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A LINEAR PROGRAMMING MODEL FOR DESIGN OF
COMMUNICATIONS NETWORKS WITH PROBABILISTIC DEMAND

by

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TABLE OF CONTENTS

1.	Introduction: A Design Synthesis Problem	1
2.	Approaches to Determine Trunking and Switching Requirements to Meet Demand for Service	6
3.	A Formal Specification of a Network Hierarchy	10
3.1.	Designation of Direct and Final Links	10
3.2.	Classifying Point-to-Point Offered Loads	13
4.	The Formulation of a Nonlinear Supply Model	15
4.1.	Blocking Probabilities and Overflow Traffic	15
4.2.	An Assumption on Marginal Capacities	16
4.3.	A Nonlinear Integer Programming Formulation for the Network Hierarchy of Section 3	18
4.3.1.	Sum of All Type 2 Parcels Offered to Final IJ	18
4.3.2.	Sum of All Type 3 Parcels Offered to Final IJ	19
4.3.3.	A Constraint on the Sum of All Traffic Offered to Final IJ	20
4.3.4.	Accounting for Total Switched Traffic	22
4.3.5.	Cost Assumptions and the Nonlinear Model	23
5.	A Linear Programming Approximation to the Nonlinear Program P	26
5.1.	The Convexity Properties of the Blocking Probabilities	26
5.2.	The Key Approximation and the Linear Program	29
5.3.	Solving the Linear Program LP ¹ Through Bounded Variable Reductions	37

Table of Contents	2
6. Numerical Experiments on Two Examples	39
6.1. First Example: A Network Based on California Field Data	39
6.2. The Second Example: Figure 1's Network Hierarchy	45
7. Conclusions	49
Appendix: Proof of Proposition 1 and Its Three Corollaries	52
References	58

ABSTRACT

In this paper marginal investment costs are assumed known for two kinds of equipment stocks employed to supply telecommunications services: trunks and switching facilities. From the supply viewpoint customer demands for service and quality of service are specified probabilistically between pairs of junctions in a network according to different hours of the day. Traffic itself may flow over direct "high usage" routes or over alternate routes according to a specified network routing hierarchy, a structuring which has classically led to economies of equipment in supplying customer service*

In this paper a network hierarchy is defined which includes important cases occurring in the field and also appearing in the literature. A different use of the classical concept of the marginal capacity of an additional trunk at prescribed blocking probability leads to a linear programming supply model which can be used to compute the sizes of all the high usage trunk groups. The sizes of the remaining trunk groups are approximated by the linear programming model, but can be determined more accurately by alternate methods once all high usage group sizes are computed.

The approach applies to larger scale networks than previously reported in the literature and permits direct application of the duality theory of linear programming and its sensitivity analyses to the study and design of switched probabilistic communications networks with multiple busy hours during the day. Numerical

results are presented for two examples based on field data, one of which having been designed by the multi-hour engineering method.

1. Introduction; A, Design Synthesis Problem

In this paper we treat telecommunications networks where customer demands for service are specified probabilistically between pairs of junctions according to different hours of the day. Telephone traffic may flow over the direct route which joins two distinct junctions or over an alternate route which is defined in a prespecified network routing hierarchy. Networks which permit alternate routing of traffic are termed switched because switching operations are required to alternately route a call. The network routing hierarchy permits traffic which is blocked on a direct route to be switched through other junctions in further attempts to connect the original pair of junctions. The switching process tends to smooth out the peaks of traffic loads which occur throughout the network at different times of the day. Consequently, less equipment may be required to service the overall traffic load on the network than for a similar network without alternate routings.

An example of a network routing hierarchy is given below in Figure 1. It consists of junctions A through H and two different kinds of links joining certain pairs of junctions. A link is merely a dimensionless entity whose existence indicates that telephone calls, collectively termed traffic, may flow in either direction between the two junctions which it joins, without involving any other junction than these two. A dashed line designates a direct link while a solid line designates a final link. If there is a direct link between a call-origination junction and a call-destination junction, then a telephone

connection is first attempted on this link, the first choice route. Should the first choice connection fail, then an attempt is made to alternately route the call by way of final links, and in this case the traffic is referred to as overflow traffic. Arrows in Figure 1 indicate the overflow routing scheme. In case no direct link exists between a call origin and destination, then the call is also routed along the final links. Should a connection on final routes fail, we say that the call is "lost", and the caller must try to place the call again.

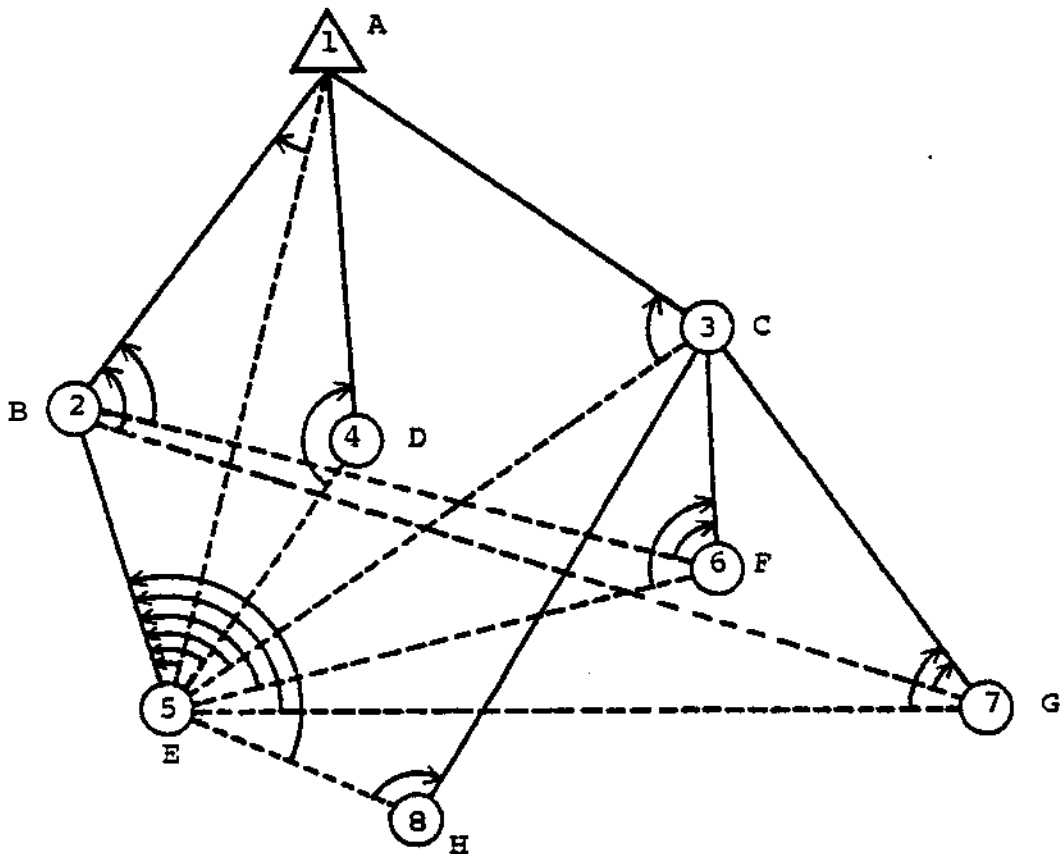


Figure 1. A Network Hierarchy with Direct (—) and Final (---) Links Where Overflow From a Direct Link Onto a Final Link is Indicated by an Arrow.

The basic problem attacked in this paper is one of design synthesis; solve for least-cost equipment charges in a given network routing hierarchy which are sufficient to meet altered point-to-point customer demands for service during different times of day to within a prescribed blocking probability. The emphasis is on the provision of a telecommunications service by an optimal use of available equipment. The model we develop includes a probabilistic specification of customer demand by time of day and includes alternate routings, where each direct link has a uniquely specified alternate route in the hierarchy. It is a nonlinear integer program P, which takes as a basic "unit" of equipment the concept of a "trunk". The terminology requires clarification.

In this paper a trunk shall merely refer to a channel which is required in order for a telephone call to transpire. As such, it is a dimensionless quantity. The call carrying capacity of a trunk depends on the probabilistic mechanism underlying customer calling patterns. For example, during a fixed hour a trunk could carry 60 one-minute serially placed telephone calls. Under this discipline the total carried load during the hour is 3600 call-seconds, denoted 36 CCS. Expressed another way, we observe that the probability of a call being blocked is zero. On the other hand should a demand for 60 one-minute calls occur simultaneously, then the offered load is still 36 CCS, but only 0.6 CCS is actually carried. The blocking probability is now 59/60.

A collection of trunks joining two distinct junctions is merely referred to as a trunk group. It is convenient to view a link as a trunk group. According to network engineering

principles, it is quite reasonable to assume that customer originated calls are generated by a Poisson process and are assigned sequentially to a trunk group. These assumptions yield an important property which is fundamental to our development of a good linear programming approximation to the nonlinear integer program P, namely, that the carried load on the last trunk is monotonically decreasing with the number of trunks, see Messerli [13]. The necessary results upon which the linear programming construction is based are proved in an Appendix.

When the hours at which the final groups reach capacity are in some sense stable and when the hour at which total network switching reaches capacity is stable, then an optimal solution to the linear program exists for which all direct group sizes themselves are integers. This assumption has some practical significance and has been employed in an example in Eisenberg [5].

The nonlinear and linear supply models of this paper employ certain concepts of unit costs with respect to both trunking and switching. The definition of "cost" shall be limited to the incremental investment cost of providing a trunk on the direct route between two junctions and the incremental investment cost of providing a trunk along the uniquely specified alternate route connecting these two junctions. In addition, we shall include unit switching investment costs per CCS as a crude approximation for switching investments stemming from switching calls from one trunk group to another.

Finally, we present linear programming calculations for two network hierarchies occurring in the field, one of which has been designed using nonlinear steepest descent methods, see Eisner [6].

2. Approaches to Determine Trunking and Switching Requirements to Meet Demand for Service

Over the past 30 years it appears that there have been at least two basic approaches to the design synthesis problem discussed in the previous section.

The basic thrust of our paper proceeds according to what we term the first approach to the design problem. It incorporates specific probability distributions for each parcel of traffic, where a parcel is merely that portion of traffic which follows specific routes in the network. Different parcels experience different blocking probabilities, even on the very same trunk group. For example a given trunk group may accommodate customer originated traffic governed by the Poisson probability distribution, and the group may also accommodate overflow traffic which is "peaked", in the sense that the mean of the distribution is less than its variance. Investigations of the blocking probabilities of individual parcels have been made by Wilkinson [20], Katz [12], and more recently by Deschamps [4].

The pioneering work representing a probabilistic approach which has had widespread use throughout the telecommunications industry is the 1954 paper by Truitt [19]. The generally accepted name of the method reflects the fact that economic considerations are also an integral part. The method is termed the "ECCS method", where the letter "E" stands for "economic". The method was introduced by Truitt for the simplest of routing hierarchies consisting of a triad of junctions with one overflow possibility,

and one specific time of day (single hour). The solved-for variables are the specific sizes of all trunk groups.

