index of Controllability for Linear Deterministic Processes

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1. Abstract

Process control research has concentrated on designing controllers to improve the performance of existing processes. Assuming an optimal but realizable controller, it is intuitively evident that there are some inherent limits on how well a process can be controlled (e.g., time delays). This limit suggests that one can develop an index which measures the "controllability" of a process. We have developed such an index in this study.

Controllability is defined as the ability of the process to move quickly and smoothly from one operating condition to another and to deal effectively with disturbances. Our index is defined as the minimum time necessary to overcome the worst expected disturbance and/or setpoint change. It accounts for the presence of process time delays and constraints, each of which can profoundly affect one's ability to control a process. While not a focus of this study, it can also account for nonlinearities.

In our model, disturbances **come** from a known family of possibilities. Using measurements, the optimal controller has to decide when disturbances have hit, identify which they are, and determine their magnitude. With delays disturbances can enter the process well in advance of their detection. Between the time the controller detects that disturbances have occurred and when it then identifies which they are, the

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controller for our index positions the process to minimize its worst potential performance. With each new measurement, the controller recomputes its next move and is thus expressed in the form of a multilevel optimization problem.

A multilevel optimization problem can be very difficult to evaluate. We show how to simplify this problem for some important special cases.

2. Introduction

Process design is done largely without any consideration regarding controllability. Heuristics such as overdesign, SISO rules of thumb, and simulation have been used to evaluate controllability, but these sometimes fail (Holt, 1984). Past experience has shown that understanding the effects of design on control are desirable. Doyle (Doyle, 1986) has indicated that good control research is not how to design control systems that can get the best possible performance out of a large MIMO system but rather *"the most important role a control engineer can play in the process industry as a whole is to decide how to build plants."*

It has been shown previously (Morari, 1983) that the best performance that any controller may achieve on a process (i.e. its controllability) is a function of the process itself. One can evaluate controllability independent of the controller by using an optimal controller. Morari (Morari, 1983) defines controllability as "the ability of the process to move fast and smoothly from one operating condition to another and to deal effectively with disturbances." Controllability is difficult to define mathematically and is dependent on the specific problem under investigation.

Existing measures can be categorized as steady state or dynamic. Steady state measures include those based on structural controllability and the relative gain array (RGA). Structural controllability indicates whether a plant can be taken to the origin from any initial point. It is guaranteed if the rank of the controllability matrix is full. RGA is a measure of interaction, and it guides the designer into choosing what are the pairings that minimize interactions of MIMO systems (Stephanopoulos, 1984).

Dynamic measures include Bandwidth, and the process reaction curve for SISO processes. For MIMO systems, these measures include Minimum Necessary Delay (Perkins, 1985), those based on the Internal Model Control (IMC) structure (Garcia, 1982), and Singular Value Analysis (SVA) for MIMO systems Control systems that have large bandwidth are capable of tracking setpoint changes at high frequencies (Friedland, 1986). The process reaction curve assumes that most processes have a sigmoidal shaped response curve which can be approximated as a first order process (time constant t_G) with a delay(fy). An heuristic indicates that for $t^\wedge ty$ ratios less than 3, processes have poor controllability and for ratios greater than 7 processes have good controllability (Reimann, 1986).

The Minimum Necessary Delay is a simple to calculate scalar measure. It indicates the minimum time after which all outputs can be specified independent of the inputs. IMC was used by Holt (Holt, 1985) to obtain bounds on the controllability of linear systems. It has been used for systems containing nonminimum phase elements (time delays and right half plane zeroes). Finally, SVA provides a closed loop measure of sensitivity of the process to model errors. Processes with small condition numbers (for a frequency range) tolerate model/plant mismatch better than processes with large condition numbers.

MND and IMC are usually sufficient in the case where the process dynamics are dominated by nonminimum phase elements since these elements deteriorate the plant the most. However, these methods are limited to linear systems and cannot handle constraints or measurement delays. Since most chemical processes are nonlinear, have measurement delays, and operate at constraints for economic reasons, a new index is needed.

In this research we have developed an index to evaluate controllability in the time domain that can be used to design and/or retrofit chemical processes for improved control.

3. Description of Index: Properties Desired

As was mentioned above, controllability is very difficult to quantify. There are many aspects to be considered so that a tradeoff must be made between what can be solved and what is to be solved. We desire an index that

- is independent of
    1. the controller
    2. the disturbances/setpoint changes
- and can take into account
    1. process constraints
    2. nonlinearities
    3. time delays
    4. model uncertainty
    5. stochastic models

The following sections dicuss how some of these properties can be obtained.

3.1. Optimal Controller

We obtain an index independent of the controller by choosing a minimum time optimal controller. That is, instead of evaluating the process with any number of different controllers, we have chosen the *best* controller possible, making the results unique for a given process.

3.2. Disturbance/Setpoint Change

We assume that a set of disturbances and/or setpoint changes that are likely to occur is available. Out of this set, we pick the worst disturbance to reject or the worst setpoint change to track, whichever results in the largest minimum time.

3.3. Process Constraints and Nonlinearities

Although the frequency domain is very powerful for analysis, it has its limitations: constraints on the inputs and states are difficult to handle, the methods are limited to linear systems, and results are sometimes difficult to interpret Since our problem is formulated in the time domain, it can handle process contraints and nonlinearities in the model.

Since our problem is formulated in the time domain, adding constraints on the inputs and their derivatives can be easily done. For example, constraints on the first derivative of the inputs, that is, constraints on how quickly a valve can open or close, are dealt with by augmenting the states of the system with these inputs and letting their derivatives become new input variables.

3.4. Time Delays

Time delays are very common in chemical engineering processes. Delays in applying control or in obtaining measurements can adversely affect controllability. Process delays, in general, deteriorate dynamic performance, since the controller cannot affect and/or cannot know about the state of the process immediately. Input delays occur when the manipulated variables cannot affect the process immediately. A common example of input delays is that of transportation lag, where a fluid takes a finite time (the delay) to flow through a pipe.

Measurements in chemical engineering are often characterized by time delays (e.g. composition analysis). These measurement delays deteriorate the attainable controllability in the sense that the controller only has old information about the state of the process. The controller is not able to react immediately to a disturbance. The effect of measurement delays on the dynamic performance of the process has usually been ignored in past indices. Section (6) illustrates how single and multiple input and measurement delays can be included into our controllability index.

3.5. Model Uncertainty

Since all models are only approximations of the true process, the index should also take into account model uncertainty. Although very important, this aspect has not been studied in this work.

3.6. Stochastic Models

Most chemical processes have disturbances that enter the process in a stochastic manner and measurements that are corrupted by noise. Systems described by stochastic models will be dealt with in a companion paper (Carvallo, 1988a).

4. Index Formulation for Case without Delays

Given a set of expected inputs, setpoint changes to track, and disturbances to reject, we define our index as *the minimum time necessary for the process to overcome the worst expected input (disturbance and/or setpoint change)*. If delays and noise do not exist, this optimization problem can be formulated mathematically in the time domain as the following optimal control problem:

Problem P1

$$\max_{r, d_k \in D} \; \min_{u_k \in U} \; t_f \tag{1}$$

$$s.t. \quad x_{k+1} = f(x_k, u_k, d_k) \tag{2}$$

$$x_0 \; given \tag{3}$$

$$g(t_f, x_{t_f}, u_{t_f}, r) = 0 \tag{4}$$

$$h(x_k, u_k) \leq 0 \tag{5}$$

where the set D contains all possible inputs (disturbances and setpoint changes) coming into the process, and U is the bounding set on the manipulated variables. More will be said about the form of the disturbances in the following section. In the objective (1), the outer *max* operator searches for the worst disturbance (worst in a minimum time sense), and the inner *min* operator searches for the optimum minimum control policy to reject that worst disturbance.

Problem P1 is independent of the controller because we have chosen an optimal controller. That is, instead of evaluating the process with any number of different controllers, we have chosen the *best* controller possible, making the results unique for a given process.

This optimal control computation is carried out in the time domain so that constraints, and nonlinearities **can be handled.** Constraint **(4)** may consist of driving **the** output error to zero (e.g., $r - Cx_{t_f} = 0$). Also, we almost certainly want **the** process to arrive and stay at the final condition. This requirement can be expressed mathematically as $x_{t_f} = f(x_{t_f}, u_{t_f}, d_{t_f})$.

If we assume that the nominal point is optimal, then the time to drive the process back to the nominal point (or to a new nominal point) will result in lost profit. We have chosen minimum time to minimize this lost profit. Minimum time controllers are usually not implemented in practice because the computations are difficult. Since we are not doing these computations on-line, computation time is not the main issue. The authors (Carvallo, 1988b) have developed a mixed integer linear program formulation for solving efficiently in one pass the minimum time optimal linear control problem.

