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December, 1983

DRC-70-21-R3

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June 1983

The research underlying this report was supported by Grant ECS-8205425 of the National Science Foundation and Contract N00014-82-K-0329 NR047-607 with the U.S. Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the U.S. Government

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Abstract

We discuss a new conceptual framework for the convexification of discrete optimization problems, and a general technique for obtaining approximations to the convex hull of the feasible set. The concepts come from disjunctive programming and the key tool is a description of the convex hull of a union of polyhedra in terms of a higher- dimensional polyhedron. Although this description was known for several years, only recently was it shown by Jeroslow and Lowe to yield improved representations of discrete optimization problems. We express the feasible set of a discrete optimization problem as the intersection (conjunction) of unions of polyhedra, and define an operation that takes one such expression into another, equivalent one, with fewer conjuncts. We then introduce a class of relaxations based on replacing each conjunct (union of polyhedra) by its convex hull. The strength of the relaxations increases as the number of conjuncts decreases, and the class of relaxations forms a. hierarchy that spans the spectrum between the common linear programming relaxation, and the convex hull of the feasible set itself. Instances where this approach presents advantages include critical path problems in disjunctive graphs, network synthesis problems, certain fixed charge network flow problems, etc. We illustrate the approach on the first of these problems, which is a model for machine sequencing.

1. Introduction

Most discrete optimization problems are solved by some kind of enumerative procedure. These procedures use relaxations of the feasible set, and of the subsets into which the latter is broken up, in order to derive bounds on the objective function value of these subsets. Their efficiency depends crucially on the strength of these bounds, which in turn hinges on the strength of the relaxation used. The most commonly used relaxation is the linear program obtained by removing the integrality conditions, sometimes amended with cutting planes. However, some integer programming problems have more than one formulation, and the various formulations may give rise to linear programming relaxations of varying strengths. This was known for a long time about the [illegible] for which the disaggregation of the capacity constraints involving the 0-1 variables produces a considerably stronger linear program than the aggregated one. To the disaggregation of the capacity constraints, Rardin and Choe [11] have recently added a disaggregation of the flow variables of fixed charge [illegible] flow problems, either from arc into path flows, or from single commodity into multicommodity flows, which often yields a stronger linear program than the one in the original variables.

Approaching the problem from another standpoint, that of mixed integer representability of various functions and sets, Jeroslow and Lowe [10] have recently shown how certain mixed integer formulations using a larger number of variables than the common formulation, give rise to stronger linear programming relaxations. Their approach essentially uses disjunctive programming, and our work is closely related to theirs.

Disjunctive programming is optimization over disjunctive sets. A disjunctive set is a set defined by inequalities connected to each other by the operations of conjunction ($\wedge$, juxtaposition, "and") or disjunction ($\vee$, "or"). Since inequalities define halfspaces, a disjunctive set can also be viewed as a collection of halfspaces joined together by the operations of intersection ($\cap$) or union ($\cup$). A disjunctive program is then a problem of the form $\min\{cx \mid x \in F\}$, where F is a disjunctive set.

Any integer or mixed integer program can be stated as a disjunctive program, usually in more than one way. Conversely, any bounded disjunctive program can be stated as a pure or mixed integer 0-1 program. This is not always true, though, of an unbounded disjunctive program: the set $x_j \leq 0 \vee x_j \geq 1$, for instance, cannot be represented by the use of integer variables unless x_j is bounded.

Besides this - not too important - difference in the domain of applicability of the two problem classes, it is often convenient to view integer programming problems as disjunctive programs. Apart from the fact that this is the most natural and straightforward way of stating many problems involving logical conditions (dichotomies, implications, etc.), the disjunctive programming approach seems to be fruitful both theoretically and practically. On the theoretical side, it provides some neat structural characterizations which offer new insights. On the practical side, it produces a variety of cutting planes, including facets of the convex hull of feasible points, which are hard to obtain by other means. In some cases, like set covering and partitioning, these cutting planes have been shown to be considerably stronger than those derived by other means, and have been successfully used in algorithms. In this paper we show that disjunctive programming also provides strong relaxations of an integer program. For background on disjunctive programming, see the surveys [4], [9], [12].

In this paper we introduce a general framework in which various linear programming relaxations can be classified, ranked, strengthened at a given computational cost, and viewed from a unifying perspective. In fact, we provide a family of relaxations of a (pure or mixed) integer 0-1 program (P) whose members form a hierarchy in terms of their strength, or tightness. The members of this hierarchy span the whole spectrum between the usual linear programming relaxation and the convex hull of the feasible set of (P). This is obtained by viewing (P) as a disjunctive program and making use of the rich variety of representations available for the latter. Our main tool is the operation of taking the convex hull of various disjunctive sets.

The paper is organized as follows. Section 2 discusses some basic properties of disjunctive sets and their equivalent forms, and describes a procedure for systematically generating these forms from each other. Section 3 deals with characterizations of the convex hull of a disjunctive set, and their relationship to mixed integer representations of such a set. Section 4 introduces the hull relaxation of a disjunctive set, which gives rise to the hierarchy of relaxations mentioned earlier. Section 5 illustrates these concepts and procedures on the disjunctive graph formulation of the machine sequencing problem.

2. Disjunctive Sets and Their Equivalent Forms

We denote a halfspace by

$$H^+ = \{x \in \mathbb{R}^n | ax \geq a_o\},$$

where $a \in \mathbb{R}^n$, $a_o \in \mathbb{R}$. While the intersection of a finite collection of halfspaces, i.e., a set of the form

$$P = \bigcap_{i \in M} H_i^+ = \{x \in \mathbb{R}^n | a^i x \geq a_{io},\ i \in M\}$$

is known as a polyhedron, we call the union of a finite collection of halfspaces, i.e., a set of the form

$$D = \bigcup_{i \in M} H_i^+ = \{x \in \mathbb{R}^n | \bigvee_{i \in M} (a^i x \geq a_{io})\},$$

an <u>elementary</u> disjunctive set.

A disjunctive set F can be expressed in many different forms, that are logically equivalent and can be obtained from each other by considering F as a logical expression whose statement forms are inequalities, and applying the rules of propositional calculus. Among these equivalent forms, the two extreme ones are the <u>conjunctive normal form</u> (CNF)

$$F = \bigcap_{i \in T} D_i,$$

where each D_i is an elementary disjunction, and the <u>disjunctive normal form</u> (DNF)

$$F = \bigcup_{i \in Q} P_i,$$

where each P_i is a polyhedron.

The usual statement of most discrete optimization problems is in the form of an intersection of elementary disjunctions, that is in CNF. We give a few examples.

The feasible set of a mixed integer 0-1 program, given by the constraints

$$a^i x \ge b_i,\ i \in M;\quad 0 \le x_j \le 1,\ j \in N;\quad x_j \le 0 \vee x_j \ge 1,\ j \in I \subseteq N;$$

is in CNF, and can be written as $F = \bigcap_{i \in T} D_i$, with $T = M \cup N_1 \cup N_2 \cup I$ (where $N_1 = N_2 = N$), and D_i defined as $\{x | a^i x \ge b_i\}$ for $i \in M$; $\{x | x_i \ge 0\}$ for $i \in N_1$; $\{x | -x_i \ge -1\}$ for $i \in N_2$; and $\{x | -x_i \ge 0 \vee x_i \ge 1\}$ for $i \in I$.

The DNF of the same set is $F = \bigcup_{S \subseteq I} P_S$, where P_S is the set of those x satisfying $a^i x \ge b_i$, $i \in M$; $0 \le x_j \le 1$, $j \in N$; $x_j \ge 1$, $j \in S$; and $-x_j \ge 0$, $j \in I \setminus S$.

Similarly, the feasible set of a linear complementarity problem given by

$$a^i x + b^i y = c^i,\ i \in M;\quad x_j \ge 0,\ y_j \ge 0,\ j \in N;\quad x_j \le 0 \vee y_j \le 0,\ j \in N;$$

is in CNF, and so is the feasible set of the machine sequencing problem [1]

$$t_j - t_i \ge d_{ij}, \qquad (i,j) \in Z,$$

$$t_i \ge 0, \qquad i \in V,$$

$$t_j - t_i \ge d_i \vee t_i - t_j \ge d_j, \qquad (i,j),\ (j,i) \in W,$$

where each inequality of Z defines a precedence relation between two jobs, and each disjunctive pair (i,j), (j,i) $\in$ W states the condition that jobs i and j cannot overlap.

On the other hand, the feasible set of the set covering problem defined by the m X n matrix $A = (a_{ij})$, $a_{ij} \in \{0,1\}$ ¥ i,j, can be stated in CNF either in the same way as shown for the general mixed integer program, or else by letting $T = M = \{1,\ldots,m\}$ and $F = \bigcap_{i \in T} D_i$, with

$D_i = \{x \mid \bigvee_{j \in N_i} (x_j \geq 1)\}$, $i \in T$, where $N_i = \{j \in N \mid a_{ij} = 1\}$. The DNF of the same problem, on the other hand, is $F = \bigcup_{C \in \mathcal{C}} \{x \mid x_j \geq 1, j \in C\}$, where $\mathcal{C}$ is the set of all covers.

