FINITE ELEMENT METHODS OF THE LEAST SQUARES TYPE FOR REGIONS WITH CORNERS

by

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Abstract

This paper treats problems with corner singularities«. It is shown that if appropriate weights are used in the least squares formulation, then optimal error estimates can be derived in unweighted L_2 norms.

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§1. Introduction. Least squares methods have been useful for approximating solutions to elliptic systems which are not strictly coercive. The Helmholtz equation is perhaps the most important example ([1] - [2]). The main advantage of the least squares formulation in this context, as compared with a standard Galerkin formulation, is the fact that it always produces a Hermitian positive definite system of algebraic equations. This is particularly attractive for three dimensional problems where storage considerations make iterative methods like S.O.R. desirable. The Stein-Ostrowski Theorem [3] states that for Hermitian systems this iterative method converges if and only if the system is positive definite.

The least squares formulation does however have one glaring defect, namely the extreme regularity on the solution that is needed to obtain optimal convergence. For example, the usual least squares approximation to

$$\Delta\phi + q\phi = f \quad \text{in} \quad \Omega \qquad (1.1)$$

$$\phi = 0 \quad \text{on} \quad \partial\Omega \qquad (1.2)$$

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is to require that

$$f\{|\mathrm{grad}\ 4 - \underline{u}|^2 + |\mathrm{div}\ \underline{u} + q|> - f|^2\} \qquad (1.3)$$
$$n$$

be minimized as $ and $\underline{u}$ vary over appropriate finite dimensional spaces. For such an approach to lead to optimal approximations one needs, among other things, the following regularity property [1]. There is a number 0 < C < « such that for any function f in the Sobolev space $H^1(0)$ there is a unique solution $ of (1.1) - (1.2) such that

$$\|\phi\|_{H^3(\Omega)} \leq C\,\|f\|_X \qquad d.4)$$

This result is valid for only smooth regions ft, and in particular it is not valid if Q has corners. Moreover, numerical experiments indicate that something like (1.4) may actually be necessary. For example, a series of numerical experiments [4] with regions with cracks have shown that this approach produces substandard results even with a rather extreme mesh refinement near the corner.

In this paper we consider am alternate least squares approximation in weighted spaces. These are spaces where the analogs of (1.4) are valid if the appropriate weights are used. Moreover, our analysis shows that this will lead to optimal results in unweighted L_2 norms. Numerical confirmation of these results are reported elsewhere [4].

For simplicity we shall consider planar regions ft with only one corner having interior angle 6_{Fl} as shown in Figure 1. Our results are restricted to the case $0 £ 8_Q < 2ir$ because of the crucial role played in the analysis by the Hardy-Littlewood inequality and continuous embedding in weighted Sobolev spaces• The latter is known to be valid only when Ct has the cone property, i.e.$_f$ 6 4* 2«. Moreover, the regularity results used for the weighted Sobolev spaces have only been developed for planar or conical regions in [6], hence our restriction to the planar case. Neumann and mixed problems could be treated analogously. For brevity we consider here only the Dirichlet problem.

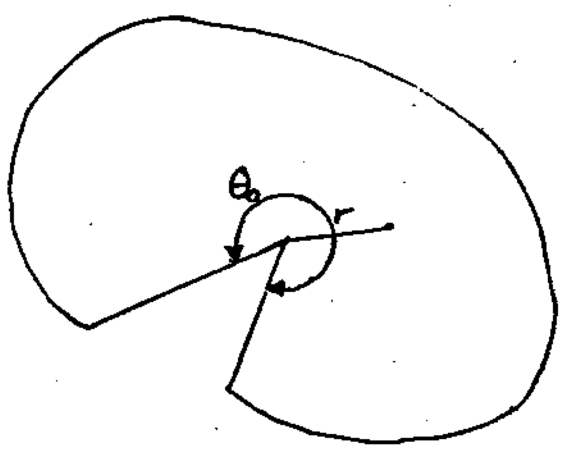

Figure 1. The planar region Ω

§2.. Formulation of the approximation. For the number $\alpha > 0$ define

$$||\underline{v}||^2_{1,\alpha} = ||r^{\alpha/2}\operatorname{div}\underline{v}||^2_{L^2(\Omega)} + ||\underline{v}||^2_{[L^2(\Omega)]^2} \quad (2.1)$$