As defined in Problem PI, the index cannot take into account measurement delays, and/or model uncertainty. A method to handle measurement delays will be described in the next section.

4.0.1. Example: Distillation Column

We illustrate with this example the impact of constraints and of different disturbances (here in the form of setpoint changes at time $k = 0$) on the time to overcome a disturbance. A continuous linear model for a high purity binary distillation column was obtained from (Morari, 1988). For a discretization time of 5 minutes, we obtained the following linear discrete model

$$x_{k+1} = \begin{bmatrix} 0.9355 & 0.0 \\ 0.0 & 0.9355 \end{bmatrix} x_k + \begin{bmatrix} 5.660*10^{-2} & -5.570*10^{-2} \\ 6.978*10^{-2} & -7.068*10^{-2} \end{bmatrix} u_k \qquad (6)$$

where u_k and x_k are given by $[L \ \ V]^T$ and $[y_d \ \ x_b]^T$ respectively. L, V, y_d, and x_b are the reflux and vapor flowrates, and top and bottom compositions.

For a specific setpoint change from 0 to r, Problem PI is easily formulated and solved as the following MILP (Carvallo, 1988b)

<u>Problem P2</u>

$$\min_{u_k \in U} \quad t_f + \epsilon \|u_k\|_1 \tag{7}$$

$$\text{s.t.} \quad t_f = 1 + \sum_{i=1}^{n_{binary}} z[i]\, 2^{i-1} \tag{8}$$

$$\|(\Phi - I)x_k + \Gamma u_k\|_1 + \|r - Cx_k\|_1 \leq (|S_k| - \sum z[i] + \sum (1 - z[i]))M \tag{9}$$

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{10}$$

$$x_0 \text{ given} \tag{11}$$

$$S_k = \{\, i \mid 1 + \sum_{i=1}^{n_{binary}} 2^{i-1} = k \,\} \tag{12}$$

$$z[i] \in (0,1). \tag{13}$$

Since the discrete minimum time optimal control problem has usually more than one control policy to achieve the same minimum time, the term $\epsilon\|u\|_1$ with $\epsilon\|u\|_1 \ll t_f$ is added to the objective 7 in order to pick the policy with the minimum effort. Constraint (9) assures that the final conditions are met *only* at the final time *t_f*. The 1-norm used above was chosen for simplicity since the 2-norm would require the solution of a more complex Mixed Integer Nonlinear Programming (MINLP) problem. The resulting MILP's were solved using ZOOM with the modeling language GAMS (Kendrick, 1985). Carvallo et. al. (Carvallo, 1988b) show computational details for solving Problem P2.

Table 4-1 shows the resulting time to overcome different setpoint changes and different bounds on the inputs. The setpoint changes are for top and bottom compositions. The large difference between the minimum times for r=(0.02,0.03) and r=(0.02, 0.0) is as predicted by Skogestad (Morari, 1988). The direction of r=(0.02,0.0) in the disturbance (setpoint) space gives a high disturbance condition number, indicating difficulty for the controls to handle it. Finally, note with the last three the high sensitivity of the minimum time to the manipulated variable bounds.

r	||u^|| bounds	time (minutes)
(0.02,0.03)	1	15
(0.02,0.0)	1	120
(0.02,0.0)	1.5	60
(0.02,0.0)	2	40

Table 4-1: Computed Minimum Times for Distillation Example

5. Disturbance Model

The objective function (1) of Problem PI requires a search over the disturbances in the set *D.* However nothing has been said about the disturbances.

It is not desirable to get an index that works for only one disturbance. On the other hand, it is not practical to calculate the index for all possible disturbances since that situation is not common. In fields like mechanical and electrical engineering, disturbances are usually assumed as white noise. In chemical engineering, the mechanism of how the disturbances affect the process is often known (Morari, 1980). White noise, which can be thought of as an impulse in the time domain, is not very descriptive of disturbances in chemical engineering, whereas a step is more commonly encountered.

Our controllability index (Problem PI) is based on a worst case analysis. For the disturbances considered, we want the index to include those disturbances that deteriorate the process the most. Process control disturbances causing major difficulties are usually of low frequency. We shall therefore assume that disturbances to be rejected can be modeled deterministically as step changes of unknown magnitude ($d_k = d^m$).

Based on this assumption, we can define the occurrence of a single abrupt change (step disturbance of large magnitude) in a process as an *event.* We can partition the problem type into three classes depending whether more than one event can occur simultaneously. These three classes are illustrated in figure (5-1) for a process with two disturbances. Class /, referred as the single-event class assumes that the likelihood that more than one abrupt change occurring simultaneously is very low, and that one can reject an old event before facing a new one.

Class // assumes that multiple events can occur simultaneously and that tlic^e multiple simultaneous

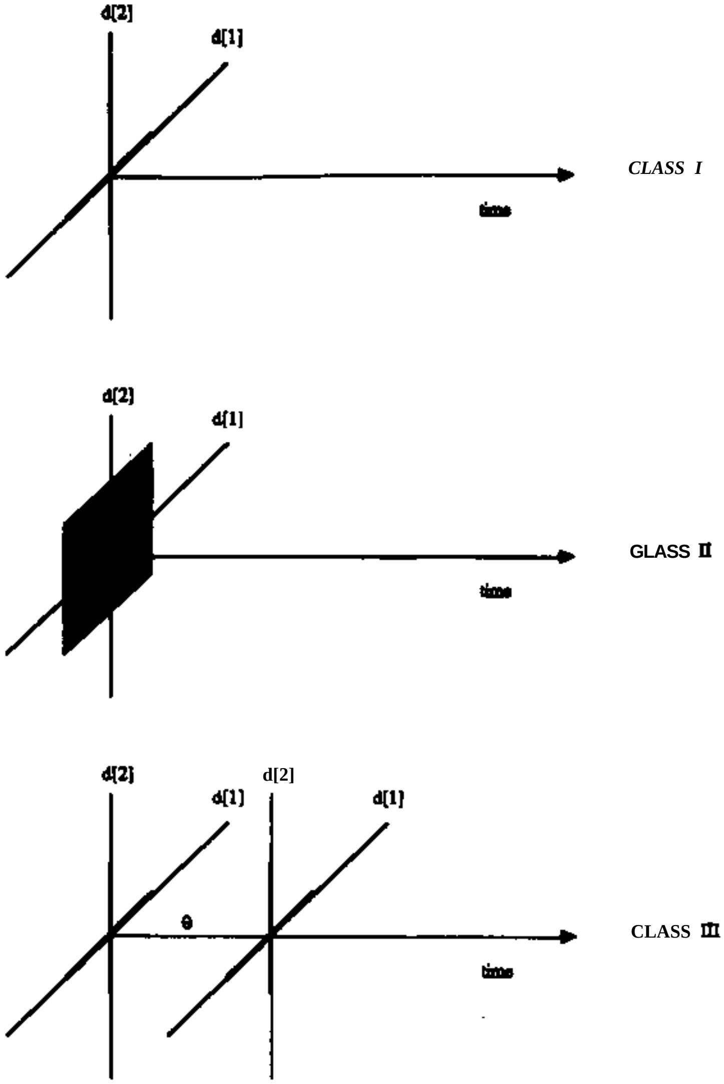


Figure 5-1: Classes of Events for a Two Disturbance Process

events can be rejected before new ones occur. This class we call simultaneous multiple-event. Class /// allows a second event to occur 6 time units after event 1 and before the first event is rejected. We will not consider class /// events in this study.

If we consider more than two disturbances, we may construct more hypothetical classes of how the events may occur. For example, we may consider a combination of classes / and // occurring at different times. However, more complex cases are less likely to occur and therefore will not be considered.

Computing the index for class / events is simpler than for class // events. For the example shown in figure (5-1) (2-dimensional disturbance vector), die outer max operator of Problem PI requires solving only 4 optimal control profiles for class /, that is, two profiles for each disturbance direction. The multiple event case requires a search over die bounded 2-dimensional space *D* (shaded region of class // in figure (5-D).

In the remainder of this paper we shall show how to compute the index for the single-event class of problems. This class is a fair representation of many control situations in chemical processes. We will also formulate how to compute the index for the multiple-event class and solve an example with some major simplifications.

