Designing Near Optimal Trajectories of Structural Systems By Fourier-Based Methods

by

V. Yen and M.L. Nagurka

EDRC-24-09-88

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V. YEN

M.L. NAGURKA

Department of Mechanical Engineering
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

March 1988

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ABSTRACT

This report describes research aimed to develop a general analytical and numerical tool for designing the near optimal trajectories of lumped-parameter dynamical systems. The approach is based on a method which approximates each generalized coordinate of a dynamical system by the sum of a fifth-order polynomial and a finite term Fourier-type series. Optimality is achieved by adjusting the free boundary conditions and the coefficients of the Fourier-type series such that a performance index is minimized. The adjusted variables can be used to determine the near optimal trajectories of the generalized coordinates and the control variables.

Reported herein is a specialization of this approach for linear structural systems, represented by simultaneous second-order differential equations. In contrast to standard linear optimal control approaches which typically require the solution of differential Riccati equations, the method presented is a near optimal approach in which the necessary and sufficient condition of optimality is represented by simultaneous linear algebraic equations, which can be solved directly. Simulation results show that the approach is computationally efficient, numerically robust, and capable of solving for the optimal control of linear systems of high order andlor of linear systems with fixed (or highly penalized) terminal configuration variables.

Future work is planned to continue investigating this innovative idea of Fourier-based optimal control. In particular, it is proposed to (i) utilize quadratic programming techniques to extend the approach to problems with linear constraints on configuration andlor control variables, (ii) develop sensitivity functions to determine the influence of system parameters on the near optimal control, (Hi) generalize the approach to apply to general linear systems, represented by simultaneous first order differential equations (state equations), and (iv) solve for the near optimal trajectories of nonlinear dynamical systems via sequential linearization. The results of this project will lead to a powerful methodology for solving optimal control problems, applicable to high order and nonlinear problems, which have previously been difficult, if not impossible, to solve.

1 INTRODUCTION

The optimal control of general dynamical systems *(i.e.,* linear and nonlinear systems) is an important topic of modern control. Substantial theory for the optimal control of dynamical systems has been developed [Schultz and Melsa, 1967; Rosenbrock, 1970; Takahashi, *etal.,* 1970; Kwakernaak and Sivan, 1972; Owens, 1981; Patel and Munro, 1982; Friedland, 1986].

Problems of optimal control are typically solved using the classical results of the calculus of variations. Essentially, the variational formulation transforms the optimal control problem into a two-point boundary value problem (2PBVP) described by differential equations. The solution of these differential equations is typically complicated since the boundary conditions are most often specified at different ends *(i.e.,* state variables are known at the initial conditions and co-state variables are known at the terminal conditions). In general, 2PBVPs are extremely difficult to solve.

Various computational algorithms have been proposed to solve such 2PBVPs *(e.g.,* see [Keller, 1975; Pereyra, 1984] and the extensive bibliography in Appendix A of [Lee and Markus, 1986]). These algorithms include gradient-based methods such as steepest-descent and variation of extremal (Newton-type) techniques. For these methods, a termination criterion is usually found by trial-and-error and convergence very often depends on the initial guess. A more serious drawback of gradient methods is their sensitivity to computational errors, which often leads to their failure in solving high order optimal control problems.

Dynamic programming represents another class of methods. From a computational perspective, it is not well suited for solving optimal control problems since it is necessary to quantize the admissible state and control values into a finite number of levels. The optimal solution is then obtained by trying all combinations of admissible state and control values. The problem is that the storage requirements typically grow very fast, and can be prohibitively large, for large order systems *(i.e.,* where the order of the state and control is large). Bellman [1957], the father of dynamic programming, called this problem the "curse of dimensionality.[11] In addition, dynamic programming is not effective in handling problems with free terminal time.

In contrast to variational methods and dynamic programming, mathematical programming techniques represent a distinct approach toward the solution of (linear and nonlinear) optimal control problems. In general, these techniques convert an optimal control problem into an algebraic optimization problem. A survey of work done prior to 1970 can be found in [Tabak, 1970]. A more recent survey can be found in [Kraft, 1980]. Theoretical aspects of determining the optimal control via mathematical programming are also covered in [Canon, *et al,* 1970; Tabak and Kuo, 1971].

A direct application of mathematical programming is to discretize the state equations using a finite difference method. A linear or nonlinear programming algorithm can then be used to determine the values of state and control variables at every time interval such that a performance index is minimized. A difficulty with this approach is that the finite difference approximation leads to a system of algebraic equations which is typically of very large order. As a result, the optimization is computationally intensive and can pose serious problems in obtaining a realistic solution.

Modified approaches involving mathematical programming have been proposed. In [Hicks and Ray, 1971] and [Sirisena and Tan, 1974] the control variables are represented by the sum of known basis functions. Mathematical programming algorithms are then used to determine the optimal values of the coefficients of the basis functions that minimize the performance index. To evaluate the performance index, such control parameterization methods require the integration of the state equations which is usually time consuming and sensitive to numerical errors. Furthermore, constraints on terminal states *{e.g.,* fixed terminal conditions) are not easily satisfied.

Mathematical programming approaches based on state parameterization have been described [Johnson 1969; Nair, 1969; Yen and Nagurka, 1987a]. Here, state trajectory parameters of dynamical systems are adjusted by mathematical programming. For example, Yen and Nagurka [1987a] represent the time history of each generalized coordinate by an auxiliary polynomial and a finite-term Fourier-type series. The free variables, such as the (free) coefficients of the polynomial and the Fourier-type series, are adjusted by a nonlinear programming method such that the performance index is minimized. The effectiveness of this technique has been demonstrated by simulation studies [Nagurka, *et al.,* 1987; Yen and Nagurka, 1987a]. A challenge of state parameterization relates to the problem of trajectory inadmissibility, *i.e.,* due to constraints on the control structure an arbitrary representation of the state trajectory may not be achievable.

Finally, state and control parameterization approaches have been suggested. In [Vlassenbroeck and Van Doreen, 1988] the state and control variables are both expanded in Chebyshev series and an algorithm is provided for approximating the system dynamics, boundary conditions and performance index. Here the Chebyshev coefficients are the free variables of the algebraic optimization problem. One distinct advantage of this approach is that it can handle linear as well as nonlinear problems. Also, the authors report promising numerical experiences with the method. The drawback of this approach is the tedious analytical formulation required for different optimal control problems (including unconstrained problems). The development of a general computational tool based on this method is formidable.

Of the different mathematical programming approaches, state parameterization offers two major advantages. First, boundary condition requirements on state variables can be handled directly. Second, if the trajectory inadmissibility problem can be overcome, the state equations can be used as algebraic equations and the performance index can be evaluated efficiently. As a result, state parameterization promises significant computational advantages relative to other approaches.

In summary, the application of algorithms based on varational methods and/or dynamic programming is practical for few real systems *(e.g.,* low order and/or linear systems). Mathematical programming methods represent an alternative and promising strategy for determining the near optimal (or suboptimal) solution for general dynamical systems. However, in many cases the solution of optimal control problems involving high order, nonlinear systems remains a serious research challenge.

Significant attention has focused on methods for solving the "simpler" problem of linear optimal control (most probably a result of the difficulty in solving the nonlinear problem). Linear optimal control represents the basis for many important applications including optimal control of structures. For the special case of a linear dynamical system with a quadratic performance index the optimal control is generally found by the Hamilton-Jacobi approach. Mathematically, this approach is a variational method which generally requires the solution of a matrix differential Riccati equation with a terminal condition. Various algorithms have been proposed to solve this type of equation; an impressive reference list has been prepared by [Ramesh, *et aL,* 1987]. These algorithms usually suffer from computational "bottlenecks" which arise in solving for the optimal control of high order systems.