and let $\underline{W}^1_\alpha(\Omega)$ be the closure of $[C^\infty(\bar\Omega)]^2$ in this norm. Let

$$S_\delta \subseteq \overset{\circ}{H}{}^1(\Omega) \qquad \underline{V}_h \subseteq \underline{W}^1_\alpha(\Omega) \quad (2.2)$$

be finite dimensional spaces. We seek

$$\phi_\delta \in S_\delta \qquad \underline{u}_h \in \underline{V}_h \quad (2.3)$$

which minimize

$$\int_\Omega \{|\operatorname{grad}\psi^\delta - \underline{v}^h|^2 + r^\alpha|\operatorname{div}\underline{v}^h + q\psi^\delta - f|^2\} \quad (2.4)$$

over $\psi^\delta \in S_\delta$, $\underline{v}^h \in \underline{V}_h$ where r is the distance to the vertex (Fig. 1). Appropriate choices for $\alpha > 0$ will be discussed in the next section.

An equivalent statement of this variational principle involves the bilinear form

$$B_\alpha((\psi,\underline{v}),(\xi,\underline{w})) = \int_\Omega (\operatorname{grad}\psi-\underline{v})\cdot(\operatorname{grad}\xi-\underline{w}) + r^\alpha(\operatorname{div}\underline{v}+q\psi)(\operatorname{div}\underline{w}+q\xi) \quad (2.5)$$

and the functional

$$F_\alpha(\xi,\underline{w}) = \int r^\alpha f(\operatorname{div}\underline{w} + q\xi) \quad (2.6)$$

In particular, the minimum of (2.4) is characterized by the following variational principle. Find functions (2.3) such that

$$B_\alpha((\phi_\delta,\underline{u}_h),(\psi^\delta,\underline{v}^h)) = F_\alpha(\psi^\delta,\underline{v}^h) \quad (2.7)$$

holds for all $\psi^\delta \in S_\delta$ and $\underline{v}^h \in \underline{V}_h$.

Observe that $B_\alpha(\cdot,\cdot)$ and $F_\alpha(\cdot)$ are continuous on $\mathcal{X} = \overset{\circ}{H}{}^1(\Omega) \times \underline{W}^1_\alpha(\Omega)$. Indeed, taking

$$|||(\phi,\underline{u})||| = \{\|\operatorname{grad}\phi\|^2_{[L^2(\Omega)]^2} + \|\phi\|^2_{L^2[\Omega]} + \|\underline{u}\|^2_{1,\alpha}\}^{1/2}, \tag{2.8}$$

as the norm on $\mathcal{X}$, then as an easy consequence of the Schwarz inequality we obtain with $\alpha \geq 0$ and $q \in L^\infty(\Omega)$

$$|B_\alpha((\phi,\underline{u}),(\psi,\underline{v}))| \leq C|||(\phi,\underline{u})|||\ |||(\psi,v)||| \tag{2.9}$$

and for $f \varepsilon L^2(\Omega)$ in (2.6).

$$|F_\alpha(\psi,\underline{v})| \leqq C|||(\psi,\underline{v})||| \tag{2.10}$$

Thus (2.7) has a meaning, and in fact, is equivalent to a Hermitian nonnegative definite system of algebraic equations once a basis is chosen for $S_\delta \times \underset{\sim}{U}_h$. If we assume that (1.1) - (1.2) has a unique solution for each $f \varepsilon L^2(\Omega)$, then the algebraic system is positive definite. This will be the case, for example, if q is never equal to an eigenvalue of the Laplacian with Dirichlet boundary conditions.