6. Index Formulation for Case with Delays

The solution to the minimum time optimal control problem described in Problem P2, or any optimal control problem is sometimes misleading, particularly for processes with delays. This is illustrated in figure (6-1). The control objective is to minimize the Integral Square Error (ISE) for a process to track a, square setpoint trajectory. The time of the setpoint changes are at future unknown times. The resulting output profile, shown in figure (6-1.a), has the inputs reacting before they know about the setpoint changes. Since the controller should not be predicting the setpoint changes which here are our disturbances, the above performance cannot be achieved. The nonpredicting optimum profile is shown in figure(6-l.b).

6.1. An Implementable Controller

An optimal *realizable* controller such as illustrated in figure 6-2 must prevent the control law from being based on prediction. To prevent prediction we shall see here that we must solve a discrete moving horizon optimization problem (Jang, 1987). The horizon is usually defined as an on-line tuning parameter (i.e, when ISE is the objective). Since we use a minimum time optimal controller, the minimum time necessary to bring die process to the desired final condition is a natural choice for the horizon. The optional term e l l ^ i « yean be added to the objective function (as in (7) in Problem P2) so that, whenever more than one control profile can achieve minimum time, the one that requires minimum effort will be chosen.

As shown in figure (6-3), we obtain measurements every A i ^ ^ time units and compute the u_k profile from the current time *k* to the horizon *tj* (current estimated minimum time). We assume the time to complete these computations is small and can be ignored for chemical processes. The newly computed

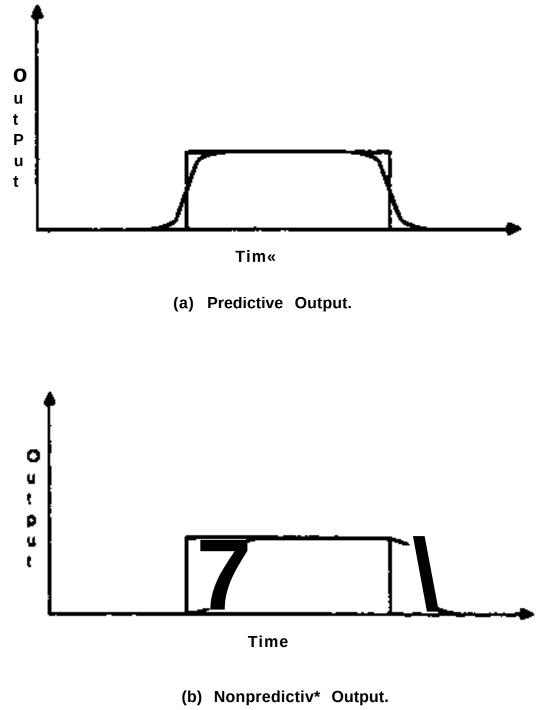


Figure 6-1: Example to Illustrate the Problem of Prediction

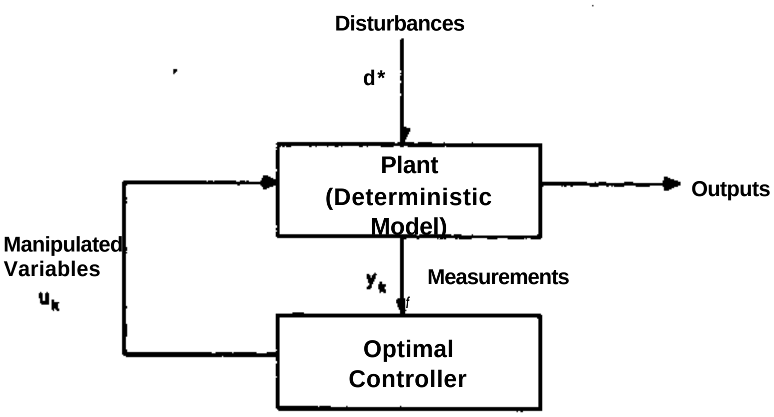


Figure 6-2: Diagram for Optimal Controller

control is applied for tsk_{meas} time units. At this point a revised control is available due to the complerion of the next measurement and compute cycle. In the limit where Ajfc,,,^ approaches zero (e.g., continuous measurement), the resulting output profile will approach the output profile shown in figure (6-l.b).

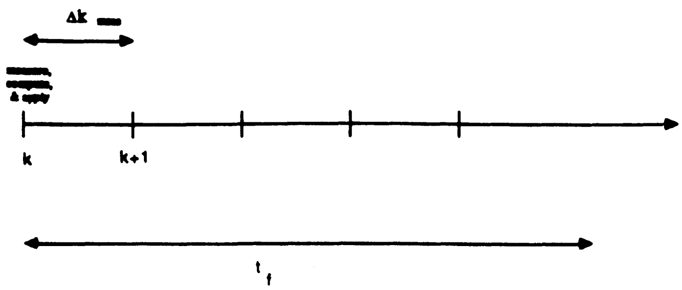


Figure 6-3: Illustration of Optimal Controller

Based on the above information, after an event has been detected and at each time k, a simple algorithm for the implementation of our controller is as follows

Algorithm 2

1. Estimate d_k
2. If d_k differs from that used to solve the previous optimal control problem, then solve Problem P1 to reject d_k. Apply u_k from k to k+1. Otherwise apply u_k from the previous optimal control problem.
3. Iterate until the terminal condition is reached (zero error at rest).

7. Detection and Identification Times

7.1. Definitions

In order to unify the terminology used when referring to our controller, it is helpful to refer to figure (7-1). After a disturbance enters the process (defined as time zero), time elapses due to delays and/or process/measurement noise before the controller is able to *detect* that something has happened (detection time, t_D). After detecting the change, we start *discriminating* among disturbance models, with their parameters being *estimated* simultaneously at a lower level. The time when we know exactly which disturbance (model) entered the process is the *identification* time (t_I). Following t_I, we can bring the process to the desired terminal state in minimum time, t_f.

In making the index independent of the controller, we need an optimal controller that will calculate the smallest t_f possible. Times t_D and t_I are fixed for linear systems. For nonlinear systems, one may be able to affect t_D, t_I, and t_f by manipulating u_k, making the problem more involved.

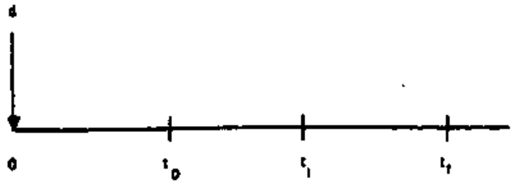


Figure 7-1: Detection, Identification, and Rejection of an Event

1.2. Control Strategy

There are three intervals that we need to consider in order to determine the optimal controls: from the time the disturbance occurred (initial time) until it is detected, between detection and identification, and after identication.

12.1. Initial Time to Detection

In the interval between 0 and t_D there is zero control action since the disturbance has not been yet detected.

12.2. Between Detection and Identification

The control policy for the interval between t_D and *tj* is more difficult to define because the actual disturbance that entered the process is not known and therefore cannot be rejected. After the event has been detected, and before we can identify which disturbance entered the process, we have several options for control: (i) do not do anything, (ii) make a move assuming that of all the possible disturbances, the worst has entered the process, or (iii) drive the process to a position that will be in the best "worst" position, i.e., that no matter which disturbance is identified, the worst time taken to recover from the current time *k* to y is the least it can be.

The following example illustrates why controller (iii) is the correct one. Assume that two disturbances d^l and &- are such that they can only be rejected by controls that have an opposite effect on the process. That is, the controls needed to reject d^l and & in minimum time drive the process in opposite directions. Also, assume that d^2 takes much longer to be rejected. If d^l enters the process, controller (ii) will try to reject d^2 from t_D to *tj*. After d^l is identified, the process will be far away from the terminal state.

If d^2 enters the process, controller (i) will not move from t_D to t_f. When d^2 is identified (ome r_7). controller (i) will take longer to reject $d^l{}_9$ than if controller (iii) were applied. Controller (iii) will place the process in such a way that at time *tj* it will typically but not always reject d^l and d^2 at the same *minimum* final time *tp*

One may argue that both controllers (ii) and (iii) will tend to reject the worst disturbance out of the given set of possible disturbances. The important difference is that the worst case controller (ii) will concentrate on the worst disturbance *alone.* Even though controller (iii) will tend to reject the worst disturbance, it will not overreact to it It will keep the process prepared to react to any of the possible disturbances, and not die worst disturbance alone.

Figure (7-2) illustrates controller (iii) for a two output *(y[l]* and *y[2])* dynamic process with output measurements delayed by 6[1] and 5[2], and subject to three possible events $(d^l{}_f\, d^2$, and $d^3)$. At time zero, the process is at rest at the origin. Since measurements are delayed by 5[1], the event will be detected at t_D «8[1]+1. Further, let us assume that the event can be identified at r_7»5[l]+2 so that $tj\text{-}t_D$ = 1. Assume that from t_D to t_l we do not know which of the three events occurred (possible events obtained from section (7.3.2). We need to calculate the controls u_k $(k\text{-}t_D = tf\text{-}l)$ that will place the process in such a position that the maximum time to recover from any disturbance to the origin will be the least it can be.