Although the CNF and the DNF are the two extremes of the spectrum of equivalent forms of a disjunctive set, they share a property not common to all forms: each of them is an intersection of unions of polyhedra. We will say that a disjunctive set that has this property is in regular form (RF). Thus the RF is

(2.1) $$F = \bigcap_{j \in T} S_j ,$$

where for $j \in T$,

(2.2) $$S_j = \bigcup_{i \in Q_j} P_i , \quad P_i \text{ a polyhedron}, \quad i \in Q_j .$$

The CNF is the RF in which every S_j is elementary, i.e., every polyhedron P_i is a halfspace. The DNF, on the other hand, is the RF in which $|T| = 1$. Notice that if F is in the RF given by (2.1), (2.2), each S_j is in DNF. A disjunctive set S_j in the DNF (2.2) will be called improper if $S_j = P_t$ for some $i \in Q_j$, proper otherwise. Any disjunctive set S_j such that $|T_j| = 1$ is improper. S_j is convex (and polyhedral) if and only if it is improper.

Next we define an operation which, when applied to a disjunctive set in RF, results in another RF with one less conjuncts, i.e., an operation which brings the disjunctive set closer to the DNF. There are several advantages to having a disjunctive set in DNF, i.e., expressed as a union of polyhedra; beyond this, the motivation for the basic step introduced here will become clearer below when we discuss relaxations of disjunctive sets.

<u>Theorem 2.1</u>. Let F be the disjunctive set in RF given by (2.1), (2.2). Then F can be brought to DNF by $|T| - 1$ applications of the following <u>basic step</u>, which preserves regularity:

For some $k, \ell \in T$, $k \neq \ell$, bring $S_k \cap S_\ell$ to DNF, by replacing it with

$$(2.3) \qquad S_{k\ell} = \Big(\bigcup_{i \in Q_k \setminus Q_\ell} \; \bigcup_{j \in Q_\ell \setminus Q_k} (P_i \cap P_j)\Big) \cup \Big(\bigcup_{i \in Q_k \cap Q_\ell} P_i\Big)$$

<u>Proof</u>. First we show that $S_{k\ell}$ is the DNF of $S_k \cap S_\ell$. By the distributivity of $\cup$ and $\cap$, we have

$$S_k \cap S_\ell = \Big(\bigcup_{i \in Q_k} P_i\Big) \cap \Big(\bigcup_{j \in Q_\ell} P_j\Big)$$

$$= \bigcup_{i \in Q_k} \bigcup_{j \in Q_\ell} (P_i \cap P_j) .$$

But for every $i_0 \in Q_k \cap Q_\ell$,

$$\bigcup_{j \in Q_\ell \setminus \{i_0\}} (P_{i_0} \cap P_j) \cup P_{i_0} = \bigcup_{j \in Q_k} (P_j \cap P_{i_0}) ,$$

and thus $S_k \cap S_\ell = S_{k\ell}$ as defined in (1.3).

The set F given by (2.1), (2.2) is the intersection of $|T|$ unions of polyhedra. Every application of the basic step replaces the intersection of p unions of polyhedra (for some positive integer p) by the intersection of p-1 unions of polyhedra. Regularity is thus preserved, and after $|T| - 1$ basic steps F becomes a single union of polyhedra, i.e., is in DNF. ||

Remark. If $S_k = P_{i_0}$ for some $i_0 \in Q_k$, i.e., S_k is improper, then

(2.4) $$S_{k\ell} = \begin{cases} S_\ell & \text{if } i_0 \in Q_\ell \\ \bigcup_j (P_{i_0} \cap P_j) & \text{otherwise .} \end{cases}$$

Every basic step reduces by one the number of conjuncts S_j in the RF to which it is applied. On the other hand, it is also of interest to know the effect of a basic step on the number of polyhedra whose unions are the conjuncts of the RF. When the basic step is applied to a pair of conjuncts S_k, S_ℓ that are both proper disjunctive sets, namely unions of polyhedra indexed by Q_k and Q_ℓ, respectively, then the set $S_{k\ell}$ resulting from the basic step is the union of p polyhedra, where

$$p = |Q_k \setminus Q_\ell| \times |Q_\ell \setminus Q_k| + |Q_k \cap Q_\ell| .$$

This is to be compared with the number of polyhedra in the unions defining S_k and S_ℓ, which is $|Q_k| + |Q_\ell|$. Obviously, more often than not a basic step applied to a pair of proper disjunctive sets results in <u>an increase in the number of polyhedra</u> whose union is taken. On the other hand, when one of the two disjunctive sets, say S_k, is improper, then $S_{k\ell}$ is the union of <u>at most as many polyhedra</u> as S_ℓ.

Given a disjunctive set in CNF with t conjuncts, where the i^{th} conjunct is the union of q_i halfspaces, and given the same disjunctive set in DNF, as the union of q polyhedra, we have the bounding inequality

$$q \le q_1 \times \ldots \times q_t .$$

Because performing a basic step on a pair S_k, S_ℓ such that S_k is improper, results in a set $S_{k\ell}$ that is the union of no more polyhedra than is S_ℓ, it is often useful to carry out a parallel basic step, defined as follows:

For F given by (2.1), (2.2), and $S_k = \emptyset$ for some $k \in Q_k$ (i.e., S_k improper), replace $\bigcap_{j\in T} S_j$ by $\bigcap_{j\in T\setminus\{k\}} S_{kj}$, where each S_{kj} is defined by (2.4).

Note that if some of the basic steps of Theorem 2.1 are replaced by parallel basic steps, the total number of steps required to bring F to DNF remains the same.

Next we turn to the operation of taking the convex hull of a disjunctive set, which plays a central role in the construction of the family of relaxations that we are about to introduce.

3. The Convex Hull of a Disjunctive Set

We have two characterizations of the convex hull of a disjunctive set, each of which requires the set to be in DNF. The first one is described by the following two theorems.

Theorem 3.1. [3, 4, 9]. Let

$$(3.1) \qquad F = \bigcup_{i\in Q} P_i, \quad P_i = \{x \in R^n \mid A^i x \geq a_0^i\}, \quad i \in Q,$$

where each A^i is an $m_i \times n$ matrix, each a_0^i is an m_i-vector, and Q is an arbitrary index set. Let $Q^* = \{i \in Q \mid P_i \neq \emptyset\}$, and let

$$C(Q^*) = \left\{ x \in R^n \;\middle|\; \begin{array}{l} \alpha x \geq \alpha_0 \text{ for all } (\alpha, \alpha_0) \in R^{n+1} \text{ such that} \\ \alpha = u^i A^i, \; \alpha_0 \leq u^i a_0^i, \; i \in Q^*, \\ \text{for some } u^i \in R^{m_i}, \; u^i \geq 0, \; i \in Q^*. \end{array} \right\}.$$

Then

$$\text{cl conv } F = C(Q^*).$$

For the next Theorem we need a definition. An inequality Qfx > d_o is said to define (or induce) a facet of a polyhedron P of dimension n, if $ax \geq a_o$ for all xeP, and $atx = ct_o$ for n affinely independent points xeP.

Theorem 3.2 [3, 4]. Let the set F defined by (2.1) be full-dimensional, and let Q be finite. Then the inequality #x $\geq at_o$> where a_o t 0, defines a facet of clconv F if and only if a ^ 0 is a vertex of

$$F^{\#} = \left\{ y \in R^n \;\middle|\; \begin{array}{l} y =^* u^{\wedge 1}, \quad ieQ^* \\ \text{for some } u^i \geq 0, \quad ieQ^* \\ \text{such that } u^i a^i_o \geq ct_o \end{array} \right\}$$

Analogous results are known for the cases where F is less than full dimensional and/or a_o = 0 (see [3]).

This characterization can be used to derive strong cutting planes whenever Q is small or, although Q is large, the special structure of the polyhedra P^i makes it easy to find vertices of F . Such cutting planes have been derived in [2, 4, 5, 7, 12] and have been successfully used to solve, for instance, set covering [6] and set partitioning [8] problems.

The second characterization expresses the convex hull of a disjunctive set as the projection into H^n of a higher dimensional polyhedron. It is this second characterization that we are going to use extensively in this paper. Since this result is from an unpublished technical report, we provide the proof here. As before, we denote Q* = {ieQ|P^i ^ 0).

Theorem 3.3 [3]. Let F be given by (3.1), and let 5(Q*) be the set of all those x e B^n such that there exist vectors (y_i , y_i) e TR^{n+1} , ieQ*, satisfying

$$\sum_{i\in Q^*} y^i = 0$$

$$A^i y^i - a_o^i y_o^i \geq 0 \quad , \quad i\in Q^*$$

(3.2)

$$\sum_{i\in Q^*} y_o^i = 1$$

$$y_o^i \geq 0 \quad , \quad i\in Q^* .$$

Then

$$\text{cl conv } F = S(Q^*).$$

Proof. (i) We first show that conv $F \subseteq S(Q^*)$. Let $\bar{x} \in$ conv F; then

$$\bar{x} = \sum_{i\in Q^*} z^i \lambda_i$$

for some points $z^i \in P_i$, $i\in Q^*$, and scalars $\lambda_i \geq 0$, $i\in Q^*$, such that $\sum_{i\in Q^*} \lambda_i = 1$. Setting $y^i = z^i\lambda_i$ and $y_o^i = \lambda_i$, $i\in Q^*$, we obtain a set of vectors (y^i, y_o^i), $i\in Q^*$, that together with $\bar{x}$ satisfy (3.2); hence $\bar{x} \in S(Q^*)$.