1.1 Scope

The research reported in this technical report specializes the Fourier-based state parameterization approach of [Yen and Nagurka, 1987a] to linear, time-invariant, structural systems with quadratic performance indices. The method exploits the linearity of the system model and the quadratic nature of the performance index to guarantee identification of a global minimum, while being computationally very efficient. The problem of trajectory inadmissibility is solved by the introduction of "artificial" control variables. The effectiveness of the approach is demonstrated via computer simulation studies. In these studies, the results of the Fourier-based approach achieve impressive accuracy in matching Riccati-based solutions with significantly less computational effort.

This report is organized as follows. The methodology of the Fourier-based optimal control approach is described in Section 2. In particular, Section 2.1 presents the application of the approach to actively controlled structural systems with quadratic performance indices, and represents a review of the methodology presented in [Yen and Nagurka, 1987b]; Section 2.2 generalizes the algorithm described in Section 2.1 to structural systems where the number of control variables is less than the number of degrees-of-freedom. Simulation results are presented at the end of both Sections 2.1 and 2.2. Section 3 discusses some important characteristics of the Fourier-based approach. Section 4 outlines the directions of future research, and conclusions are given in Section 5.

2 METHODOLOGY

The dynamical system considered here is a linear structure, whose behavior is governed by the system of differential equations:

$$\underline{M}\ddot{\underline{x}}(t) + \underline{C}\dot{\underline{x}}(t) + \underline{K}\underline{x}(t) = \underline{B}\underline{u}(t) \qquad (1)$$

with known initial conditions $\underline{x}(0) = \underline{x}_0$, $\dot{\underline{x}}(0) = \dot{\underline{x}}_0$, where $\underline{x}$ is an $N \times 1$ configuration vector (*i.e.,* a column vector of N configuration variables), $\underline{u}$ is an $L \times 1$ control vector, $\underline{M}$ is an $N \times N$ positive definite mass matrix, $\underline{C}$ is an $N \times N$ positive semidefinite structural damping matrix, $\underline{K}$ is an $N \times N$ positive semidefinite stiffness matrix, and $\underline{B}$ is an $N \times L$ control influence matrix.

In this report, it is assumed that $L \leq N$, *i.e.,* the number of control variables is less than or equal to the number of configuration variables. The derivation in Section 2.1 considers the case $L = N$, *i.e.,* the configuration and control vectors have the same dimension, and $\underline{B}$ is nonsingular. For this case, the structure is actively controlled. Section 2.2 addresses the case $L<N$.

The design goal is to find the optimal control $\underline{u}(t)$ in the time interval $[0, t_f]$ such that the quadratic performance index

$$J = \begin{bmatrix} \underline{x}(t_f) \\ \\ \dot{\underline{x}}(t_f) \end{bmatrix}^T \underline{H} \begin{bmatrix} \sim xU_f) \\ \\ iit_f) \end{bmatrix} + \int_0^{t_f} [\underline{x}^T \underline{Q}_1 \underline{x} + \dot{\underline{x}}^T \underline{Q}_2 \dot{\underline{x}} + \dot{\underline{x}}^T \underline{Q}_3 \underline{x} + \underline{u}^T \underline{R}\, \underline{u}]\, dt \qquad (2)$$

is minimized. Here, Q_v £>2, Q_v and // are real, positive, semidefinite matrices and $\underline{R}$ is a positive definite matrix. In addition, // and $\underline{R}$ are symmetric. (*T* denotes transpose.) It is assumed that the configuration and control vectors are not bounded, the terminal configuration $\underline{x}(t_f)$ and its rates $\dot{\underline{x}}(t_f)$ and $\ddot{\underline{x}}(t_f)$ are free, and the terminal time t_f is fixed. Problems with constrained boundary conditions are addressed in Section 3.1.

2.1 Actively Controlled Structural Systems

For actively controlled systems the number of control variables is equal to the number of degrees-of-freedom of the structure. In this case, the control influence matrix, 5, is a nonsingular square matrix.

2.1.1 Development

The basic idea of the Fourier-based optimal control approach is to approximate each configuration variable by the sum of an auxiliary polynomial and a finite term Fourier-type series, *i.e.*, for / = i, . . . , N_9

$$XiU) = d\hat{}t) + \sum_{k=1}^{K} 2j^a ik^{cos} \frac{}{t_f} + \sum_{k=1}^{w} Z,^b ik^{sln} \frac{}{t_f} \qquad (3)$$

where *K* is the number of terms included in the Fourier-type series and d_t is a fifth-order polynomial in time

$$d_i(t) = d_{i0} + d_n t + d_{i2}t^2 + d_{i3}t^3 + d_{i4}t^4 + d_{i5}t^5 \qquad (4)$$

The six coefficients of this auxiliary polynomial can be written in terms of (i) six boundary conditions, *i.e.*, initial conditions $x_t\{0)$, $\dot{x}JO)$, and $\ddot{x}JO)$, and terminal conditions $x\hat{}$) , $\dot{x}/tj)$, and $\ddot{x}/tj)$, and (ii) coefficients of the Fourier-type series. (Note that of the six boundary conditions only *xJO)* and $\dot{x}_t\{0)$ are assumed known.) Explicit expressions for these coefficients are given in Appendix 1.

Equation (3) can be rearranged and presented in the form

$$x_i(t) = p_i + \rho_1 \ddot{x}_{i0} + 9z^{x}if + 9z^{x}\dot{i}f + P4^{x}\ddot{i}f + \overset{K}{\underset{*=1}{X}}{}^{a*a}i* + \sum_{J[\{=]}^{K} \beta_k b_{ik} \qquad (5)$$

where $x_{I0} = x_{\{}(0)$, $x_{if} = x_{;}(t_f)$, and similarly for the corresponding time derivatives, and where

$$Pi = {}^{x}i0 + {}^{\dot{x}}i0t + I^{r}[-10x_{i0} - 6\dot{x}_{i0}t_f](t/t_f)^3 +$$

$$[15x_{i0} + S\dot{x}gotMt/t/ + [-6x_{i0} - 3\dot{x}_{iO}tMtlt_f)^5 \qquad (6)$$

$$pj = \overset{1}{\wedge}[(f/r/ - 3(t/t_fF + 3(f/r/ - (t/t_f)^5] \qquad (7)$$

$$p_2 = [100/r/ - 15(r/r/ + 6(f/r/] \qquad (8)$$

$$p_3 = r_/[-4(r/r_/)^3 + l(t/t_fY - 3(t/t_f)^5] \qquad (9)$$

$$p_4 = \frac{1}{2}lf_f[\{tlt_fy - 2(t/t_f)^4 + (t/t_f)^5] \qquad (10)$$

$$a_k = -1 + Akh\&Ht/t)^2 - 2(t/t)^3 + (t/t)^4] + cos\frac{}{t_f} \qquad (11)$$

$$S_v = 2kn[-it/t_f) + lO(t/t_f)^3 - 15(t/t_f)^4 + 6(t/t_f)^5] + \wedge m\frac{2k\pi t}{f} \qquad (12)$$

$$v$$

By differentiating equation (5), the configuration variable rates can be expressed as:

$$xff) = q_t + aJn + c_2x_{if} + a_3i_{/7} + a_4x_{if} + \sum_{k=1}^{K} Yy_k a_{ik} + \sum_{k=1}^{K} Y3_k b_{ik} \qquad (13)$$

