THE PERFECTLY MATCH/VQLE SUBGRAPH POLYTQPE

by

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ABSTRACT

The following type of problem arises in practice: in a node-weighted graph G, find a minimum weight node set that satisfies certain conditions and, in addition, induces a perfectly matchable subgraph of G. This has led us to study the convex hull of incidence vectors of node sets that induce perfectly matchable subgraphs of a graph G, which we call the perfectly matchable subgraph polytope of G. For the case when G is bipartite, we give a linear characterization of this polytope, i.e., specify a system of linear inequalities whose basic solutions are the incidence vectors of perfectly matchable node sets of G. We derive this result by three different approaches, using linear programming duality, projection, and lattice polyhedra, respectively. The projection approach is used here for the first time as a proof method in polyhedral combinatorics, and seems to have many similar applications Finally, we completely characterize the facets of our polytope, i.e., we separate the essential inequalities of our linear defining system from the redundant ones.

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1. Introduction

Given a graph $G = (V,E)$, it is often of interest to identify those node sets of G that are perfectly matchable, i.e., those $S \subseteq V$ such that $\langle S \rangle$, the subgraph of G induced by S, has a perfect matching. We call the convex hull of the incidence vectors of perfectly matchable node sets of a graph G, the perfectly matchable subgraph polytope (PMS polytope) of G.

The identification of the perfectly matchable node sets of a graph G would of course become much easier if the PMS polytope of G could be linearly described, i.e., if one had a system of linear inequalities whose basic solutions are precisely the extreme points of the PMS polytope of G. The existence of such a linear system follows from the by now classical result that the convex hull of a finite set of points in $\mathbb{R}^n$ is the intersection of a finite number of halfspaces in $\mathbb{R}^n$, i.e., the solution set of a finite system of linear inequalities in n variables. But the identification of such a linear system defining a polytope given by the set of its extreme points (that are either explicitly listed or specified by some definition, like here) is usually a hard task, which has so far been solved only for a few cases. In this paper we give such a linear characterization of the IMS polytope of a bipartite graph. The case of a general graph will be addressed in another paper.

The question examined here arose in the context of a real world problem that had to do with the optimal scheduling of drivers for a municipal bus company. This particular application, which gave the initial motivation for our research, is described in section 2 of the paper. Section 3 introduces the system of linear inequalities defining the IMS polytope of a bipartite graph and gives a first proof of the validity of this linear characterization, based on linear programming duality theory. Section 4 gives an alternative proof, using a projection technique that is of interest in itself, since it may serve as a proof method in situations analogous to, but different from, the one examined here. Finally, section 5 gives a third proof, based on the theory of lattice polyhedra.

Section 6 of the paper focuses on the question of redundancy in the system introduced in section 3, and gives a complete characterization of the facets of the IMS polytope of a bipartite graph.

2. Motivation: A Bus Driver Scheduling Problem

The following problem was brought to our attention by Mr. A. Roes of the Operations Research group of Nederlandse Spoorwegen, the Dutch Railway Company.

A municipal bus company had to schedule the tours of duty of its drivers, so as to cover a daily set of trips to be executed. A set covering approach was used, i.e., the problem was formulated as

$$\min\{cx \mid Ax \geq e,\ x \in \{0,1\}^n\},$$

where A is an m X n 0-1 matrix whose j^{th} column represents a potential daily (tour of) duty for a driver, with $a_{ij} = 1$ if duty j covers trip i, $a_{ij} = 0$ otherwise, while c_j is the cost of duty j, and e = (1,...,1). In a typical case the matrix A had about 150-200 rows and 3000-4000 columns.

However, the way the columns of A, i.e., the potential duties, were generated, suggested another approach. Initially, a set of "early parts" (morning half-tours) and "late parts" (afternoon half-tours) of duty were generated independently of each other, then all the compatible early part-late part pairs were explicitly generated as potential full day duties. The number of early parts and late parts was typically about 150 and 200 respectively, and the 3-4000 columns of A arose from the fact that only 10-13% of the 30,000 pairs were compatible (because of starting and ending properties in space and time). If the number of early parts and late parts is n_1 and n_2, respectively, and the ratio of compatible early part-late part pairs to all such pairs is r, then $n = r \times n_1 \times n_2$; i.e., n is usually much larger than $Q_1 + V$

Now let $A^1 = (a^1_{ij})$ and $A^2 = (a^2_{ij})$ be $m \times n_1$ and $m \times n_2$ matrices, respectively, defined by

$$a^1_{ij} = \begin{cases} 1 & \text{if early part } j \text{ covers trip } i \\ 0 & \text{otherwise} \end{cases}$$

and

$$a^2_{ij} = \begin{cases} 1 & \text{if late part } j \text{ covers trip } i \\ 0 & \text{otherwise,} \end{cases}$$

and let c^1 and c^2 be the cost vectors of early parts and late parts, respectively. Further, let $G = (V_1 \cup V_2, E)$ be the bipartite graph whose node sets V_1 and V_2 correspond to the early parts and the late parts, respectively, and whose edges correspond to compatible early part-late part pairs. Then the above problem can be reformulated as follows:

(i) Find $x^1 \in \{0,1\}^{n_1}$ and $x^2 \in \{0,1\}^{n_2}$ to

(2.1) minimize $c^1x^1 + c^2x^2$

subject to

(2.2) $A^1x^1 + A^2x^2 \geq e$

and

(2.3) (x^1, x^2) is the incidence vector of some $S \subseteq V_1 \cup V_2$ such that < S > has a perfect matching.

(ii) Find a minimum-weight perfect matching in the graph < S > with edge-weights

$$c_{ij} = c_i + c_j, \qquad (i,j) \in E.$$

Here, as before, < S > denotes the subgraph of G induced by the node set S.

Problem (ii) is of course polynomially solvable; whereas problem (i) replaces the original 3-4000 variable set covering problem by a 350-variable set covering problem with side condition (2.3).

The solvability of the problem thus hinges on whether one can conveniently represent condition (2.3).

3. A Linear Characterization of the PMS Polytope

Let G = O ^ U V ^ E) be a bipartite graph with parts V_1 and V_2, i.e., with node set V = V^UV^ and edge set E such that every eeE joins some node of V_1 to some node of V_2.

Let 7?J(G) be the family of perfectly matchable node sets of G, i.e.,

$\mathcal{M}$(G) « [S£V|< S > has a perfect matching}.

For any SSV, the incidence vector (characteristic vector) of S is x e{0,l}$^{1 \cdot j}$ such that x_j - 1, jeS, x_j = 0, jeV\S. Let £(G) be the set of incidence vectors of members offlJ(G),and for any set T, let conv T denote the convex hull of T.

Our objective in this section is to give a linear system of inequalities defining conv %(G), i.e., the EMS polytope of G.

Whenever it is not confusing, we will write $\mathcal{M}$ for $\mathcal{M}(G)$ and $\mathcal{X}$ for $\mathcal{X}(G)$.

Many problems involving matchings, in particular in bipartite graphs, can be shown to be special cases of certain matroid problems. For instance, if $G = (V_1 \cup V_2, E)$ is the bipartite graph introduced above and for $k = 1, 2$, $\mathcal{J}_k$ is the family of those edge sets that meet every node in V_k at most once, then the system $M_k = (E, \mathcal{J}_k)$ is a matroid; and the intersection of the two matroids M_1 and M_2 is the independence system $(E, \mathcal{J}_1 \cap \mathcal{J}_2)$, where $\mathcal{J}_1 \cap \mathcal{J}_2$ is simply the family of all (not necessarily perfect) matchings in G. The matching polytope of G is then the convex hull of incidence vectors of all members of $\mathcal{J}_1 \cap \mathcal{J}_2$.

Another example, more closely related to our problem is the following. In an arbitrary graph H with node set N, let $\mathcal{J}$ be the family of those subsets of N covered by some matching. Then the system $(N, \mathcal{J})$, as shown by Edmonds and Fulkerson [4], is a matroid.

In such cases as the above, results on matroid polyhedra due to Edmonds [2, 3] lead to linear characterizations of the type that we are interested in. However, these results are not applicable to our case, since the PMS polyhedron of a graph (bipartite or not) does not have a matroidal structure. To see this, it is sufficient to recall the fact that every $S \in \mathcal{J}(G)$ is of even cardinality.

We now briefly state our notational conventions. An edge joining nodes i and j is denoted (i,j). For $S, T \subseteq V$, the set of edges joining nodes in S to nodes in T is denoted (S,T). For $S \subseteq V$, F(S) denotes the set of nodes adjacent to some node in S. Clearly, if $S \subseteq V_1$, then $F(S) \subseteq V_2$ and vice versa. For the sake of brevity, we write F(i) for F({i}).

For any $x \in \mathbb{R}^{|V|}$ and any $S \subseteq V$, we let $x(S) = \Sigma(x_i : i \in S)$.