(ii) Next we show that $S(Q^*) \subseteq$ cl conv F. Let $\bar{x} \in S(Q^*)$ and let $(\bar{y}^i, \bar{y}_o^i)$, $i\in Q^*$, be vectors that together with $\bar{x}$ satisfy (3.2). Let

$$Q_1^* = \{i\in Q^* \mid \bar{y}_o^i > 0\} \quad , \quad Q_2^* = \{i\in Q^* \mid \bar{y}_o^i = 0\}.$$

For $i\in Q_1^*$, $\bar{y}^i/\bar{y}_o^i$ is a solution to $A^i x \geq a_o^i$, i.e., $(\bar{y}^i/\bar{y}_o^i) \in P_i$; therefore

$$\bar{y}^i/\bar{y}_o^i = \sum_{j\in V_i} v^{ij}\mu_{ij} + \sum_{k\in W_i} w^{ik}\nu_{ik}$$

for some extreme points v^{ij} and extreme direction vectors w^{ik} of P_i, indexed by V_i and W_i respectively, and some scalars $\mu_{ij} \geq 0$, $j\in V_i$, $\nu_{ik} \geq 0$, $k\in W_i$, satisfying $\sum_{j\in V_i} \mu_{ij} = 1$. Setting $\bar{\mu}_{ij} = \mu_{ij}\bar{y}_o^i$, $\bar{\nu}_{ik} = \nu_{ik}\bar{y}_o^i$, we obtain

$$\bar{y}^i = \sum_{j \in V_i} v^{ij}\rho_{ij} + \sum_{k \in W_i} w^{ik}\sigma_{ik}$$

with $\rho_{ij} \geq 0$, $j \in V_i$, $\sigma_{ik} \geq 0$, $k \in W_i$, and $\sum_{j \in V_i} \rho_{ij} = \bar{y}^i_o$.

For $i \in Q^*_2$, either $\bar{y}^i = 0$, or else $\bar{y}^i$ is a nontrivial solution to the homogeneous system $A^i y \geq 0$; hence

$$\bar{y}^i = \sum_{k \in W_i} w^{ik}\sigma_{ik}$$

for some extreme direction vectors w^{ik} of P_i, indexed by W_i, and some scalars $\sigma_{ik} \geq 0$, $k \in W_i$.

Thus we have

$$\begin{aligned}\bar{x} &= \sum_{i \in Q^*} \bar{y}_i \\ &= \sum_{i \in Q^*_1}\left(\sum_{j \in V_i} v^{ij}\rho_{ij} + \sum_{k \in W_i} w^{ik}\sigma_{ik}\right) + \sum_{i \in Q^*_2}\left(\sum_{k \in W_i} w^{ik}\sigma_{ik}\right) \\ &= \sum_{i \in Q^*_1} \sum_{j \in V_i} v^{ij}\rho_{ij} + \sum_{i \in Q^*} \sum_{k \in W_i} w^{ik}\sigma_{ik}\end{aligned}$$

with

$$\sum_{i \in Q^*_1} \sum_{j \in V_i} \rho_{ij} = \sum_{i \in Q^*_1} \bar{y}^i_o = 1,$$

i.e., $\bar{x}$ is the convex combination of finitely many points and directions of F. Hence $\bar{x} \in$ cl conv F.

(iii) Since

$$\text{conv } F \subseteq \mathcal{S}(Q^*) \subseteq \text{cl conv } F$$

and $\mathcal{S}(Q^*)$ is closed, while cl conv F is the smallest closed set containing conv F, clearly $\mathcal{S}(Q^*) =$ cl conv F.||

In order to use this characterization of the convex hull, one needs to know which P_i are nonempty. This inconvenience is considerably mitigated by the fact, to be shown below, that the information in question becomes irrelevant if the systems $A^i y^i \geq a^i_0$ satisfy a condition that is often easy to check. Let $(3.2)_Q$ be the constraint set obtained from (3.2) by substituting Q for Q*, and let S(Q) be the set obtained from S(Q*) by the same substitution. For any polyhedron P, let rec P denote the recession cone of P, i.e.,

$$\text{rec } P := \{y \mid x + \lambda y \in P, \forall x \in P, \forall \lambda \geq 0\}.$$

If S_1 and S_2 are sets, we denote

$$S_1 + S_2 = \{x \mid x = y^1 + y^2 \text{ for some } y^1 \in S_1, y^2 \in S_2\}.$$

Theorem 3.4. S(Q) = S(Q*) if and only if

$$(3.3) \qquad \bigcup_{i \in Q \setminus Q^*} \{y \in R^n \mid A^i y \geq 0\} \subseteq \sum_{i \in Q^*} \text{rec } P_i .$$

Proof. For $i \in Q \setminus Q^*$, $A^i y - a^i_0 y_0 \geq 0$, $y_0 \geq 0$ implies $y_0 = 0$. Therefore

$$S(Q) = S(Q^*) + C,$$

where C is the expression (union of polyhedral cones) on the lefthand side of (3.3). Clearly, $S(Q^*) + C = S(Q^*)$ if and only if $C \subseteq \text{rec } S(Q^*)$. But from Theorem 3.3,

$$\text{rec } S(Q^*) = \text{rec cl conv } F$$

$$= \sum_{i \in Q^*} \text{rec } P_i ,$$

hence S(Q) = S(Q*) if and only if (3.3) holds. ∎

Corollary 3.5. If for every $i \in Q$, some subset of the set of inequalities $A^i x^i \geq a_0^i$ defines a bounded nonempty polyhedron, then $g(Q) = S(Q^*)$.

Thus the disjunctive program $\min\{cx \mid x \in F\}$, where F is given by (3.1), is equivalent to the linear program $\min\{cx \mid x \in S(Q^*)\}$. Furthermore, there is a 1-1 correspondence between vertices of the polyhedra P_i, $i \in Q^*$, and basic solutions of the system (3.2). More specifically [3]:

(i) If $\bar{x}$ is a vertex of P_i for some $i \in Q^*$, then the vector with components $(\bar{y}^i, \bar{y}_0^i) = (\bar{x}, 1)$, $(\bar{y}^k, \bar{y}_0^k) = (0, 0)$, $k \in Q \setminus \{i\}$, together with $\bar{x}$, is a basic solution of the system (3.2).

(ii) If $\hat{x}$ together with $(\hat{y}^k, \hat{y}_0^k)$, $k \in Q$, is a basic solution of (3.2), then $(\hat{y}^i, \hat{y}_0^i) = (\hat{x}, 1)$ for some $i \in Q^*$, $(\hat{y}^k, \hat{y}_0^k) = (0, 0)$ for $k \in Q \setminus \{i\}$, and $\hat{x}$ is a vertex of P_i.

Thus all basic solutions of the system (3.2) (or $(3.2)_Q$) satisfy the condition $y_0^i \in \{0, 1\}$, $i \in Q$. On the other hand, a solution of (3.2) (or $(3.2)_Q$) satisfying this condition need not be basic. It is then natural to ask the question, what do such solutions represent? The next theorem addresses this issue.

We denote by $S_I(Q)$ the set of those $x \in \mathbb{R}^n$ for which there exist vectors $(y^i, y_0^i) \in \mathbb{R}^{n+1}$, $i \in Q$, satisfying the constraints of $(3.2)_Q$ and the condition $y_0^i = 0$ or 1, $i \in Q$; i.e.,

$$S_I(Q) := \{x \in S(Q) \mid y_0^i \in \{0, 1\},\ i \in Q\}.$$

Theorem 3.6. Let $F = \bigcup_{i \in Q} P_i$, $Q^* = \{i \in Q \mid P_i \neq \emptyset\}$, and $Q^{**} = \{i \in Q^* \mid P_i \not\subseteq P_j,\ \forall j \in Q^* \setminus \{i\}\}$. If F satisfies

(3.4) rec ?± =* rec P_j , ¥ i,jeQ**

and

(3.5) [y|A^ky $\geq$ 0} c rec ?± , ¥ keQ\Q*, ieQ**

then

SjCQ) = F.

Proof. With or without (3.4) and (3.5), S^Q) 2F. Indeed, if x e ?± for some ieQ, then x together with the vectors $(y^i, y^i_o) = (x, 1)$, $(y^k, y^k_o) =$ (0, 0), keQ\(i}, satisfies the constraints defining $S^I(Q)$. It remains to be shown that if (3.4) and (3.5) hold, §].(Q) £F.

Suppose (3.4) and (3.5) are satisfied and let x e $S^I(Q)$. Then there exists keQ**, Q'cQ** and Q" c Q\Q*, such that

$$x = y^k + \sum_{ieQ'UQ''} y^i ,$$

and x together with the vectors (y^k, 1), (y^1, 0), ieQ'UQ*, and (y^, $y^{\wedge}_o$) = (0, 0), jeQXQ'UQ'uOc}, satisfies $(3.2)_n$. But then y^keP, and y^ie rec P. for ieQ' (from (3.4)) and for ieQ'' (from (3.5)). Thus x e P_k.||

While the condition of Theorem 3.6 is not necessary, it is as weak a sufficient condition as one can get without breaking up Q** into further subsets, for some of which the equality in (3.4) can be weakened to inclusion.