and

$$xft) = r_t + \wedge\wedge + \$_2x_{if} + \wedge i \wedge + 4>_4j?_y + \sum_{k=1}^{\wedge} \varepsilon_k a_{ik} + \sum_{k=1}^{A:} \zeta_k b_{ik} \qquad (14)$$

where

$$<7_t = \dot{P}i \quad , \; r_t = \dot{p} \backslash \qquad (15a,b)$$

$$a, = \dot{p}! \quad ,<!>, = \ddot{p}! \qquad (16a,b)$$

$$«y_2 = \dot{P}2 \quad ' \; \wedge 2 = \dot{P}{*}2 \qquad (17a,b)$$

$$a_3 = \dot{p}_3 \quad , \; 03 = \ddot{p}_3 \qquad (18a,b)$$

$$o_4 = \dot{P}_4 \quad . \; 4>4 = \dot{P}{*}4 \qquad (19a,b)$$

$$y_k = \dot{a}_k \quad , \quad e_k = \ddot{a}_k \qquad (20a,b)$$

$$8^* = \dot{P}^* \;.\; C^* = \ddot{P}^* \qquad (21a,b)$$

The parameters defined in equations (7)-(12) and (16)-(21) are configuration independent and are functions of time only, since the terminal time t_f is assumed known. Furthermore, since the initial conditions x_{i0} and $\dot{x}_{i0}$ are given, p_i, q_i, and r_i are known functions of time.

From equation (5), the configuration variable $x_t\{t)$ can be written in compact form as

$$xff) = _{Pi} + 2^T li \qquad (22)$$

where

$$2^r = [\ Pi \quad P_2 \quad Ps \quad P_4 \quad {}^a i \ \bullet \bullet \bullet \ \% \quad Pi \ \bullet \bullet \bullet \ P^* \ ] \qquad (23)$$

and

$$li = [\ ** > \quad \wedge \quad *\dot{i}f \quad *\ddot{i}f \quad {}^a i\backslash \ \bullet \bullet \bullet \ ** \quad *ii \ \bullet \bullet \bullet \ {}^b iK \ \ F \qquad (24)$$

are vectors of dimension m = 4 + 2 AT. Similarly,

$$*,\bullet = Qi + \wedge^r Z- \qquad (25)$$

$$*f = \wedge + \wedge / \qquad (26)$$

where

$$\underline{o}^T = [\ a_x \quad a_2 \quad a_3 \quad a_4 \quad Y_2 \ . \ . \ . \ y_K \quad 8_X \ . \ . \ . \ 5\wedge \ ] \qquad (27)$$

$$\$^7 = [\ 4\triangleright i \quad \triangleleft 2 \quad \triangleleft 3 \quad \wedge 4 \quad \wedge \ . \ . \ . \ \% \quad Ci \ \bullet \bullet \bullet \ C\wedge \ 1 \qquad (28)$$

The configuration variables for the iV degrees of freedom can be written in terms of a configuration vector $\underline{x}(t)$, *i.e.*,

$$\underline{x}(t) = \pounds(r) + 2^* «Z \qquad (29)$$

where

$$\underline{x}(t) = [x_x(t) \quad x_2(t) \ . \ . \ . \ x_N(t)]^T \qquad (30)$$

$$\underline{p}(t) = [\wedge(0 \quad P_2(f) \ . \ . \ . \ p\wedge r)F \qquad (3D$$

$$\underline{y} = \begin{bmatrix} 11 \\ \underline{y}_2 \\ \\ \\ \underline{y}_N \end{bmatrix} \qquad (32)$$

and

$$\underline{\rho}^* = \begin{bmatrix} 2^r & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{p}^T & \cdots & \underline{0} \\ \\ \\ \underline{0} & \cdots & & \underline{\rho}^T \end{bmatrix} \qquad (33)$$

Note that % is a column vector of dimension *Nm* and that £* is a matrix of dimension *N x Nm.* Similarly, configuration rate vectors,

$$\dot{\underline{x}}(t) = qif) + \underline{o}^*(t)Y \qquad (34)$$

and

$$\ddot{\underline{x}}(t) = \underline{r}(t) + \underline{\phi}^*(t)\underline{y}, \qquad (35)$$

can be introduced, where *q_(t)* and *r(t)* are defined analogously to equation (31) and o* and <J* are defined analogously to equation (33).

Since *x(t), x(t)* , and *'x(t)* are known functions of *y_*, the control vector *u(t)* can be expressed from equation (1) as a function of *y_*. Ultimately, the interest is to express the performance index as a function of *y_*. Toward this end, the performance index of equation (2) is decomposed into two parts:

$$J = J_x + J_2 \qquad (36)$$

where J_x is the cost associated with the terminal configuration and its rate and J_2 is the cost associated with the trajectory. The terminal configuration and its rate can be written as a linear transformation of *y_*, *i.e.,*

$$\begin{bmatrix} - \\ \dot{x}(t_f) \end{bmatrix} = \underline{Z}\underline{y} \qquad (37)$$

where $\underline{Z}$ is a $2N x\, mN$ matrix with elements 1 and 0, specified according to

$$z_{ij} = \begin{cases} 1 \;, & j=(i-1)m+2 \quad for \quad i=1,\ldots,N \\ & j=(i-N-1)m+3 \quad for \quad i=N+1,\ldots,2N \\ 0 \;, & otherwise \end{cases} \qquad (38)$$

From equation (37), the cost J_x is

$$J_1 = \begin{bmatrix} \underline{x}(t_f) \\ \dot{\underline{x}}(t_f) \end{bmatrix}^T \underline{H} \begin{bmatrix} \underline{x}(t_f) \\ \dot{\underline{x}}(t_f) \end{bmatrix} = fZJEZ\_1 \qquad (39)$$

From equation (1) (for the control vector) and equations (29), (34), and (35) (for the configuration vector and its rates), the cost J_2 is

$$J_2 = \int_0^{t_f} [\underline{x}^T \underline{Q}_1 \underline{x} + \dot{\underline{x}}^T \underline{Q}_2 \dot{\underline{x}} + \dot{\underline{x}}^T \underline{Q}_3 \underline{x} + \underline{u}^T \underline{R} \underline{u}] dt \qquad (40)$$

$$= \int_0^{t_f} [\underline{y}^T \underline{\Lambda} \underline{y} + \underline{y}^T \underline{\Gamma} + \underline{\Omega}^T \underline{y} + \underline{\Sigma}] dt \qquad (41)$$

where

$$\underline{\Lambda} = \underline{\phi}^{*T} \underline{F}_1 \underline{\phi}^* + \underline{\sigma}^{*T} \underline{F}_2 \underline{\sigma}^* + \underline{\rho}^{*T} \underline{F}_3 \underline{\rho}^* + \underline{\phi}^{*T} \underline{F}_4 \underline{\sigma}^* + \underline{\phi}^{*T} \underline{F}_5 \underline{\rho}^* + \underline{\sigma}^{*T} \underline{F}_6 \underline{\rho}^* \qquad (42)$$

$$\underline{\Gamma} = \underline{\phi}^{*T} \underline{F}_1 \underline{r} + \underline{\sigma}^{*T} \underline{F}_2 \underline{q} + \underline{\rho}^{*T} \underline{F}_3 \underline{p} + \underline{\phi}^{*T} \underline{F}_4 \underline{q} + \underline{\phi}^{*T} \underline{F}_5 \underline{p} + \underline{\sigma}^{*T} \underline{F}_6 \underline{p} \qquad (43)$$

$$\underline{\Omega}^T = \underline{r}^T \underline{F}_1 \underline{\phi}^* + \underline{q}^T \underline{F}_2 \underline{\sigma}^* + \underline{p}^T \underline{F}_3 \underline{\rho}^* + \underline{r}^T \underline{F}_4 \underline{\sigma}^* + \underline{r}^T \underline{F}_5 \underline{\rho}^* + \underline{q}^T \underline{F}_6 \underline{\rho}^* \qquad (44)$$

$$\underline{\Sigma} = \underline{r}^T \underline{F}_1 \underline{r} + \underline{q}^T \underline{F}_2 \underline{q} + \underline{p}^T \underline{F}_3 \underline{p} + \underline{r}^T \underline{F}_4 \underline{q} + \underline{r}^T \underline{F}_5 \underline{p} + \underline{q}^T \underline{F}_6 \underline{p} \qquad (45)$$