Further important extensions of the ECCS-method occurred in three directions. First, more accurate refinements of the overflow distributions themselves were made following the "equivalent random method" of Wilkinson [20]. Second, more complicated network hierarchies were introduced, see for example Rapp [15]. The third advance involved incorporating traffic overflows and constraints on blocking probabilities for more than one time of day in the same cost minimization model, see Rapp [15] and Eisenberg [5]. It appears that it is necessary to consider overflow traffic for multiple times of day in order to determine trunk group sizes which meet stated blocking probability constraints. In addition, networks based on field data have been reported in Eisenberg [5] and Eisner [6] where potential costs savings may be realized by incorporating multiple times of day.

The second major approach to determine levels of telecommunications equipment appeared in the 1956 paper of Kalaba and Juncosa [11]. Their approach is based on a linear programming model for a classical routing problem having variable link capacities, and as such is a large scale one. Several contrasts to the first approach (embodied in the ECCS method) are apparent.

First, the parcels of traffic in the Kalaba-Juncosa model are deterministic. Traffic originating at junction i and terminating at junction j is a given constant, a_{ij} . Second, demands are specified for each year (or other relevant time period), in contrast to a specification for multiple "hours"

within a fixed time period. Consequently, link capacities may be specified for ensuing future periods, but the impact of multiple busy periods within a given period has not been modeled.

In spite of severe deterministic assumptions the pioneering linear programming model of Juncosa and Kalaba can theoretically accommodate all conceivable routing possibilities, for their traffic variables are indexed by an origin-destination point pair and also a specific through-switched point, over all possible triads.

About 5 years after the Juncosa-Kalaba paper, a series of papers written by Gomory and Hu on communication network flows appeared in the SIAM Journal [8], [9], [10]. Their work occurred over a 4-year period and expanded significantly the size of the linear programming network models that could be treated computationally. They were able to combine features of generalized linear programming decomposition techniques with efficient Ford-Fulkerson methods for solving network subproblems. Gomory and Hu also stressed the importance of including communications demands indexed by time, such as time of day, t . They proceeded under the reasonable assumption that the time value takes on only a finite number of values. Alternatively, one could employ a continuous load curve with time-of-day varying demand.

Gomory and Hu illustrated their computational approach on a 10-node, 20-arc network with demands for two different time periods, and a given set of unit capacity (expansion) costs.

Based on discussions with engineers in the field, principally from the Long Lines Company of A.T. & T., we have found that

both approaches have had significant impact in the actual design of telecommunications networks. The completely deterministic approach (the second approach) has been particularly important in delineating first choice, second choice, etc. alternate routes between pairs of junctions to be used in defining a network hierarchy. Once a network hierarchy is established, economies of scale are then achievable according to optimal use of the underlying probability distributions of originating and alternately routed customer traffic.

Defining a network hierarchy is an essential feature of our approach, and we proceed now to this task using elementary graph theoretic terminology.

3, A Formal Specification of a Network Hierarchy

3.1. Designation of Direct and Final Links

Let there be given N distinct junctions, termed points $p = 1, 2, \dots, N$, where N is a positive integer. When specifying a pair of points, it shall always be understood that i is distinct from j . An arc is defined to be the ordered pair of points \overline{ij} where i is the originating point and j is the terminating point. Let there be specified a subset G of all possible ordered pairs to be termed the collection of arcs of the network. We say that the points i, j are joined by an arc if $(i, j) \in G$. In general, not all pairs of points are in G , i.e. the network is typically not a complete graph. We say that traffic is permitted to flow over only the arcs of the network.

If \overline{ran} is in the arc set G , then we denote the special route ft_{mn}^{\wedge} by

$$ft_{mn}^{\wedge} = m; \overline{mn}, n. \quad (1)$$

Routes having more than one arc are similarly defined as an ordered list of s points for s an integer, $s \geq 1$ which are pairwise disjoint when $s > 1$ together with the corresponding arcs employed to join i to j :

$$i; \overline{ii_1}; i_1; \dots; \overline{i_s j}; j,$$

a notation which shall mean

$$i \xrightarrow{\overline{ii_1}} i_1 \xrightarrow{\overline{i_2 i_3}} i_3 \xrightarrow{\overline{i_4 i_5}} i_5 \dots$$

when $s = 1$. In graph theoretic terminology a route is simply a path of length $s + 1$.

Let there be specified a collection of arcs \mathcal{A} having the following property. For every pair of points (m, n) , there is a unique route $ft_{m,n}$ consisting either of one arc as in (1) or consisting of a_{mn} number of points $[m, n]_1, \dots, [m, n]_{a_{mn}}$, $a_{mn} \geq 1$ and the corresponding $a_{mn} + 1$ connecting arcs, denoted by:

$$S_{m,n} = m; \overline{[m, n]_1}; \overline{[m, n]_2}; \dots; \overline{[m, n]_{a_{mn}}}; n,$$

The notation, $T_j \in ft_{m,n}$ shall mean that \overline{ij} is one of the $a_{mn} + 1$ arcs of the unique route $ft_{m,n}$. The uniquely determined positive integer-valued position of arc \overline{ij} in the list of arcs in $ft_{m,n}$ is denoted by

$$\eta(\overline{ij}, \overline{mn}).$$

Arcs in \mathcal{A} shall be termed final arcs and indicated by solid lines as in Figure 1 of Section 1. The collection \mathcal{A} is the edge set of the given specific spanning tree of the network.

There is also given another collection of arcs denoted \mathcal{H} , none of which is in \mathcal{A} . These arcs are termed high usage arcs, $\overline{k\ell}$, and connect certain pairs of points (k, ℓ) . High usage arcs are indicated by dashed lines in Figure 1. Since \mathcal{A} itself is a spanning tree, it follows that for any $\overline{k\ell} \in \mathcal{H}$ there is a unique route $ft_{k,\ell}$ defined according to (2), where necessarily $a_{ki} \geq 1$, since $k\ell \notin \mathcal{A}$. This uniquely determined route shall be termed the alternate route for high usage arc $\overline{k\ell}$. Thus,

each high usage arc has an alternate route consisting solely of final arcs,, and we shall say that traffic can overflow from a high usage arc to its alternate route. The relationships between \mathcal{A} and \mathcal{B} shall be termed a network hierarchy. Observe that $G = \mathcal{A} \cup \mathcal{B}$.

According to the basic idea of a trunk discussed in Section 1, it follows that a trunk group joining point m to point n can service calls from m to n or calls from n to m . In particular, the trunk group should satisfactorily service the total offered load arising from both traffic directions. These engineering-based considerations motivate a simple graph theoretic definition of link MN . Given any two distinct points m, n the link MN shall be the union of the arcs \overline{mn} and \overline{nm} provided of course, both arcs are in the arc set G . When \overline{kf} and \overline{fk} are both in \mathcal{B} , the link KL shall be identified with the trunk group servicing total offered load from arcs \overline{kf} and \overline{fk} . Similarly, link IJ in \mathcal{A} shall service traffic on both arcs \overline{IJ} and \overline{JI} . We shall say that each link MN consists of x^{MN} number of trunks where x^{MN} is a non-negative variable to be solved for.

The terminology of "high usage" and "final" corresponds to telephone usage in the field and therefore we shall refer to "high usage arcs or links" rather than "direct arcs or links".

Some of these definitions are illustrated in Figure 2 below which is a portion of Figure 1 of Section 1.

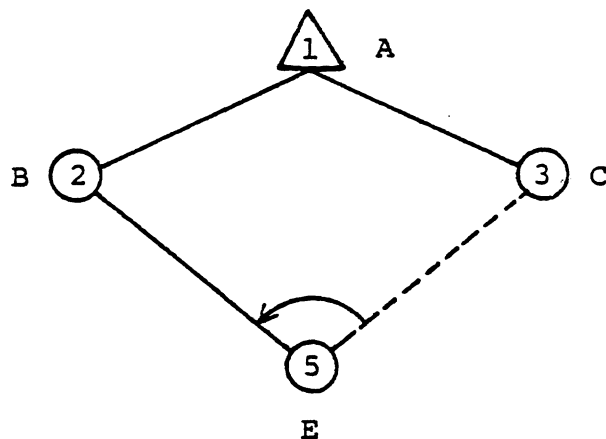


Figure 2. Portion of Figure 1 Indicating the Alternate Route for 5; $\overline{53}$; 3, where $\sigma_{53} = 2$, $[53]_1 = 2$, $[53]_2 = 1$, and $\eta(\overline{21}, \overline{53}) = 2$.

3.2. Classifying Point-to-Point Offered Loads

For each pair (m,n) there is a non-negative demand for traffic denoted a_{mn} , from m to n termed originating traffic. Traffic is usually stated in units of erlangs, or in hundred call seconds per hour [CCS] as discussed in Section 1.

Let \overline{ij} be a fixed final arc, $\overline{ij} \in \mathfrak{F}$. Traffic parcels offered to \overline{ij} consist of three types.

Type 1 Parcel: The originating traffic parcel a_{ij} is called type 1 parcel of traffic.

Type 2 Parcels: Traffic overflowing from high usage arcs onto final \overline{ij} is called type 2 traffic. Formally, there exists $\overline{kl} \in \mathfrak{H}$ such that $\overline{ij} \in \mathfrak{R}_{kl}$. We say originating traffic on high usage arc kl overflows to \overline{ij} . Introduce,

$$H_{ij} = \{\overline{kl} \in H \mid ij \in R_{kl}\}. \quad (3)$$

For example, in Figure 1, with $i, j = 1, 2$

$$H_{12} = \{\overline{13}, \overline{35}, \overline{41}, \overline{42}, \overline{57}, \overline{72}, \overline{75}, \overline{85}\}.$$

Type 3 Parcels: Type 3 traffic occurs between points m, n where $\overline{mn} \notin G$, but where nevertheless demand a_{mn} is positive.

In this case demand is serviced by a route consisting only of final arcs in G . For this case we assume $\overline{ij} \in R_{mn}$ and say

that originating traffic a_{mn} requires final arc TJ for completion* Introduce

$$F_{ij} = \{\overline{mn} \notin G \mid \overline{ij} \in R_{m,n}\}. \quad (4)$$

For example, from Figure 1,

$$F_{13} = \{\overline{16}, \overline{17}, \overline{18}, \overline{21}, \overline{43}\}.$$

Having established a particular network hierarchy, we are now in a position to specify probability distributions for customer originated traffic and to determine the expected overflow traffic from a high usage group to a final group in its uniquely specified alternate route. These specifications together with the network hierarchy then lead to a nonlinear supply model formulation, a task we address in the next section.

4. The Formulation of a. Nonlinear Supply Model

4.1. Blocking Probabilities and Overflow Traffic

The call discipline is one of the factors in determining the relationship between the offered load to a trunk group and its carried load. Another key factor in determining carried loads is the assumption that customer originated traffic is Poisson distributed with arrival rate denoted by A , see Messerli [13]. Fortunately, there is strong evidence to suggest that the number of calls occurring in a fixed, small time interval can be adequately modeled as a Poisson probability distribution. With these assumptions the distinction between a trunk group's offered load and carried load can now be made precise.