The essential fact about Theorem 3.6 is the following immediate consequence, which was proved earlier in a different way by Jeroslow and Lowe [10].

Corollary 3.7. If each P_i is nonempty and bounded, then $S_I(Q)$ = F.

Thus not only is S(Q) the convex hull of the union of the nonempty, bounded polyhedra P^, ieQ, but Sj(Q) is a valid mixed-integer representation of such a union of polyhedra. As Jeroslow and Lowe [10] have recently noticed, this representation is better than the usual one, since its linear programming relaxation is 5(Q), the convex hull of the union, which is often not true of the usual representation. By the latter we mean the representation of F = $\underset{ieQ}{U} P_L$. as the set $A_T(Q)$ of those x e R^n_x satisfying

$$l\wedge x - (a^*_o - L^i)6_i \wedge L^1, \quad ieQ$$

$$\underset{ieQ}{I} 6_i = 1$$

$$6_i \in \{0, 1\} \quad , \quad ieQ .$$

where each L^i is a lower bound (vector) on $A^i x$,

If we denote by A(Q) the set obtained from $A_I(Q)$ by relaxing the conditions 6_i e{0, 1} to $6_i \geq 0$, ieQ, A(Q) is not necessarily the convex hull of F, In other words, while S(Q) = conv $S_I(Q)$ whenever all P_i are non-empty and bounded, for A we only have the relation

$$A(Q) \ 2 \ \text{conv} \ \Delta_I(Q)$$

which often holds as strict inclusion, as will be illustrated later.

We need one more result before introducing the family of relaxations of a disjunctive set. Namely, we want to use Theorem 3.3 to characterize the convex hull of an elementary disjunctive set.

<u>Theorem 3.8</u>. Let D = $\underset{ieQ}{U} HI^l$ » (x cH^nl $\underset{ieQ}{V}$ (a^ $\geq$ a_{lo})}. Then

$$\text{cl conv } D = \begin{cases} \mathbb{R}^n & \text{if D is proper} \\ H_k^+ & \text{if D is improper, with } D = H_k^+. \end{cases}$$

<u>Proof</u>. If D * H_k^+ for some keQ, cl conv D * H_k^+ since IL_k^+ is closed and convex. Suppose now that D is proper, and let $\bar{x}$ be an arbitrary but fixed point in H^a. From Theorem 3.3, $\bar{x}$ e cl conv D if and only if the system

$$\sum_{ieQ} y^1 = \bar{x}$$

$$a^i y^i - a_{io} y_o^i \geq 0 \quad , \quad i\epsilon Q$$

$$\sum_{i\epsilon Q} y_o^1 - !$$

$$yj \geq 0 \quad , \quad ieQ$$

has a solution. From the Theorem of the Alternative, this is the case if and only if the system

$$-u_0 V + v = 0 \quad , \quad i\epsilon Q$$

(3.6)

$$u_o^i a_{10} - v_o \geq 0 \quad ,, \quad ieQ$$

$$v\bar{x} - v_o < 0$$

$$\%^i \geq 0 \quad > \quad i\epsilon Q,$$

where u_o^i e H, ieQ, v_o 6 H, and veR^n, has no solution*

Since D is proper, there exists no keQ such that $H_i^+ C H_k^+$, ¥ ieQ; hence there exist no scalars $u_o^i \geq 0$, ieQ, such that $\Pi a_o^i = u_o^k a_o^k$, ¥ icQ. Thus (2,6) has no solution for any $\bar{x}$, and hence $\bar{x}$ e cl conv D for all $\bar{x}$ e H^n, i.e., cl conv D = m^a.||

The convex hull of a proper elementary disjunctive set is thus $\mathbb{R}^n$, i.e., replacing such a set with its convex hull is tantamount to throwing away all the constraints that define it. This of course is not true for more general disjunctive sets, as will become clear soon.

The system (3.2) which defines the convex hull of a disjunctive set in DNF is easy to write down, but is unwieldy when the set Q is large; and for a mixed integer program whose feasible set F is expressed as a disjunctive set in DNF, Q tends to be large. Thus an attempt to use Theorem 3.3 to generate the convex hull of the feasible set is in general not too promising.

On the other hand, the feasible set of most discrete optimization problems, when given as a disjunctive set in CNF, has conjuncts that are the unions of small numbers of halfspaces, often only two. Performing some basic steps one obtains a set in RF whose conjuncts are still the unions of small numbers of polyhedra. Note that if a disjunctive set is in the RF given by (2.1), (2.2), each conjunct S_j is in DNF; hence we know how to take its convex hull. Naturally, taking the convex hull of each conjunct is in general not going to deliver the convex hull of the disjunctive set, but can serve as a relaxation of the latter. This takes us to the class of relaxations announced at the beginning of this paper.

4. A Hierarchy of Relaxations of a Disjunctive Set

Given a disjunctive set in regular form

$$F = \bigcap_{j \in T} S_j$$

where each S_j is a union of polyhedra, we define the hull-relaxation of F, denoted h-rel F, as

$$\text{h-rel } F := \bigcap_{j\in T} \text{cl conv } S_j .$$

The hull-relaxation of F is not to be confused with the convex hull of F: its usefulness comes precisely from the fact that it involves taking the convex hull of each union of polyhedra <u>before</u> intersecting them.

Next we relate the hull-relaxation of a disjunctive set to the usual linear programming relaxation of the feasible set of a mixed integer program. Obviously, the hull-relaxation of any disjunctive set is polyhedral, since the intersection of polyhedra is a polyhedron. Suppose now that we have a disjunctive set in CNF,

$$F_o = \bigcap_{j\in T} D_j ,$$

where each D_j is the union of halfspaces. Let $T^* = \{j\in T \mid D_j \text{ is improper}\}$, and denote

$$P_o = \bigcap_{j\in T^*} D_j ,$$

with $P_o = \mathbb{R}^n$ if $T^* = \emptyset$. P_o can be viewed as the "polyhedral part" of F_o, i.e., the intersection of those elementary disjunctive sets that are halfspaces.

<u>Lemma 4.1</u>.

$$\text{h-rel } F_o = P_o .$$

<u>Proof</u>.

$$\text{h-rel}\left(\bigcap_{j\in T} D_j\right) = \text{h-rel}\left(P_o \cap \left(\bigcap_{j\in T\setminus T^*} D_j\right)\right)$$

$$= \text{cl conv } P_o \cap \left(\bigcap_{j\in T\setminus T^*} \text{cl conv } D_j\right)$$

by the definition of the hull-relaxation. But cl conv $P_o = P_o$ and from Theorem 3.6, cl conv $D_j = \mathbb{R}^n$ for all $j \in T \setminus T^*$. This yields the equality stated in the Lemma.||

When the feasible set of a (pure or mixed integer) 0-1 program is stated in CNF (which is the usual way of stating it), T^* is the index set of all the conjunctive, i.e., ordinary linear constraints, and $T \setminus T^*$ is the index set of the disjunctions $x_j \leq 0 \vee x_j \geq 1$. Thus P_o is the linear programming feasible set, and the hull-relaxation of a (pure or mixed-integer) 0-1 program stated in CNF is <u>identical to the usual linear programming relaxation</u>.

The next question we address is what happens if one applies the hull-relaxation to a disjunctive set that is <u>not</u> in CNF. Specifically, we look at the effect of a basic step in the sense of relating the hull-relaxation of the RF before the basic step to that of the RF after the basic step.

<u>Lemma 4.2</u>. For j = 1, 2, let

$$S_j = \bigcup_{i \in Q_j} P_i,$$

where each P_i, $i \in Q_j$, $j = 1, 2$, is a polyhedron. Then

$$\text{cl conv}(S_1 \cap S_2) \subseteq (\text{cl conv } S_1) \cap (\text{cl conv } S_2). \tag{4.1}$$

<u>Proof</u>. Certainly $S_1 \cap S_2 \subseteq (\text{cl conv } S_1) \cap (\text{cl conv } S_2)$, and since cl conv $(S_1 \cap S_2)$ is the smallest closed convex set to contain $S_1 \cap S_2$, (4.1) follows.||

<u>Theorem 4.3</u>. For i = 0, 1,...,t, let

$$F_i = \bigcap_{j \in T_i} S_j^i$$

be a sequence of regular forms of a disjuntive set, such that

(i) F_0 is in CNF, with $P_0 = \cap\{S_j^0 | S_j^0$ is improper$\}$;

(ii) F_t is in DNF;

(iii) for $i=1,\ldots,t$, F_i is obtained from F_{i-1} by a (possibly parallel) basic step.