Next we state the linear system defining the MS polytope of G, i.e., the convex hull of X.

<u>Theorem 3.1</u>. Let P be the convex polytope consisting of those $x \in \mathbb{R}^{|V|}$ satisfying

(3.1) $\quad 0 \le x_i \le 1, \qquad i \in V$

(3.2) $\quad x(V_1) - x(V_2) = 0$

and

(3.3) $\quad x(S) - x(\Gamma(S)) \le 0, \qquad \forall\, S \subseteq V_1.$

Then $P = \operatorname{conv} X$.

<u>Proof</u>. It is easy to see that $\operatorname{conv} X \subseteq P$. For let x be any vertex of conv X; then x is the incidence vector of some $T \in \mathcal{T}$, hence (3.1) holds trivially. Further, (3.2) is the requirement that $|T \cap V_1| = |T \cap V_2|$, and (3.3) simply states that for any $S \subseteq V_1$, T must contain at least as many nodes of $\Gamma(S)$ as of S. Both of these requirements are readily seen to be necessary conditions for < T > to have a perfect matching, and together they constitute the "easy" part of the well-known König-Hall theorem [11], [6].

To prove the converse, namely that $P \subseteq \operatorname{conv} X$, we will show that every vertex of P belongs to X. This will be done by showing that for any vector $c = (c_i : i \in V)$ of real node costs there is an optimal solution x^* to the linear program

(L) $\quad \max\{cx \mid x \in P\}$,

such that $x^* \in X$. Since every vertex of P is the unique optimal solution to such a linear program for some c, this will give the result.

We define a vector $\bar{c} = (\bar{c}_{ij} : (i,j) \in E)$ of edge costs by letting $\bar{c}_{ij} = c_i + c_j$ for all $(i,j) \in E$. For any matching $M \subseteq E$, if S is the set of nodes covered by M, then M is a perfect matching in < S >, and

(3.4) $\Sigma(\tilde{c}_{ij}:(i,j)\in M) = \Sigma(c_i:i\in S)$.

Conversely, for any $S \in \mathcal{M}$ and any perfect matching M in $< S >$, M is also a matching in G, and (3.4) holds. Therefore the problem of maximizing cx over $x \in \mathcal{X}$ can be solved by finding a maximum-weight matching (in terms of the edge-weights $\tilde{c}$) in G.

Let M* be such a matching, and let x* be the incidence vector of the node set S* covered by M*. We will show that x* is an optimal solution to the linear program (L), by constructing a feasible solution to the dual of (L) having the same objective function value as (L).

Since edge-variables are two-indexed, we amend our notational conventions by writing, for $S, T \subseteq V$, $u(S,T) = \Sigma(u_{ij}:i\in S,\ j\in T)$, and $u(i,T) = u(\{i\}, T)$, $u(S,j) = u(S,\{j\})$.

The graph G being bipartite, the incidence vector u* of the matching M* is an optimal solution to the linear program

$$(L_1)\qquad \begin{array}{lll} \max \tilde{c}u & & \\ u(i,V_2) \leq 1 & & i\in V_1 \\ u(V_1,j) \leq 1 & & j\in V_2 \\ u \geq 0 & & \end{array}$$

whose dual is

$$(D_1)\qquad \begin{array}{lll} \min t(V_1) + t(V_2) & & \\ t_i + t_j \geq \tilde{c}_{ij} & & (i,j)\in E \\ t \geq 0 & & \end{array}$$

Let t* be an optimal solution to (D_1). By linear programming duality, (3.5) $\quad$ £($\tilde{c}_{ij:}$(i,j)eM*) = t*^) + t*(V_2).

We aow write down the linear program (D), dual to (L):

min yO^) + y(V_2)

(3.6) $\quad$ y_t + £(z_s:Ss:V_1, ieS) $\geq c_i$ $\quad$ $i \epsilon V_1$

(3.7) $\quad$ $_{7j}$ - Z(z_s:SSV$_r$ je r(S)) $\geq c_j$ $\quad$ $j \epsilon V_2$

(3.8) $\quad$ $y_{\pm}$, $_{7j}$ $\geq$ 0, $\quad$ $i \epsilon V_1$, $j \epsilon V_2$

(3.9) $\quad$ $z_g \geq 0$, $\quad$ S C V ^ $\quad$ z_{V_1} unconstrained.

How let y_i^* = t_i^* for ieV^ y_j^* » t_j^* for jeV$_2$; and z_g • 0 for all SSV$_1$. Then (3.8) - (3.9) are satisfied, and

(3.10) $\quad$ y*(V_x) + y*(V_2) = t*(Vp + t*(V_2)

= S($\tilde{c}_{ij}$:(i,j)rtf*) - ex*.

Next we will describe a procedure for redefining the value of z_ for certain subsets SSV-$_i$ in such a way as to satisfy (3.6)-(3.7), without changing the value of any y_i^*. Therefore, the vector (y*, z) obtained in this way will be the optimal solution to (D) required for the completion of our proof.

At all stages of the procedure, the vector (y*, z) will satisfy the following two symnetric properties:

(3.11) $\quad$ If for somei ^

y_i^* + E(z_S:S SV$_1$, ieS) » c_i - e $\quad$ for some e > 0, then for every $j \in F(i)$,

$$y_j^* - \Sigma(z_S : S \subseteq V_1,\ j \in T(S)) \geq c_j + \varepsilon.$$

(3.12) If for some $j \in V_2$

$$y_j^* - \Sigma(z_S : S \subseteq V_1,\ j \in T(S)) = c_j - \varepsilon \qquad \text{for some } \varepsilon > 0,$$

then for every $i \in F(j)$,

$$y_i^* + \Sigma(z_S : S \subseteq V_1,\ i \in S) \geq c_i + \varepsilon.$$

These conditions state that if the current solution violates the inequality associated with some node by an amount ε, there is a surplus of at least ε at every adjacent node. By the initial definition of (y^*, z) and in view of the inequalities $t_i^* + t_j^* \geq \bar{c}_{ij}$, conditions (3.11)-(3.12) are satisfied initially.

Define

$$S_0 = \{i \in V_1 \mid y_i^* + \Sigma(z_S : S \subseteq V_1,\ i \in S) < c_i\},$$

$$T_0 = \{j \in V_2 \mid y_j^* - \Sigma(z_S : S \subseteq V_1,\ j \in T(S)) < c_j\}.$$

Note that by (3.11) and (3.12), no $i \in S_0$ and $j \in T_0$ are adjacent.

If at any stage of the procedure $S_0 = T_0 = \emptyset$, then (3.6) and (3.7) are satisfied and we are done. If $S_0 \neq \emptyset$, let $s = 0$ and perform Reduction 1. If $S_0 = \emptyset$ but $T_0 \neq \emptyset$, let $t = 0$ and perform Reduction 2.

<u>Reduction 1</u>. Let

$$\varepsilon = \min_{i \in S_s} c_i - y_i^* - \Sigma(z_S : S \subseteq V_1,\ i \in S)$$

and define $z_{\omega_s} = \epsilon (> 0)$. Then (3.11) and (3.12) are still satisfied (since (3.11) was satisfied before), but the set

$$S_{s+1} = \{i \in V_1 \,|\, y_i^* + \Sigma(z_S : S \subseteq V_1,\ i \in S) < c_i\}$$

is a proper subset of S_s.

If $S_{s+1} = \emptyset$, Reduction 1 is complete; otherwise set $s \leftarrow s + 1$ and repeat Reduction 1.

~~Reduction 2~~. Let

$$\epsilon = \min_{j \in T_t} c_j - y_j^* + \Sigma(z_S : S \subseteq V_1,\ j \in \Gamma(S)).$$

Then $\epsilon > 0$. Define $z_{\omega_1} \leftarrow z_{\omega_1} - \epsilon$, $\bar{S}_t = V_1 \backslash \Gamma(T_t)$, and $z_{\bar{S}_t} = z_{\bar{S}_t} + \epsilon$. Note that the effect of this change is to decrease $c_j - y_j^* + \Sigma(z_S : S \subseteq V_1,\ j \in \Gamma(S))$ by ϵ for $j \in T_t$ and to leave it unchanged for $j \in V_2 \backslash T_t$, and also to decrease $y_i^* + \Sigma(z_S : S \subseteq V_1,\ i \in S) - c_i$ by ϵ for $i \in \Gamma(T_t)$ but to leave it unchanged for $i \in V_1 \backslash \Gamma(T_t)$.

Conditions (3.11) and (3.12) still hold (since (3.12) was satisfied before), and the new z_ω still satisfy (3.9); but the set

$$\{j \in V_2 \,|\, y_j^* - \Sigma(z_S : S \subseteq V_1,\ j \in \Gamma(S)) < c_j\}$$

is a proper subset of T_t.