Assume that calls are assigned sequentially to a trunk group consisting of n trunks. Let A denote the average customer arrival rate according to the Poisson distribution. The only assumption required on customer calling time is that it has finite mean $1/\mu$. Otherwise, it may be arbitrarily distributed. Under these conditions the probability that all of the n trunks in the group are busy is given by the classical Erlang B-formula:

$$B(n,a) = \frac{a^n/n!}{\sum_{k=0}^n (a^k/k!)}, \quad (5)$$

for $n = 0, 1, \dots$, where $a = A/\mu$ with its units termed erlangs. The history of the original Erlang formula and its important generalizations may be found in Gnedenko-Kovalenko [7] and Syski [18].

An erlang is thus a measure of the flow of traffic per unit time. In the traffic engineering literature an erlang is one call-hour per hour, or equivalently 36 CCS per hour. The "hour" as the unit of time is so standard, it is usually dropped, and one says an erlang is 36 CCS. The value "a" in the Erlang formula is termed the offered load to the given trunk group. The expected overflow traffic is then $aB(n,a)$.

4.2. An Assumption on Marginal Capacities

The important benefits of being able to compute changes in equipment stock to meet changes in demand were recognized much earlier by Kalaba and Juncosa [11], Gomory and Hu [8], [9], [10] and others. Fortunately, incremental studies on the network hierarchy introduced in Section 3 permit certain simplifying assumptions that make computations attractive. These assumptions relate to the concept of the marginal capacity of an additional trunk at a prescribed blocking probability. The resulting supply model is an optimization which is much simpler than would be possible when constructing a network ab initio. The assumptions and model are now presented.

When traffic intensity a_{ij}^+ is offered to a given high usage arc TJ consisting of x_{ij}^- number of trunks, then the expected amount which overflows to arc $\overline{i[ij]_1}$ in 3 is $a_{ij}^- B(x_{ij}^-, a_{ij}^+)$, according to (5) above. On a final arc, however, the three types of traffic parcels introduced in Section 3.2 comprise the offered load: originating traffic, overflows from

high usage arcs, and originating traffic on other final arcs which require the particular final arc for completion. The basic model seeks optimal sizes of links, rather than arcs, as defined in Section 3.1 to accommodate two-way traffic. The following definition and key assumption emphasizes this approach.

Definition. For each final link or trunk group IJ consisting of x_{--} number of trunks let $P(x_{--}, Q_{IJ}(t))$ denote the blocking probability at time t , where the offered load $Q_{IJ}(t)$ consists of types 1 through 3 traffic parcels (Section 3.2), Define

$$p' = \max_{t, IJ \in \mathcal{A}} \{P_{IJ}(x_{IJ}, Q_{IJ}(t))\}.$$

The quantity $p = 1 - p'$ is termed the quality of service of the network.

Marginal Capacity Assumption

For each final link IJ , there exist two positive constants Y_{IJ} and b_{--} such that if $T^+ > 0$, then

$$\max_t P_{IJ}(x_{IJ} + \langle T^+ / Y_{IJ} \rangle, Q_{IJ}(t) + S) \leq p' \quad (6a)$$

and if $0 < T^- < b_{--}$, then

$$\max_t P_{IJ}(x_{IJ} - \lfloor T^- / Y_{IJ} \rfloor, Q_{IJ}(t) - T^-) \leq p', \quad (6b)$$

where $\langle x \rangle$ is the smallest integer greater than or equal to x , termed the integer round-up of x and where $\lfloor x \rfloor$ is the largest integer less than or equal to x termed the integer part of x . Y_{IJ} is termed the marginal capacity of an additional trunk at, blocking probability p' .

Inequality (6a) states that when $\langle T^+ / Y_{I4J} \rangle$ number of trunks are added to the trunk group servicing final arcs \overline{ij} and \overline{ji} , then at least an additional amount of traffic $p r^+$ is carried. Inequality (6b) states that when t^+ / Y_{j1} number of trunks are removed from the trunk group, then the decrease in carried traffic is at most $p f^+$.

We assume throughout that each high usage group KL consists of x_{KL}^+ (integer) number of trunks, and that each final group IJ consists of x_{IJ}^+ number of trunks, establishing what we term the existing network. It is further assumed that the existing network can supply all service demanded $a_{mn}(t)$ for all $p^i \wedge s$ (m,n) and all times of day t with the provision of a quality of service p .

4.3. A Nonlinear Integer Programming Formulation for the Network Hierarchy of Section 3

The first task is to develop an expression for the sum of the traffic parcels of Section 3.2 offered to a final link IJ of the existing network. The type 1 parcel is simply $a^+ j(t) + a_{ji}(t)$.

4.3.1. Sum of All Type 2 Parcels Offered to IJ

For any $TJ \in 3$ it follows from the marginal capacity assumption that the overflow from $\overline{kf} \in H_{ij}^+$ is at least

$$a_{ki}(t) B(x_{KL}, a_{ki}(t) + a_{kj}(t)) p < 1 < I7 > * z_{i-1} \quad (7a)$$

providing H_{ij}^+ is non-empty. Likewise for the final arc in the opposite direction, \overline{ji} , the overflow from $\overline{kf} \in H_{ji}^-$ is at most

$$a_{kl}(t)B(x_{KL}, a_{kl}(t)) + \rho_{\langle \overline{ve}(t) \rangle} p^{(ii(3i, k4) - 1)}. \quad (7b)$$

Summing the overflows in (7a) over all $\overline{kl} \in H_{ij}$, then summing the overflows in (7b) over all $\overline{kl} \in H_j^{\wedge}$ and adding these two sums yields a lower bound for the total type 2 traffic parcels offered to trunk group IJ. Let this sum be denoted by $L_{IJ}^{(2)}(t)$, i.e.,

$$L_{IJ}^{(2)}(t) = \frac{\sum_{\overline{kl} \in H_{ij}} a_{kl}(t)B(x_{KL}, a_{kl}(t) + a_{lk}(t)) \rho^{(\eta(\overline{ij}, \overline{kl}) - 1)} + \frac{\sum_{\overline{kl} \in H_{ji}} a_{kl}(t)B(x_{KL}, a_{kl}(t) + a_{lk}(t)) \rho^{(\eta(\overline{ji}, \overline{kl}) - 1)} \quad (8)$$

for each final trunk group IJ. For the case that H_{ij} is empty, we automatically take the appropriate summand in (8) to be zero. This case does not occur in Figure 1. An upper bound on the total overflow traffic, type 2, to IJ is obtained by deleting both ρ -terms in expression (8).

4.3,2. Sum of All Type 3 Parcels Offered to IJ

For any $\overline{mn} \in P_{ij}$ it follows from the marginal capacity assumption that the expected portion of originating traffic parcel $a_{mn}(t) * \overline{mn} / G^{\wedge}$ offered to trunk group IJ is:

$$a_{mn}(t) \rho^{(\eta(\overline{ij}, \overline{mn}) - 1)},$$

provided that F_{ij} is non-empty. Trunk group IJ is also offered the same expression for the load stemming from $\overline{mn} \in F_{ji}$, again providing F_{ji} is non-empty.

Summing all these parcels of traffic over $\overline{mn} \in F_{ij} \cup P_{ji}$ yields the type 3 sum:

$$L_{IJ}^{(3)}(t) = \sum_{\overline{mn} \in F_{ij}} \dots + \sum_{\overline{mn} \in P_{ji}} \dots + \dots (kL) - 1, \quad (9)$$

again with the proviso that a sum over an empty set is defined to be zero.

4.3.3. A Constraint on the Sum of All Traffic Offered to Final IJ

The maximum total expected offered load E_{IJ} which final group IJ of the existing network can service at blocking probability $1-p$ is the maximum, over all times of day t , of the sums of the three types of expected offered load parcels. Accordingly,

$$E_{IJ} = \max_t \{ a_{ij}(t) + a^*(t) + L^*(t) + L^*(t) \}. \quad (10)$$

Our modeling approach is concerned with (1), modified offered loads $\tilde{a}_{mn}(t)$ for all pairs (m,n) , (2), modifications of the number of trunks \tilde{x}_{KL} and \tilde{x}_{IJ} , respectively of high usage group KL and final group IJ, and (3), a modification in the network service quality \tilde{p} . Under these three kinds of modifications, we may define quite analogously to (8) and (9) the expressions

$$\tilde{L}_{IJ}^{(2)}(t) \quad \text{and} \quad \tilde{L}_{IJ}^{(3)}(t),$$

and analogous to (10) write

$$\tilde{E}_{IJ} = \max_t \{ \tilde{a}_{ij}(t) + \tilde{a}_{ji}(t) + \tilde{L}_{IJ}^{(2)}(t) + \tilde{L}_I^{\wedge}(t) \}. \quad (11)$$

If $\tilde{E}_{--} - E_{T-} > 0$, then according to the marginal capacity assumption, case (6a)₃, only $[(\tilde{E}_{-j} - E_{u^{\wedge}}) / Y_{IJ}]$ number of trunks need be added to final group IJ, where Y_{IJ} is the marginal capacity of an additional trunk at blocking probability $1-\tilde{p}$. Let y_{IJ} denote the integer number of trunks required in group IJ in order to service initial demand E_{--} at the new service quality p . Hence we obtain a feasibility requirement on the modified IJ trunk group size, x_{--} ,

$$\tilde{E}_{IJ} - E_{IJ} \leq Y_{IJ} (\tilde{x}_{IJ} - y_{IJ}) \quad (12)$$

where \tilde{x}_{IJ} is integer.

If $\tilde{E}_{TT} - E_{--} < 0$, then we invoke a stronger version of the marginal capacity assumption regarding case (6b). We require that $T \equiv |\tilde{E}_{IJ} - E_{IJ}| > a$ quantity which depends on the \tilde{x}_{IJ} and certain \tilde{x}_{KL} variables, lie within the 0 to b_{IJ} range required in order for (6b) to hold. In other words, when $[(\tilde{E}_{-j} - E_{TJ}) / Y_{IJ}]^{num_t}$ of trunks are removed from Y_{IJ} the resulting modification

$$\tilde{x}_{IJ} = y_{IJ} - [|\tilde{E}_{IJ} - E_{IJ}| / Y_{IJ}]$$

may be offered the modified load at blocking probability $(1-\tilde{p})$. It follows that the same feasibility requirement as (12) holds for this case too.

The system of inequalities (12), one inequality for each

final group IJ , shall determine a set of constraints for the nonlinear supply model, and we shall write these constraints in greater detail when actually specifying the model. But, first we need to take account of the total switched traffic in the network.

4.3.4, Accounting for Total Switched Traffic

Let us work with the modified loads $\tilde{a}^{\wedge}(t)$, modified number of trunks \tilde{x}_{KL} and \tilde{x}_{rj}^* and modified service quality p .