Then

$$P_0 = \text{h-rel } F_0 \supseteq \text{h-rel } F_1 \supseteq \ldots \supseteq \text{h-rel } F_t = \text{cl conv } F_t .$$

Proof. The first equality holds by Lemma 4.1, since F_0 is in CNF. The last equality holds by the definition of a hull-relaxation, since F^t is in DNF, i.e., $|T^t| = 1$. Each inclusion holds by Lemma 4.2, since for $k = 1,\ldots,t$, F^k is obtained from F^{k-1} by a basic step.||

For any F^i in the above sequence, we can obtain from the hull-relaxation a mixed-integer programming representation of F^i by using Theorem 3.6. However, this representation requires one 0-1 variable for every polyhedron P_h^i in the expression

$$(4.2) \qquad F_i = \bigcap_{j\in T_i} S_j^i , \quad S_j^i = \bigcup_{h\in Q_j^i} P_h^i , \quad P_h^i = \{y\in R^n | A^h y \geq a_0^h\}, \quad h\in Q_j^i, \ j\in T_i,$$

which is usually much more than the number of 0-1 variables needed to represent the CNF of the same set, i.e.,

$$(4.3) \qquad F_0 = \bigcap_{r\in T_0} S_r^0 , \qquad S_r^0 = \bigcup_{s\in Q_r} H^s .$$

The next theorem gives a mixed integer representation of F_i which uses the same number of variables as that of F_0. For F_0 as defined in (4.3), let $T_1 = \{r\in T_0 | S_r^0$ is proper$\}$.

Theorem 4.4. Let F_o be the disjunctive set in CNF given by (4.3), and let F_i be the disjunctive set in RF given by (4.2), obtained from F_o by a sequence of basic steps, and satisfying the conditions of Theorem 3.6. Then F_i is the set of those $x \in \mathbb{R}^n$ for which there exist vectors $(y^h, y_o^h) \in \mathbb{R}^{n+1}$, $h \in Q_j$, $j \in T_i$, and scalars δ_{rs}, $s \in Q_r$, $r \in T_o'$, satisfying

$$(4.4)\qquad \left.\begin{array}{l} \left.\begin{array}{r} x - \sum\limits_{h \in Q_j} y^h = 0 \\ A^h y^h - a_o^h y_o^h \geq 0 \\ y_o^h \geq 0 \end{array}\right\} h \in Q_j \\ \sum\limits_{h \in Q_j} y_o^h = 1 \end{array}\right\} j \in T_i$$

$$(4.5)\qquad \sum_{h | P_h \subseteq H_s^+} y_o^h - \delta_{rs} = 0 \qquad s \in Q_r,\ r \in T_o'$$

$$(4.6)\qquad \begin{array}{ll} \sum\limits_{s \in Q_r} \delta_{rs} = 1, & r \in T_o' \\ \delta_{rs} \in \{0,1\},\ s \in Q_r, & r \in T_o'. \end{array}$$

Proof. From Theorem 3.6, for each $j \in T_i$ the constraints (4.4) define the convex hull of S_j^i, and if amended with the condition $y_o^h \in \{0,1\}$, $h \in Q_j$, they define S_j^i itself. We will show that the constraints (4.5), (4.6) enforce precisely this condition, and therefore all constraints together define $F_i = \bigcap_{j \in T_i} S_j^i$.

For any given δ satisfying (4.6), the unique set of y_o^k satisfying (4.5) is defined by

(4.7) $y_{ij}^h = \begin{cases} 1 & \text{if } \delta_{rs} = 1,\ \forall (r,s): H_{rs}^+ \supseteq P_h \text{ and } s \in Q_r \\ 0 & \text{otherwise.} \end{cases}$

Indeed, $\delta_{rs} = 0$ implies $y_{ij}^h = 0$ for all $h \in Q_r$, $j \in T_i$, such that $H_{rs}^+ \supseteq P_h$, which means that those constraints (4.5) for which $\delta_{rs} = 1$ must be satisfied by setting $y_{ij}^h = 1$ for precisely those $h \in Q_i$, $j \in T_i$ for which this is prescribed by (4.7). ||

Theorem 4.4 provides a way of representing any disjunctive set in regular form as the feasible set of a mixed-integer program with the same number of 0-1 variables as would be required to represent the same disjunctive set in CNF.

In order to make best use of the hierarchy of relaxations defined in Theorem 4.3, one would like to know which basic steps result in a strict inclusion as opposed to an equality. The next theorem addresses this question.

Theorem 4.5. For j = 1, 2, let

$$S_j = \bigcup_{i \in Q_j} P_i,$$

where each P_i, $i \in Q_j$, j = 1, 2, is a polyhedron. Then

(4.8) $\quad \text{cl conv}(S_1 \cap S_2) = (\text{cl conv } S_1) \cap (\text{cl conv } S_2)$

if and only if every extreme point (extreme direction) of (cl conv S_1) $\cap$ (cl conv S_2) is an extreme point (extreme direction) of $P_i \cap P_k$ for some $(i, k) \in Q_1 \times Q_2$.

Proof. Let T_L and T_R denote the lefthand side and righthand side, respectively, of (4.8). Then

$$T_L = \text{cl conv} \left(\bigcup_{i \in Q_1} \bigcup_{k \in Q_2} (P_i \cap P_k) \right);$$

Thus $x \in T_L$ if and only if there exist scalars $\lambda_j \geq 0$, $j \in V$ and $\mu_\ell \geq 0$, $\ell \in W$, such that $\sum_{j \in V} \lambda_j = 1$ and

$$x = \sum_{j \in V} v_j \lambda_j + \sum_{\ell \in W} w_\ell \mu_\ell ,$$

where V and W are the sets of extreme points and extreme direction vectors, respectively, of the union of all ^ ^ ^ J (i» k) $\subset Q_1 \times Q_2$«

On the other hand, $x \in T_K$ if and only if there exist scalars $\lambda'_j \geq 0$, $j \in V'$ and $\mu'_\ell \geq 0$, $\ell \in W'$, such that $\sum_{j \in V'} \lambda'_j = 1$ and

$$x = \sum_{j \in V'} v'_j \lambda'_j + \sum_{\ell \in W'} w'_\ell \mu'_\ell ,$$

where v' and w' are the sets of extreme points and extreme direction vectors, respectively, of T_R. If the condition of the theorem holds, i.e., if $V' \subseteq V$ and $W' \subseteq W$, then $T_K \subseteq T_L$, and since by (4.1) $T_L \subseteq T_K$, we have $T_L = T_K$ as claimed. If, on the other hand, $V' \setminus V \neq \emptyset$ or $W' \setminus W \neq \emptyset$, then there exists $x \in T_K \setminus T_L$, hence (4.1) holds as strict inclusion.||

One immediate consequence of this Theorem is

Corollary 4.6. Let

$$K = \{x \in \mathbb{R}^n \mid 0 \leq x_j \leq 1, \ j = 1, \ldots, n\},$$

and

$$S_j = \{x \in K \mid x_j \leq 0 \vee x_j \geq 1\}, \qquad j = 1, \ldots, n.$$

Then

$$(4.7) \qquad \operatorname{conv} \bigcap_{j=1}^{n} S_j = \prod_{j=1}^{n} \operatorname{conv} S_j .$$

Thus basic steps that replace a set of disjunctive constraints of the form

$$x_j \leq 0 \vee x_j \geq 1, \quad j \in T$$

by a disjunctive constraint of the form

$$\bigvee_{S \subseteq T} \begin{pmatrix} x_j \leq 0, & j \in S \\ x_j \geq 1, & j \in T \setminus S \end{pmatrix}$$

before caking the hull-relaxation, do not produce a stronger relaxation: taking the convex hull before or after the execution of such basic steps produces the same result. In order to obtain a stronger hull-relaxation, the basic steps to be performed must involve some other constraints than those of the above form.

Next we illustrate on some examples various situations when taking the convex hull before or after a basic step does make a difference.

Example 4.1. (Fig. 3.1) Let $P_1 = \{x \in \mathbb{R}^2 \mid x_1 = 0,\ 0 \leq x_2 \leq 1\}$, $P_2 = \{x \in \mathbb{R}^2 \mid x_1 = 1,\ 0 \leq x_2 \leq 1\}$, $P_3 = \{x \in \mathbb{R}^2 \mid -x_1 + x_2 \geq 0.5,\ x_1 \geq 0,\ x_2 \leq 1\}$, $P_4 = \{x \in \mathbb{R}^2 \mid x_1 - x_2 \geq 0.5,\ x_1 \leq 1,\ x_2 \geq 0\}$, and let $F = S_1 \cap S_2$, with $S_1 = P_1 \cup P_2$, $S_2 = P_3 \cup P_4$. Then