$\underline{F}_1, \ldots, \underline{F}_6$ are constant matrices that depend on structural parameters and the performance index. If, for notational convenience, $\underline{S}_{jBV} = \underline{S}^{-1}$, then

$$\underline{F}_1 = MF\&\pounds B*,M. \qquad (46)$$

$$\underline{F}_2 = g\&RB\wedge C\_ + Q_2 \qquad (47)$$

$$\underline{F}_3 = \underline{K}^T \underline{B}_{inv}^T \underline{R} \underline{B}_{inv} \underline{K} + \underline{Q}_1 \qquad (48)$$

$$\underline{F}_4 = \underline{M}^T \underline{B}_{inv}^T (\underline{R}+\underline{R}^T) \underline{B}_{inv} \underline{C} = 2\underline{M}^T \underline{B}_{inv}^T \underline{R} \underline{B}_{inv} \underline{C} \qquad (49)$$

$$\underline{F}_5 = \underline{M}^T \underline{B}_{inv}^T (\underline{R}+\underline{R}^T) \underline{B}_{inv} \underline{K} = 2\underline{M}^T \underline{B}_{inv}^T \underline{R} \underline{B}_{inv} \underline{K} \qquad (50)$$

£5 = £^r2L≤*±*^r)*~to~£ + £~3~ = IQT∆R^K_ + & (51)

Since 2 is independent of time, equation (41) can be written as

'2 = /A*~Z~ + /r + Q*^r1 + r (52)

where

$$\underline{A}^* = f'A\underline{d}t \qquad (53)$$

$$\underline{\Gamma}^* = \int_0^{t_f} \underline{\Gamma} dt \qquad (54)$$

n*=JW (55)
0

$$\underline{\Sigma}^* = \int_0^{t_f} \underline{\Sigma} dt \qquad (56)$$

Since $J = J_X+J_2$ is quadratic in terms of £, the necessary and sufficient condition for global minimum /, determined from

dJ
ii -°- • (57)

is

[A* + _A*^T + l^HZ]^ = -r - Ω* (58)

Equation (58) represents a system of linear algebraic equations with the number of equations equal to the number of unknown variables, *i.e.*, the elements of %. It can be solved using any of a variety of linear equation solvers, such as Gaussian elimination routines. In solving this equation for £, the integrals of equations (53) - (55) must be evaluated. This can be done numerically or analytically. The integrals have been evaluated in closed-form. (See Appendix 2 for integral tables.) The fact that the integrals have been evaluated analytically makes the Fourier-based approach an integration-free method. As a result, the computational cost is independent of the length of time of the trajectory, making the approach substantially more efficient than standard approaches (except possibly for the case of exceedingly small time intervals.)

An important feature of equation (58) is that the coefficient matrix of £ is independent of initial conditions. The integrals A* are independent of initial conditions (whereas the integrals F* and Q* are functions of initial conditions, terminal time, and system parameters.) Thus, for the same optimal control problem with different initial conditions, the coefficient matrix remains the same; only the right-hand side constant vector needs to be recomputed. As a result, numerical algorithms such as LQ decomposition (and linear algebraic equation solvers based on matrix inversion) are particularly efficient for recalculation of y_ for different initial conditions.

2.1.2 Summary of Algorithm

The following methodology applies for the system of equation (1) (with £ as an *N x N* nonsingular matrix) and the performance index of equation (2).

Step 1: Select K_9 the number of terms to be included in the Fourier-type series. From equations (29), (34), and (35), set up the relationships between the configuration vector and its rates, $\underline{x}(t)$, $\dot{\underline{x}}(t)$, and $\ddot{\underline{x}}(t)$, and the unknown vector, % (consisting of the unknown boundary conditions and the coefficients of the Fourier-type series).

Step 2: Establish the necessary and sufficient condition for minimum /, as shown by equation (58). This involves evaluating the integrals of equations (53), (54), and (55). (The integral of equation (56) is only needed for the evaluation of the performance index.)

Step 3: Solve equation (58) for y_ using a linear equation solver. Determine $\underline{x}(t)$, $\dot{\underline{x}}(t)$, and $\ddot{\underline{x}}(t)$ from equations (29), (34), and (35), respectively, and $\underline{u}(t)$ from equation (1).

2.1.3 Examples

Example 1:

Problem Statement: Consider the linear, two degree-of-freedom, mechanical system shown in Figure 1. The displacements x_x and x_2 are measured with respect to the equilibrium positions of the masses. For this system, the equation of motion can be written in matrix form as:

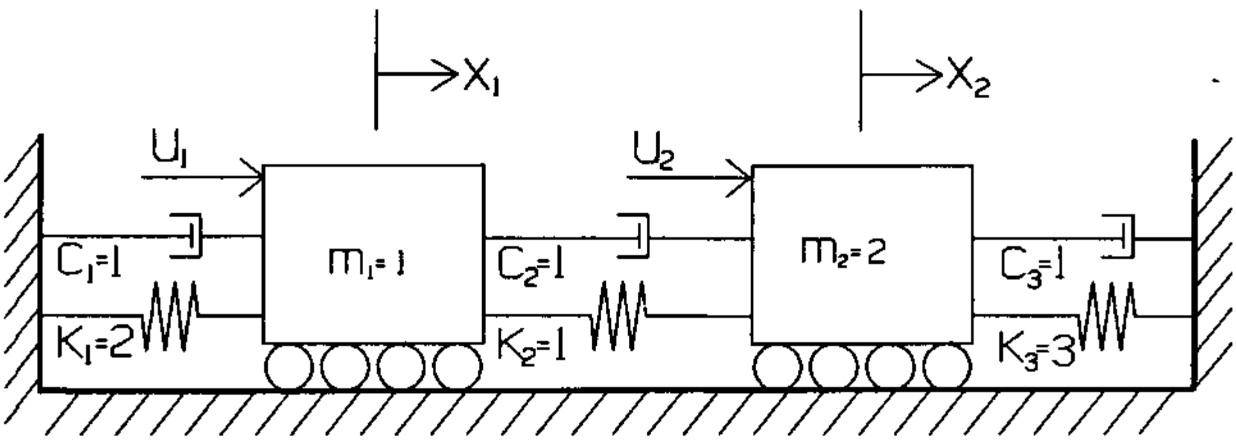

Figure 1. Two Degree-of-Freedom Mechanical System.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \ddot{\underline{x}} + \begin{bmatrix} 1 & \text{[illegible]} \\ 1 & \text{[illegible]} \end{bmatrix} \dot{\underline{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \underline{x} = \underline{u} \qquad \textbf{(59)}$$

with initial conditions $\underline{x}(0) = [\,1 \quad 1\,]^T$ and $\dot{\underline{x}}(0) = [\,1 \quad 1\,]^T$, where $\underline{x} = [\,x_1 \quad x_2\,]^T$. (It is assumed that the units for displacement $\underline{x}$, velocity $\dot{\underline{x}}$, acceleration $\ddot{\underline{x}}$, and control $\underline{u}$ are m, m/sec, m/sec^2, and N, respectively.)