For the example illustrated in figure (7-2), at time $k\text{-}t_D$ this controller can be expressed mathematically as follows for a linear problem:

Problem P3

$$\min_{u_k} \quad \max \quad \min_{\bar{u}^m_{k,t}} \quad t^{?}_{kf} \tag{14}$$

•'• $\bar{K}$∧ - (y∧)A?+Yv'-ruj (is)

s.t. 3c£+1 - <D*£ + r«£ + A3? (16)

$$C\bar{x}^m_{k,t^m_{kf}} = 0 \tag{17}$$

$$\bar{x}^m_{k,t^m_{kf}} = \Phi\bar{x}^m_{k,t^m_{kf}} + \Gamma\bar{u}^m_{k,t^m_{kf}} + A3? \tag{18}$$

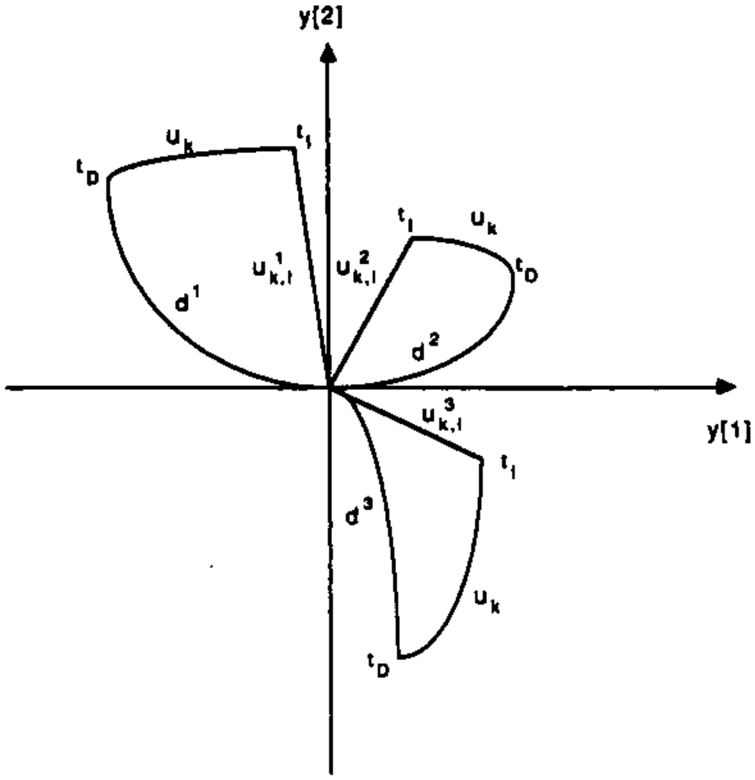


Figure 7-2: Output Profiles for Three Different Disturbances

$$|u_k|, |\bar{u}^m_{k,l}| \leq u_{max} \tag{19}$$

where * - /^, 3^ are the possible disturbance estimates at time *k,* t^{TM}_{kf} is the minimum time that it would take the process to recover from disturbance 3£, U£J (£ < / ≤ $t^{TM}_{k}j$) are the controls to reject 3£\ and $\bar{x}$£ ***(k < I <*** t^{TM}_{kf}***)*** **are used to model the effect of 3^ on the states. Note that Problem P3 assumes that all disturbances occur at time zero.**

The objective function (14) of Problem P3 was obtained as follows. The inner *min* operator in (14) calculates the controls ū£J *[k < I £* t^{TM}_{kf}*)* to reject each possible disturbance 3^. The inner *max* operator in (14) will pick the disturbance out of all the possible disturbances *m* that takes the longest to be rejected. Finally, the outer *min* operator chooses the current control u_k that minimizes the time for the system to recover from the worst disturbance.

All constraints in Problem P3 are for each disturbance a£\ Constraint (15) gives the *m* different initial conditions for the *m* different discrete models (16) resulting from using the same control u_k from *ktok-hl.* For the more general case where $k > t_D$, that is, when $t_f > r_D$+l, the controls *Uj* in (15) have already been

applied from earlier steps j for $t_D£j<$ k.

The terminal constraints (17) and (18) require that the system be back at the nominal point (17) at rest (18). Constraint (19) defines bounds on all manipulated variables.

The objective function (14) of Problem P3 is very difficult to solve as formulated. Fortunately, it can be reformulated as a simpler MHJP as follows. The max operator in the objective (14) of Problem P3 will pick up the *worst* disturbance *(3™k°™*)* for the process to recover in minimum time f^rs*. The process will then recover from the other disturbances (m * *worst)* at the same time or faster. For those disturbances 3^ for which $t_k^{TM}{}_f$ < $t\wedge^{rst}$ we can in principle find a set of controllers¯a£J from $t£J^{st}$ - *1* ^ to $t\wedge^{st}$ so that the process will remain at rest. We can then define a common final time at each time k (iJT*) and simplify Problem P3 to the following MILP

<u>Problem P4</u>

$$\min_{u_k, \bar{u}_{k,l}^m \in U} \quad t_{kf}^{worst} + \varepsilon \|u_k\|_1 \tag{20}$$

$$\text{s.t.} \quad t_{kf}^{worst} = 1 + \sum_{j=1}^{n_{binary}} z[j] 2^{j-1} \tag{21}$$

$$\|(\Phi - I)\bar{x}_{k,l}^m + \Gamma \bar{u}_{k,l}^m + \Lambda \bar{d}_k^m\|_1 + \|C\bar{x}_{k,l}^m\|_1 \le (|S_l| - \sum_{j \in S_l} z[j] + \sum_{j \notin S_l} z[j])M \tag{22}$$

$$\bar{x}_{k,k+1}^m = (\sum_{j=0}^{k} \Phi^j)\Lambda \bar{d}_k^m + \sum_{j=t_D}^{k} \Phi^{k-j}\Gamma u_j \tag{23}$$

$$\bar{x}_{k,l+1}^m = \Phi \bar{x}_{k,l}^m + \Gamma \bar{u}_{k,l}^m + \Lambda \bar{d}_k^m \tag{24}$$

j= (25)

$$z[j] \in (0,1). \tag{26}$$

For the more general problem with $t_f > t_D + l$, the controller algorithm can be summarized as follows.

for $t_D \le k \le t_f - 1$

1. Obtain all possible disturbances 3£ consistent with the measurements.
2. Solve for the control policy u_k and $\overline{u}$^, so as to minimize the worst time to recover from 3£\
3. Apply u_k from k to *+1.

723. After Identification

The controller for the interval between *tj* and y is given by a minimum time optimal control problem to reject the *identified* disturbance. The controller chosen for the interval between t_D and t_f may contain the control policy for the interval between *tj* and *y* If not only one more minimum time optimal control problem must be solved.

7.3. Detectability and Identifiability

We make the assumption that a past state of the system is known. This assumption requires that at some earlier time we observed the state of the system in the classical sense. We further assume we have a perfect model of the process - i.e., we are dealing with a deterministic model.

From the point where we know the state, we can in principle keep track of all inputs u_k into the system." Further if the measurements allow us to identify exactly all the disturbances which have occurred, we can maintain our knowledge of the states by stepping the model forward in time using these inputs and disturbances. With delayed measurements, we cannot compute exactly the current state; however, we can compute past states up to the point for which we have been able to identify the disturbances which have occurred.

We are assuming in this paper that all disturbances can be characterized with a finite (small) number of parameters. For example a Class / disturbance which is a step will be characterized by its time of occurrence and by its magnitude.

We assume a linear discrete models with delays in the measurements

$$x_{k+1} = \Phi x_k + \Gamma u_k + \Lambda d_k$$

$$y_k = \sum_{i=1}^{n_\delta} C_i x_{k-\delta[i]_{meas}}, \qquad \delta[i+1]_{meas} > \delta[i]_{meas} \geq 0 \quad \forall i \tag{27}$$

where k is the current time. Since any known state can be made the origin of the model, we assume x_k and u_k are all zero until the disturbance occurs at time £=0.

7J.I. Disturbance Detection

We now discuss a method to determine detection times for model (27).