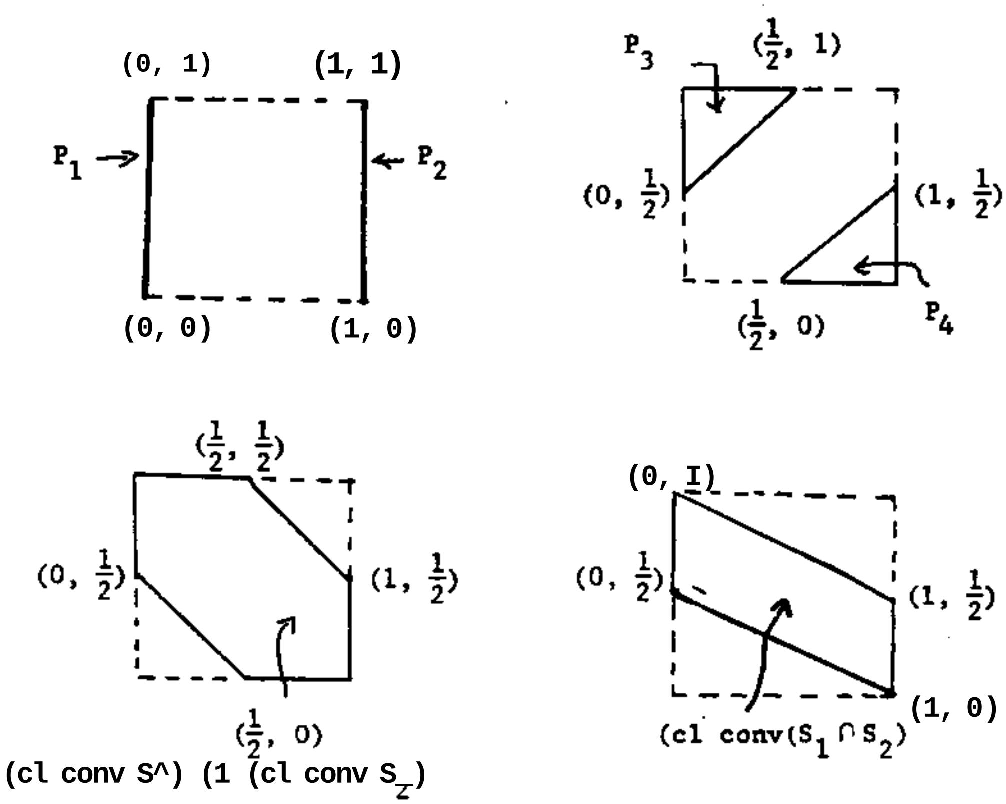


Fig. 3.1

cl conv Sj^ = £x cR [sic] |0 ≤ x_1 ≤ 1, 0 ≤ x_2 ≤ 1}

cl conv S_2 = (x elR^2|0.5 ≤ x_x + x_2 ≤ 1.5, 0 ≤ ^ ≤ 1, 0 ≤ x_2 ≤ l},

and

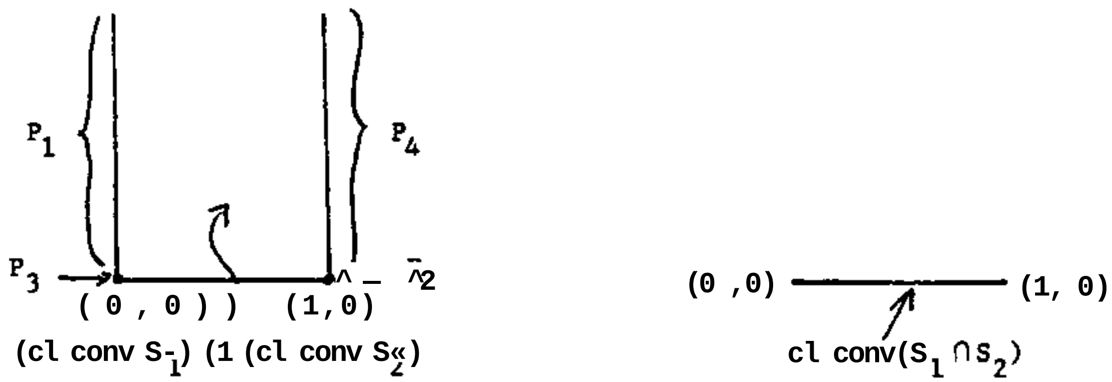
(cl conv S_1) 0 (cl conv S_2) = cl conv S_2.

On the other hand, S_x 0 S_2 = (P_x UP_3> fl (P_2 U P_4> (since ?$_l$ 0 P_4 = P_2 0 P_3 = 0), and

cl conv($S_i$riS_2) = {x cR^2| l ≤ x_x + 2x_2 ≤ 2, 0 ≤ x_x ≤ l}.

Here (4.1) holds as strict inclusion, because the vertices (0.5, 0) and (0.5, 1) of (cl conv S) C\ (cl conv S_2) are not vertices of either P_1 PlP^ or P2^P4* although the first one is a vertex of P_4 and the second one a vertex of P_3.

Example 4.2. (Fig. 3.2) Let ?$_1$ = [x e IR^2 ^ » 0, x_2 ≥ 0},
P_2 = (x €]R^2|x_1 = 1, x_2 = 0}, P_3 = {x €lR^2|x_1 = 0, x_2 = 0},
P_4 = (x eIR |x_x = 1, x_2 ≥ 0}, and let F = S ^ S ^ with S_l = P_1UP_2,
S_2 = P_3U P_4. Then



Fig, 3,3

cl conv S_x = cl conv S_2 = [x eE^2|0 ≤ x_x ≤ 1, x_2 ≥ 0},

= (cl conv S_x) 0 (cl conv S_2),

whereas

cl conv(S_1 ∩ S_2) = cl conv((P_x UP_3) 0 (P_2 UP_4))

= (x elR^2|0 ≤ x_t ≤ 1, x_2 = 0}.

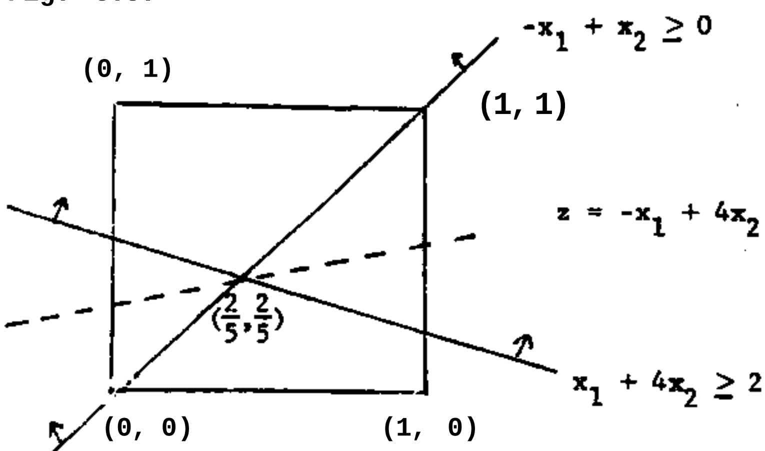
Here (4.1) holds as strict inclusion because (0, 1) is an extreme direction vector of $(\text{cl conv } S_1) \cap (\text{cl conv } S_2)$, but not of $P_1 \cap P_3$ or $P_2 \cap P_4$.

It is an important practical problem to identify typical situations when it is useful to perform some basic step, i.e., to intersect two conjuncts of a RF before taking their convex hull. The usefulness of such a step can be measured in terms of the gain in strength of the hull-relaxation versus the price one has to pay in terms of the increase in size. Since the convex hull of an elementary disjunctive set is H^n, i.e., taking the convex hull of such sets does not constrain the problem at all, one should intersect each elementary disjunctive set S_j in the given RF with some other conjunct S_k before taking the hull-relaxation. This can be done at no cost (in terms of new variables) if S_k is improper. Often intersecting a single improper conjunct S_k with each proper disjunctive set S_j appearing in the same RF, i.e., executing a single parallel basic step before taking the hull relaxation, can substantially strengthen the latter without much increase in problem size. As to shich improper conjunct S_k to select, a general principle that one can formulate is that the more restrictive is S_k with respect to each S_j, the better suited it is for the purpose. The next example illustrates this.

<u>Example 4.3</u> Consider the 0-1 program

$$\text{(P)} \qquad \min \{z = -x_1 + 4x_2 \mid -x_1 + x_2 \geq 0;\ x_1 + 4x_2 \geq 2;\ x_1, x_2 \in \{0, 1\}\}$$

illustrated in Fig. 3.3.



<u>Figure 3.3</u>

The usual linear programming relaxation gives the optimal solution $\bar{x}_1 = \bar{x}_2 = 2/5$, with a value of $\bar{z} = 6/5$. This of course corresponds to taking the hull-relaxation of the CNF of the feasible set of (P), which contains as conjuncts the improper disjunctive sets corresponding to each of the inequalities of (P) (including $0 \le x_1 \le 1$, $0 \le x_2 \le 1$) and the two proper disjunctive sets $S_1 = \{x \in R^2 \mid x_1 \le 0 \vee x_1 \ge 1\}$, $S_2 = \{x \in R^2 \mid x_2 \le 0 \vee x_2 \ge 1\}$. If P_0 is the intersection of all the improper disjunctive sets, the hull relaxation of the CNF of (P) is $F_0 = P_0 \cap \text{conv}\, S_1 \cap \text{conv}\, S_2$.

Let us write $K = \{x \in R^2 \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$, and $P_0 = P_{01} \cap P_{02}$, with $P_{01} = \{x \in K \mid -x_1 + x_2 \ge 0\}$, $P_{02} = \{x \in K \mid x_1 + 4x_2 \ge 2\}$. Now suppose we intersect each of S_1 and S_2 with P_{01} before taking the convex hull, i.e., use the hull relaxation $F_1 = P_{02} \cap \text{conv}(P_{01} \cap S_1) \cap \text{conv}(P_{01} \cap S_2)$. We find that $\text{conv}(P_{01} \cap S_1) = \text{conv}(P_{01} \cap S_2) = \{x \in K \mid -x_1 + x_2 \ge 0\}$, and hence $F_1 = F_0$, i.e., these particular basic steps bring no gain in the strength of the relaxation.