The problem is to design the optimal control trajectory $\underline{u} = [\,u_1 \quad u_2\,]^T$ such that the **performance index:**

$$J = \begin{bmatrix} \underline{x}(1) \\ \dot{\underline{x}}(1) \end{bmatrix}^T \underline{H} \begin{bmatrix} \underline{x}(1) \\ \dot{\underline{x}}(1) \end{bmatrix} + \int_0^1 [\,\text{[illegible]} \; \dot{\underline{x}}^T \underline{Q}_2 \dot{\underline{x}} + \dot{\underline{x}}^T \underline{Q}_3 \underline{x} + \underline{u}^T \underline{R} \underline{u}\,]\,dt \qquad \textbf{(60)}$$

is minimized, where

$$\underline{H} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix} \qquad (61)$$

and

$$\underline{Q}_1 = \underline{Q}_2 = \underline{Q}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; \underline{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (62)$$

Riccati Solution: This problem can be solved by standard linear optimal control methods employing the Hamilton-Jacobi approach. A standard solution method is based on the Riccati equation for which a vector ($\underline{X}$) of state variables ($\underline{x}$ and $\dot{\underline{x}}$) is introduced. The equation of motion can be rewritten as:

$$\dot{\underline{X}} = \begin{bmatrix} \underline{0} & \underline{I} \\ -\underline{M}^{-1}\underline{K} & -\underline{M}^{-1}\underline{C} \end{bmatrix} \underline{X} + \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} \underline{u} = \underline{A}\underline{X} + \underline{B}\underline{u}, \quad \underline{X} = \begin{bmatrix} \underline{x} \\ \dot{\underline{x}} \end{bmatrix} \qquad (63)$$

and the performance index can be rewritten as:

$$J = \underline{\lambda}^T(t_f)\underline{H}\underline{\lambda}(t_f) + \int_0^{t_f}(\underline{\lambda}^T\underline{Q}\underline{\lambda}+\underline{u}^T\underline{R}\,\underline{u})dt \tag{64}$$

where

$$\underline{Q} = \begin{bmatrix} \underline{Q}_1 & \frac{1}{2}\underline{Q}_3 \\ \frac{1}{2}\underline{Q}_3 & \underline{Q}_2 \end{bmatrix} \tag{65}$$

The Riccati equation is a differential equation in terms of the $N \times N$ matrix $\underline{S}(t)$

$$\dot{\underline{S}}(t) = -\underline{S}(t)\underline{A} - \underline{A}^T\underline{S}(t) - \underline{Q} + \underline{S}(t)\underline{E}\,\underline{R}^{-1}\underline{E}^T\underline{S}(t) \tag{66}$$

$$\underline{S}(t_f) = \underline{H} \tag{67}$$

which is a system of $4N^2$ first order differential equations. After solving for $\underline{S}(t)$, the optimal control can be found from

$$\underline{u}(t) = -\underline{R}^{-1}\underline{E}^T\underline{S}(t)\underline{\lambda}(t) \tag{68}$$

Actually, it can be shown that $\underline{S}(t)$ is symmetric; hence, $N(2N+1)$, and not $4N^2$, first order differential equations must be solved.

Fourier-Based Solution: Alternatively, the problem can be solved using the proposed Fourier-based approach. The following steps of the algorithm are outlined.

Step 1: Set $K = 1$, *i.e.*, employ the crudest approximation involving a one-term Fourier-type series. Thus, the vector of free parameters

$$\underline{y} = [\ddot{x}_{10} \quad x_{1f} \quad \dot{x}_{1f} \quad \ddot{x}_{1f} \quad a_{11} \quad b_{11} \quad \ddot{x}_{20} \quad x_{2f} \quad \dot{x}_{2f} \quad \ddot{x}_{2f} \quad a_{21} \quad b_{21}]^T \tag{69}$$

contains twelve unknowns. Evaluate the known functions of time, *i.e.*,

$$\underline{R} = [\ p_1 \quad p_2 \quad p_3 \quad p_4 \quad \alpha_1 \quad \beta_1\ ]^T \tag{70}$$

and

$$\underline{Z}^T = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_1 & p_2 & p_3 & p_4 & \alpha_1 & \beta_1 \end{bmatrix} \tag{71}$$

where p_v p_v p_v p_2, p_3, p_4, a_v and $\$_x$ are determined from equations (6) - (12) with t_f = 1 *sec* and *K*= 1.

Step 2: In equation (58) *H* is given in equation (61). To evaluate A*, F\ and ft* , the constant matrices F_19 ... , *F*^ of equations (46) - (51) are first calculated, and then integral tables are used. (See Appendix 2.) In this example, A* is

A* =

0.0648	-0.7185	0.2783	-0.0008	-13286	0.6863	0.0075	-0.4178	0.1500	-0.0117	0.2298	0.1624
-0.0756	193649	-11.7253	0.7685	•163086	•6.7604	02727	8.0758	-4.0284	03845	-15.8425	-0.9889
-0.0515	-1.7968	6.3472	-0.4366	11.1558	3.0831	-0.1901	0.4240	2.0828	-02119	9.5107	02190
0.0289	0.1256	-0.1735	0.0848	-23119	-0.4465	0.0189	-0.3060	0.0552	0.0075	-0.7749	0.0844
-23119	14.4565	-33007	-13286	1413065	•10.1163	-0.7749	172106	-53057	02298	15.0259	-103180
0.4465	-6.7604	3.6200	-0.6863	10.1163	15.8826	-0.0844	-0.9889	0.8082	-0.1624	103180	0.9013
0.0107	-0.7035	0.2924	-0.0240	03788	02894	03298	-22022	0.8169	0.0214	-6.7566	2.4618
0.5108	9.9329	•6.7665	0.6702	-28.7150	-1.4086	-0.9403	623238	•38.8952	22938	-323242	-25.7965
-03516	03050	33400	-0.3718	172050	02256	0.1625	-153000	22.0508	-13829	27.1773	123218
0.0344	-03441	0.1484	0.0107	-1J558	0.1797	0.0806	1.0319	-0.8587	03298	-8.7126	-1.9857
-13558	30.0831	•11.1680	03788	21.3476	-19.9299	-8.7126	283620	-13557	-6.7566	547.6921	-20.1317
-0.1797	-1.40*6	1.2212	4.2S94	19.9299	urn	1.9t57	-25.7965	133696	-2.4618	20.1317	i23728

(72)

and the sum (F* + Q*) is

$$(T+Q^*) = \begin{bmatrix} 1.4082 \\ -68.6030 \\ 28.9642 \\ -1.5542 \\ 5.9907 \\ 24.0617 \\ 6.2483 \\ -192.5749 \\ 87.6462 \\ -3.8273 \\ -52.7877 \\ 80.7157 \end{bmatrix} \qquad (73)$$

From equation (38),

$$
Z = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad (74)
$$

Step 3>: **The system of linear algebraic equations (58) is solved for** $\underline{y}$

$$
1 = \begin{bmatrix} -8.850858 \\ 1.798765\text{X}10\text{-}^{1} \\ -7.128376\text{xl0-}^{2} \\ 7.458418 \\ \text{-5.219254x10-*} \\ \text{-4.694757xl(H} \\ -7.384817 \\ 4.287209\text{X}10\text{"}^{1} \\ -1.881872\text{X}10\text{"}^{1} \\ 4.281012 \\ 3.608875\text{xl0"}^{5} \\ \text{2.970233xl(H} \end{bmatrix} \qquad (75)
$$

From £, the configuration vector, its rates, and the control vector can be determined.