If $T_{t+1} = \emptyset$, Reduction 2 is complete; otherwise set $t \leftarrow t + 1$ and repeat Reduction 2.

After at most $|S_0| \leq |V_1|$ iterations of Reduction 1 and at most $|T_0| \leq |V_2|$ iterations of Reduction 2 we obtain a vector (y^*, z) satisfying (3.6)-(3.9), and thus the proof of the theorem is complete.||

At this point some remarks are in order.

First, there is a certain lack of symmetry in the linear system (3.1)-(3.3) defining conv X, in that it contains inequalities only for subsets S of V_1, but not for subsets T of V_2. The analogous inequalities for subsets of V_2 would be

$$(3.13) \qquad x(T) - x(\Gamma(T)) \leq 0, \ \forall\, T \subseteq V_2.$$

These are clearly valid and could have been included in the system, but they can also be derived from (3.1)-(3.3). For if $T \subseteq V_2$ and we define $S = V_1 \setminus \Gamma(T)$, then $\Gamma(S) \subseteq V_2 \setminus T$; and by subtracting (3.2) from the inequality $x(S) - x(\Gamma(S)) \leq 0$, we obtain $x(V_2 \setminus \Gamma(S)) - x(\Gamma(T)) \leq 0$. But since $\Gamma(S) \subseteq V_2 \setminus T$ implies $T \subseteq V_2 \setminus \Gamma(S)$, and since $x \geq 0$, this last inequality implies $x(T) - x(\Gamma(T)) \leq 0$.

If we had included the inequalities (3.13) in our system defining conv X, then Reductions 1 and 2 could have been made completely symmetric by using the new dual variables that would have been introduced.

Second, suppose $S \subseteq V_1$ is such that the graph $\langle S \cup \Gamma(S) \rangle$ is disconnected, with components $\langle S_k \cup \Gamma(S_k) \rangle$, $k = 1,\ldots,q$. Then the inequality $x(S) - x(\Gamma(S)) \leq 0$ is the sum of the q inequalities $x(S_k) - x(\Gamma(S_k)) \leq 0$, $k = 1,\ldots,q$, hence redundant. Now suppose $\langle S \cup \Gamma(S) \rangle$ is connected and K is the node set of the component of G containing $\langle S \cup \Gamma(S) \rangle$, with $K_i = K \cap V_i$, $i = 1,2$, but the graph $\langle (K_1 \setminus S) \cup (K_2 \setminus \Gamma(S)) \rangle$ is disconnected, with components $\langle T^k \rangle$, $k = 1,\ldots,q$. Let $T_i^k = T^k \cap V_i$, $i = 1,2$. Then for $k = 1,\ldots,q$, we have $\Gamma(T_1^k \cup S) \supseteq T_2^k \cup \Gamma(S)$, or else removing the node set $S \cup \Gamma(S)$ from G would not make $\langle T_1^k \cup T_2^k \rangle$ a maximal connected subgraph. Also, $\Gamma(T_1^k \cup S) \subseteq T_2^k \cup \Gamma(S)$, or else $\langle T_1^k \cup T_2^k \rangle$ would not be connected. Thus we conclude that $\Gamma(T_1^k \cup S) = T_2^k \cup \Gamma(S)$, But then adding the q inequalities $x(T^US) - x(r(T^US)) \leq 0$, $k = 1,\ldots,q$, and subtracting (q - 1) times the equation (3.2), yields the inequality $x(S) - x(r(S)) \leq 0$, which is therefore redundant.

We have thus shown than Theorem 3.1 remains true if (3.3) is replaced by

(3.3') $x(S) - x(I^*(S)) \leq 0$ for all $S \subseteq V_1$ such that the graphs $< S \cup T(S) >$ and $<(K_1 \setminus S) \cup (K_2 \setminus T(S))>$ are connected, where $< K >$ is the component of G containing $< S \cup r_\#(S) >$, and $K_i = K \cap V_i$, $i = 1,2$.

Third, note that if c is integer valued, then so is £, and thus t* can be chosen to be integer valued. Then each iteration of Reduction 1 or 2 will result in integer e and hence in integer valued (y*, z). Thus for any integer valued c, the linear program (D), dual to (L), has integer optimal solutions. Thus our linear system defining the PMS polytope of a bipartite graph is <u>totally dual integral</u>. (This concept was introduced by Hoffman [9] and used extensively by Edmonds and Giles [5]. See also Schrijver [12].)

Fourth, if we set $c_i = 1$ for all $i \in V_1$ and $c_j = 0$ for all $j \in V_2$, then the value of (an optimal solution to) (L), and hence of (D), is the cardinality of a maximum matching in G. Now suppose G has no matching that covers all $i \in V_1$; then if (y*, z*) is an optimal integer solution of (D),

$$y^*(V_1) + y^*(V_2) \geq (= \max\{cx \mid x \in P\}) < |V_1|.$$

Since each y_i^* is a nonnegative integer, this implies that $y_i^* = 0$ for some $i \in V_1$. But since (y*, z*) must satisfy (3.6), there must be some $S \subseteq V_1$ such that $z_S^* > 0$. Now suppose the optimal solution (y*, z*) is chosen such that the number of positive components of z* is minimum, and let $S \subseteq V_1$ be such that $z_S^* > 0$. Then $|S| > |\Gamma(S)|$; for if not, then by adding z_S^* to y_i^* for $i \in S$, subtracting z_S^* from y_j^* for $j \in \Gamma(S)$, and then setting $z_S = 0$, we could obtain a new optimal solution to (D) with fewer positive components of z, a contradiction. Thus we obtain the hard part of the König-Hall Theorem, namely that if $G = (V_1 \cup V_2, E)$ has no matching that covers all of V_1, then there exists $S \subseteq V_1$ such that $|S| > |\Gamma(S)|$. Furthermore, this last result combined with our second remark gives a strengthened version of the hard part of the König-Hall Theorem: for G such that $|V_1| = |V_2|$ to have a perfect matching, it is sufficient that the condition $|S| \leq |\Gamma(S)|$ be satisfied for every $S \subseteq V_1$ such that $\langle S \cup \Gamma(S) \rangle$ and $\langle (K_1 \setminus S) \cup (K_2 \setminus \Gamma(S)) \rangle$ are connected, where K is the node set of the component of G containing $\langle S \cup \Gamma(S) \rangle$, and $K_i = K \cap V_i$, $i = 1,2$.

Fifth, any optimal solution (y*, z*) to (D) can be seen to have the following property. There exists a nested sequence of sets $\emptyset \subseteq U_n \subseteq U_{n-1} \subseteq \ldots \subseteq U_1 \subseteq U_0 \subseteq V_1$ such that for any $S \subseteq V_1$, $z_S^* > 0$ if and only $S = U_i$ for some $i \in \{0,\ldots,n\}$. This is so because if we did s iterations of Reduction 1, we will have defined sets $\emptyset \subseteq S_s \subseteq S_{s-1} \subseteq \ldots \subseteq S_1 \subseteq S_0$. If we did t iterations of Reduction 2, we will have defined sets $S^0 \subseteq S^1 \subseteq \ldots \subseteq S^t$. Further, from (3.11) and (3.12), $S_0 \subseteq S^0$. Combining these sequences gives the claimed sequence $(U_i: i = 0,1,\ldots,n)$.

Finally, we have shown that for any optimal solution t* to the node covering problem (D^1), there is an optimal solution (y*, z*) to (D) for which y* = t*. Of course the converse is also true: if (y*, z*) is an optimal solution to (D), then setting t* = y* gives an optimal solution to the node covering problem (D^1).

4. An Alternative Derivation via Projection

In this section we give an alternative derivation of the linear system defining the IMS polyhedron of a bipartite graph, based on a polyhedral interpretation of Benders's partitioning theorem [1]. This approach is of more general interest than its particular use in this paper, since it provides a technique for projecting a polyhedron in $\mathbb{R}^n$, or some (not necessarily polyhedral) subset of a polyhedron in $\mathbb{R}^n$, into some specified subspace of $\mathbb{R}^n$.

To be specific, let Q be an arbitrary subset of $\mathbb{R}^q$, and let

$$Z = \{(u,x) \in \mathbb{R}^{p+q} \mid Au + Bx \leq b,\ u \geq 0,\ x \in Q\}$$

where A, B and b are $m \times p$, $m \times q$, and $m \times 1$ matrices, respectively, such that $Z \neq \emptyset$. The projection of Z into the subspace of the x-variables is defined as

$$X = \{x \in \mathbb{R}^q \mid \text{there exists } u \in \mathbb{R}^p \text{ such that } (u,x) \in Z\}.$$

We are interested in describing the set X in a way similar to Z, i.e., by a set of linear inequalities plus, of course, the condition $x \in Q$. The following theorem accomplishes this.