Suppose instead that we intersect S_1 and S_2 with P_{02} before taking the convex hull, i.e., use the hull relaxation $F_2 = P_{01} \cap \text{conv}(P_{02} \cap S_1) \cap \text{conv}(P_{02} \cap S_2)$. Then $\text{conv}(P_{02} \cap S_1) = \{x \in K \mid x_1 + 4x_2 \ge 2\}$, $\text{conv}(P_{02} \cap S_2) = \{x \in K \mid x_2 = 1\}$, and $F_2 = \{x \in K \mid x_2 = 1\}$, which is a stronger relaxation than F_0. Using the relaxation F_2 instead of F_0, i.e., solving $\min\{z = -x_1 + 4x_2 \mid x \in F_2\}$, yields $\bar{x}_1 = \bar{x}_2 = 1$, with $\bar{z} = 3$, which happens to be the optimal solution of (P).

Note that P_{01} cuts off only one vertex of $\text{conv}(S_1 \cap K) = \text{conv}(S_2 \cap K) = K$, whereas P_{02} cuts off two vertices of K. ||

When basic steps are used that intersect <u>proper</u> disjunctive sets before taking their convex hull, the number of variables in the hull relaxation increases. Especially attractive are those situations where the increase in problem size is mitigated by the presence of some structure that makes it possible to solve the increased linear programs efficiently. This is the case in the machine sequencing problem discussed in the next section, as well as in certain network synthesis and fixed charge network flow problems.

5. An Illustration: Machine Sequencing via Disjunctive Graphs

In this section we illustrate the concepts and methods discussed in sections 1-4 on the example of the following well known job shop scheduling (machine sequencing) problem: n operations are to be performed on different items using a set of machines, where the duration of operation i is d_i. The objective is to minimize total completion time, subject to (i) precedence constraints between the operations, and (ii) the condition that a machine can process only one item at a time, and operations cannot be interrupted. The problem is usually stated [1] as

$$\min t_n$$

$$(P)\qquad \begin{array}{ll} t_j - t_i \geq d_i, & (i,j) \in Z \\ t_i \geq 0, & i \in V \\ (t_j - t_i \geq d_i) \vee (t_i - t_j \geq d_j), & (i,j) \in W^+ \end{array}$$

where t_i is the starting time of job i (with n the dummy job "finish"), V is the set of operations, Z the set of pairs constrained by precedence relations, and W^+ the set of pairs that use the same machine and therefore cannot overlap in time. It is often useful to represent the problem by a disjunctive graph $G = (V, Z, W)$, with vertex set V and two kinds of directed arc sets: conjunctive (or usual) arcs, indexed by Z, and disjunctive arcs, indexed by W. The set W consists of pairs of disjunctive arcs and is of the form $W = W^+ \cup W^-$, with $(i,j) \in W^+$ if and only if $(j,i) \in W^-$. The subset of nodes corresponding to each machine, together with the disjunctive arcs joining them to each other, forms a disjunctive clique, A selection $S \subset W$ consists of exactly one member of each pair of W: i.e., there are 2^q possible selections, where $q = \frac{1}{2}|W|$: G is illustrated in Fig. 5.1, where the disjunctive arcs are shown by dotted lines. If $\mathcal{S}$ denotes the set of selections, for every $S \in \mathcal{S}$, $G_S = (V, Z \cup S)$ is an ordinary directed graph; and the problem (P(S)) obtained from (P) by replacing the set of disjunctive constraints indexed by W^+ with the set of conjunctive constraints indexed by S is the dual of a longest path (critical path) problem in G_S. Thus solving (P) amounts to finding a selection $S \in \mathcal{S}$ that minimizes the length of a critical path in G_S.

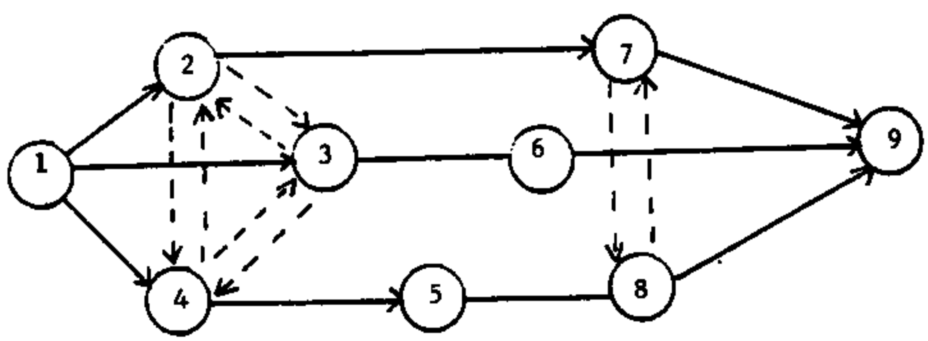


Fig. 5.1

The usual mixed integer programming formulation of (P) represents each disjunction

(5.1) $$t_j - t_i \ge d_i \quad \vee \quad t_i - t_j \ge d_j$$

by the constraint set

$$t_j - t_i - (d_i - L_{ij})y_{ij} \ge L_{ij}$$

(5.2) $$-t_j + t_i + (d_j - L_{ji})y_{ij} \ge d_j$$

$$y_{ij} \in \{0,1\},$$

where L_{ij} is a lower bound on $t_j - t_i$. Unless one wants to use a very crude lower bound L_{ij}, one has to derive lower and upper bounds, L_i and U_i, respectively, on each t_i, $i \in V$, and set $L_{ij} = L_j - U_i$. L_j can be taken to be the length of a longest path from node 1 (the source) to node j in the (conjunctive) graph $G_\emptyset = (V, Z)$, and U_j the difference between the length of a critical path in G_S for some arbitrary selection $S \subset \mathcal{S}$, and the length of a longest path from node j to node n (the sink) in $G_\emptyset$.

The constraint set (5.2) accurately represents (5.1) (amended with the bounds $L_k \le t_k \le U_k$, k = 1,2), but its linear programming relaxation $(5.2)_L$, obtained by replacing $y_{ij} \in \{0,1\}$ by $0 \le y_{ij} \le 1$, has no constraining power, as shown by the next theorem.

Theorem 5.1. If the disjunction (5.1) is proper, then every t_i, t_j that satisfies

(5.3) $$L_i \le t_i \le U_i \quad , \quad L_j \le t_j \le U_j$$

also satisfies $(5.2)_L$.

Proof. It suffices to show that the four extreme points (L_i, L_j), (L_i, U_j), (U_i, L_j), (U_i, U_j) of the two-dimensional box defined by (5.3) satisfy $(5.2)_L$ for some y_{ij}. We first write $(5.2)_L$ in the form

$(5.2)_L$ $$(L_j - U_i)(1 - y_{ij}) + d_i y_{ij} \le t_j - t_i \le -d_j(1 - y_{ij}) + (U_j - L_i)y_{ij}$$

$$0 \le y_{ij} \le 1$$

and note that (L_i, U_j) and (L_j, U_i) satisfy (5.2) for $y_{ij} = 1$ and $y_{ij} = 0$, respectively. To show that (L_i, L_j) satisfies $(5.2)_L$ for some y_{ij}, we substitute (L_i, L_j) into $(5.2)_L$ and obtain

$$(5.4) \qquad \frac{d_j - L_i + L_j}{d_j - L_i + U_j} \leq y_{ij} \leq \frac{U_i - L_i}{U_i - L_j + d_i} ,$$

To see that (5.4) is feasible, note that the right hand side increases with U_i; so (5.4) is feasible if it is for the smallest admissible value of U_i, which is $L_j + d_j$ (for smaller U_i (5.1) becomes improper). Substituting $L_j + d_j$ for U_i we obtain that (5.2) is feasible whenever $L_i + d_i \leq U_j$, which is a condition for (5.1) to be proper.

An analogous argument shows that (U_i, U_j) satisfies (5.2) for some y_{ij}. ||

Consider now the mixed integer representation of (5.1) associated with the hull-relaxation of the feasible set of (P). If the latter is given in CNF, as is usually the case, applying the hull-relaxation to this form yields nothing, since the convex hull of the disjunctive set defined by (5.1) is R^2, the space of (t_i, t_j). If we perform a parallel basic step of the type defined in section 3 and introduce into each disjunct of (5.1) the lower and upper bounds on t_i and t_j, this replaces every elementary disjunctive set D_{ij} defined by a pair of constraints (5.1), by a disjunctive set

$$S_{ij} = \left\{ (t_i, t_j) \middle| \begin{pmatrix} t_j - t_i \geq d_i \\ L_i \leq t_i \leq U_i \\ L_j \leq t_j \leq U_j \end{pmatrix} \vee \begin{pmatrix} t_i - t_j \geq d_j \\ L_i \leq t_i \leq U_i \\ L_j \leq t_j \leq U_j \end{pmatrix} \right\} .$$