Let $S(t)$ denote the total switched traffic throughout the network at time t . We shall now show that

$$\begin{aligned} \tilde{S}(t) \ll & \sum_{ij \in E} \tilde{L}_{ij} \left(\tilde{a}_k^{\wedge}(t) B(\tilde{x}_{KL}, \tilde{a}_k^{\wedge}(t) + \tilde{a}_{jk}^{\wedge}(t)) \right)^{\eta(\overline{ij}, \overline{kL})-1} \tilde{p}^* \\ & + \sum_{ij \in F} -2 \left(\tilde{a}_{mn}^{\wedge}(t) \right)^{\eta(\overline{ij}, \overline{kL})-1} \tilde{p}^*. \end{aligned} \quad (13)$$

The amount of overflow traffic from high usage arc \overline{kL} destined for final arc TJ is $\tilde{a}_k^{\wedge}(t) B(\tilde{x}_{KL}, \tilde{a}_k^{\wedge}(t) + \tilde{a}_{jk}^{\wedge}(t))$. However, before this particular parcel reaches \overline{ij} it must be consecutively switched at points $k, [k^{\wedge}]^{\wedge}, \dots, [k^{\wedge}]^{\wedge} / [JJ, kT]^{-1}$ comprising the alternate route g^{\wedge} of \overline{icZ} , if $Ti(\overline{ij}, \overline{kC}) \wedge 2$. Therefore, in this case the total amount of switched traffic is:

$$\tilde{a}_{kL}^{\wedge}(t) B(\tilde{x}_{KL}, \tilde{a}_{kL}^{\wedge}(t) + \tilde{a}_{jk}^{\wedge}(t)) \sum_{q=0}^{\eta(\overline{ij}, \overline{kL})-1} \tilde{p}^q. \quad (14)$$

The same analysis applies to type 3 traffic. The total

traffic switched due to originating loads $\tilde{a}_{mn}(t)$, \overline{mn} not in the arc set G , requiring $\overline{ij} \in \mathcal{F}$ for completion is

$$\tilde{a}_{mn}(t) = \sum_{q=0}^{\eta(\overline{ij}, \overline{k\ell})-1} \tilde{\rho}^q. \quad (15)$$

We now sum (14) over all $\overline{k\ell} \in H_{ij}$ and then over all $\overline{ij} \in \mathcal{F}$, with the convention that the summation is zero whenever H_{ij} is empty. Similarly, (15) is summed over all $\overline{mn} \in F_{ij}$ and then over all $\overline{ij} \in \mathcal{F}$, with the convention that the respective term is zero when F_{ij} is empty. Finally, summing these two sums yields (13).

4.3.5. Cost Assumptions and the Nonlinear Model

Analogous to Eisenberg [5] and Elsner [6] we shall invoke simplifying cost assumptions for trunks and switching. We shall employ unit marginal investment costs per trunk and shall use the same cost for augmenting a trunk group as for diminishing a trunk group.¹ We shall denote the marginal cost per trunk for trunk group MN by $c_{MN} > 0$.

Changes in switching investment costs shall be approximated by using a marginal switching investment cost c per CCS of switched traffic, as for example in Eisenberg [5].

In the absence of real data and analogous to Eisenberg [5] we can merely set $c_{IJ} = c_{KL} = \$1000$ for each final trunk and high usage trunk, and also set $c = \$62$ (per CCS).

¹In practice, one rarely takes away existing equipment, but merely waits until the normal growth in message volume takes up the current slack.

We are now ready to state the basic nonlinear programming supply model.

Program P. Assume an existing network. Section 3 has demands
 $a_{mn}(t)$ for all pairs (m,n), integer trunk group sizes x^{\wedge} and
 x_{TJ}^{\wedge} high usage and final groups respectively, and an overall
network service probability p with marginal capacities Y_{TJ}^*
Let modified positive demands be denoted by $\tilde{a}_{mn}(t)$, and let \tilde{p}
denote a modified service probability with marginal capacity
 Y_{XJ} . Assume c^{\wedge} and $c_{i,j}$ are costs per trunk on high usage
group KL and final group IJ and that c denotes the switching
cost per CCS. Let E_{IJ} be defined according to (10). Compute

$$M^{\wedge} = \min_{\substack{\text{Finals} \\ IJ}} \sum_{IJ} c_{TJ}^{\wedge} x_{TJ}^{\wedge} + \sum_{KL} c^{\wedge} r^{\wedge} r^{\wedge} + \lambda \quad (16a)$$

from among non-negative integers \tilde{x}_{TJ} , $\tilde{c}_{i,j}$ for all finals IJ
and high usages KL and real \tilde{S} which satisfy:

$$\begin{aligned} & \tilde{a}_{ij}(t) + \tilde{a}_{ji}(t) + \sum_{k \in H_{ij}} \tilde{a}_{kl}(t) B(\tilde{x}_{KL}, \tilde{a}_{kl}(t) + \tilde{a}_{kl}(t)) \tilde{\rho}^{(\eta(\overline{ij}, \overline{kl}) - 1)} \\ & + \sum_{k \in H_{ji}} \tilde{a}_{kl}(t) B(\tilde{x}_{KL}, \tilde{a}_{kl}(t) + \tilde{a}_{kl}(t)) \tilde{\rho}^{(\eta(\overline{ji}, \overline{kl}) - 1)} \\ & + \sum_{m \in F_{ij}} \tilde{a}_{mn}(t) \tilde{\rho}^{(\eta(\overline{ij}, \overline{mn}) - 1)} + \sum_{m \in F_{ji}} \tilde{a}_{mn}(t) \tilde{\rho}^{(\eta(\overline{ji}, \overline{mn}) - 1)} \\ & - E_{IJ} \leq Y_{IJ} (\tilde{x}_{IJ} - Y_{IJ}) \end{aligned} \quad (16b)$$

for each final IJ and each t , where Y_{IJ} is the required
number of trunks in IJ for a \tilde{p} service probability, the

B-function given in (5), and

$$\frac{\sum_{ij \in \bar{G}} \sum_{kl \in \bar{H}_{ij}} \tilde{a}_{kl}(t) B(\tilde{x}_{kl}, \tilde{a}_{kl}(t) + \tilde{a}_{lk}(t)) \tilde{\rho}^{(\eta(\bar{ij}, \bar{kl}) - 1)} + \sum_{ij \in \bar{G}} \tilde{a}_{ij}(t) \tilde{\rho}^{(\eta(\bar{ij}, \bar{ij}) - 1)} \leq \tilde{S} \quad (16c)$$

for each t . (This completes Program P.)

Observe that the system of inequalities (16b) is merely (12) with full detail of the terms f_{ij} showing the \tilde{x}_{kl}^{TM} and $5L_{ij}$ as variables. On the other hand (16c) merely defines the maximum switched traffic in the network according to (13).

It is obvious that Program P is consistent because the \tilde{x}_{ij} variables may be taken arbitrarily large as well as the \tilde{S} variable, P must have a finite minimum. Otherwise some \tilde{x}_{ij} or \tilde{x}_{kl} necessarily become arbitrarily large and since all cost coefficients are positive, the objective function would arbitrarily increase which is a contradiction.

Program P is a nonlinear integer programming problem which can be well approximated for practical purposes by a continuous convex program. In fact, even more can be done. Program P can be approximated by a finite linear program based on the special convexity property and monotonicity property of the Erlang B-function, see Messerli [13]. We focus now on how the linear programming approximation is constructed.

5* A Linear Programming Approximation to the Nonlinear Program P5,1. The Convexity Properties of the Blocking Probabilities

In engineering practice the definition of the "load on last trunk" with respect to a trunk group of size $n + 1$ which is offered the load "a" is defined by:

$$D(n,a) = B(n,a) - B(n + 1,a) \quad (17)$$

where the Erlang B-function is defined in (5), for $n \gg Q_y l_3 \dots$ where $B(0,a) \gg i$. Observe that $D(n,a) > 0$ for each non-negative integer n . Messerli [13] gives a proof that for any fixed $a > 0$, $D(n,a)$ is strictly decreasing in the non-negative integer variable n ,

$$D(n + 1,a) < D(n,a) \quad (18)$$

for $n = 0, 1, \dots$

For "a" fixed define the polygonal function $B^{\wedge}(*,a)$ from the non-negative reals to the non-negative reals by

$$f(x,a) = -D(n,a)x + (n + 1)B(n,a) - nB(n + 1,a), \quad (1.9)$$

where n is the integer part, $[x]$, of x . Note that

$f(r,a) = B(r,a)$ for each non-negative integer r .

The graph of the polygonal function $f(-,a)$ reveals its convexity and monotonicity properties, which are basic for the construction of the linear program.

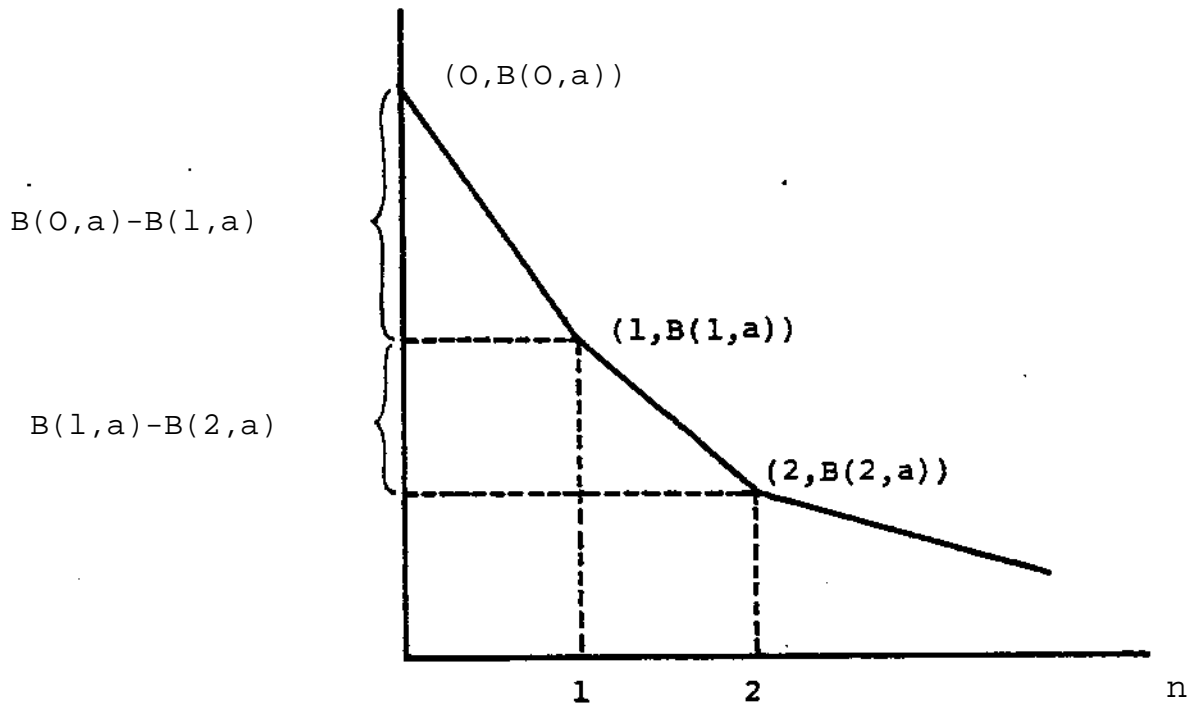


Figure 3. The Polygonal Function Determined by the Erlang B-Function on Non-Negative Integers

For each non-negative integer n the left-hand side of (19) defines an affine function on the non-negative reals. The following cumulative-type expression for this affine function follows from Charnes-Cooper [1], pages 352-353.