Before stating the result, we recall that a polyhedral cone C is the intersection of a finite number of halfspaces through the origin, and a pointed cone is one of which the origin is an extreme point. A ray of a cone C is the set $R(y)$ of all nonnegative multiples of some $y \in C$, called the direction (vector) of $R(y)$. A vector $y \in C$ is extreme, if for any y^1, $y^2 \in C$, $y = \frac{1}{2}(y^1 + y^2)$ implies y^1, $y^2 \in R(y)$. A ray $R(y)$ is extreme if its direction vector y is extreme. A pointed polyhedral cone has a finite number of extreme rays, and is the conical hull of its extreme rays. Of course, for every nonzero $x \in R(y)$, we have $R(x) = R(y)$ and consequently every cone that contains more than the origin has an infinite number of extreme direction vectors. However the smallest set of vectors of which a cone is the conical hull, consists of one direction vector from each extreme ray.

For a cone C we let extr C denote such a (finite) set of extreme direction vectors. Note that extr C is uniquely determined up to positive multiples.

<u>Theorem 4.1</u>. Let Z and X be defined as above, and let

$$W = \{v \in \mathbb{R}^m | vA \geq 0,\ v \geq 0\}\ .$$

Then

$$X = \{x \in \mathbb{R}^q | (vB)x \leq vb,\ \forall\ v \in \text{extr } W;\ x \in Q\}.$$

<u>Proof</u>. The polyhedral cone W is a subset of $\mathbb{R}^m_+$, hence pointed. Therefore W is the conical hull of its extreme rays, and any $x \in \mathbb{R}^q$ satisfies the inequality $(vB)x \leq vb$ for every extreme direction v of W, if and only if it satisfies it for all $v \in W$.

Now let $\bar{x} \in X$; then $\bar{x} \in Q$ and there exists $\bar{u} \in \mathbb{R}^p$ such that $\bar{u} \geq 0$ and $A\bar{u} + B\bar{x} \leq b$. Further, let $v \in W$; then $vB\bar{x} \leq vb - vA\bar{u} \leq vb$, since $\bar{u} \geq 0$ and $vA \geq 0$ imply $vA\bar{u} \geq 0$. Thus $(vB)\bar{x} \leq vb$, $\forall\ v \in$ extr W.

Conversely, suppose $\hat{x} \in \mathbb{R}^q$ satisfies $x \in Q$ and $(vB)x \leq vb$, $\forall\ v \in$ extr W. Then there exists no $v \in \mathbb{R}^m$ such that $vA \geq 0$, $v \geq 0$ and $v(b - B\hat{x}) < 0$. Therefore, from Farkas's well known Lemma, there exists some $\hat{u} \in \mathbb{R}^p$ such that $\hat{u} \geq 0$ and $A\hat{u} \leq b - B\hat{x}$. But then $\hat{x} \in X$.||

Note that, if $W = \{0\}$ (like for instance in the case when $A \leq 0$), then $X = \{x \in \mathbb{R}^q | x \in Q\}$.

We now turn to our problem of giving a linear characterization of the PMS polytope of a bipartite graph G. Although we are looking for a linear system in terms of the variables x_i associated with the nodes of G, we will start with the much easier task of giving a linear characterization in terms of variables associated with both nodes and edges. Such a linear system of course defines a polyhedron in a higher dimensional space than the one that we are looking for, however by projecting this polyhedron into the space of the node variables we will obtain the system of Theorem 3.1.

Recall that the IMS polytope of G is conv *Z,* where *X* is the set of incidence vectors of perfectly matchable node sets of G. Let, as before, a variable x_i be associated with node i of G, and let a variable u_{ij} be associated with edge (i,j) of G. As in section 3,we write u(S,T) = £(u$_{ij}$:isS, jeT), $u(i,T) = u(\{i\},T)$, and $u(S,j) = u(S,\{j\})$.

It is not hard to see that a 0-1 vector x e m$^{|V|}$ 'is the incidence vector of some perfectly matchable node set of G if and only if there exists some integer ue 1R$^{|E|}$ », such that

$$u(i,r(i)) - x_i = 0 \qquad i \in V_1$$

(4.1) $$u(r(j),j) - x_j = 0 \qquad j \in V_2$$

$$u_{ij} \geq 0, \qquad (i,j) \in E.$$

Furthermore, since the coefficient matrix of (4.1) is totally unimodular, the integrality condition on u can be omitted, and the 0-1 condition on x can be replaced by

(4.2) $$0 \leq x_i \leq 1, \qquad i \in V.$$

Thus (4.1) and (4.2) provide a linear characterization of conv *%* in terms of node and edge variables. One way of obtaining a linear characterization in terms of the node variables only, is then to project the polyhedron defined by (4.1), (4.2) into the subspace of the node variables.

To this end, we first rewrite (4.1)-(4.2) as a system of linear inequalities. This can be done in several ways, and we choose to (a) change the sign of the equations jsV_1; (j3) replace all equations by inequalities of the form $\leq$; and (y) add all the inequalities thereby obtained for ieV_1 and jeV_2, and change the direction of the resulting inequality. This yields the system

$$
\begin{array}{lll}
 & -u(i,\Gamma(i)) + x_i \leq 0 & isV_1 \\
 & u(T(j),j) - Xj \leq 0 & jeV_2 \\
(4.3) & -x(V_x) + x(V_2) \leq 0 & \\
 & u_{ij} \geq 0 & (i,j) \epsilon E \\
 & 0 \leq x_\pm \leq 1 & icV
\end{array}
$$

which is equivalent to (4.1)-(4.2). Note that the coefficient matrix of (4.3) is still totally unimodular.

We now apply Theorem 4.1 to this system. The set Q and the matrices A, B and b that define Z of Theorem 4.1 are in this case as follows:

$$Q = [x \; elR^{|V|} | - x(V_1) + x(V_2) < 0, \; 0 < x_i < 1, \; isV\};$$

A is the node-edge incidence matrix of G, with the signs of the rows indexed by V_1 changed;

B is a diagonal matrix of order |v|, with +1 for the diagonal entries indexed by V^, and -1 for those indexed by l_2', and, finally,

b is the 0 vector with |v| components.

Now the cone W of Theorem 4.1 is

$$W = \left\{ vernal \;\middle|\; \begin{array}{ll} -v_i + v_j \geq 0, & i\epsilon V_1, \; j\epsilon V_2, \; (i,j)\epsilon E \\ v_i \geq 0, & isV \end{array} \right\}$$

and in order to project the polyhedron defined by (4.3) into the subspace of the node variables, we have to characterize the extreme rays of W.

<u>Theorem 4.2</u>. The vector $v \in W$ is extreme if and only if there exists $\alpha > 0$ such that either

$$(4.4) \qquad v_i = \begin{cases} \alpha & \text{for exactly one } i = i_* \in V_2 \\ 0 & \text{for all } i \in V_1 \cup V_2 \setminus \{i_*\} \end{cases}$$

or

$$(4.5) \qquad v_i = \begin{cases} \alpha & i \in S \cup T(S) \\ 0 & \text{otherwise} \end{cases}$$

for some $S \subseteq V_1$ such that $< S \cup T(S) >$ is connected.

<u>Proof</u>. Sufficiency. Let $v \in W$ be of the form (4.4), and assume for the sake of contradiction that v is not extreme, i.e., $v = \frac{1}{2}(v^1 + v^2)$ for some $v^1, v^2 \in W \setminus R(v)$. Then $v^1_i = v^2_i = 0$, $\forall\, i \in V_1 \cup V_2 \setminus \{i_*\}$, and $v^1, v^2 \in R(v)$. Thus v is extreme.

Now let $v \in W$ be of the form (4.5), and again assume that $v = \frac{1}{2}(v^1 + v^2)$ for some $v^1, v^2 \in W$. Then $v^1_i = v^2_i = 0$ for $i \in (V_1 \setminus S) \cup (V_2 \setminus T(S))$, and

$$(4.6) \qquad v^1_i + v^2_i = 2\alpha, \qquad i \in S \cup T(S).$$

Note that from (4.6), for any $i \in S$, $j \in T(S)$, $v^1_i > v^1_j$ if and only if $v^2_i < v^2_j$; but the constraints of W imply $v^k_i \leq v^k_j$, $k = 1,2$, for any such pair i,j. Hence $v^k_i = v^k_j$, $k = 1,2$, for all pairs $i \in S$, $j \in T(S)$; and since $< S \cup T(S) >$ is connected, it follows that $v^k_i = v^k_j = \alpha^k$ (constant), $k = 1,2$, for all $i,j \in S \cup T(S)$. Therefore $v^1, v^2 \in R(v)$, i.e., v is extreme.