The feasible set of (P) is then of the form

$$F = P_o \cap \left(\bigcup_{(i,j) \in W^+} S_{ij} \right) \qquad (5.7)$$

where P_o is the polyhedron defined by the inequalities (5.3) and $t_j - t_i \geq d_i$, $(i,j) \in Z$. Further, we have (since all S_{ij} are bounded, clconv S_{ij} = conv S_{ij})

$$\text{h-rel } F = P_o \cap \left(\bigcap_{(i,j) \in W^+} \text{conv } S_{ij} \right)$$

and from Theorem 3.3, the convex hull of S_{ij} is the set of those (t_i, t_j) satisfying the constraints

$$t_k = t_k^1 + t_k^2 \qquad k = i,j$$

$$t_j^1 - t_i^1 \geq d_i y_{ij}$$

$$-t_j^2 + t_i^2 \geq d_j(1 - y_{ij})$$

(5.8)

$$L_k y_{ij} \leq t_k^1 \leq U_k y_{ij} \qquad k = i,j$$

$$L_k(1 - y_{ij}) \leq t_k^2 \leq U_k(1 - y_{ij})$$

$$0 \leq y_{ij} \leq 1$$

Also, from Corollary 3.7, the set of those (t_i, t_j) satisfying (5.8) and $y_{ij} \in \{0,1\}$ is S_{ij}, since both disjuncts of S_{ij} are bounded polyhedra; and thus using (5.8) with $y_{ij} \in \{0,1\}$ for all $(i,j) \in W^+$ is a valid mixed integer formulation of (P). This representation uses the same number of 0-1 variables as the usual one, but introduces two new variables, t_k^1, t_k^2, for every original variable t_k, with associated bounding inequalities $L_k y_{ij} \leq t_k^1 \leq U_k y_{ij}$, $L_k(1 - y_{ij}) \leq t_k^2 \leq U_k(1 - y_{ij})$. At the price of this increase in the number of variables and constraints, one obtains as the hull-relaxation a linear program whose feasible set is considerably tighter than in the usual formulation, since each constraint set (5.8) defines the convex hull of S_{ij}. It is not hard to see that each of the two points (L_i, L_j) and (U_i, U_j) violates (5.8) unless it is contained in one of the two halfspaces defined by $t_j - t_i \geq d_i$ and $t_i - t_j \geq d_j$.

Let us now perform some further basic steps on the regular form (5.7) before taking the hull-relaxation. In particular, let us intersect all S_{r_U} such that i and j belong to the same disjunctive clique K. If we denote $T(K) := \cap(S_{ij} : i, j_6K, i \neq j)$, and if $|K| = p$, then

$$T(K) = \left\{ t \in \mathbb{R}^p \;\middle|\; \begin{array}{l} t_i - t_j \geq d_j \vee t_j - t_i \geq d_i, \quad i, j \in K, \ i \neq j \\ L_i \leq t_i \leq U_i, \quad i \in K \end{array} \right\} .$$

Taking the basic steps in question consists of putting T(K) in disjunctive normal form. Let < K > denote the subgraph of G induced by K, i.e., the disjunctive clique with node set K. A selection in < K >, as defined at the beginning of this section, is a set of arcs containing one member of each disjunctive pair. Thus if < K > is viewed simply as the complete digraph on K, then a selection is the same thing as a tournament in < K >. If S_k denotes the k-th selection in < K > and Q indexes the selections of < K >, then the DNF of T(K) is $T(K) = \bigcup_{k \in Q} T_k(K)$, where

$$T_k(K) = \left\{ t \in \mathbb{R}^p \;\middle|\; \begin{array}{l} t_j - t_i \geq d_i, \quad (i,j) \in S_k \\ L_i \leq t_i \leq U_i, \quad i \in K \end{array} \right\} .$$

It is easy to see that if S_k contains a cycle, then $T_k(K) = \emptyset$. Let $Q^* = \{k \in Q \mid S_k \text{ is acyclic}\}$. Every selection is known to contain a directed Hamilton path, and for acyclic selections this path must be unique. Furthermore, every acyclic selection is the transitive closure of its unique directed Hamilton path.

Let P_k denote the directed Hamilton path of the acyclic selection S_k; then S_k is the transitive closure of P_k, and the inequalities $t_j - t_i \geq d_i$, $(i,j) \in P_k$, obviously imply the remaining inequalities of $T^K(K)$, corresponding to arcs $(i,j) \in S_k \setminus P_k$. Thus a more economical expression for the DNF of T is $T(K) = \bigcup_{k \in Q^*} T^K(K)$, with

$$T_k(K) = \left\{ t \in \mathbb{R}^p \;\middle|\; \begin{array}{l} t_j - t_i \geq d_i, \ (i,j) \in P_k \\ \\ l_i \leq t_i \leq u_i, \ i \in * \end{array} \right\}.$$

Now let M be the index set of the disjunctive cliques in G, and K_m the node set of the m-th such clique. Then the RF obtained from (5.7) by performing the basic steps described above is

$$(5.9) \qquad F = P_Q \cap \left(\bigcap_{m \in M} T(K_m) \right),$$

and the hull-relaxation of this form is

$$(5.10) \qquad \text{h-rel}\, F = P \cap \left(\bigcap_{m \in M}{}^{\text{conv}}\, T(K) \right).$$

For $m \in M$, let Q_m^* index the acyclic selections in $< K_m >$; and for $k \in Q_m^*$, let S_k^m and P_k^m denote the k-th acyclic selection in $< K_m >$, and its directed Hamilton path, respectively. Then introducing a continuous variable x_k^m for every acyclic selection S_k^m and a 0-1 variable y_{ij} for every disjunctive pair of arcs $\{(i,j), (j,i)\}$, and using Theorem 4.4, we obtain the following mixed integer formulation of problem (P) based on the hull-relaxation (5.10).

$$
(P)\quad
\begin{array}{llll}
\min t_n & & & \\
t_j - t_i & \geq d_i & & (i,j) \in Z \\
t_j - \sum_{k \in Q^*_m} t^k_j & = 0 & & j \in K_m,\ m \in M \\
-t^k_{j(1,k)} + t^k_{j(2,k)} - d_{j(1,k)} \lambda^m_k & \geq 0 & \} & \\
\quad \cdots & \cdots & & k \in Q^*_m,\ m \in M \\
-t^k_{j(p_k-1,k)} + t^k_{j(p_k,k)} - d_{j(p_k-1,k)} \lambda^m_k & \geq 0 & & \\
t^k_{j(1,k)} - t^k_{j(p_k,k)} + (u_{j(p_k,k)} - L_{j(1,k)}) \lambda^m_k & \geq 0 & \} & \\
\sum_{k \in Q^*_m} \lambda^m_k & = 1 & & m \in M \\
\sum_{k \mid (i,j) \in S_k} \lambda^m_k + y_{ij} & = 1 & \} & (i,j) \in W^+ \\
\sum_{k \mid (j,i) \in S_k} \lambda^m_k - y_{ij} & = 0 & \} &
\end{array}
$$

$$t_j,\ t^k_j \geq 0,\ \forall\ j,k;\ \lambda^m_k \geq 0,\ \forall\ k,m;\ y_{ij} \in \{0,1\},\ (i,j) \in W^+ .$$

Theorem 5«2# Problem (P) is equivalent to (P): if t is a feasible solution to (P), there exist vectors t^k and scalars X3, keQ*> meM, and a vector y, satisfying the constraints of (P); and conversely, if t, t^k, λ^m_k, $k \in Q^*_m$, meM, and y satisfy the constraints of (P), then t is a feasible solution to (P).

Proof, (P) is the representation of (P) given in Theorem 4.4, with the set F_i replaced by F as defined in (5.9), and with the difference that the upper bounding inequalities $-t_j^k + U_j \lambda_k \geq 0$, $j \epsilon K_m$, are replaced by the single inequality $t^k_{j(1,k)} - t^k_{j(p_k,k)} + (U_{j(p_k,k)} - L_{j(1,k)})\lambda_k \geq 0$. The role of the upper bounding inequalities is to force each t_j^k to 0 when $\lambda_k^m = 0$, and the inequality that replaces them in $\mathcal{P}$ does precisely that: together with the inequalities associated with the arcs of P_k^m, it defines a directed cycle in $< K_m >$ and thus $\lambda_k^m = 0$ forces to 0 all t_j^k, $j \epsilon K_m$.||

The linear programming relaxation of ($\mathcal{P}$) is much stronger than the linear programming relaxation of the common mixed integer formulation of (P). Preliminary computational experience on a few small problems indicates that the value of this stronger linear programming relaxation tends to be much higher than that of the usual linear programming relaxation. For example:

	Value of		
	Usual LP	Strong LP	IP
Problem 1	18	25.1	31
Problem 2	8	10.7	13
Problem 3	20	25.8	35

On the other hand the linear programming relaxation of ($\mathcal{P}$), unlike that of the usual mixed integer formulation of (P), is not a longest path problem. This is a serious disadvantage, which has to be overcome by finding a solution method that takes advantage of the structure of ($\mathcal{P}$). While this is in general still an unsolved problem, an important aspect of it has been successfully solved. Namely, if ($\mathcal{P}$) is to be solved by projection on the space of the y-variables, i.e., by Benders's partitioning method, then in order to generate the inequalities of the Benders master problem one has to solve the dual of the linear program obtained from ($\mathcal{P}$) for various 0-1 values of y. We have recently found a way of deriving a solution to this problem from a solution to the longest path problem that corresponds to it in the usual formulation of (p). But the discussion of this algorithm is left to another paper.

Acknowledgement

I wish to acknowledge the useful conversations that I had with Bob Jeroslow on the subject matter of this paper.

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