For a fixed non-negative integer n

$$-D(n,a)x + (n + 1)B(n,a) - nB(n + 1,a) = 1 + \sum_{r=0}^n (c_r - c_{r+1})(x-r) \tag{20}$$

for every real non-negative x , where $c_0 = 0$ and $c_r = -D(r,a)$ for $r = 0, 1, \dots$

As strongly suggested by Figure 3, the following proposition yields a uniquely determined system of supporting hyperplanes

for the epigraph K of the function $f(-, a)$. The proposition and its three corollaries shall be proved in an appendix.

Proposition 1. Let K be the epigraph of $\hat{B}(\cdot, a)$, $K = \{(z, x) \in]\mathbb{R}^2 \mid x \geq 0 \text{ and } z \geq f(x, a)\}$. Let L be the set of all (z, x) in \mathbb{R}^2 which satisfy the semi-infinite system of linear inequalities

$$z - 1 \geq \sum_{r=0}^n (c_r - c_{r-1}) J U - r \quad (21)$$

for $x \geq 0$ and $n = 0, 1, 2, \dots$

Then $K = L$ and K is non-empty.

Corollary 1. Let \bar{x} be non-negative real. Then $(\hat{B}(\bar{x}, a), \bar{x})$ satisfies each inequality of (21) strictly except for (i), the inequality indexed by $[\bar{x}]$ i.e., the inequality

$$z - 1 \geq f(c_{\bar{x}} - c_{\bar{x}-1}, \bar{x})$$

which it satisfies as an equality, and (ii) possibly the inequality indexed by $[\bar{x}] - 1$ when $\bar{x} \geq 1$. The latter inequality is satisfied as an equality if and only if \bar{x} is a positive integer.

Corollary 2. Let V be a positive integer and set $K^V = K \cap \{(z, x) \mid 0 \leq x \leq V\}$. Let L^V be the set of all (z, x) which satisfy

$$z - 1 \geq \sum_{r=0}^n (c_r - c_{r-1}) (x - r) \quad x \geq 0$$

for $n = 0, 1, \dots, V-1$. Then $K^V = L^V$.

Corollary 3. $(z,x) \in K^1$ is an extreme point of K^1 if and only if x is a non-negative integer and $z = B(x,a)$.

In view of Figure 3, which reflects the basic integer convexity property (18), these results are intuitively clear. They are formally proved in the appendix.

5.2. The Key Approximation and the Linear Program

We now replace in Program P the B-function by the polygonal \hat{B} -function, and the integrality conditions on the x_{KL}, \tilde{x}_{IJ} variables are removed. Finally, upper bounding constraints $SL_{KL} \leq V_{KL}$ are imposed, where the v_{KL} are large positive integers.

The next step replaces each term $a^{(\wedge \wedge * K_L, \wedge k \wedge \wedge + \tilde{a}_{LK}(t))}$ in (16b) and (16c) with the new variable z_{KL}^t and requires that

$$\tilde{a}_{KL}(t) \hat{B}(\tilde{x}_{KL}, \tilde{a}_{KL}(t) + \tilde{a}_{LK}(t)) \leq z_{KL}^t.$$

The new approximation program so obtained, denoted P^f , is the following.

Program P^1 . Same assumptions as in P. Let V_{KL} be large positive integers for high usage links. Compute

$$M_p = \min_{\substack{I \\ \text{Finals} \\ IJ}} \sum_{IJ} \tilde{x}_{IJ}^{ex} + \sum_{\substack{L \\ \text{High Usages} \\ KL}} \sum_{\wedge \wedge L} c_{XTJ} \tilde{x}_{ICT} + c_s \tilde{s} \quad (22a)$$

from among reals $\tilde{x}_{XJ}, \tilde{x}^{\wedge \wedge}, z_{KJ}^t, \tilde{s}$ which satisfy:

$$x_{IJ}(t) \leq \tilde{x}_{IJ}, \text{ where } x_{ZJ}(t) = \quad (22b)$$

$$\begin{aligned} & (\tilde{a}_{ij}(t) + \tilde{a}_{jx}^*(t) + \frac{\sum_{kl \in H_{ij}} \tilde{a}_{kl}(t) z_{kl}^t \tilde{\rho}^{(\eta(\overline{ij}, \overline{kl}) - 1)} \\ & \quad + \frac{\sum_{kl \in H_{ji}} \tilde{a}_{kl}(t) z_{kl}^t \tilde{\rho}^{(\eta(\overline{ji}, \overline{kl}) - 1)} \\ & \quad + \frac{\sum_{mne \in F_{ij}} \tilde{a}_{mn}(t) \tilde{\rho}^{(\eta(\overline{ij}, \overline{mn}) - 1)} + \sum_{\overline{mne \in F_{ji}}} \tilde{a}_{mn}(t) \tilde{\rho}^{(\eta(\overline{ji}, \overline{mn}) - 1)} \\ & \quad = \tilde{E}_{IJ} + \{Y_{IJ} Y_{IJ}^T\} / Y_{IJ} \end{aligned}$$

for each final IJ , and time t , and $s(t) \leq \bar{s}$ (22c)

where

$$s(t) = \frac{1}{3} \sum_{ij \in \bar{U}} \sum_{kl \in H_{ij}} z_{kl}^t \tilde{\rho}^{(\eta(\overline{ij}, \overline{mn}) - 1)} + \frac{\sum_{ij \in \bar{U}} \sum_{mne \in F_{ij}} \tilde{a}_{mn}(t) \tilde{\rho}^{(\eta(\overline{ij}, \overline{mn}) - 1)}$$

for each t , and

$$\tilde{a}_{kl}(t) \hat{B}(\tilde{x}_{KL}, \tilde{a}_{kl}(t) + \tilde{a}_{lk}(t)) \leq z_{kl}^t \quad (22d)$$

for each high usage arc \overline{kl} , and time t

and $0 \leq \tilde{x} \leq 1 \vee$, for each (22e)

high usage link KL .

It is obvious now in view of Corollary 2 that P^F is equivalent to the finite linear program denoted LP^F , obtained by replacing (22i?) with the finite system of linear inequalities:

$$\begin{aligned}
-\tilde{a}_{k\ell}(t)D(\nu, \tilde{a}_{k\ell}(t) + \tilde{a}_{\ell k}(t)) \tilde{x}_{KL} + z_{k\ell}^t &\geq \tilde{a}_{k\ell}(t)(\nu + 1)B(\nu, \tilde{a}_{k\ell}(t) + \tilde{a}_{\ell k}(t)) \\
&- \tilde{a}_{k\ell}\nu B(\nu + 1, \tilde{a}_{k\ell}(t) + \tilde{a}_{\ell k}(t)) \quad (23)
\end{aligned}$$

for $\nu = 0, 1, \dots, V_{KL} - 1$, and each $\overline{k\ell} \in \#$, and each t . It is equally obvious that Program LP' is consistent and has a finite minimum since the \tilde{x}_{KL} variables are bounded and all cost coefficients are positive. Hence P' itself has optimal solutions.

An important observation about optimal solutions to LP' is best made in the following formal terms.

Proposition 2. Let $\{(\tilde{x}_{IJ}^*), (\tilde{x}_{KL}^*), (z_{k\ell}^{t*}), \tilde{S}^*\}$ be an optimal solution to Program P'. Then

- (i) for each final group IJ, (22b) is satisfied exactly for some t -- denote the set of such t 's by T_{IJ} ,
- (ii) (22c) is satisfied exactly for some t' -- denote the set of such t' 's by T ,
- (iii) for each $\overline{k\ell} \in \#$ and IJ there is at least one $t \in T_{IJ}$ such that (22d) is satisfied exactly, and there is at least one $t' \in T$ such that (22d) is satisfied exactly, and
- (iv) $\{(\tilde{x}_{IJ}^*), (\tilde{x}_{KL}^*), \hat{a}_{k\ell}(\tilde{x}_{KL}^*), \tilde{a}_{k\ell}(t) + \tilde{a}_{\ell k}(t), \tilde{S}^*\}$ is also an optimal solution to P'.

Proof: Since each c_{IJ} is positive, (22b) cannot be satisfied strictly for every t . Otherwise, \tilde{x}_{IJ} can be decreased without affecting any other variables while maintaining feasibility, and

with a lower total cost. Let $T_{XJ} = \{t \mid \{(2^{\wedge}), (5^{\wedge}), (zF^*), \tilde{S}^*\}$ satisfies (22b) exactly\}.

Similarly, since $c > 0$, (22c) must be satisfied exactly for some t , and we denote this set of t 's by T .

To prove part (iii), let $\bar{k}T$ be any member of $3i$ and IJ be any final group. If to the contrary (22d) were satisfied strictly, with respect to $k-t$, for each $t \in T_{IJ}$, then $z_{k,t}^{J''}$ may be decreased for each $t \in T_{XJ}$ without violating (22d) and hence feasibility. But $\tilde{a}^{\wedge}(t) > 0$ for every t , and therefore the term $a^{\wedge} \wedge i$ *** the left-hand side of (22b) decreases strictly, and this decreases the entire left-hand side of (22b). Therefore \tilde{x}_{IJ}^* itself can be decreased giving a lower total cost since $c_{..} > 0$ and no other variables in the cost function are altered. This is a contradiction, and therefore (22d) is satisfied exactly for at least one $t \in T_{IJ}$.

An identical argument proves the last statement of part (iii).

Part (iv) follows from the fact that for all those t for which (22d) is strictly satisfied, \tilde{z}^* may be decreased to its lower bound without affecting feasibility. ||

We now use Corollary 1 of Proposition 1 to discuss the cost effects due to using an optimal solution of P^1 as a solution to the integer program P . If $\tilde{x}^{-j^{\wedge}}$ is not an integer, then $(B(\langle \tilde{x}^{\wedge} \rangle, \tilde{a}_k^{\wedge}(t) + \tilde{a}^{\wedge}(t)) j O \tilde{C}_K^* \wedge)$ is in the epigraph of $\hat{B}(\cdot, \tilde{a}_{k,t}^{\wedge}(t) j \tilde{a}^{\wedge}(t))$ for each t , where $\langle \% L \rangle$ is the integer round-up. The round-up introduces an increase in the total cost

associated with high usage KL , $(\tilde{x}_{JQ} > \tilde{x}_{KJ}^{*})^{c_{KL}}$ where
 $0 < \tilde{O}_{IJ} - \tilde{x}_{KL}^{*} < !$. An off-setting cost effect from final
 groups IJ and switching \tilde{S}^* occurs because from the monotonicity
 of $\$(\cdot)^{\wedge}(t) + \wedge - tk^{\wedge}$, each z^{\wedge} does not increase.

Finally, in order to insure quality of service \tilde{p} , non-integer final group sizes \tilde{x}_{IJ} should be rounded up, thereby increasing total costs. Numerical estimates of these various off-setting cost effects due to round up of trunk group sizes determined by Program P^f have not been obtained. It appears to us that such estimates must stem from numerical experiments on field data. Certainly, as strongly suggested by Figure 3 and Proposition 1 and its Corollaries, integer programming pathologies from straightforward rounding processes do not occur.

There are special assumptions that can be placed on Program P^1 which guarantee the existence of an optimal solution to P^T such that all of the high usage group sizes (\tilde{x}_{JL}) are integers. These do not necessarily comprise high usage group specification of an optimal solution to the nonlinear integer program P . Nevertheless, they provide a starting point for determining final group sizes by other methods which do not depend on the marginal capacity assumption, such as Wilkinson's Equivalent random method [18], [20].