Necessity. Let v be an extreme vector of W, and let $T = \{i \in V \mid v_i > 0\}$. Define $S = T \cap V_1$, and consider first the case where $S = \emptyset$. Then if

T = U₁»..-fj_t} with t > 1, and if e_j denotes the unit vector inlu$^{|V|}$» with 1 in position j, we have

$$v = v_{j_1} e_{j_1} + \dots + v_{j_t} e_{j_t}$$

$$= j(v \ + v\ ),$$

where v^1 * $2v_{j_1} e_{j_1}$, v^2 » $2(v_{j_2} e_{j_2} + \dots + v_{j_t} e_{j_t})$, with v^1, v^2eW, and v^1iR(v), v^2 iR (v). Thus if |T| > 1, v is not extreme, contrary to the assumption. We conclude that if S = 0, then |T| « 1 and thus v is of the form (4.4).

Now consider the case when S ^ 0. Then F(S) STflV₂, or else there exists ieS, j eF(S) such that v_i > 0, v_j = 0, i.e., v violates some constraint of W. Also, T(S) aT(lV₂, or else there exists j_oeV₂\r(S) such that v_{j} > 0. But then for any e satisfying 0 < e < v_{j_o}, the vectors v^1 and v^2, obtained from v by replacing v_{j_o} with $v^1_{j_o} = v_{j_o} + €$ and $v^2_{j_o} = v_{j_o} - e$, respectively, satisfy the equation v « $j(v^1 + v^2)$, although v^1,v^2 €«\R(v), contrary to the assumption that v is extreme. We therefore have F(S) = TdV$_2$ i.e., T = SUr(S).

We claim that < T > is connected. For suppose not, and let K be the node set of a component of < T >. Then v • $\frac{1}{2}(v^1 + v^2)$, where

$$v^1_i = \begin{cases} 0 & i \in K \\ v_i & i \in V_1 \cup V_2 \backslash K \end{cases}$$

and

$$v^2_i = \begin{cases} 2v_i & i \in K \\ v_i & i \in V_1 \cup V_2 \backslash K, \end{cases}$$

while at the same time v^1,v^2eW\R(v), contrary to the assumption that v is extreme. Thus < T > = < SLir(S) > is connected.

Finally, we claim that $v_i = a$, ieT, for some constant a > 0. For suppose not; then again v = $\frac{1}{2}(v^1 + v^2)$, with v^1 and v^2 defined by

$$v^1_i = \begin{cases} \min\{v_j : j \in T\} & i \in T \\ 0 & i \in V_1 \cup V_2 \setminus T \end{cases}$$

and

$$v^2_i = \begin{cases} 2v_i - \min\{v_j : j \in T\} & i \in T \\ 0 & i \in V_1 \cup V_2 \setminus T \end{cases}$$

while $v^1, v^2 \in W \setminus R(v)$, contrary to the assumption that v is extreme. This proves that if $S \neq \emptyset$, then v is of the form (4.5). ||

Having described the extreme rays of W, we can now apply Theorem 4.1 to the system (4.3). The extreme direction vectors of the form (4.4) give rise to inequalities $x_i \geq 0$, $i \in V_2$, which are redundant (since they are part of the definition of Q). The extreme vectors of the form (4.5) give rise to an inequality $x(S) - x(F(S)) \leq 0$ for every $S \subseteq V_1$ such that $\langle S \cup T(S) \rangle$ is connected.

If G is connected, then the inequality $x(V_1) - x(V_2) \leq 0$, which can also be written as $x(V_1) - x(r(V_1)) \leq 0$, obtained from the extreme vector of W that corresponds to $S = V_1$, together with the inequality $-x(V_1) + x(V_2) \leq 0$ of (4.3), gives rise to the equation $x(V_1) - x(V_2) = 0$. If G is disconnected with components $\langle K_1 \rangle, \ldots, \langle K_t \rangle$, where $K_i = S_i \cup F(S_i)$, $i = 1, \ldots, t$, then the equation $x(V_1) - x(V_2) = 0$ is obtained by first adding the inequalities $x(S_i) - x(F(S_i)) \leq 0$, $i = 1, \ldots, t$, and then combining the resulting inequality, $x(V_1) - x(V_2) \leq 0$, with the inequality $-x(V_1) + x(V_2) \leq 0$ of (4.3).

Thus applying Theorem 4.1 to the system (4.3), we obtain the linear characterization of the RiS polytope of G given in Theorem 3.1, except for those inequalities (3.3) such that $\langle S \cup T(S) \rangle$ is disconnected, which are missing. But these inequalities are redundant, as shown in the remarks following Theorem 3.1, where the system (3.3) was replaced by (3.a').

5. A Third Derivation via Lattice Polyhedra

Lattice polyhedra were introduced by Hoffman and Schwartz [10] (see also [7], [8]) as a class of integer polyhedra that generalizes both matroid polyhedra and bipartite matching polyhedra. We will show that the PMS polytope of a bipartite graph can also be expressed in this form.

A lattice $\mathcal{L}$ is a partially ordered set closed under two associative and commutative binary operations, $\wedge$ and $\vee$, and such that

$$\text{for } a,b \in \mathcal{L},\ a \wedge b \leq a,b \leq a \vee b;$$
(5.1)
$$a \leq b \Rightarrow \quad a = a \wedge b,\ \ b = a \vee b.$$

To define a lattice polyhedron, we further need a set $\mathcal{U}$ and a mapping $f:\mathcal{L} \to 2^{\mathcal{U}}$ that satisfies for every W_1, W_2, $W_3 \in \mathcal{L}$,

(5.2) $\quad W_1 \leq W_2 \leq W_3$ implies $f(W_1) \cap f(W_3) \subseteq f(W_2)$

(5.3) $\quad f(W_1) \cap f(W_2) \subseteq f(W_1 \vee W_2) \cap f(W_1 \wedge W_2)$

(5.4) $\quad f(W_1) \cup f(W_2) \subseteq f(W_1 \vee W_2) \cup f(W_1 \wedge W_2)$.

and a submodular function $r:\mathcal{L} \to \mathbb{Z}_+$ (the set of nonnegative integers). The basic result on lattice polyhedra [10] can then be stated as follows.

Theorem 5.1. For any nonnegative integer $d \in \mathbb{R}^{|\mathcal{U}|}$, the convex polyhedron whose points are those $x \in \mathbb{R}^{|\mathcal{U}|}$ satisfying

(5.5) $\quad 0 \leq x \leq d$

and

(5.6) $\quad \Sigma(x_i : i \in f(W)) \leq r(W), \qquad \forall\, W \in \mathcal{L},$

has only integer vertices. Moreover, the linear system (5.5), (5.6) is totally dual integral.

To apply this theorem to our case, we let £ be the collection of all $W \subseteq V_2$ ordered by set inclusion, and we define the operations v and A to be U and 0 , respectively. Then sd is well known to be a lattice. We let U_m= V, the node set of G.

For Wejd we define f(W) » SUW, where S « T^OO is the maximal subset of V_1 such that F(S) $SW_{\#}$ Equivalently, S consists of all those nodes of V_1 adjacent only to nodes in W.

Now for $W_\pm$ e=£, i = 1,2,3, condition (5.2) requires that $W_1 \subset W_2 \subset W_3$ imply

$$(W_X \cup S_X) \cap (W_3 \cup S_3) \subset (W_2 \cup S_2),$$

where $S_\pm$ = r ^{1} ^) , i = 1,2,3. Since $W_1 C W_2 C W_3$ implies S ^ S ^ S ^ this condition is satisfied.

Further, for $W_\pm$ esd, i - 1,2, (5.3) requires that

$$(W_X \cup S_X) \cap (W_2 \cup S_2) c\wedge \cap W_2 \cup r^{"1}(W_1 \cap W_2),$$

where, again, $S_i = r^{"1}(W_i)$, i- 1,2. Since (^ U S ^ 0 (W_2 U S_2) - (W_x fl W_2) U (S_x 0 S_2), and since it is easily checked that S^^ H S_2 « T"^{1} (WJ^HWJ), this requirement is also satisfied.

Finally, for $W_\pm$ e=d, i = 1,2, (5.4) requires that .

$$W_1 \cup S_1 \cup W_2 \cup S_2 \subseteq W_1 \cup W_2 \cup r^{"1}(W_i \cup W_2)\,,$$

Since S_t U S_2 c r"ViUW$_2$), this condition is also satisfied.

Next, we have to choose a nonnegative integer function r on sd, that is submodular. For W esd, we define r(W) = \w\, which clearly satisfies this requirement (and is in fact modular).

We can now apply Theorem 5.1 to derive our linear characterization of the PMS polyhedron of a bipartite graph. To this end, we *set* $d_i = 1$, ieV, in (5.5), and use the above definitions to rewrite (5.6) as

(5.6') $x(r"^{1}(W)) + x(W) \leq |W|, \quad ¥ W \subseteq V_2.$

If we now complement the variables x_i, ieV_2, i.e., define new variables x^ = $x_£$, ie^, x^ = 1 - x^ isV_2, then the system (5.5), (5.6 ') becomes

(5.7) $0 \leq x_t^7 \leq 1,$ ieV

(5.8) $x'(F^{-1}(W)) - x'(W) \leq 0,$ f W £ V_2

and Theorem 5.1 asserts that the convex polytope P* defined by (5.7), (5.8) has integer vertices.