Comparison: **For this example, the Riccati equations were solved using a fourth-order Runge-Kutta method with a time step of 0.01** ***sec.*** **Running in Turbo Pascal (Version 3.02A) on an IBM PC/XT with an 8087 co-processor, the computational time was 76** ***sec.*** **(The symmetry mentioned above was not exploited.) In comparison, the Fourier-based approach required less than 3** ***sec*** **to establish and solve the linear algebraic equations for the vector of free variables,** ***y_,*** **using a Gauss-Jordan routine.**

The time responses of the control variables, u_x **and** M_2**, and the displacements,** x_x **and** x_2**, are shown in Figures 2 and 3, respectively. The results show that the Fourier-based optimal trajectories determined using only a one-term Fourier-type series agree quite well with the optimal trajectories from the Riccati solution.**

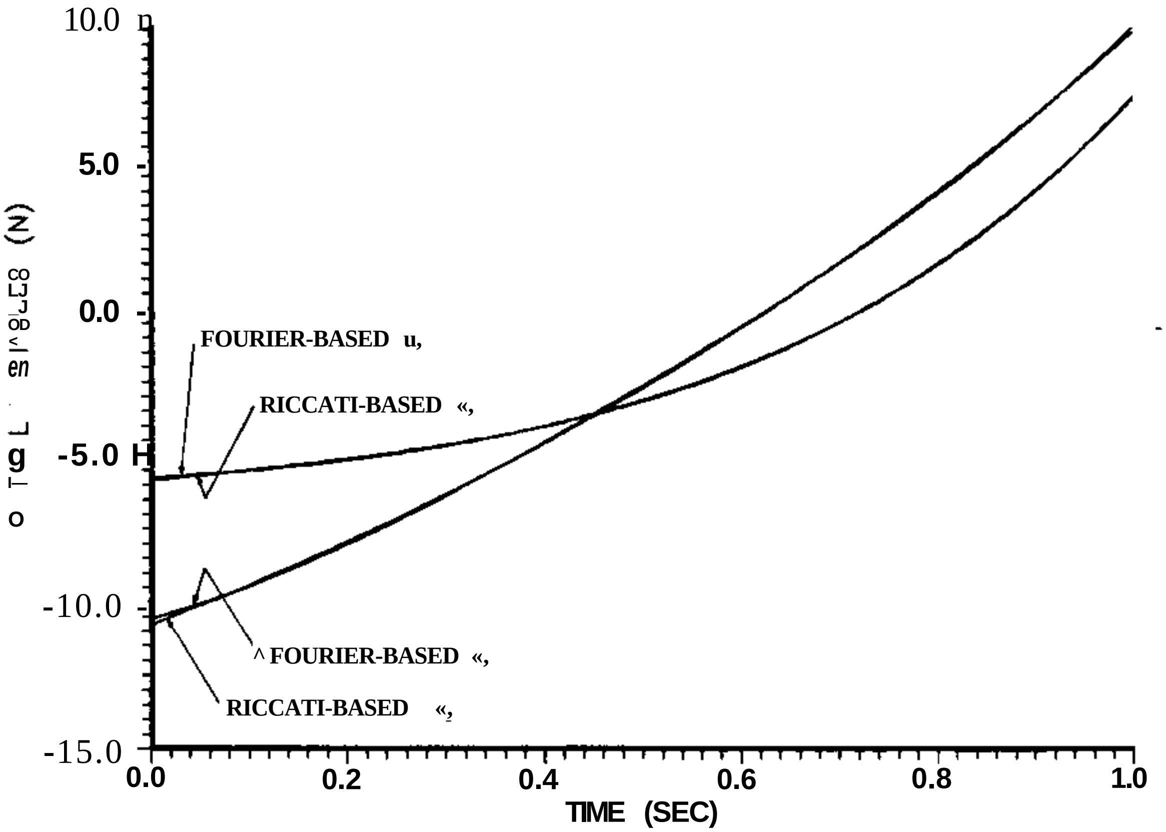

Figure 2. Riccati-Based and Fourier-Based Optimal Solutions of the Control Variables for Example 1.

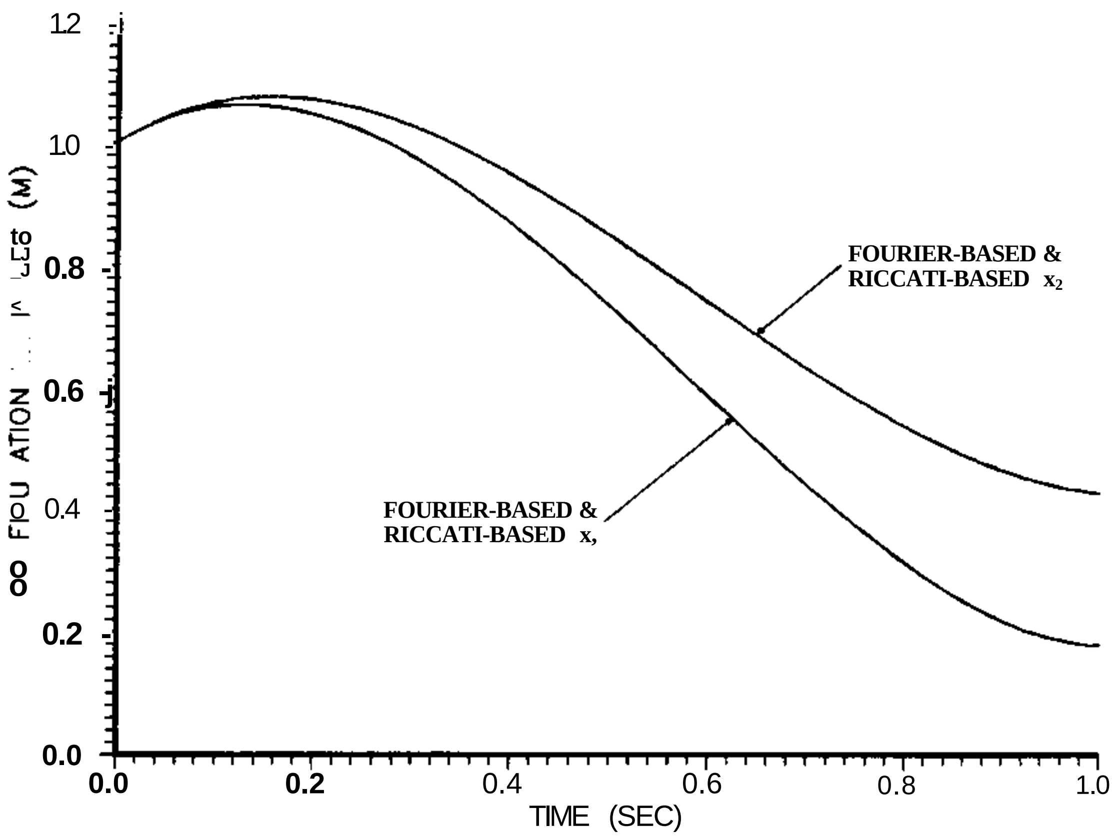

Figure 3. Riccati-Based and Fourier-Based Optimal Solutions of the Configuration Variables for Example 1.