One of the special assumptions is the following.

Definition. A final group IJ is said to have a stable busy hour $t^{\wedge j}$ if and only if for any specification (\wedge_{KL}) which satisfies (22e) for all those $\bar{k}^T \in H_i \cup H_j^{\wedge}$ [see (3), Section 3]

and (z^f) which satisfy (22d) exactly,

$$X_{IJ}(t_{IJ}) = \max_t X_U(t). \quad (24)$$

The entire network is said to have a stable switching busy hour t if under the conditions above

G

$$S(t_a) = \max_t S(t). \quad (25)$$

Proposition 3. Assume that each final group IJ has a stable busy hour t_{IJ} and that a stable switching busy hour t_g exists. Then there exists an optimal solution to Program P^f such that all high usage group sizes are integers.

Proof: By Proposition 2 an optimal solution to P^1 exists of the form

$$\chi = \{X_F, X_H, (z_{kl}^{t*}), \tilde{S}^*\},$$

where

$$X_F = \{\tilde{x}_{IJ} \mid \text{all final groups } IJ\},$$

$$X_H = \{\tilde{x}_{KL}^* \mid \text{all high usage groups } KL\}$$

and $z_{kl}^{t*} = \wedge \wedge \wedge \wedge (\wedge L \wedge T C \wedge \wedge + 5) i k(t)$ for all $\wedge \in *$ and times t . For any non-negative integer ν define

$$z_{kl}^{t,\nu} = \tilde{a}_{kl}(t) B(\nu, \tilde{a}_{kl}(t) + \tilde{a}_{lk}(t)).$$

Then by Proposition 1 and its Corollaries

$$z_{kl}^{t*} = \mu_{KL} z_{kl}^{t,\nu_{KL}} + (1 - \mu_{KL}) z_{kl}^{t,\nu_{KL}+1} \quad (26)$$

for each $\overline{ki} \in J_i$ and time t , where $v^{\wedge} = f\%iJ$ and $\circ f. \mu_{KL} \leq 1$.

Assume now that some high usage size \tilde{x}_{RS}^* is not an integer-

For each t define

$$X_{IJ}^1(t) = [\tilde{a}_{rs}(t) z_{rs}^{t, \nu_{RS}} \rho^{\tilde{\rho}(\eta(\overline{ij}, \overline{rs})-1)} + \tilde{a}_{sr}(t) z_{sr}^{t, \nu_{RS}} \rho^{\tilde{\rho}(\eta(\overline{ji}, \overline{sr})-1)} \\ + \sum_{\overline{ki} \in H_{ji}^1} \tilde{a}_{kl}(t) z_{kl}^{t, \nu_{kl}} \rho^{\tilde{\rho}(\eta(\overline{ij}, \overline{kl})-1)} + \sum_{\overline{kl} \in H_{ji}} \tilde{a}_{kl}(t) z_{kl}^{t, \nu_{kl}} \rho^{\tilde{\rho}(\eta(\overline{ji}, \overline{kl})-1)} + \dots] / Y_{IJ},$$

where the " \dots " denotes the remaining terms of the expression

$X_{IJ}(t)$ which do not involve any of the subscripted z -variables,

Let $X_{IJ}^2(t)$ be defined exactly as $X_{IJ}^1(t)$ above, except that $z_{RS}^{t, \nu_{RS}}$ and $z_{SR}^{t, \nu_{RS}}$ are replaced by $z_{RS}^{t, \nu_{RS}+1}$ and $z_{SR}^{t, \nu_{RS}+1}$, respectively. This notational specification is

repeated with respect to the inequalities (22c), obtaining

respectively: $S^1(t)$ and $S^2(t)$ for each t

Define for each t :

$$X_{IJ}^*(t) = \mu_{RS} X_{IJ}^1(t) + (1-\mu_{RS}) X_{IJ}^2(t) \quad (27a)$$

and

$$\tilde{S}^*(t) = \mu_{RS} S^1(t) + (1-\mu_{RS}) S^2(t) \quad (27b)$$

where now $0 < \mu_{RS} < 1$ since \tilde{x}_{RS}^* is not an integer.

Let X_{IJ}^* , X_{IJ}^1 , and X^{\wedge}_T denote the maxima of $X_{IJ}^*(t)$, $X_{IJ}^1(t)$, and $X_{IJ}^2(t)$ respectively with respect to t . Then of course $X_{IJ}^* = \hat{X}_{IJ}^*$. $\wedge Y$ the existence of a stable final group

busy hour t_{TT} , it follows by definition that $X_{XJ}^* = X_{XJ}^*(t_{XJ})$,
 $X_{IJ}^1 = X_{IJ}^1(t_{IJ})$, and $X_{IJ}^2 \gg X_{IJ}^2(t_{IJ})$. Hence by (27a), we have

$$\tilde{x}_{IJ}^* = \mu_{RS} X_{IJ}^1 + (1 - \mu_{RS}) X_{IJ}^2. \quad (28)$$

Similarly, defining $S^* = \max_t S(t)$, $S^+ = \max_t S^+(t)$, and

$S^2 = \max_t S^2(t)$, the existence of a stable network switching busy
hour t implies

$$\tilde{S}^* = \mu_{RS} S^1 + (1 - \mu_{RS}) S^2. \quad (29)$$

We now consider total costs associated with the three
feasible solutions to P^f , indexed with "rf", "1", and "2". Let
C denote the portion of total cost which is common to these
three feasible solutions. Then

$$\begin{aligned} \sum_{IJ}^L G_{IJ} + C_{RS} S + c \tilde{S}^* + C &= \mu_{RS} \left[\sum_{IJ} c_{IJ} X_{IJ}^1 + c_{RS} \nu_{RS} + c S^1 + C \right] \\ &+ (1 - \mu_{RS}) \left[\sum_{XJ} E_{XJ} c^j + C_{ggd} + 1 + c S^2 + c \right]. \end{aligned} \quad (30)$$

But from optimality of the "rf"-solution the sum to the left of
the equality sign of (30) is less than or equal to each of the
bracketed terms to the right of the equality sign. Therefore,
by (30) itself it follows that both "1"-solution and the "2"-
solution are in fact optimal solutions for Program P^1 . Either
one of them may be chosen, and the process repeated, namely
taking any remaining non-integral \tilde{x}_{KL}^* and purifying it to an
integer. Since no high usage group sizes which are already
integer are affected, it follows that the process terminates in
a finite number of steps with an optimal solution to P^1 all of

whose high usage group sizes \tilde{x}_{KL} , are integers, $||$

The assumptions of Proposition 3 were employed in one of the examples in Eisenberg [5], pp. 13-14.

Because of the linear inequality system (23)₃, Program LP¹ may be quite large and for practical purposes it would be useful to be able to solve a smaller problem in place of LP^f. The monotonicity of the \hat{S} -function, essentially Corollary 1 of Proposition 1 suggest a useful procedure.

5* 3. Solving the Linear Program LP^f Through Bounded Variable Reductions

Let LP_{all}¹ be the bounded variable version of LP^f obtained by replacing (22e) with

$$\tilde{x}_{KL} \leq \beta_{KL}$$

for each high usage group, and in (23) restrict u to:

$v = \sum_{KL} T_{rr}^{gmm} f_{ijrr} \sim 1$ where i_{KL}^{TM} and j_{KL}^{3L} are non-negative integers such that $\alpha_{KL} - 1 - \epsilon_{KL} > 2$.

Proposition 4. Under the above bounded variable assumptions:

- (i) any optimal solution $\{(\epsilon_{KL}^*), (\epsilon_{I}^*), (zfy, S^*)\}$ of LP_{BD}¹ is feasible for LP¹

and

- (ii) if for each KL

$$\alpha_{KL} < \tilde{x}_{KL} < \beta_{KL}, \quad (31)$$

then this optimal solution is also optimal for Program LP^1 .

Moreover, there exist I^\wedge, J^\wedge and an optimal solution of LP^1 such that with respect to \tilde{x}_{KL}^* of that solution, (31) holds.

Proof: By the argument used in the proof of (iv) of Proposition

2, $((x^\wedge), (x_{\uparrow}^{*j}), (\bar{z}_k^\wedge), S)$ is optimal for LP^\wedge where

$\bar{z}_{kl}^t = \tilde{a}_{kl}(t) \tilde{x}_{kl}^* + \tilde{a}_{lk}(t)$ for each $k < t \in **$ By Corollary 1,

for each KL , $(\bar{z}_{kl}^t)_{k < t \in **}$ satisfies (23) for every non-negative

integer. Since $\bar{z}^\wedge \in z^{\wedge*}$ and (22b) and (22c) are already

satisfied, it follows that (\bar{z}^\wedge, S^*) satisfies all

the constraints of LP^f . This proves (i)-

The first part of (ii) follows from linear programming duality theory. Because of (31) the two dual variables stemming respectively from the two bounding constraints on \tilde{x}_{KL}^* are both zero*. Hence one may delete these constraints in LP^f and the same dual optimal solution prevails. Therefore by duality (\bar{z}^\wedge, S^*) is optimal for the relaxed-variable constrained program LP^1 . The remaining statement of part (ii) follows from Corollary 1 and the fact that the non-negative integers t^\wedge satisfy $t^\wedge - 1 - t^\wedge * 2$. ||

In the next section we present results of numerical experiments on two examples, one of which has been previously solved and published, see Eisenberg [5] and Eisner [6].

6. Numerical Experiments on Two Examples

6.1. First Example: A Network Based on California Field Data

We apply Program P of Section 4.3.5 to the network given in Eisenberg [5] and Elsner [6], which in turn is based on Gardena, California field data. The hierarchical structure of the network is given in Figure 4 below.

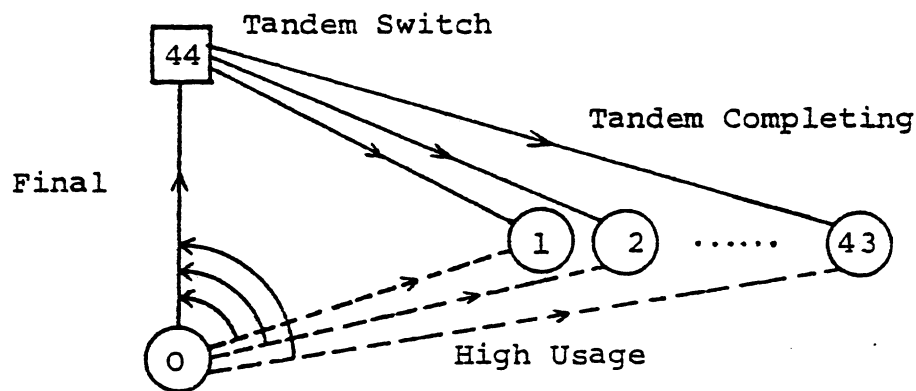


Figure 4. A Network Hierarchy Based on Gardena, CA Data, Eisenberg [5].

In this network there is only one originating office, labelled 0, and 43 terminating offices labelled 1 through 43. Traffic flow on each trunk group is one way as indicated, and there are two times of day, t_1 (hour 1) and t_2 (hour 2). The overflow hierarchy is indicated in Figure 4.