A disturbance is detected when any of the current measurements becomes nonzero. We can find $t_D[m]$ mathematically as follows. We first calculate the *nfi*[1] measurement profile j£ given that *d^d occurs at time 0. This profile can be calculated from (27) recursively, and is given by**[3]

$$fi-*k*'' \qquad (28)$$

where matrix

$$\Theta_k = 0 \qquad 0 \le k \le \delta[1]$$

$$\Theta_k = \sum_{i=1}^{n_s} C_i \Phi^{k-\delta[i]-1} \Lambda \qquad k > \delta[1]. \qquad (29)$$

The detection time for each disturbance *d"* is given by

$$t_D[m] = \min k \quad s.t. \quad ibfli > 0 \qquad (30)$$

which cannot be earlier than 5[1] 4-1. Detection of *d"* occurs at the first time *k* when it is not in the null space of ©£.

73.2. Disturbance Identification

We will now discuss **how** to determine the identification time and of the magnitudes and times of occurrence of the possible **disturbances** in **the** interval from t_D to t_{Jt} We define a disturbance as being identified when **we** can **determine** it uniquely from the measurements. We assume that any disturbance that enters the process can be identified when all measurements are available. Therefore, an upper bound on *tj* is given by $5[n_g]+1$.

Figure 7-3 defines identification times for a process where the output profiles y^{TM}_k are given by (28) and (29). It illustrates that there may be a period after the detection time for which different disturbances have

---

[3]The profiles y^{TM}_k in (28) and throughout this section assume that $u_k = 0$ in (27). However, our results are also valid for u_k * 0 since for linear systems we can redefine y_k and x_k so that they are independent of u_k.

identical profiles. The end of this period is the identification time, and it may be different for each disturbance. For example, if d^3 enters the process, it will be identified 1 time unit after t_D but if either d^1 or d^2 occur, they will be identified only after 6 time units because until then they cannot be distinguished from each other.

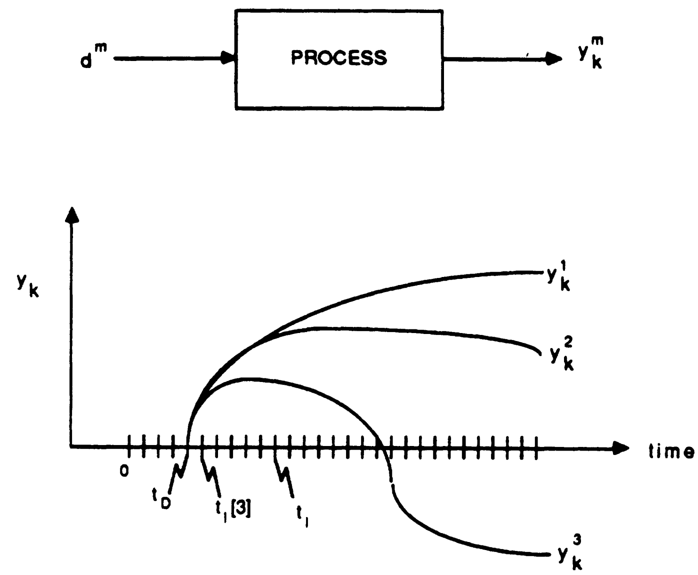


Figure 7-3: Output Profiles for Different Disturbances

We can find t_I for a disturbance d^m as follows. The disturbance is identified when the $n_d + 1$ parameters, corresponding to the time of occurrence of the disturbance and the elements of the vector d^m, can be determined uniquely from the available measurements. However, since we can precompute $t_D[m]$ for each disturbance from the previous section, we can calculate the time of occurrence for each disturbance from (30). Thus we only need to compute n_d parameters uniquely at t_I.

Since there are n_d parameters to be identified, if $rank(C_1 \Lambda) \geq n_d$ then $t_I = t_D = \delta[1] + 1$. If $rank(C_1 \Lambda) < n_d$, one must obtain new measurements.

Define

$$y_{\delta[1]+1,k} = \Theta_{\delta[1]+1,k} d^m \qquad (3i)$$

with

$$y_{\delta[1]+1,k} = [y_{\delta[1]+1}^T, \ldots, y_k^T]^T$$

$$\Theta_{\delta[1]+1,k} = [\Theta_{\delta[1]+1}^T, \ldots, \Theta^T \qquad (32)$$

The identification time t_I occurs at the minimum time k such that the rank of ©sn i+i * is greater than or equal to n_d.

Finally, we determine the magnitudes of the possible disturbances from equation (31).

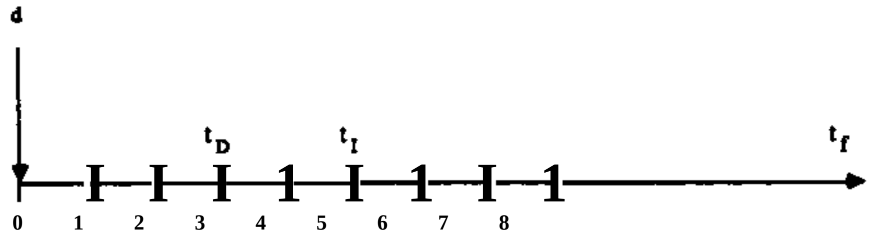
7.4. Multilevel Index Formulation

We can formulate the index as a multilevel optimization problem based on the controller of section. (7.2). At each level (time) the control policy for the next sample period is computed.

Figure (7-4) is useful in order to develop our multilevel formulation. We assume that the disturbance enters at time k - 0, that it will be detected at time t_D » 3, and that it will be identified at time tj = 5. From 0 to t_D - 3, u_k - 0 since the disturbance has not been detected. From t_D « 3 to r;-1 « 4, possible disturbance estimates 3^ are available at each time (level) k. We calculate controls from Problem P4 in order to drive the process in such a way that the worst time to recover from 3^ is the least it can be. The controls u_k are calculated and applied at each time k. Note that at k * tj— 1 * 4, as shown in figure 7-4, t_{worst} ^t_f. At /k = 5, the disturbance is identified and one final optimal control problem solution discovers the remaining controls u_h 5 £ k £ tj and our index, y.

Figure (7-5) shows the resulting multilevel formulation for finding the index where D is the disturbance search space. It can be easily extended to include more levels. Also, note that prediction is avoided since controls at lower levels (earlier times - e.g., u^) are fixed at higher levels (future times) and therefore cannot be used to reject disturbances at future times (i.e., 3^).

Since the index in figure 7-5 is highly embedded, it is difficult to compute as shown. Also, the presence of binary variables at the lower levels of the formulation makes the writing of necessary conditions for optimality difficult. We will solve it by an exaustive search on the set D.



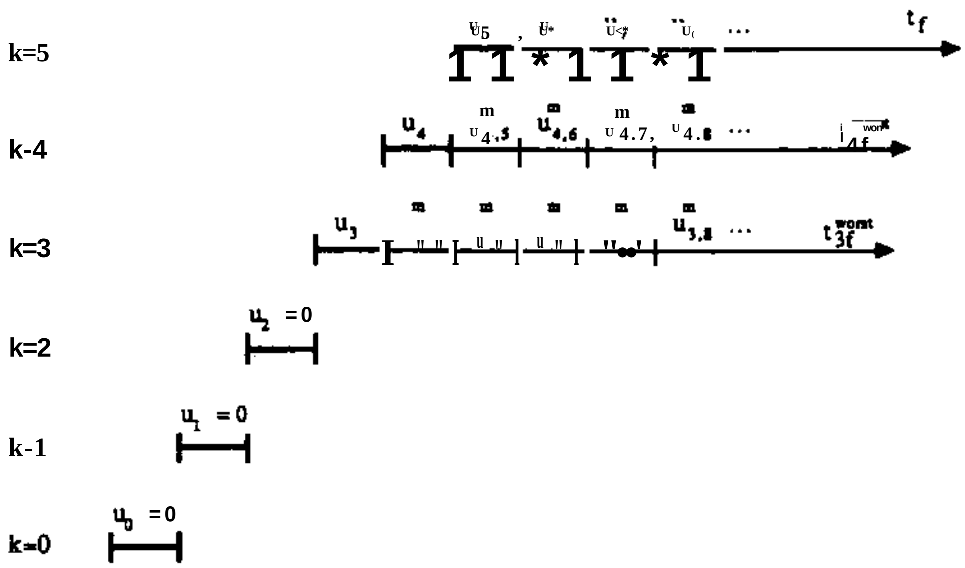


Figure 7-4: Illustration of Multilevel Formulation

8. Single-event Case with Measurement Delays

As mentioned before, the single-event disturbance model (class *I)* is a realistic one for many chemical processes. Given that only one of the disturbances has "hit" the process, we need a method to estimate the following about the disturbance: the time at which it can be detected, the time at which it can be identified, and the magnitudes and times of occurrence of the possible disturbances in the interval from t_D to *tj.* We then use this information and apply the controller from section 7.2.