The linear system of Theorem 3.1 differs from the above in three respects. First, there is an inequality (5.8) for every WCV_2, not just those for which W * T(S) for some S£V_1. Suppose that *W £* T(S) for any SsVj and let *w'* = rOT"^{1}^)). Then tf'ctf and the inequality (5.8) for W^7 is $x^/(r"^1(W))$ - x'O*') $\leq$ 0, which together with (5.7) implies the inequality (5.8) for W. Hence all such inequalities can be dropped without affecting the integrality of the polytope.

Second, (5.8) does not contain the inequalities (3.3) corresponding to sets SCV_1 such that T(S) = T(T) for some proper superset TSV_1 of S. But if such T exists, then the graph < (K^S) U (^^(S)) > is disconnected, where < K > is the component of G containing S and T, and K_i = KflV$_i$, i = 1,2. This is so because the nodes in T\S ^ 0 are not adjacent to any node in K_2\F(S) = K_2r(T). As discussed in the remarks following Theorem 3.1, the inequalities (3.3) corresponding to such sets SCV_1 are redundant.

Third, the equation (3.2) is not present in the system (5.7), (5.8). This is a genuine difference between the two polytopes, P defined by the system (3.1)-(3.3), and P*- defined by (5.7), (5.8). However, the equation (3.2) defines a face of P*, and since the vertices of a face are vertices of the polyhedron, it follows that P also has integer vertices. This provides the third proof of the fact that P • conv *Z.*

6. Facets of the PMS Polytope

In this section we address the question as to which of the inequalities defining the IMS polytope of a bipartite graph are essential. This is obviously a matter of practical interest, as the number of inequalities in the system (3.3) is rather large.

The facets of a polyhedron P are its maximal (relative to inclusion) non-empty proper faces. If dim P is the dimension of P, then the dimension of a facet of P is dim P - 1. An inequality $c^*x \le a_o$ is called facet-inducing (for P), if it is satisfied by all x e P, and the polyhedron PO [xjax = a_o} is a facet of P, i.e., has dimension dim P - 1.

In the remarks following Theorem 3.1, we have pointed out that some of the inequalities defining the PMS polytope of G are redundant, and that the system (3.1), (3.2), (3.3) can in fact be replaced by the smaller system (3.1), (3.2) and (3.3') . In this section we show that most of the inequalities of the latter system are essential, i.e., facet-inducing.

First, we have to determine the dimension of our polytope. Let again

lVi

P denote the set of x e l[1] ' satisfying (3.1)-(3.3), shown in Theorem 3.1 to be the IMS polytope of G = O ^ U V ^ E) .

The equality set of the system (3.1)-(3.3) is the set of those members that are satisfied with equality by all x e P. A basis of the equality set is a maximal subset whose coefficient matrix is of full row rank.

We can now apply Theorem 5.1 to derive our linear characterization of the RiS polyhedron of a bipartite graph. To this end, we set $d_i = 1$, ieV, in (5.5), and use the above definitions to rewrite (5.6) as

(5.6') $x(r^{-1}(W)) + x(W) \leq |W|, \quad \forall W \subseteq V_2.$

If we now complement the variables x^ ieV"2, i.e., define new variables x_i' =* x£, icV^ x^ = 1 - x^ ieV2, then the system (5.5), (5.6') becomes

(5.7) $0 \leq$ x^ ≤ 1, ieV

(5.8) $x'(r^{-1}(W)) - x'(W) \leq 0$, ¥ W£V2

and Theorem 5.1 asserts that the convex polytope P* defined by (5.7), (5.8) has integer vertices.

The linear system of Theorem 3.1 differs from the above in three respects. First, there is an inequality (5.8) for every W£V2, not just those for which W » F(S) for some S*£V_1. Suppose that *W £* T(S) for any SSV1 and let *w'* = $r(r^{-1}(W))$. Then *w'dW* and the inequality (5.8) for W7 is $x'(r^{-1}(W))$ - x'Cw') $\leq$ 0, which together with (5.7) implies the inequality (5,8) for *W.* Hence all such inequalities can be dropped without affecting the integrality of the polytope.

Second, (5.8) does not contain the inequalities (3.3) corresponding to sets SCV-$_1$ such that F(S) = F(T) for some proper superset T£V_1 of S. But if such T exists, then the graph < (KAS) U (K^V^S)) > is disconnected, where < K > is the component of G containing S and T, and K_i = KPiV_i, i » 1,2. This is so because the nodes in T\S ^ 0 are not adjacent to any node in K_2\F(S) = KAr(T). As discussed in the remarks following Theorem 3.1, the inequalities (3.3) corresponding to such sets SCV_1 are redundant.

Third, the equation (3.2) is not present in the system (5.7), (5.8). This is a genuine difference between the two polytopes, P defined by the system (3.1)-(3.3), and P* defined by (5.7), (5.8). However, the equation (3.2) defines a face of P*, and since the vertices of a face are vertices of the polyhedron, it follows that P also has integer vertices. This provides the third proof of the fact that P = conv *Z.*

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The facets of a polyhedron P are its maximal (relative to inclusion) non-empty proper faces. If dim P is the dimension of P, then the dimension of a facet of P is dim P - 1. An inequality $ax \leq a_0$ is called facet-inducing (for P), if it is satisfied by all $x \in P$, and the polyhedron $P \cap \{x|ax = a_0\}$ is a facet of P, i.e., has dimension dim P - 1.

In the remarks following Theorem 3.1, we have pointed out that some of the inequalities defining the PMS polytope of G are redundant, and that the system (3.1), (3.2), (3.3) can in fact be replaced by the smaller system (3.1), (3.2) and (3.3'). In this section we show that most of the inequalities of the latter system are essential, i.e., facet-inducing.

First, we have to determine the dimension of our polytope. Let again P denote the set of $x \in R^{|V|}$ satisfying (3.1)-(3.3), shown in Theorem 3.1 to be the IMS polytope of $G = (V = \cup V_i, E)$.

The equality set of the system (3.1)-(3.3) is the set of those members that are satisfied with equality by all $x \in P$. A basis of the equality set is a maximal subset whose coefficient matrix is of full row rank.

For any graph G, we define $\tilde{G}$, the set of <u>adjacency vectors</u> of G, to be the set of all incidence vectors of pairs of nodes which are joined by an edge. Thus $\tilde{G}$ has as many elements as G has edges, and each $x \in \tilde{G}$ has exactly two components equal to 1 and all other components equal to 0. The following Lemma will be useful in the rest of this section.

<u>Lemma 6.1</u>. Let $\tilde{F}$ be the set of adjacency vectors of a forest $F = (V,E)$ with k components. Then $\tilde{F}$ is linearly independent, $|\tilde{F}| = |V| - k$, and every $x \in \tilde{F}$ satisfies $x(K_1) = x(K_2)$ for every component (tree) $< K >$ of F, where K_1 and K_2 are the parts of K.

<u>Proof</u>. Elementary.||

<u>Theorem 6.2</u>. Let $\mathcal{K}$ be the set of components of $G = (V_1 \cup V_2, E)$, and for every $< K > \in \mathcal{K}$, let $K_i = K \cap V_i$, $i = 1,2$. Then the system

$$(6.1) \qquad x(K_1) - x(K_2) = 0, \qquad \forall < K > \in \mathcal{K},$$

is a basis of the equality set of (3.1)-(3.3).

<u>Proof</u>. It is clear that the equations (6.1) are linearly independent and belong to the equality set of (3.1)-(3.3). Let F be an edge maximal spanning forest of G, and $\tilde{F}$ the set of its adjacency vectors. Since every pair of adjacent nodes is perfectly matchable, $\tilde{F} \subseteq \mathcal{X}$. By Lemma 6.1, $\tilde{F}$ is linearly independent and each $x \in \tilde{F}$ satisfies (6.1). Since $|\tilde{F}| = |V| - k$, where $k = |\mathcal{K}|$, no basis of the equality set can contain more than k equations. But k is the number of equations in (6.1), so (6.1) is a basis.||

<u>Corollary 6.3</u>. If $G = (V_1 \cup V_2, E)$ has k components, $\dim P = |V| - k$.

<u>Proof</u>. The dimension of a polyhedron in $\mathbb{R}^{|V|}$ is $|V|$ minus the rank of the equality set.||

We now turn to the identification of facet inducing inequalities. The following result will be of use in this task. We recall from section 2 the definitions of 57J(G) as the collection of perfectly matchable node sets of G, and *Z(G)* as the set of incidence vectors of such node sets.