Base Demand

We assume that the network is constructed ab initio, namely all the initial demands between pairs of offices are zero and all initial trunk sizes are zero. According to (10) then, it follows

that $E_0^{44} = E_{44} = 0$ for $t = 1, 2, \dots, 43$.

Incremental Demand

Following Eisenberg [5], but in the notation of Section 4 we set $\tilde{a}_{0,44}(t) = 0$ and $\tilde{a}_{44}^*(t) = 0$ for $t = t_x, t_2$ and $* = 1, 2, \dots, 43$. The rest of the positive incremental demands ($\tilde{a}_{0,l}^*(t)$) in CCS are given in columns 2 and 3 of Table 1 below. Following Eisenberg we take a marginal capacity of 30 CCS for all final groups and a quality of service, 0.99. Unit costs are #1000 per trunk and #62 switching cost per CCS, With these specifications Program P of Section 4.3.5 becomes the following one.

$$\text{Find } M \ll \min 1000 [\tilde{x}_{0,44} + \sum_{l=1}^{44} (\tilde{x}_{44,l} + \tilde{x}_{0,l} J + 62\tilde{S})]$$

subject to

43

$$\tilde{a}_{0,44}(t) \leq 0,44 \quad \text{for } t = t_1, t_2$$

$$\tilde{a}_{0,l}(t) B(\tilde{x}_{0,l}, \tilde{a}_{0,l}(t)) \rho \leq 30\tilde{x}_{44,l} \quad \text{for } l = 1, \dots, 43$$

and $t = t_1, t_2$

and

$$\sum_{l=1}^{43} \tilde{a}_{0,l}(t) B(\tilde{x}_{0,l}, \tilde{a}_{0,l}(t)) \leq \tilde{S} \quad J2E \quad t = t_x, t_2,$$

where the \tilde{x}^* are all non-negative integers.

The above nonlinear integer program was approximated by the linear program derived by the methods of Section 5,2, which was then solved using suitable bounded variable reductions based on

Section 5.3, The bounds of the high usage group sizes were chosen by our prior knowledge of Eisenberg's [5] and Elsner's [6] solutions. An optimal linear programming solution so obtained is termed the incremented network. Table 1 presents an incremented network and includes the overflows from the high usage trunk groups to the final trunk group 0,44.

Table 2 compares the sizes of the high usage trunk groups occurring in our incremented network with those computed in Eisenberg [5] and those computed in Eisner [6]. Finally, Table 3 gives some overall comparisons between the three solutions.

Table 1. Specification of Incremented Offered Load Demands for Example 1 and an Optimal Linear Programming Solution with all Overflows from High Usage Groups

Trunk Group	Offered Loads (CCS)		Overflow (CCS)		High Usage Trunks	Tandem-Completing Trunks
	Hour 1	Hour 2	Hour 1	Hour 2		
1	60	140	3.75	41.98	3.99	1.40
2	119	9	16.27	0.00	4.99	0.54
3	82	20	10.26	0.05	4.00	0.34
4	305	76	20.00	0.00	11.99	0.67
5	30	0	4.78	0.00	1.99	0.16
6	59	7	9.18	0.01	3.00	0.31
7	102	56	9.80	0.90	4.99	0.33
8	256	161	21.31	1.63	10.00	0.71
9	366	230	22.41	0.84	13.99	0.75
10	469	310	20.26	0.60	17.99	0.68
11	115	115	14.60	14.60	5.00	0.49
12	144	34	16.87	0.01	5.99	0.56
13	206	335	0.00	27.72	12.04	0.92
14	310	650	0.00	86.14	19.07	0.88
15	284	319	13.72	24.99	12.00	0.83
16	93	152	7.07	33.26	5.00	1.11
17	17	24	5.45	9.60	1.00	0.32
18	74	325	0.00	0.00	12.01	0.00
19	102	158	4.43	23.04	5.99	0.77
20	137	322	1.41	71.33	8.99	2.38
21	222	247	10.74	18.10	9.99	0.60
22	252	390	6.82	59.39	12.00	1.98
23	445	194	0.00	0.00	19.25	0.00
24	176	86	19.99	0.70	7.00	0.67
25	83	29	10.64	0.23	4.00	0.35
26	98	21	17.15	0.06	4.00	0.57
27	158	74	13.24	0.29	6.79	0.44
28	124	36	0.00	0.00	7.12	0.00
29	54	25	7.25	0.70	3.00	0.24
30	38	1	8.10	0.00	2.00	0.27
31	31	17	5.15	1.20	2.00	0.17
32	140	46	15.29	0.08	6.00	0.51
33	96	30	16.20	0.26	4.00	0.54
34	122	62	17.59	1.41	5.00	0.58
35	163	57	14.96	0.06	7.00	0.50
36	163	72	14.96	0.25	7.00	0.50
37	296	238	17.13	4.75	12.00	0.57
38	33	28	5.93	4.07	1.99	0.20
39	240	3	0.00	0.00	11.65	0.00
40	136	7	13.78	0.00	5.99	0.46
41	54	4	7.25	0.00	3.00	0.24
42	52	35	6.55	2.06	2.99	0.22
43	206	9	0.00	0.00	10.47	0.00
Totals	6712	5154	430.30	430.31	318.46	25.76

Table 2. Comparison of High Usage Trunk Group Sizes Computed by the Multi-Hour Method, A Descent Method, and Linear Programming for the Gardena Network (Rounded to Nearest Integers)

Trunk Group	High Usage Group Sizes			Tandem-Completing
	From Multi-Hour Method [5]	From Descent Method [6]	From Linear Programming	Linear Programming
1	4	4	4	1
2	3	5	5	1
3	4	4	4	0
4	6	12	12	1
5	0	1	2	0
6	1	3	3	0
7	4	5	5	0
8	8	10	10	1
9	12	14	14	1
10	18	18	18	1
11	5	5	5	1
12	7	6	6	1
13	10	11	12	1
14	16	19	19	3
15	12	12	12	1
16	5	5	5	1
17	1	1	1	0
18	6	9	12	0
19	5	6	6	1
20	8	9	9	2
21	10	10	10	1
22	12	13	12	2
23	17	17	19	0
24	8	7	7	1
25	4	4	4	0
26	5	4	4	1
27	7	7	7	0
28	6	5	7	0
29	3	3	3	0
30	2	2	2	0
31	2	2	2	0
32	6	6	6	1
33	5	4	4	1
34	6	5	5	1
35	7	7	7	1
36	7	7	7	1
37	12	12	12	1
38	2	2	2	0
39	10	10	12	0
40	6	6	6	1
41	3	3	3	0
42	3	3	3	0
43	9	8	11	0
Totals	287	306	319	29

Table 4. Comparisons of Total Number of Trunks, Switching Costs, and Total Costs for the Multi-Hour, Descent, and Linear Programming Solutions of the Gardena Network

Network Characteristics	Multx-Hour [5]	Descent [6]	Linear Programming
# High Usage Trunks	287	306	319
# Final Trunks	39	NA*	14
# Tandem Compl.	NA	NA	29
Switching Cost	#44,640	NA	#26,000
Total Cost	#405,315	#38 5,500	#385,400

*NA = not available

6.2. The Second Example: Figure 1's Network Hierarchy

We solve Program LP^f of Section 5.2 applied to the network hierarchy of Figure 1 of Section 1 with the following specification of input data.

Base Demand

Traffic demand is assigned to all 56 pairs of points of Figure 1 by daytime, evening, and nighttime according to three basic kinds of pairs:

(1) each of the pairs $\overline{13}$ and $\overline{31}$ receive 500 CCS during daytime and 0 during the other two periods,

(2) each pair which includes exactly one of the nodes 1 or 3 receives 100 CCS during daytime and 0 during the other two periods,

and

(3) each pair which excludes both nodes 1 and 3 receives 75 CCS during daytime, 200 CCS during evening, and 100 CCS during nighttime.

These choices were imagined upon viewing nodes 1 and 3 as "commercial" nodes and viewing all other nodes as "residential". They represent particular choices of the inputs $a_{ij}(t)$, $\tilde{a}_{i..}(t)$, and $\tilde{a}_{.j}(t)$ of Program LP^T . Analogous to the first example we assume that the cost per trunk is \$1000, that the switching cost is \$62 per CCS, and that the quality of service is 0.99. Using these inputs and the hierarchy of Figure 1, an optimal solution to LP^f was obtained termed the base network.

Incremented Demand

Assume that an increase in demand of 20\$ occurs uniformly among all of the 56 calling pairs* With all other inputs to LP¹ remaining unchanged an optimal solution was obtained, termed (as before) the ~~incremented network~~.

Moreover, Program LP¹ was solved under three additional restrictions on the time t , namely, all high usage links be sized according to: (a) daytime loads, (b) evening loads, and (c) nighttime loads, respectively. These restricted solutions result from the requirement that the network be "engineered" according to a fixed single hour, respectively. This is in contrast to the multi-hour solutions of the base and incremented networks, and provides a test of reasonableness of the multi-hour solutions.

For purposes of computer usage, the size of LP¹ was reduced by the bounded variable restrictions of Proposition 4 of Section 5.3. For example, setting the V^{\wedge} bounds in (33) at 25 for each high usage group yields a 64 variable linear program with 1232 constraints. This program was solved by solving a finite sequence of much smaller bounded variable programs. The results are given in Table 4 below.

Table 4. Computer Results of Four Solutions of Program LP¹,
 Section 5,2: Base and Incremented Networks₃ and
 Network Single Hour Designs, Base Demand Incremented
 20\$ Uniformly, #1000 Cost/Trunk, #62 Switching Cost/CCS,
 and 0.99 Quality of Service

Final Groups	Base Network	Incremented Network	Single Hour Designs		
			Daytime	Evening	Nighttime
12	34.7	39.9	76.0	49.9	62.9
13	91.9	109.0	108.6	115.8	115.9
14	58.3	68.1	74.7	68.0	72.0
25	20.5	23.4	74.2	28.3	39.2
36	46.6	54.2	66.5	54.1	62.1
37	46.7	54.2	66.6	54.1	62.1
38	58.6	69.2	75.0	69.3	73.2
High Usage Groups					
15	2	3	11	0	0
26	17	20	9	21	12
27	17	20	9	21	12
35	7	8	12	0	0
45	18	20	9	21	12
56	18	21	10	21	12
57	18	21	10	21	12
58	18	21	10	21	12
Total Switched Traffic (ERL)	135.9	162.6	262.3	161.6	227.0
Total Cost (000)	£775.5	£915.2	£1,180.1	£926.2	£1,065.8

Discussion

We have observed that in both the base and incremented networks each final group has a stable busy hour, introduced in Section 5.2. Final groups 12 and 13 have daytime stable busy hours and all other final groups have evening stable busy hours, and this pattern is identical for both networks. Stable busy hours also occurred in the single hour designs. Consequently according to Proposition 3 all high usage group sizes of any of the 4 linear programs must be integers, and this has been verified in Table 4 . Observe also that the multi-hour (incremented network) solution has a total cost which is less than each of the single hour design total costs, although the single evening hoar solution is only 1.2\$ larger than the multi-hour solution. Apparently, the opportunity of engineering final groups 12 and 13 at another time, namely daytime, permits a slight savings in total costs.