$$\max_{d \in D} \quad \min_{u_k|k \geq 5 \in U} \quad t_f + \varepsilon \|u_k\|_1$$

$$s.t. \quad t_f = 1 + \sum_{j=1}^{n_{binary}} z[j]_5 2^{j-1}$$

$$\|(\Phi - I)\bar{x}^{ident}_{5,l} + \Gamma u_l + \Lambda \bar{d}^{ident}_5\|_1 + \|C\bar{x}^{ident}_{5,l}\|_1 \leq (|S_l| - \sum_{j \in S_l} z[j]_5 + \sum_{j \in S_l} z[j]_5)M$$

$$\bar{x}^{ident}_{5,l+1} = \Phi \bar{x}^{ident}_{5,l} + \Gamma u_l + \Lambda \bar{d}^{ident}_5$$

$$\bar{x}^{ident}_{5,6} = (\sum_{j=0}^{5} \Phi^j)\Lambda \bar{d}^{ident}_5 + \sum_{j=3}^{5} \Phi^{5-j}\Gamma u_j$$ where u_3 and u_4 are *given by*

$$\min_{u_4, \bar{u}^m_{4,l} \in U} \quad t^{worst}_{4f} + \varepsilon \|u_4\|_1$$

$$s.t. \quad t^{worst}_{4f} = 1 + \sum_{j=1}^{n_{binary}} y[j]_4 2^{j-1}$$

$$\|(\Phi - I)\bar{x}^m_{4,l} + \Gamma \bar{u}^m_{4,l} + \Lambda \bar{d}^m_4\|_1 + \|Cx^m_{4,l}\|_1 \leq (|S_l| - \sum_{j \in S_l} z[j]_4 + \sum_{j \notin S_l} z[j]_4)M$$

$$\bar{x}^m_{4,l+1} = \Phi \bar{x}^m_{4,l} + \Gamma \bar{u}^m_{4,l} + \Lambda \bar{d}^m_4$$

$$\bar{x}^m_{4,5} = (\sum_{j=0}^{4} \Phi^j)\Lambda \bar{d}^m_4 + \sum_{j=3}^{4} \Phi^{4-j}\Gamma u_j$$ where u_3 *is given by*

$$\min_{u_3, \bar{u}^m_{3,l} \in U} \quad t^{worst}_{3f} + \varepsilon \|u_3\|_1$$

$$s.t. \quad t^{worst}_{3f} = 1 + \sum_{j=1}^{n_{binary}} y[j]_3 2^{j-1}$$

$$\|(\Phi - I)\bar{x}^m_{3,l} + \Gamma \bar{u}^m_{3,l} + \Lambda \bar{d}^m_3\|_1 + \|C\bar{x}^m_{3,l}\|_1 \leq (|S_l| - \sum_{j \in S_l} z[j]_3 + \sum_{j \in S_l} z[j]_3)M$$

$$\bar{x}^m_{3,l+1} = \Phi \bar{x}^m_{3,l} + \Gamma \bar{u}^m_{3,l} + \Lambda \bar{d}^m_3$$

$$\bar{x}^m_{3,4} = (\sum_{j=0}^{3} \Phi^j)\Lambda \bar{d}^m_3 + \Gamma u_3$$

where $S_l = \{ j \mid 1 + \sum_{j=1}^{n_{binary}} 2^{j-1} = l \}$, *and* $z[j]_k \in \{0,1\}$, for $k = 3,4,5$.

Figure 7-5: Multilevel Formulation for Single-event Index

Once the controller is designed, the index in Problem PI is calculated by finding the disturbance, out of the **allowed set *D*, that gives the worst** minimum **time** to **be** rejected. If there are n_d disturbances, at most $2n_d$ **control profiles have to be** solved. For linear systems whose manipulated variables are bounded symmetrically **(upper and lower** bounds **have** the same magnitude), only n_d control profiles have to be solved.

8.1. Detection and Identification

For class / disturbances, a lower bound for the detection time t_D is 5[1] + 1. We can express the output profile y_k^{TM} at time k - 5[1] + 1 by substituting d^n^a^e_m for the disturbance in (28) or

$$ys[\backslash)+i \ - \ (C_{\{}A)[m]d[ml \tag{33}$$

where $(C_x A)[m]$ is the m[m] column of the matrix $C_{\{}$ A. Since class / disturbances only occur at the axis of the set D, the condition t_D > 8[1] +1 can only occur when the null space of CjA is aligned with the axis.

Identification times and disturbance estimates can be obtained from section 7.3.2. We estimate the magnitudes of the possible disturbances from equation (31) with $(f^*$ » $a\%e_m$.

Equations (28) and (29) can be used to define processes for which disturbances take longer to be identified. For example, for $d[l]$ and $d[2]_t$ we can define

$$Ay_k(l_92) \ = \ y^l{}_k\text{-}y^2{}_k \tag{34}$$

where

$$y_k^1 = \sum_{i=1}^{n_\delta} C_i \Phi^{k-\delta[i]-1} \Lambda[1] d[1]$$
$$y_k^2 = \sum_{i=1}^{n_\delta} C_i \Phi^{k-\delta[i]-1} \Lambda[2] d[2] \tag{35}$$

and $A[m]$ is the m[m] column of A. We can substitute (35) into (34) to obtain

$$\Delta y_k(1,2) = \sum_{i=1}^{n_\delta} C_i \Phi^{k-\delta[i]-1} (\Lambda[1] d[1] - \text{A[2M2]}). \tag{36}$$

We want processes such that A>^(1,2) - 0 for large k. A trivial solution results when any two columns of A are **dependent, i.e., when** A[i] - <xA(/] for some cc The nontrivial solution comes from solving

$$\sum_{i=1}^{n_s} C_i \Phi^{k-\delta[i]-1} v - 0 \qquad (37)$$

where v is any nonzero vector. An example of a process with the above properties is one with two dynamically identical units in parallel (see example in section 8.3).

8.2. Method to Compute the Index

The formulation in figure 7-5 can be used to calculate the index for Class / problems as follows. The disturbance search space D is defined as in class / of figure 5-1. Also, we substitute the disturbance estimates at each time k with 3J[1] - $o£e_m$.

Another feature of Problem P4 can be used to simplify the formulation in figure 7-5. It applies to processes for which (37) holds. For such processes, the control policy between t_D and t_I is the same t regardless of what disturbance entered the process because the same disturbances $<\%=o\%e_m$ will be estimated between t_D and tj independent of which event occurred. Since it is also reasonable to assume that the detection and identification times are the same for all disturbances, the index will be given by the ^calculated from Problem P4 at time k - r$_;$-1 - 5. This is true because the disturbance(s) picked up by the outer max operator in the objective function (1) is also the worst disturbance(s) picked up by the max operator in the objective function (14) of Problem P3. If we let a ^ - a j [1] - ^ in the formulation of figure 7-5, we can obtain the index from the following formulation

$$
\begin{aligned}
&\max_{d \in D} \ \min_{u_k = \bar{u}^m_{5,j} \in U} \quad t_f^{worst} + \varepsilon \|u_k\|_1 = t_f + \varepsilon \|u_k\|_1 \\
&\text{s.t.} \quad t_{5f}^{worst} = 1 + \sum_{j=1}^{n_{binary}} z[j] 2^{j-1} \\
&\|(\Phi - I)\bar{x}^m_{5,j} + \Gamma \bar{u}^m_{5,j} + \Lambda[m]\bar{\alpha}^m_5\|_1 + \|C\bar{x}^m_{5,j}\|_1 \le \Big(|S_l| - \sum_{j \in S_l} z[j] + \sum_{j \notin S_l} z[j]\Big) M \\
&\bar{x}^m_{5,j+1} = \Phi \bar{x}^m_{5,j} + \Gamma \bar{u}^m_{5,j} + \Lambda[m]\bar{\alpha}^m_5 \\
&\bar{x}^m_{5,6} = \Big(\sum_{j=0}^{5} \Phi^j\Big)\Lambda[m]\bar{\alpha}^m_5 + \sum_{j=3}^{5} \Phi^{5-j}\Gamma u_j \\
&S_l = \Big\{ j \mid 1 + \sum_{j=1}^{n_{binary}} 2^{j-1} = l \Big\} \\
&z[j] \in \{0,1\}.
\end{aligned}
\tag{38}
$$

where the controls u_k are computed for $3 \le k \le 5$.

Since a search is required in at most $2n_d$ disturbances, we solve the index by assuming a disturbance and find the minimum time to reject it. We then pick the worst minimum time as our index.