<u>Theorem 6.4</u>. For any S S V^ the equality

(6.2) x(S) - x(r(S)) = 0

is satisfied by the incidence vectors of precisely those T eyTJ(G) such that

(6.3) (VjXS, T(S)) 0M = 0

for every perfect matching M of < T > .

<u>Proof</u>. Let x be the incidence vector of some T eJ7J(G) . Clearly, x satisfies (6.2) if and only if)S 0 T \ = JT(S) 0 T | . NOW if (6.3) holds for at least one perfect matching M of < T >, then M matches the nodes of S 0 T with those of F(S) PIT, hence x satisfies (6.2). On the other hand, if (6.3) is violated by some perfect matching M^7 of < T >, then M' matches the nodes of S O T with a proper subset of the nodes of T(S) PIT, hence |SP!TJ < |T(S) PIT| and (6.3) is violated by x. We conclude that (6.3) holds for at least one perfect matching of < T > if and only if it holds for all perfect matchings of < T >; and this is the case if and only if the incidence vector x of T satisfies (6.2).||

For any SCV-, let G_c denote the graph obtained from G by removing the edge set (Vj\3,r(S)), i.e., let

G_S = < S UT(S) > U < (Vj\S) U (V_2\T(S)) > .

Then Theorem 6.4 implies

<u>Corollary 6.5</u>. For any S£V_1,

$$\mathcal{Z}(G) \cap \{x \mid x(S) - x(r(S)) = 0\} - Z(G_S) .$$

Theorem 6.4 and Corollary 6.5 essentially say that for any SsV$_1$, the polyhedron [x e P|x(S) - x(r(S)) • 0} is itself a PMS polytope, namely the one for the subgraph G_c of G obtained by deleting the edges in (V!\S,r(S)).

We are now ready to state the main result on facets of conv *Z(G)*, i.e., of P.

~~Theorem 6.6~~. Let 0 *f* S ø Vl. Then the inequality

(6.4) x(S) - x(r(S)) ≤ 0

is facet inducing if and only if G$_S$ has exactly one more component'than G.

~~Proof~~. The inequality (6.4) is facet inducing, i.e., the set PH f>|x(S) - x(F(S)) « 0} is a facet of P, if and only if it has dimension d = dim P - 1. From Theorem 3.1 and Corollary 6.5,

Pn[x|x(S) - x(r(S)) « 0} -

= conv *Z(G)* (1 £x|x(S) - x(r(S)) = 0} = conv %(Gg).

From Corollary 6.3, dim P = |v| - k, and dim conv £(G_g) = |v| - kg, where k and k$_S$ denote the number of components of G and G_c, respectively. Thus (6.4) is facet inducing if and only if k$_S$ = k + l.||

At this point there is at least one feature of Theorem 6.6 that requires immediate comment. In the remarks following Theorem 3.1 we have stated that any inequality (6.4) such that < S U T(S) > is disconnected, is redundant; yet from Theorem 6.6, such an inequality may still be facet-inducing, provided that the graph Gg has exactly one more component than G, a condition that is not incompatible with < S U F(S) > being disconnected. So it seems that some facet inducing inequalities are redundant. This is indeed the case, due to the fact that dim P < |v|, i.e., that the equality set of the system (3.1)-(3.3) is nonempty. Every one of the equalities satisfied by all x e P can be added, after multiplication with some arbitrary constant, to any of the inequalities of (3.1)-(3.3), to yield another valid inequality. This way infinitely many inequalities may induce the same facet of P, whereas in any minimal linear system defining P, every facet of P is obviously represented by only one (facet inducing) inequality. Thus we have to address the question as to which among the facet inducing inequalities of (3.1)-(3.3) induce distinct facets.

Before answering this question, it will be useful to restate Theorem 6.6 in the following slightly different form.

Theorem 6.6'. The inequality (6.4), where $S \subseteq V_1$ is facet inducing if and only if G has a unique component $\langle K^* \rangle$ such that $0 \neq S^* \cap K_1^*$, where $S^* = S \cap K^*$ and $K_i^* = K^* \cap V_i$, $i = 1,2$, and the graphs $\langle S^* \cup T(S^*) \rangle$ and $\langle (K_1^* \setminus S^*) \cup (K_2^* \setminus T(S^*)) \rangle$ are connected.

This form of the theorem (which can easily be derived from the other one) implies that for all components $\langle K \rangle$ of G other than $\langle K^* \rangle$, either $S \cap K = 0$ or $S \cap K = V_i$.

Theorem 6.7. Facet inducing inequalities

$$(6.5) \qquad x(S) - x(r(S)) \leq 0$$

and

$$(6.6) \qquad x(T) - x(T(T)) \leq 0$$

induce the same facet of P if and only if G has a component $\langle K^* \rangle$ such that

$$(6.7) \qquad 0 \neq S \cap K^* = T \cap K^* \neq K^* \cap V_r$$

Proof. Since (6.5) and (6.6) are facet inducing, if G has a component $\langle K^* \rangle$ satisfying (6.7), then K^* is unique, and $x \in P$ satisfies (6.5) with equality if and only if it satisfies (6.6) with equality, i.e., the two inequalities induce the same facet.

Conversely, if no such $\langle K^* \rangle$ exists, then there exists $u \in V_1 \setminus S$, $v \in F(S)$, such that $(u,v) \in E$ and either $u,v \in T \cup F(T)$, or $u,v \notin T \cup F(T)$. Then the adjacency vector of $\{u,v\}$ satisfies (6.6) with equality, but (6.5) with strict inequality; i.e., the two inequalities induce different facets.||

We now turn to the inequalities (3.1).

<u>Theorem 6.8</u>. The inequality $x_v \geq 0$ is facet inducing if and only if v is not a cutnode or an isolated node of G.

<u>Proof</u>. If v is neither a cutnode nor an isolated node of G, there exists an edge-maximal spanning forest F of G in which v has degree 1. Then the set $\tilde{F}$ of adjacency vectors of F contains a unique $\hat{x}$ such that $\hat{x}_v = 1$. Therefore, using Lemma 6.1, $\tilde{F} \cup \{0\} \setminus \{\hat{x}\}$ is a set of dim P affinely independent members of P, all satisfying $x_v = 0$. Thus, denoting $Q = \{x \in P \mid x_v = 0\}$, we have dim Q $\geq$ dim P - 1. On the other hand, $\hat{x} \in P \setminus Q$, hence Q is a proper face of P; therefore dim Q = dim P - 1 and so $x_v \geq 0$ is facet inducing.

Conversely, if node v is isolated, then $x_v = 0$ for every $x \in P$, and thus $x_v \geq 0$ does not induce a proper face. If v is a cutnode, let L be the node set of a component created by deleting v, and let $L' = L \cup \{v\}$. Then every $x \in P$ such that $x_v = 0$ also satisfies $x(L \cap V_1) - x(L' \cap V_2) = 0$. But let $\hat{x}$ be the incidence vector of $\{v,w\}$ for any $w \in L$ adjacent to v. Then $\hat{x}$ satisfies $x(L \cap V_1) - x(L' \cap V_2) = 0$, but $\hat{x}_v \neq 0$. Hence the inequality $x_v \geq 0$ does not induce a maximal proper face of P.||

<u>Theorem 6.9</u>. Facet inducing inequalities $x_v \geq 0$ and $x(S) - x(F(S)) \leq 0$ define the same facet of P if and only if $\{v\} = K_1^* \setminus S$ and $F(v) \subset F(K_1^* \cap S)$, where K^* is the node set of the unique component of G satisfying

$\emptyset \neq S \cap K^* \subseteq K_1^*$ (= $K^* \cap V_1$).

<u>Proof</u>. If the conditions hold, then the inequality $x_v \geq 0$ can be obtained from $x(S) - x(F(S)) \leq 0$ by subtracting the equations $x(K_1) - x(K_2) = 0$, where $K_i = K \cap V_i$, $i = 1,2$, for all those components

$< K >$ of G such that $K_i \cap S \neq 0$. Therefore the two inequalities induce the same facet. The converse can be shown by an argument analogous to the one used to prove the necessity of Theorem 6.7, and the details are omitted.]|

<u>Theorem 6.10</u>. The inequality $x_v \leq 1$ is facet inducing if and only if v either has at least two neighbors, or belongs to a two node component of G. In the first case, no other inequality (3.1) or (3.3) induces the same facet. In the second case, only the inequality $x_u \leq 1$, where u is the other node of the component containing v, induces the same facet as $x_v \leq 1$.

<u>Proof</u>. Sufficiency. If v has two distinct neighbors, u and w, define $\bar{x}$ by

$$\bar{x}_i = \begin{cases} 2 & \text{if } i = v \\ 1 & \text{if } i = u \text{ or } w \\ 0 & \text{if } i \in V \setminus \{u,v,w\}. \end{cases}$$

Then $\bar{x} \notin P$, but $\bar{x}$ satisfies all the constraints (3.1)-(3.3) except for the inequality $x_v \leq 1$. Therefore this inequality is essential, hence facet inducing, and no other inequality of (3.1)-(3.3) induces the same facet.