§3. Analysis of errors. Because of the singularities in the solution to (1.1) - (1.2), piecewise linear functions are perhaps the most practical choices for $\mathcal{S}_\delta$ and $\mathcal{U}_h$, and in this section we shall restrict attention to this case. The grids for the space $\mathcal{S}_\delta$ of scalars need not coincide with the grids for the space $\mathcal{U}_h$ of vectors, and as we shall see subsequently, there will be important reasons for this.

We recall that approximation theory asserts that there is a positive number C satisfying the following [5].

Given any $\underline{u} \in \underline{W}^k_\alpha$ and $\phi \in H^{\ell}(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$ there are $\hat{\underline{u}}_h \in \mathcal{U}_h$ and $\hat{\phi}_\delta \in \mathcal{S}_\delta$ such that

$$\|\underline{u}-\hat{\underline{u}}_h\|_{t,\alpha} \leq Ch^{\mu}\|\underline{u}\|_{k,\alpha} \qquad \mu = \min\{k-t, \tfrac{\pi}{\theta_0}-1+\tfrac{\alpha}{2}\} \tag{3.1}$$

$$\|\phi-\hat{\phi}_\delta\|_{t,0} \leq C\delta^{\ell-t}\|\phi\|_{\ell,0}{}^{(1)} \tag{3.2}$$

for $0 \leq t \leq k \leq 2$, $t \leq \ell$ and $0 < h < \underline{h}_0$, $0 < \delta \leq \delta_0$. The goal of this section is to develop similar estimates for the errors

$$\underline{e} = \underline{u} - \underline{u}_h, \qquad \epsilon = \phi - \phi_\delta \tag{3.3}$$

in the least squares approximation.

Crucial to our error analysis is the regularity of the solutions to (1.1) - (1.2) in appropriate Sobolev spaces. For simplicity we shall assume that the interior angle is re-entrant,

(1) Here $\|\cdot\|_{k,\alpha}$ denotes the norm associated with $\underline{W}^k_\alpha$. In addition, $\|\cdot\|_{t,0}$ denotes the (unweighted) norm on $H^t(\Omega)$.

i.e., % < e_Q < 2w. Due to [5] for given f € L^2.(Q) and q not being an eigenvalue there exists exactly one solution *h* of (1.1) - (1.2) in H^Q) OH[1],,,!!)), 9 * 1 ^ Therefore due to Kondratiev [6] the following important regularity result holds.

Theorem 1; Let g be not an eigenvalue of (1.1), (1.2). Then there are

$$a > 2t + 2 - 2s_Q, \quad s_Q = £-, \quad C_R > 0 \tag{3.4}$$

such that for any f 6 w£,(Q) (t = 0,1) the solution 4> of (1.1) - (1.2) satisfies

$$\|\phi\|_{t+2,a} \leq C_R \|f\|_{t,a} \quad 1 \circ J^f \ «t,0 \tag{3.5}$$

Moreover, for 1 < s < 1 + s_Q

$$\|\phi\|_{s,0} \leq C_R \|f\|_{0,0} \tag{3.6}$$

It is an easy consequence of (2.7) that {u^ ^} is a best approximation to {grad 4>#4>3 in the norm generated by B_a(«,')• Thus we have the following consequence of Theorem 1 and the approximation properties (3.1) - (3.2) together with (2.9).

Lemma 1. There is a constant C > 0 depending only on C_R and

$$a \geq 4 " \frac{2\pi}{\omega} \tag{3.7}$$

such that

$$B_a((\epsilon,e),(\epsilon,e))^{1/2} \text{ -Ill.}(\epsilon_f Q) \text{ III}_a \leq C(h + 6^{-\circ}) \|f\|_{1,0} \tag{3.8}$$

Our next estimate is in the weighted dual norm defined as follows

$$\|u\|_\alpha^* = \sup_\eta \frac{\int r^\alpha \psi \eta}{\|\eta\|_{1,\alpha+4}} \tag{3.9}$$