8.3. Example

We will calculate the controllability index for the process shown in figure (8-1). The process consists of two first-order units in parallel (1,2) whose outputs feed another first-order unit (3). Each unit has time constant 1 and is fed by a manipulated variable u_k and a disturbance d_k For a discretization time of 0.5, the following discrete linear model can be used to describe the above process

$$
\begin{aligned}
x_{k+1} &= \begin{bmatrix} 0.6065 & 0.0000 & 0.0000 \\ 0.0000 & 0.6065 & 0.0000 \\ 0.3032 & 0.3032 & 0.6065 \end{bmatrix} x_k + \begin{bmatrix} 0.3935 & 0.0000 & 0.0000 \\ 0.0000 & 0.3935 & 0.0000 \\ 0.0902 & 0.0902 & 0.3935 \end{bmatrix} u_k \\
&+ \begin{bmatrix} 0.3935 & 0.0000 & 0.0000 \\ 0.0000 & 0.3935 & 0.0000 \\ 0.0902 & 0.0902 & 0.3935 \end{bmatrix} d_k \\
|u_k| &\le \begin{bmatrix} 1.0 \\ 1.1 \\ 1.0 \end{bmatrix}
\end{aligned}
\tag{39}
$$

where states 2 and 3 are used as measurements and are delayed by 3.5 and 0.5 time units respectively. We will find that $t_D = 2$ and $t_f = 8$ for this example. We can tell from $x[3]_k$ at $t = t_D = 2$ time units that a

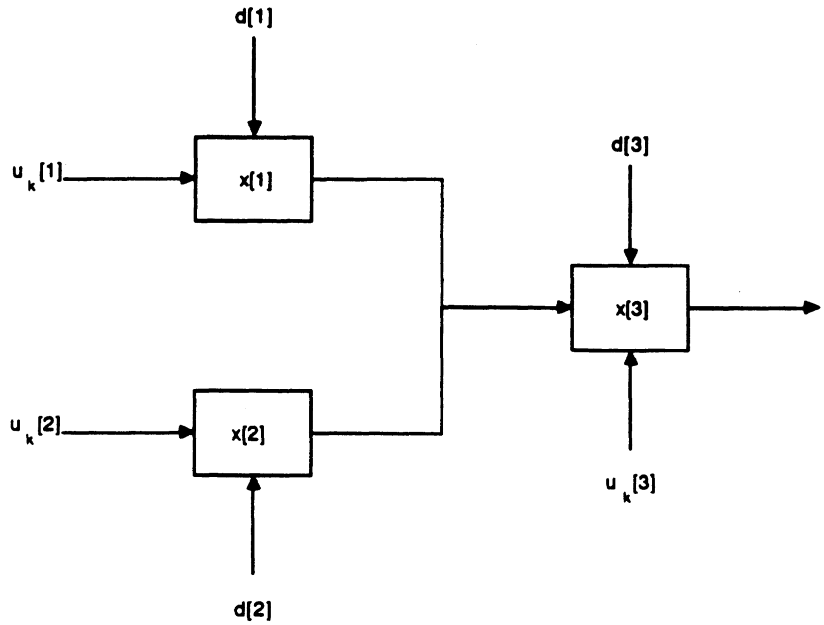


Figure 8-1: Process for Example 1

disturbance hit, but only after receiving the measurement on $x[2]_k$ at $k=t_I=8$ time units can we decide which disturbance hit.

Since $d[1]$ and $d[2]$ give the same output record y_k^1 from $t_D=2$ to $t_I=8$ as given in (37), we can simplify the required computations using (38) as our controller. Also, we can eliminate $d[3]$ as a possible worst disturbance candidate to compute the index since it can be identified at $k=2$. In order to find the index, we then need to compute the two optimal control profiles to reject $d[1]$ and $d[2]$ since the upper and lower bounds on u_k are identical.

If we assume that the magnitudes of $d[1]$ and $d[2]$ have maximum values of 0.9, the resulting indices with corresponding worst disturbances are as follows. For controller (iii), disturbances $d[1]$ and $d[2]$ are the worst disturbances and the index is 5. The worst case controller (ii) gave an index of 6 and was caused by disturbance $d[2]$. Finally, for an optimal controller without an estimator, the index was 6.5 with $d[1]$ as the worst disturbance. One can see that controller (iii) performs best as expected, and the controller without an estimator performs worst. If the disturbance estimates changed in the interval between t_D and t_I, we would use the formulation in figure 7-5 in order to calculate our index.

9. Multiple-event Case with Measurement Delays

Computing the index in Problem PI for class // where multiple disturbances occur simultaneously is more difficult. In the single-event case the outer search on the index (outer max operator in objective (1)) is done for n_d disturbances only. In the multiple-event case we need to search the ^dimensional space for the worst disturbance (shaded region in figure 5-1, class *II).* The single-event index *(I_{SE})* is therefore a special case of the multiple event index *(I^£)* so that I_{SE} £/^/£.

The outer search for the worst disturbance can be simplified as follows. Since bounds on the disturbances are usually available, and the global solution to Problem PI will be given by disturbances at the bounds, we can reduce the search space to the boundary of *D.* Further, we can discretize this outer boundary and define the set of all discrete points as D*. The more discretization points we use, the more disturbances that need to be evaluated and the more accurate the index will be. We would then be interested in a discretization such that a finer mesh will not affect our index.

The objective 14 of Problem P3 requires estimates of all possible disturbances at each time *k.* This estimates result from a search in the ^dimensional space for all possible disturbances 3£. Following, we will formulate the problem and due to the low likelihood that two or more abrupt events will occur simultaneously, we will solve an example with a simplified instance of the multiple-event class.

9.1. Detection and Identification

The inner search is simplified as follows. The controller from section (7.2) must be solved to reject any disturbance from the set D*. This controller needs all possible disturbance estimates 3^ between t_D and r;. These possible disturbances are determined similar to class /. Detection and times are equal to 5[11-1 for all disturbances except for the one which lies on the null space of C_yA. t_D and t_f can be obtained from sections (7.3.1) and (7.3.2) respectively. In the interval from t_D to r7, we will estimate all possible disturbances from (31).

9.2. Method to Compute the Index

The multilevel formulation shown in figure (7-5) can also be used for class // disturbance models. The main difference is that *D* is changed to *D** where D* is defined as the discretized boundary *of* the disturbance search space. An example with a multiple-event disturbance follows.

93. Example: CSTR

A nonlinear continuous dynamic model for the Continuous Stirred Tank Reactor (CSTR), shown in figure (9-1), is **given by (Hicks,** 1971)

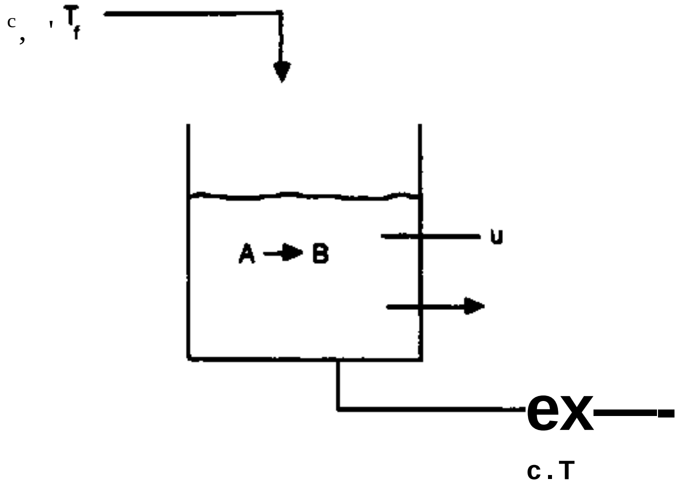


Figure 9-1: Continuous Stirred Tank Reactor

$$\dot{c} - (cy-c)/9-\bar{r}$$

$$f \quad - \quad (T_f-T)/Q+J7-au(T-T_c) \qquad \textbf{(40)}$$

$$7 = 300(e^{-252(V\dot{7})})c$$

where c, cy are the outlet inlet concentrations, *T*, T_j^* are the outlet inlet temperatures, 6 is the reactor time constant, *7* is the reaction rate, / is the heat of reaction constant, and **a** is the constant for heat convection. Substituting the following normalized variables

$x[1] = c/C_{f_s}$	$d[1] - c_f i c_{fs}$	$u[1] = u$	
$x[2] = T/Jc_{f_s}$	$d[2] = T_f/Jc_{f_s}$	$u[2] = i/e$	**(41)**
$y_c = T_c/Jc_{f_s}$	$r - 7/c_{fs}$		

where Cf_s is the steady state value for cy, in (40) gives

ill] - $u[2](d[l]-x[l])-r$

4̇2] – «[2](42] – 42]) + $r - au[1](x[2] - y_c)$ **(42)**

$r = 300 e^{-25.2/x[2]} x[1]$

For the steady state,

41]-0.408126	d[1] 1-00000	«[1]-757.86	
42] = 3.297630	42] -3.29763	u[2]« 0.099	**(43)**
$y_c = 2.9$	a- 1.95X10"4		

and for a discretization time of 0.5 time units, we may obtain the following linearized discrete model

$$x_{k+1} = \begin{vmatrix} .8834 & -.0623 \\ .0659 & .9439 \end{vmatrix} x_k + \begin{vmatrix} 1.244\text{-}10^{"6} & .2784 \\ -3.769\text{-}10^{"5} & .0101 \end{vmatrix} u_k + \begin{vmatrix} .04657 & -.00159 \\ .00168 & .04812 \end{vmatrix} d_k \qquad \textbf{(44)}$$

The controls are constrained by

-757.86 S $u[l]_k \leq$ 742.14

(45)

-0.099 £ $u[2]_k$ £ 0.201.