7. Conclusions

It is suggested in this paper that linear programming be used to solve for changes in equipment requirements necessary to provide for altered demands for telecommunications services and altered demands for service qualities. Obtaining solutions to this basic problem is a major goal of a supply model which seeks to minimize total incremental investments subject to these constraints.

The linear programming model distinguishes high usage trunk groups from final trunk groups according to the role each plays in the network hierarchy. Solutions to the model yield incremental investments in both of these categories of equipment and also additional switching investments. Caution must be exercised however in the final selection of the sizes of the final trunk groups because of the use of the marginal capacity concept in the linear programming model.

The integrality question settled in Proposition 3 of Section 5 shows that an important subset of the variables of the approximate program P^1 can be solved for as integers by elementary linear programming. In practice, the actual values of the remaining variables, namely the final group sizes, should be determined by methods that do not depend on the marginal capacity assumption, principally Wilkinson's Equivalent Random Method [18], [20]. This approach is needed because of various peakedness effects that occur in the probability distributions of alternatively routed traffic parcels. Program P^1 , under special assumptions, provides integral numbers of high usage groups to which the

Wilkinson method applies. In general, the costs due to straightforward integer rounding of high usage groups tend to be offsetting, and round-off procedures easily maintain feasibility and hence overall network quality of service.

A related class of nonlinear integer programs which are solvable as linear programs is treated in Meyer [14], where various unimodularity assumptions are made* These assumptions do not apply in general to the class of network problems treated in this paper. However, the column-generation procedures of Meyer might be very useful for solving the linear programs with bounded variable reductions treated in Section 5.3.

We shall leave the linear programming duality developments for a later paper, where we shall pursue our conjecture that sensitivity and post-optimality analyses will be indeed useful for network design synthesis. Fortunately, by Proposition 1 and its Corollaries it appears that a much smaller list of active dual variables will be forthcoming than the total number of constraints in program LP¹.

Future work should also incorporate more than one alternate route in the network hierarchy, even though for many networks in the field the first and second choice routes are preeminent. Many networks given in the literature are included within the linear programming models of this paper. Large scale network optimizations made available through the modeling approach of this paper should enhance an effective integration of the supply model with a disaggregated econometric demand model for telecommunications services.

We conclude with an observation shared by Edward A. Silver and Stephen A. Smith, expressed in personal correspondence, that there is an interesting equivalence between telephone engineering and replenishment inventory systems, see [16] and [17]. Perhaps the design of more complex telecommunications network hierarchies may have application to the design of more complex replenishment inventory systems.

APPENDIX: Proof of Proposition 1 and Its Three Corollaries

Proposition 1. Let K be defined as,

$$K = \{(z,x) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } z \leq f(x,z)\}.$$

Let L be the set of all (z,x) in \mathbb{R}^2 which satisfy the semi-infinite system of linear inequalities

$$z - 1 \leq \sum_{r=0}^n (c_r - c_{r-x}) J(x-r) \quad \text{and} \quad x \geq 0 \quad (D)$$

for $n \geq 0, 1, \dots$

Then $K \cap L$, and K is non-empty.

Proof: Nonemptiness of K is most easily seen by observing that $(1,0) \in K$ since $f(0,a) = B(0,a) = 1$.

Let (z,x) be an arbitrary point in K . Assume throughout that $ii = [x]$, the integer part of x . Applying (19) of Section 5,1 gives

$$z \geq -D(ri,a)x + (ri + 1)B(n,a) - nB(ii + 1,a),$$

and hence from (20) we have

$$1 - U \sum_{r=0}^{\bar{n}} (c_r - c_{r-1}) J(\bar{x}-r). \quad (2)$$

Thus, (z,\bar{x}) satisfies the particular inequality of (1) indexed by the non-negative integer \bar{n} .

Consider now any integer $n, n \geq \bar{n} + 1$ and write

$$f(\bar{x},a) = 1 + A \sum_{r=0}^n (c_r - c_{r-1}) (\bar{x}-r)$$

where $A_1^n = \sum_{r=n+1}^n (c_r - c_{r+x})(x-r)$. Now for any integer z , $n+1 \leq r \leq n$, it follows that $3c-r < 0$ because $n < x < n+1 \leq r$. In addition, $c_r - c_{r+x} > 0$ for each non-negative integer r , and therefore $A^n < 0$ for each integer n , $n \geq n+1$. Hence

$$z - 1 \geq \sum_{r=0}^n (c_r - c_{r+x})(x-r), \quad (3)$$

for each integer n , $n \geq n+1$.

(2) and (3) together show that (z, \bar{x}) satisfies all those inequalities of (1) indexed by n , $n \geq n$. We now check that (z, x) also satisfies those inequalities indexed by non-negative integers n , $n \leq n-1$.

If $n = 0$ there is nothing to check for there are no such n . For $n > 1$, let n satisfy $0 \leq n \leq n-1$ and write

$$f(x, a) - 1 = \sum_{r=0}^n (c_r - c_{r+x})(x-r) + A_0^n$$

where $A_0^n = \sum_{r=n+1}^n (c_r - c_{r+x})(x-r)$. For each integer r ,

$n+1 \leq r \leq n$, it follows that $x-r \geq 0$ and $c_r - c_{r+x} > 0$ as before. Hence $A_0^n \geq 0$ and hence

$$z - 1 \geq f(x, a) - 1 \geq \sum_{r=0}^n (c_r - c_{r+x})(x-r) \quad (4)$$

for each integer n , $0 \leq n \leq n-1$. The latter finite system of inequalities (4) together with (2) and (3) show that (z, \bar{x}) satisfies (1), implying $K \supseteq L$ and in particular L is non-empty.

The other inclusion $L \subseteq K$ is trivial because any (\bar{z}, \bar{x})

in L satisfies in particular

$$\bar{I} - 1 \leq \sum_{r=0}^{\bar{n}} L(c_r - c_{r-x}) J(\bar{x}-r).$$

Using (19) and (20) again shows $\bar{z}^n \wedge f(\bar{x}, a)$ i.e. $(\bar{z}, \bar{x}) \in K$.

Corollary 1. Let \bar{x} be non-negative real. Then $(f(\bar{x}, a), \bar{x})$ satisfies each inequality of (1) strictly except for (i), the inequality indexed by $[\bar{x}]_3$, which it satisfies as an equality, and (ii) possibly the inequality indexed by $[\bar{x}]-1$ when $[\bar{x}]_3 > 1$. The inequality $[\bar{x}]-1$ is satisfied as an equality if and only if \bar{x} is a positive integer.

Proof: Let $\bar{z} = B(\bar{x}, a)$. Application of (3) shows that (\bar{z}, \bar{x}) satisfies each inequality indexed by n , $n \geq \bar{n} + 1$, strictly, where $i = [\bar{x}]$. By (19) and (20) of Section 5.1, it follows that (\bar{z}, \bar{x}) , satisfies the inequality determined by \bar{n} as an equality.

It only remains to prove that the inequalities indexed by non-negative integers n , $n \in \bar{n}-2$ are satisfied strictly. There is nothing to check if $\bar{n} \leq 1$. For $\bar{n} \geq 2$, let n be any integer $0 \leq n \leq \bar{n}-2$. Then

$$H(\bar{x}, a) - 1 = \sum_{r=0}^{\bar{n}} L(c_r - c_{r-x}) (\bar{x}-r) + [A + (c_n - c_{n-1}) (\bar{x}-n)]$$

where $A = \sum_{r=n+1}^{\bar{n}-1} (c_r - c_{r-1}) (\bar{x}-r)$. Since $n \leq \bar{x} \leq n+1$, it follows that $(c_n - c_{n-1}) (\bar{x}-n) \geq 0$ and $A > 0$. Hence

$$\hat{B}(\bar{x}, a) - 1 > \sum_{r=0}^n (c_r - c_{r-1}) (\bar{x} - r)$$

for each integer n , $0 \leq n \leq \bar{n} - 2$.

The last assertion follows from examining

$$\hat{B}(\bar{x}, a) - 1 = \sum_{r=0}^{\bar{n}-1} (c_r - c_{r-1}) (\bar{x} - r) + (c_{\bar{n}} - c_{\bar{n}-1}) (\bar{x} - \bar{n})$$

where $\bar{n} = [\bar{x}] \geq 1$, for the inequality indexed by $\bar{n} - 1$ is satisfied as an equality if and only if $\bar{x} - \bar{n} = 0$. ||

It will be useful later to include upper bounds on the x -variables in the set K . The following Corollary states that in this case one only needs a finite number of the inequalities of (1).

Corollary 2. Let V be a positive integer and set $K' = K \cap \{(z, x) \mid 0 \leq x \leq V\}$. Let L' be the set of all (z, x) which satisfy

$$z - 1 \geq \sum_{r=0}^n (c_r - c_{r-1}) (x - r), \quad x \geq 0$$

for $n = 0, 1, \dots, V - 1$. Then $K' = L'$.

Proof: Let $L'' = L \cap \{(z, x) \mid 0 \leq x \leq V\}$. Then by Proposition 1, $K' = L''$. Since L'' incorporates the semi-infinite system (1), it follows immediately that $L'' \subset L'$. On the other hand, let (\bar{z}, \bar{x}) be arbitrary in L' . Then, $0 \leq \bar{x} \leq V$ and $\bar{n} \leq V$, where $\bar{n} = [\bar{x}]$. If $\bar{n} \leq V - 1$, then membership in L' implies

$$\bar{z} - 1 \geq \sum_{r=0}^{\bar{n}} (c_r - c_{r-1}) (\bar{x} - r).$$

Using (20) followed by (19) we find that $\bar{z} \geq 2 \cdot S(\bar{x}, a)$ implying $(\bar{z}, \bar{x}) \in K^1$.

On the other hand, if $\bar{r}_i = V$, then necessarily $\bar{x} = V$.
Moreover,

$$\bar{z} - 1 \geq \sum_{r=0}^{V-1} (c_r - c_{r+1}) (V-r).$$

But the right-hand sum equals by (20),

$$-1 - D(V-1, a)V + VB(V-1, a) - (V-1)B(V, a),$$

which is merely $-1 + B(V, a)$. Therefore, in this case

$$\bar{z} \geq \hat{B}(V, a)$$

and $(\bar{z}, \bar{x}) \in K^T$ also. Thus, in either case $(\bar{z}, \bar{x}) \in K^1$ which implies (\bar{z}, \bar{x}) satisfies the entire inequality system (1) by Proposition 1. Hence, $(\bar{z}, \bar{x}) \in L^{ff}$, and hence $L^1 \subset L^*$. Therefore, $L^1 \ll L^{ff}$ which yields $K^1 = L^T$.

Corollary 3. (z, x) is an extreme point of K^1 if and only if x is a non-negative integer and $z \ll B(x, a)$.

Proof: There are only two variables z and x in the linear inequality system (1), Hence extreme points can only occur on the boundary of K^1 at the intersection of a pair of linearly independent equations. By Corollary 1 a pair of linearly independent equations arise if and only if \bar{x} is a non-negative integer, and moreover, each non-negative integer does satisfy two (adjacent) linearly independent equations. This includes the special cases of the endpoints where for $(1, 0)$, the additional

inequality $x \in 0$ is used and at $(B(V,a),V)$ the inequality $x \in V$ is used.

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