Example 2:

This example considers the same problem as that of Example 1 except that matrix f£ in equation (61) has only one nonzero element, *i.e.,* H_n. The goal here is to examine the numerical robustness of the Riccati and Fourier-based methods when high penality exists on the terminal configuration. In particular, by specifying H_n as a very large positive constant, the terminal configuration ATJCI) is heavily penalized and should be driven to zero.

Simulation studies based on the Riccati and Fourier-based approaches show that the terminal configuration $x_x(l)$ becomes small as H_n becomes large. The results are summarized in Table 1, along with the resulting value of the performance index for each weighting.

It is found that for an H_n of 10^3 and 10^4 the results are in agreement when (i) solving the Riccati equation by fourth-order Runge-Kutta integration with 100 time steps, and (ii) using the Fourier-based approach (with a two-term Fourier-type series, *i.e.,* $K = 2$). For an H_n of 10^5, the number of integration time steps in solving the Riccati equation must be increased to 200 to achieve comparable accuracy. This phenomenon is again demonstrated when $H_n = 10^6$, *i.e.,* the number of time steps must be further increased to achieve similar accuracy with the Fourier-based result. In fact, 100 time steps is insufficient to achieve a numerically stable integration for this high value of weighting. In contrast, the Fourier-based approach provides accurate results for each weighting, and does not require additional computation.

The robustness of the Fourier-based approach was tested further using extremely large values of H_n. At an H_n of 10^{30}, the Fourier-based approach succeeded in predicting a very small terminal configuration, *i.e.,* an $x_x(tj)$ of $1O^{\sim 29}$ (and a performance index of 13.81355). In summary, the results indicate that the Fourier-based approach can be applied to systems with highly penalized terminal states without sacrificing the method's computational simplicity.

2.2 General Structural Systems

The Fourier-based approach is based on the parameterization of the trajectories of the generalized coordinates. The parameters, i.e., the elements of y_, include the free boundary conditions and the coefficients of the Fourier-type series. The optimal control is found by solving for the values of these parameters that minimize a performance index. (The parameterization of the trajectory is an important feature of the Fourier-based method, as described later in Section 3.)

Table 1: Results of Riccati and Fourier-Based Simulations for Different Weighting Constants for Example 2

Weighting	Steps*	$x_1(t_f)$		/	
		Riccati	Fourier	Riccati	Fourier
10^3	100	8.205xl0"3	8.205xl0"3	13.74561	13.74561
10^4	100	8.309xl0-«	8.273xl(H	13.80668	13.8067
	200	8.275xl(H		13.80670	
10^5	100	1.220x10-*	8.280xl0-s	13.82585	13.81287
	200	8^43xlO-s		13.81290	
10*	100	Unstable	8280xl0-«	Unstable	13.81349
	140	2.879x10-*		30.30681	
	200	3.746xlO-J		13.98985	
	300	1.367xlO-J		13.81685	

* Steps = Number of Time Steps Used in Fourth-Order Runge-Kutta Integration of Riccati Equation

The resulting "optimal" trajectories must be admissible, which may not be achievable for some passive dynamical systems (because of limitations on the control.) For instance, consider an arbitrary trajectory of $\underline{x}$, $\dot{\underline{x}}$ and $\ddot{\underline{x}}$. From equation (1) the corresponding control $\underline{w}$, based on an inverse dynamic approach, can be determined only if the control influence matrix, $\underline{B}$, is invertible. This implies that the structure must be actively controlled, *i.e.*, the number of degrees-of-freedom of the structure must be equal to (or greater than) the number of control variables, and the control influence matrix must be invertible. As a result, the approach presented in the previous section is only feasible for actively controlled structural systems.

This section generalizes the Fourier-based approach to the more common case of non-actively controlled structural systems which have a greater number of degrees-of-freedom than the number of control variables.

2.2.1 Development

The dynamical system of interest is again the linear structure described by equation (1). In this case, the control influence matrix, $\underline{B}$, is an $N \times L$ matrix where the number of configuration variables, N, is greater than the number of control variables, L.

Equation (1) can be written as

$$\underline{M}\ddot{\underline{x}}(t) + \underline{C}\dot{\underline{x}}(t) + \underline{K}\underline{x}(t) = \underline{B}'\underline{u}'(t) \qquad (76)$$

where

$$\underline{B}' = \underline{B}'_{(N \times N)} = \begin{bmatrix} \underline{I} & | \\ & | \underline{B}_{(N \times L)} \\ \underline{0}_{(L \times I)} & | \end{bmatrix} \qquad \textbf{(77)}$$

and

$$\underline{u}' = \underline{u}'_{(N \times 1)} = \begin{bmatrix} \underline{u}^*_{(I \times 1)} \\ \\ \underline{u}_{(L \times 1)} \end{bmatrix} \qquad (78)$$

where $I = N - L$ and the subscripts in the parentheses represent the dimensions of the matrices. By introducing the artificial control vector, $\underline{u}^*$, the new control influence matrix, $\underline{B}'$, can be inverted enabling the calculation of the control, $\underline{u}'$, for any given trajectory, *i.e.*, premultiplying equation (76) by $(\underline{B}')^{-1}$ gives

$$\underline{M}'\ddot{\underline{x}}(t) \;+\; \underline{C}'\dot{\underline{x}}(t) \;+\; \underline{K}'\underline{x}(t) \;=\; \underline{u}'(t) \tag{79}$$

where

$$\text{Af} = (\underline{B}')^{-1}\underline{M} \tag{80}$$

$$\underline{a} = (\underline{B}')^{-1}\underline{C} \tag{81}$$

$$\_ \quad \_ \quad \_ \tag{82}$$

The new constant coefficient matrices can be partitioned, as follows.

$$\underline{M}' = \begin{bmatrix} M^{*}_{\cdot(IxN)} \\ \underline{M}^{**}_{(LxN)} \end{bmatrix} \tag{83}$$

$$\underline{C} = \begin{bmatrix} \overset{*}{Q}(IxN) \\ \overset{**}{Q}(LxN) \end{bmatrix} \tag{84}$$

$$\underline{K}' = \begin{bmatrix} \underline{K}^{*}_{(IxN)} \\ \pounds L\{LxN\} \end{bmatrix} \tag{85}$$

From equation (79) the artificial control vector $\underline{u}^*$ can thus be written as

$$M^{*}\ddot{\underline{x}}\{t) \;+\; \underline{C}\dot{\underline{x}}\{t) \;+\; \underline{r}^{*}(r) \;=\; \underline{u}^{*}(t) \tag{86}$$

In actuality, the artificial control does not exist, and thus it is required that

$$\underline{M}^{*}\ddot{\underline{x}}(t) \;+\; \underline{C}^{*}\dot{\underline{x}}(t) \;+\; \underline{l?}\underline{x}(t) \;=\; \underline{0} \tag{87}$$

This indicates that only trajectories satisfying equation (87) are admissible for dynamical systems described by equation (1). In other words, the *N* degree-of-freedom system possesses *L* "active" degrees-of-freedom. Given trajectories of any *L* of the *N* generalized coordinates, the trajectories of the remaining degrees-of-freedom can be determined uniquely from equation (87), with all trajectories being admissible.

Equation (87) can be viewed as describing linear coupling between degrees-of-freedom. The trajectories of / of the N generalized coordinates can always be planned such that the artificial control variables can be made to vanish, regardless of the trajectories of the remaining generalized coordinates. Consequently, the admissibility of the trajectories is guaranteed.