If v belongs to a two node component, with u the other node, define $\hat{x}$ by

$$\hat{x}_i = \begin{cases} 2 & \text{if } i = u \text{ or } v \\ 0 & \text{if } i \in V \setminus \{u,v\}, \end{cases}$$

Then again $\hat{x} \notin P$, but $\hat{x}$ satisfies all the constraints of (3.1)-(3.3) except for $x_v \geq 0$ and $x_u \geq 0$. This shows that at least one of these two inequalities is essential. But the equation (6.1) for the component of G containing u and v gives $x_u = x_v$ for all $x \in P$; so $x \in P$ satisfies $x_u = 1$ if and only if it satisfies $x_v = 1$. Therefore $x_u \leq 1$ and $x_v \leq 1$ are bdth facet inducing, and they induce the same facet.

Necessity. If v is an isolated node, $x_v = 0$ for all $x \in P$ and the inequality $x_v \leq 1$ does not induce a nonempty face of P.

Suppose now that v has a single neighbor u, and u has a neighbor $w \neq v$. If $v \in V_1$, the inequality (3.3) for $S = \{v\}$ is $x_v - x_u \leq 0$, or $x_v \leq x_u$. If $v \in V_2$, this same inequality, though not part of (3.3), can be derived as the inequality (3.13) for $T = \{v\}$. Therefore every $x \in P$ that satisfies $x_v = 1$ also satisfies $x_u = 1$. But the converse is not true, since the adjacency vector $\bar{x}$ of $\{u,w\}$ belongs to P, while $\bar{x}_u = 1$, $\bar{x}_v = 0$. Therefore the inequality $x_v \leq 1$ does not induce a maximal proper face of P.||

From the last four theorems it follows that the set of constraints (3.1), (3.2), (3.3′) comes very close to, though is generally not exactly, a minimal linear system defining P. To make it minimal, one has to remove

- every inequality $x_v \geq 0$ such that v is either an isolated node or a cutnode;
- every inequality $x_v \leq 1$ such that v has less than two neighbors and does not belong to a two node component; and, finally,
- every inequality $x(S) - x(\Gamma(S)) \leq 0$ such that $|K_1^* \setminus S| = 1$ and $\Gamma(K_1^* \setminus S) \subseteq \Gamma(K_1^* \cap S)$, where K^* is the unique component of G such that $\emptyset \neq K^* \cap S \neq K_1^*$ $(= K^* \cap V_1)$.

This still leaves a large number of inequalities, that can be exponential in the size of G. The following example illustrates the contents of this section and also is a case where the minimal defining system for P is exponential.

Let G_n be the graph of Fig. 6.1, consisting of n pairs of nodes $\{u_i, v_i\}$, each pair joined by an edge, plus a node u_o adjacent to every v_i, $i = 1,\dots,n$, and a node v_o adjacent to every u_i, $i = 1,\dots,n$. Let $V_1 \equiv \{u_o, u_1,\dots,u_n\}$, $V_2 = \{v_o, v_1,\dots,v_n\}$.

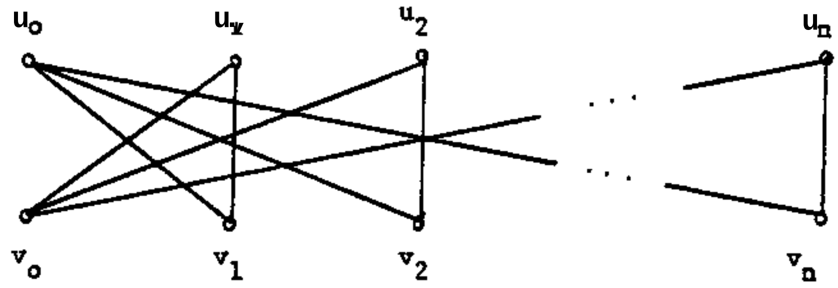

Fig. 6.1

Using the results of this section, we obtain the following minimal defining system for P:

(6.14) $$0 \le x_u \le 1, \quad u \in V_1$$
$$0 \le x_v \le 1, \quad v \in V_2$$

(6.15) $$x(V_1) - x(V_2) = 0$$

(6.16) $$x(S) - x(\Gamma(S)) \le 0, \quad \forall\, S: \emptyset \ne S \subsetneq V_1 \setminus \{u_0\}$$

(6.17) $$x_{u_0} - x(V_2 \setminus \{v_0\}) \le 0.$$

The inequalities (6.14) and the equation (6.15) are easily seen to be needed. For any nonempty $S \subseteq \{u_1, \dots, u_n\}$, $v_0 \in \Gamma(S)$, so $\langle S \cup \Gamma(S) \rangle$ is connected; and $u_0 \in V_1 \setminus S$, so $\langle (V_1 \setminus S) \cup (V_2 \setminus \Gamma(S)) \rangle$ is also connected; hence from Theorems 6.6 and 6.7, the inequalities (6.16) all induce distinct facets of P. Further, since $S \subsetneq V_1 \setminus \{u_0\}$ implies $|V_1 \setminus S| \ge 2$, the facets induced by these inequalities are also distinct from those induced by any of the inequalities (6.14) (Theorem 6.9). Finally, since $\langle \{u_0\} \cup (V_2 \setminus \{v_0\}) \rangle$ and $\langle (V_1 \setminus \{u_0\}) \cup \{v_0\} \rangle$ are connected, inequality (6.17) defines a facet of P, which is easily seen to be distinct from the facets induced by any of the other inequalities.

It remains to be shown that the omission of the remaining inequalities of (3.3) is justified. If S - ^v x ^ ^u o ^ then from Theorem 6.9 the inequality (3.3) induces the same facet as $x_{u_o} \geq 0$. Now let $u_o \in S$.

If $S = \{u_o\}$, we have the inequality (6.17). Now let $S \neq \{u_o\}$. Then T(S) « V_2, so < $(V_1 \setminus S) \cup (V_2 \setminus IXS)$ > is connected if and only if \V^\S\ ≤ 1. If |Vj\s| » 0, then S = V^ and the inequality (3.3) is implied by the equation (6.15). If $V_i \setminus S = \{u_i\}$ for some $i \in \{1,...,n\}$, then from Theorem 6.9 the inequality (3.3) induces the same facet as $x_{u_i} \geq 0$. This covers all the cases.

Notice that the number of inequalities (6.16) for G_n is $2^n - 2$, hence exponential in n.

Although the number of inequalities in our linear characterization of the IMS polytope of a bipartite graph may be large, this characterization is still computationally useful. Indeed, a linear program whose constraint set includes the system (3.1)-(3.3′) can be solved by generating the inequalities (3.3') as needed. However, the development of such a procedure goes beyond the scope of this paper.

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References

[1] J. F. Benders, "Partitioning Procedures for Solving Mixed-Variables Programming Problems." Numerische Mathematik, 4, 1962, 238-252,

[2] J. Edmonds, "Matroids and the Greedy Algorithm.11 Mathematical Programming 1. 1971, 127-136.

[3] J. Edmonds, "Matroid Intersection." Annals of Discrete Mathematics 4, 1979, 39-49.

[4] J. Edmonds and D. R. Fulkerson, "Transversals and Matroid Partition." J. Res. Nat. Bur. Stand., 69B, 1965, 147-153.

[5] J. Edmonds and R. Giles, "A Min-Max Relation for Submodular Functions on Graphs." Annals of Discrete Mathematics 1, 1977, 185-204.

[6] P. Hall, "On the Representation of Subsets." J. London Math. Soc., 10, 1935, 26-30.

[7] A. J. Hoffman, "On Lattice Polyhedra II: Construction and Examples." IBM Research Report RC 6268, 1976.

[8] A. J. Hoffman, "On Lattice Polyhedra III: Blockers and Antiblockers of Lattice Clusters." Mathematical Programming Study 8, 1978, 197-207.

[9] A.J. Hoffman, "A Generalization of Max Flow - Min Cut." Mathematical Programming, 6, 1974, 352-359.

[10] A. J. Hoffnan and D. C. Schwartz, "On Lattice Polyhedra," in A. Hajnal and Vera Sps (editors), Combinatorics (Proceedings of the 5th Hungarian Colloquium on Combinatorics, 1976). North-Holland, 1978, 593-598.

[11] D. KBnig, Theorie der endlichen und unendlichen Graphen. Akademische Verlagsgesiillschaft, Leipzig, 1936.

[12] A. Schrijver, "On Total Dual Integrality." Report AE15/80, Faculty of Actuarial Sciences and Econometrics, University of Amsterdam. To appear in Linear Algebra and its Applications.