<u>Lemma 2</u>. <u>There is *a* constant</u> $C > 0$ <u>depending only on</u> $\alpha \neq 1$ which satisfies (3.4),(3.8) and $C_S > 0$ such <u>that</u>

$$\|\operatorname{div} \underline{e} + q\xi\|_\alpha^* \leq C(\delta^{s_0} + h)\ |||(\xi,\underline{e})|||_\alpha . \tag{3.10}$$

<u>Proof</u>. Let $\eta \in W^1_{\alpha+4}$ be given, and consider the solution ζ to

$$A\zeta + q\zeta = \eta \text{ in } \Omega, \quad \zeta = 0 \text{ on } \Gamma. \tag{3.11}$$

Letting

$$\underline{p} = \operatorname{grad} \zeta$$

we have from (3.5) with α satisfying (3.4) and (3.7)

$$\|\underline{p}\|_{2,\alpha+4} \leq \|\zeta\|_{3,\alpha+4} \leq C_R\|\eta\|_{1,\alpha+4} . \tag{3.12}$$

Also

$$B_a((e,\underline{e}),<£,£)) = |r^a n(\text{div } \underline{e} + qe).$$
ft

Using orthogonality this becomes

$$B_\alpha((e,\underline{e})_r(5\text{-}\hat{C}jrE\text{-}\hat{J}\wedge)) = |r^a n(\text{div } \underline{e} + qe) \qquad (3.13)$$
ft

Thus

$$|||(e,\underline{e})||l_o \text{ III } U\text{-}ig/E\text{-}\hat{E}h) \, lll_a \, \underline{1} \, l|r^a n(\text{div } \underline{e} + qe) I \qquad (3.14)$$

Using the approximation properties (3.1) - (3.2) and taking the sup over n with II nil$_{X»a+4}$ ≤ 1 we obtain (3.10) from (3.14) since the solution € of (3.11) satisfies

$$\text{II } 511_{H^{1+<s}_{TM0}(\Omega)} < c||\text{nil},_{L^2(Q)} < \text{ell nll}_{W^{W}cri\text{-}4} i$$

The latter inequality holds due to the continuity of the imbedding of wj into L^2(<x*l) ([7] , p. 287) . The cone property for ft is needed for this result.

<u>Lemma 3</u>. <u>There is a constant</u> C > 0 <u>depending only on</u> a j≥ 4 - ^L, II qll$_{L_\infty(n)}$ <u>and</u> C^ <u>such</u> Jfchat 0

$$\text{II ell}_{L^2(ft)}^{0} \; i \; C(||\text{div } \underline{e} + qe||_a^* + 6^{S\circ}||l(e,e)HI_a) \qquad (3.15)$$

<u>Proof</u>. The first step is to solve for e in L^2(ft)

$$An + qTl = e \text{ in } ft, \; n = O \text{ on } r. \qquad (3.16)$$

For this system we have from (3.4) and (3.5)

$$^{11\,nll}2.,c \wedge {}^{C|1\,e|\,l}{}_0, a\wedge {}^{C|1\,eII}{}_0 \qquad (3.17)$$

provided $2 - \frac{2\pi}{\theta_0} < a$ for $\pi \leq \theta_0 \leq 2\pi$. Also it follows from (3.16) that

$$\|e\|_0^2 = \int_Q e(\Delta n + qn)$$

which after integrating by parts becomes

$$\|e\|_0^2 = \int_\Omega (\{-\text{grad } e \cdot \text{grad } n + qen\} \qquad (3.18)$$

But

$$B_a((e,\underline{e}),(n,\underline{0})) = \int_Q (\text{grad } e - \underline{e}).\text{grad } n + r^{a}(\text{div } \underline{e}+qe)qn\}. \quad (3.19)$$

Thus combining (3.18) - (3.19) and using orthogonality we obtain

$$\|e\|_Q^2 = \int [-\underline{e}\cdot\text{grad } n + qen + r^a(\text{div } \underline{e} + qe)\, qn - B_0((e,\underline{e}),(n-\hat{n}_h,0)).$$