Let us further assume that there are delays on both meaurements so that the detection time is 2 time units, and the identification time is 8 time units. Our objective is to calculate the minimum time necessary to reject step changes in the feed disturbances, and to bring the process to rest

In order to simplify the solution to the problem, we will use an unique estimate of the disturbance at each time, from the available subspace of *d*. A unique estimate of *d* can be obtained using equation (28). For example, an estimate of *d* that minimizes its 2-norm is the orthogonal projection which is given by the pseudoinverse (Morari, 1980)

3*= $ik^T ftflk^T T\wedge kf^1$- (46)

The estimate 2^ from (46) is by no means optimal and results in an upper bound for the actual index. Using (46), instead of solving for a more complex minimum recovery controller at each time (Problem P3), we solve for a minimum time optimal controller to reject the disturbance estimate 2^.

In order to compute the index, we discretized the outer boundary of *D* with only 4 disturbances. Table 9-1 shows final times for each of the four step disturbances using (46) and without using an estimator. The upper bound on the index as predicted by a controller without an estimator is 18.5. With an optimal controller that uses (46) to estimate *d* at each time (level), the upper bound is lowered to 17.5. The worst multiple-event disturbance was *d* » (0.59,0.58). Note that the final times are considerably smaller when (46) is used.

$d[1]$	$d[2]$	t_f	tf [1]
-0.99	0.58	9	10.5
-0.99	-0.59	9.5	16
0.59	0.58	17.5	18.5
0.59	-0.59	12	16

[1] final time without estimator

Table 9-1: Computed Indices for CSTR Example

10. Stochastic Disturbance Model

We have assumed that an infrequent disturbance enters the process as a step of unknown magnitude at time zero. Disturbances that enter the process can be identified after $n_d - n_y + 1$ samples in most cases, since there is no noise in the process. When the disturbances enter the process in a stochastic manner, it is more difficult to detect and/or identify them. Once we detect the disturbances, the computed controls will be a function of the statistical information available. We have developed a methodology to compute the index for stochastic step inputs.

We set up the problem as follows. We assume that both states and process measurements are excited by noise. Also, we add a function to the state model that contains the set of expected abrupt changes that may occur (i.e., step disturbances) at unknown times with unknown magnitudes, as follows

$$x_M = <\&x_k + Tu_k + AU[mf_k a + w_k \tag{47}$$

$$y_k = Cx_k + v_k$$

where v_k and w_k *are* independent zero-mean white noise processes with $E(w_k w^\wedge) = Q_k$ and $E(v_k v^\wedge) \ll R_t$ $AU[mf_k a$ is the disturbance model, $U[mf_k$ is a unit step function for disturbance *m* having occurred at time 6, and a is the magnitude of the step. One may also add a similar function to the observation vector (y_k) in order to model sensor failure and/or measurement bias. Also, $U[mf_k$ is not limited to step changes.

Using the above disturbance model, we developed a method for linear discrete time systems, to estimate our index. We first estimate the detection time t_D and identification time t_f. We then use statistical information (i.e., likelihoods and probabilities) to determine optimal control policies between t_D and *tj*. We use a controller similar to that in Problem P3. The results will be shown in a companion paper (Carvallo, 1988a).

11. Conclusions and Future Work

We have developed an index in the time domain which measures the "controllability" of a process. Our index is defined as the minimum time necessary to overcome the worst expected disturbance and/or setpoint change. It accounts for the presence of process time delays, nonlinearities and constraints, each, of which can profoundly affect one's ability to control a process.

A simple *formulation* to include process constraints, nonlinearities and input delays for evaluating the index was developed. It was shown that single measurement delays are easy to handle, but that including multiple measurement delays into the formulation is much more involved. The optimal controller was constrained in such a way to prevent prediction of events. The index was formulated as a multilevel optimization problem, where at each level the control policy for the next sample period is estimated.

The *evaluation* of this index is a difficult task in particular for nonlinear systems. One can attack the problem by assuming a disturbance and/or setpoint change, and then solving the inner problem sequentially. For linear systems, each becomes a mixed integer linear program. Since this computation is not to be done on-line, computing time is not the main issue.

The linear problem to handle stochastic inputs was defined, and its solution will appear in a separate publication. Finally, including uncertainty in the evaluation of the index is a difficult task, and this problem should be addressed further.

12. Nomenclature

C	coefficient matrix relating states to measurements for linear system model
C_x	coefficient matrix relating states **_g[|] to measurements for linear system model
d_k	disturbance variable vector at time *k*
U[m]$_k$ « G£*	estimated magnitude for the *nfi*[1] step disturbance in class / problems
cF[1]	rrfl[1] step disturbance vector
3£	m* step disturbance vector estimated at time *k*
$\bar{d}_k^{worst}$	step disturbance estimate at time *k* requiring longest recovery time
$\bar{d}_{t_i}^{ent}$	step disturbance identified at time *tj*
D	set of all possible disturbances and setpoint changes
*D**	set of points on the boundary of the disturbance space *D*
e_i	column vector with all elements zero except its i^{th} element equal to one
/	function for state propagation in nonlinear dynamic process
g	equality constraints for nonlinear dynamic process
h	inequality constraints for nonlinear dynamic process
I_{SE}, I_{ME}	controllability indices for single- and multiple-event disturbance problems
Δk_{meas}	number of discrete time units between measurements
Δk_{disc}	discretization time for discrete model
M	upper bound on MILP constraint in problems P2 and P4
n_d, n_r, n_x, n_y	dimension of disturbance, setpoint, state and mearsurement vectors respectively
n_{binary}	number of binary variables in minimum time MILP model

^5	**number of measurement** delays
Q_k	**variance matrix** for white **noise process** w_k
r	**vector of setpoints**
R_k	variance matrix for white noise process v^
S_k	index set defined in minimum time MELP model
t_D	time at which existence of disturbance(s) is detected
tj	time at which actual disturbance is identified
tj	final time
t_G	time constant for approximating a process response curve
t™j	final time computed at time k to reject disturbance m
ty	time delay for approximating a process response curve
u_k	control variable vector at time k
ū£i	control vector computed at time k to reject disturbance m for time / greater than $k+1$
U	set of allowed controls
$U[m]_k$	unit step disturbance m occurring at time 0
v_k, w_k	zero mean white noise processes with variances R_k and Q^ respectively
x_k	state variable vector at time k
x̄^i	state vector computed at time k which predicts effect of $d[m]$ for time / greater than $k+1$
y^	measured variable vector at time k
y™	predicted measurement variable vector at time k for disturbance m occurring

$y_{k,k+p}$	vector of measurements from time k to time $k+p$
z	vector of binary variables in minimum time MILP model
z_k	vector of binary variables estimated at time k in multilevel formulation of figure (7-5)

Subscripts

i,j	indices (defined by context)
k,l	value at (discrete) time k or l
m	disturbance models index
max	maximum value
min	minimum value

Greek

$\bar{\alpha}_k^m = \bar{d}[m]_k$	estimated magnitude for the m^{th} step disturbance in class I problems
α	unknown magnitude for step disturbance in stochastic model
δ_{meas}	vector of delay times on measurements y
Δ	Difference operator (e.g., $\Delta y(1,2) = y_k^1 - y_k^2$)
ε	very small positive scalar
Φ, Γ, Λ	Matrices for state propagation for linear, time invariant, discrete model
Θ_k	matrix relating measurements to disturbance m occurring at time $\theta[m]$
$\Theta_{k,k+p}$	matrix containing γ_k from time k to time $k+p$

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