Integrating the first term on the right by parts gives the simpler form

$$\|e\|_0^2 = \int_\Omega (r^a q + 1)(\text{div } \underline{e} + qe)n\,] - B_0((e,\underline{e}),(n-\hat{n}_h,0)) \qquad (3.20)$$

The second term on the right hand side of (3.20) gives the second term in (3.15). Thus our task is to estimate the first term. To do this we use the Hardy-Littlewood inequality [7, p.286]. This inequality states with D = grad that

$$C\|r^{a/2} D^2 n\|_{0,\Omega} \geq \|r^{\frac{a-2}{2}} Dn\|_{0,\Omega} \qquad (3.21)$$

provided $2-\frac{\epsilon}{\theta} < \sigma -$. Note that

$$\|(r^{\alpha} q+1)(\mathrm{div}\, \underline{e} + qe)\eta\|_{\alpha} \leq c\|\mathrm{div}\, \underline{e} + qe\|_{\infty} \|\eta\|_{\alpha}$$

$$= c\|r^{\alpha}|\mathrm{div}\, \underline{e} + qe|\ |r^{-\alpha}\eta|\|_{\alpha}. \qquad (3.22)$$

Thus using (3.9) we see that the right-hand side of (3.22) is bounded above by

$$\|\mathrm{div}\, \underline{e} + qe\|_{\infty}\ \|r^{\frac{\alpha+4}{2}}\mathrm{grad}(r^{-\alpha}\eta)\|_{0,\theta}$$

which in turn is bounded above by

$$\|\mathrm{div}\, \underline{e} + qe\|_{\infty}\ \|r^{\frac{\sigma-4}{2}}\mathrm{grad}\ \eta\|_{0,\theta}\ .$$

To use (3.17) we must bound the second term in the above by $\|\eta\|_{2,\sigma}$ and to do this we take

$$4-\alpha = \sigma - 2 \qquad (3.23)$$

in the Hardy-Littlewood inequality. This gives

$$\|(r^{\alpha}q+1)(\mathrm{div}\, \underline{e} + qe)\eta\|_{\alpha} \leq C\|\mathrm{div}\, \underline{e} + qe\|_{\infty}\ \|r^{\alpha/2}D^{2}\eta\|_{0,\theta}\ . \qquad (3.24)$$

But α satisfies (3.7); and thus by (3.23) and (3.17)

$$\|r^{\alpha/2}D^{2}\eta\|_{0} \leq C\|e\|_{Q} \qquad (3.25)$$

provided

$$2-\frac{\pi}{\theta} < \sigma \leq 2+\frac{\pi}{\theta} \qquad (3.26)$$

Combining (3.24)-(3.25) we obtain (3.15) from (3.20).

Note that the approximation property (3.1) for $k=2$, $t=1$ is only quasi-optimal if α satisfies (3.8).

Inserting (3.8) and (3.10) into (3.15) we obtain an L^2-estimate for ε.

<u>Theorem 2</u>. <u>There is a constant</u> C <u>depending only on</u> $\|q\|_{L_\infty}, \alpha \geq 4 - 2\pi/\theta_0$ <u>and</u> C_R <u>such that</u>

$$\|\phi-\phi_\delta\|_0 \leqq C(\delta^{s_0} + h)^2 \|f\|_1. \tag{3.27}$$

<u>Remark</u>. For optimal accuracy we take

$$\delta = h^{1/s_0}; \tag{3.28}$$

since $s_0 = \pi/\theta_0$, $\pi < \theta_0 < 2\pi$, the grid for the scalar field ϕ must be finer than that for the vector field $\underline{u}$.

We now use Lemma 1 to estimate $\underline{e} = \underline{u} - \underline{u}_h$. To do this we shall need for S_δ to have an inverse property. More precisely, we shall assume there is a number $0 < C < \infty$ independent of δ such that

$$\|\psi_\delta\|_1 \leqq C\delta^{-1}\|\psi_\delta\|_0 \quad \text{all} \quad \psi_\delta \in S_\delta. \tag{3.29}$$

<u>Theorem 3</u>: <u>Let</u> (3.28) <u>and</u> (3.29) <u>hold</u>. <u>Then</u> <u>for</u> $\alpha \geq 4 - \frac{2\pi}{\theta_0}$

$$\|\underline{u}-\underline{u}_h\|_0 \leqq Ch\|f\|_1 \tag{3.30}$$

<u>Proof</u>. Let $\psi_\delta \in S_\delta$ satisfy

$$\|\phi - \psi_\delta\|_r \leqq C\delta^{1+s_0-r} \|\phi\|_{1+s_0} \tag{3.31}$$

for $r = 0$ and $r = 1$. Then from

$$\|\underline{e}\|_0^2 - \|\text{grad }\varepsilon\|_0^2 \leq \|\underline{e} - \text{grad }\varepsilon\|_0^2$$

we obtain

$$\|\underline{e}\|_0 \leq C(\|\text{grad }\varepsilon\|_0 + |||(\varepsilon,\underline{e})|||_\alpha). \tag{3.32}$$

But

$$\|\text{grad }\varepsilon\|_0 \leq \|\text{grad}(\phi-\psi_\delta)\|_0 + \|\text{grad}(\psi_\delta-\phi_\delta)\|_0 \tag{3.33}$$

We use (3.31) to estimate the first term, and apply the inverse inequality (3.29) to the second term to get

$$\|\text{grad }\varepsilon\|_0 \leq C\delta^{s_0}\|\phi\|_{1+s_0} + C\delta^{-1}\|\psi_\delta-\phi_\delta\|_0. \tag{3.34}$$

But

$$\|\psi_\delta-\phi_\delta\|_0 \leq \|\phi-\phi_\delta\|_0 + \|\phi-\psi_\delta\|_0 \leq C\delta^{1+s_0}\|f\|_0. \tag{3.35}$$

Combining these estimates we obtain (3.30) with (3.28).

<u>Remark (i)</u>. For given $f \varepsilon H^1(\Omega)$ and mesh-sizes $\delta = h^{1/s_0}$ (3.27) (3.30) yield $\|\varepsilon\|_0 = O(h^2)$ and $\|\underline{e}\|_0 = O(h)$. Thus the least squares solution $(\phi^\delta, \underline{u}_h)$ of (2.4) converges with same order towards the exact solution $(\phi, \text{grad }\phi)$ where $\phi \varepsilon H^{1+s_0}(\Omega)$, $s_0 = \frac{\pi}{\theta}$, of (1.1)-(1.2), as in the regular case which is considered in [1];

(ii) For mixed boundary conditions

$$4> = 0 \text{ on } T_D, \quad \overset{\partial \text{L}}{1} 5 * 0 \text{ on } T_N \qquad (3.36)$$

where v denotes the outer normal at the boundary F = $\overline{P}_D$ U$\overline{T}$^. The solution <> of (1.1) behaves like $r^{\pi/2\theta}$ °i(φ) at the collition points, where :* is anayltic. Therefore the above analsysis holds as well for the mixed boundary value problem (1.1), (3.36) by choosing the weight

$$a \geq 4 - \frac{\pi}{T}$$

in the least squares scheme (2.4) and refining the mesh as 6 » $h^{2\theta/w}$

(iii) Since the solutions of crack problems behave like the solutions of mixed boundary value problems with smooth boundary, our weighted least squares method can also be applied to crack problems* Choosing a ≥ 3, 5 = h^2 we obtain with piecewise linear test and trial functions

$$|j€||_0 = 0(h^2), \qquad ||e||_0 = 0(h). \qquad (3.37)$$

The standard Galerkin procedure gives for € the same error estimate if 6 = h^2. But in order to obtain (3.37) for $\overline{e}$ with the Galerkin procedure, one has to use special singularity functions as test and trial functions.

(iv) The results in this paper do not apply to the general three dimensional case because the regularity results used are not known for three dimensional domains with arbitrary corners.

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