In the Fourier-based method, the generalized coordinate trajectory is approximated by a finite-term series. As a result, equation (87) will not, in general, be satisfied exactly. However, by minimizing, in a least squares sense, the contribution of the artificial control variables, an equation similar to equation (87) describing linear coupling can be derived, as developed below. Thus, there are two simultaneous objectives. One objective is to generate the near optimal trajectories; the second objective is to minimize, in the least squares sense, the contribution of the artificial control variables.

A performance index, /*, is proposed to represent the contribution of the artificial control variables.

$$J^* = \int_0^{t_f} \pounds (\gg \dot{i})^2 dt \qquad \textbf{(88)}$$

There are N vectors representing the free Fourier-based variables, $\%_v$ }>2, ... $_9y_N$. *I of* these N vectors, *i.e.*, z_{l9} &,…, %, are adjusted in such a way that J^* is minimized. Setting the first derivative of the performance index equal to zero

$$\frac{dJ^*}{d\underline{y}_i} = \underline{0} \qquad for \quad i = 1,2,\ldots,/ \qquad (89)$$

gives a set of *Im* equations in the *Nm* unknowns (j^, $\%_2$,…, *Y^* where each % has *m* elements, since J^* is a function of }\)

Equation (89) can be written as

$$YA = Q\text{-}IYB + Q.2 \qquad \textbf{(90)}$$

where ^*A* anc* *Is* are partitioned vectors of $\underline{y}$ according to

$$\mathbf{1} = \begin{bmatrix} 1A \\ 1B \end{bmatrix} \quad (91)$$

with

$$1A = \begin{bmatrix} li \\ 12 \\ \\ \underline{y}_I \end{bmatrix} \quad (92)$$

and

$$Is = \begin{bmatrix} \underline{y}_{I+1} \\ \underline{y}_{I+2} \\ \\ \underline{y}_N \end{bmatrix} \quad (93)$$

Equation (90) represents the coupling between the generalized coordinates that minimizes the effect of the artificial control variables on the trajectories.

The performance index, $J = J_l + J_2$, can be written in terms of ^ according to equations (39) and (52). In view of equation (91),

$$J = /(\underline{v}) = J(\underline{y}_A, \ \underline{y}_B) \quad (94)$$

which, from equation (90), can be written as

$$J = J(\underline{y}_B) \quad (95)$$

The necessary and sufficient condition of optimality can then be expressed as

$$\frac{<''}{d\underline{y}_B} = \underline{0} \ , \quad (96)$$

which represents Lm algebraic equations that can be solved to determine ^ . The remaining Fourier-based variables, $\underline{y}$^ , can then be computed from equation (90).

2.2.2 Summary of Algorithms

Two algorithms are described below. Both algorithms apply for the system represented by equation (1) with 5 as an $N \times L$ matrix with $N > L$.

Algorithm 1

The first algorithm is used to derive the linear coupling between the Fourier-based variables, $y_{Jt9}k = 1, 2, \ldots, N_9$ *i.e.*, the algorithm is designed to derive equation (90) which is needed to minimize the influence of the artificial control variables.

Step 1: Rewrite the governing differential equation as equation (76) by introducing an $/ \times 1$ artificial control vector, $\underline{w}^*$.

Step 2: Define a performance index /*, as shown in equation (88), representing the contribution of the artificial control variables. This performance index can be obtained from equation (2) by making //, Q_v £2, and Q_3 null matrices and making the only nonzero elements of $\underline{R}$ the first / diagonal elements with these elements equal to 1.

Step 3: Carry out Steps 1 and 2 of the algorithm in Section 2.1.2 to establish the necessary condition for minimum J^*. Here, in contrast to equation (58), the necessary condition can be written as:

$$\underline{\Lambda}_* \underline{y} = \underline{L}^* \tag{97}$$

where

$$\underline{A}^* = \begin{bmatrix} A^*A \ (Imxlm) & A^*B \ (ImxLm) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{bmatrix} \tag{98}$$

$$\underline{\Gamma}_* = \begin{bmatrix} \Gamma\text{-}^*A \ (Fmxl) \\ \vdots \end{bmatrix} \tag{99}$$

and £ is given by equation (91). (The dots in the matrices of equations (98) and (99) represent elements which are not used for this derivation.)

Step 4: Since only the first *Im* equations are used (from equation (89)), equation (97) is equivalent to

$$\underline{\Lambda}_{*A}\underline{y}_A + \underline{\Lambda}_{*B}\underline{y}_B = \underline{I}_M \quad (100)$$

which can be expressed in the form of equation (90) with

$$\underline{D}_x = -\underline{K}\backslash \quad \underline{K}_B \quad (101)$$

$$£2 = + A\overset{-1}{M} \quad I_M \quad (102)$$

Equation (90) guarantees the minimization of the influence of the artificial control variables on the trajectory, for any given *Ys* •

Algorithm 2

The second algorithm is used to find the (near) optimal trajectory for the optimal control problem defined by equation (1) with performance index (2).

Step 1: Carry out Step 1 of the previous algorithm.

Step 2: Carry out Step 1 of the algorithm presented in Section 2.1.2.

Step 3: Express the performance index as the sum of equations (39) and (52) such that

$$/ = J_x + J_2 = /(\underline{A}* + \_Z^T\underline{HZ})Y + fT + Q*^T2 + 5* \quad (103)$$

Step 4: From equations (90) and (91), equation (103) can be written (see Appendix 3) as

$$J = :£\underline{A}"2* + aj\underline{r}* + \underline{r}* \quad (104)$$

where

$$\underline{\Lambda}^{**} = \underline{D}_1^T(\underline{\Lambda}_{11}\underline{D}_1+\underline{\Lambda}_{12}) + \underline{\Lambda}_{21}\underline{D}_1 + \underline{\Lambda}_{22} \tag{105}$$

$$\underline{\Gamma}^{**} = [\underline{D}_1^T(\underline{\Lambda}_{11}+\underline{\Lambda}_{11}^T) + \underline{\Lambda}_{12} + \underline{\Lambda}_{21}]\underline{D}_2 + \underline{D}_1^T\underline{E}_1 + \underline{E}_2 \tag{106}$$

$$\underline{\Sigma}^{**} = \underline{D}_2^T\underline{\Lambda}_{11}\underline{D}_2 + \underline{D}_2^T\underline{E}_1 + \underline{\Sigma}^* \tag{107}$$

where the partitioned matrices are defined from

$$\underline{\Lambda}^* + \underline{Z}_J^T\underline{H}\underline{Z} = \begin{bmatrix} \underline{\Lambda}_{11}\ (lm\times lm) & \underline{\Lambda}_{12}\ (lm\times Ln) \\ \underline{\Lambda}_{21}\ (Lm\times lm) & \underline{\Lambda}_{22}\ (Lm\times Lm) \end{bmatrix} \tag{108}$$

$$\underline{\Gamma}^* + \underline{Q}^* = \underline{E} = \begin{bmatrix} \underline{E}_1\ (lm\times 1) \\ \underline{E}_2\ (Lm\times 1) \end{bmatrix} \tag{109}$$

Step 5: Vector $\underline{y}_B$ can be computed from

$$\frac{dJ}{d\underline{y}_B} = (\underline{\Lambda}^{**}+\underline{\Lambda}^{**T})\underline{y}_B + \underline{\Gamma}^* = \underline{0} \tag{110}$$

to minimize the given performance index. Vector $\underline{y}$ can then be computed from equation (90), using the previous algorithm.

2.2.3 Example

Example 3:

The equation of motion of this example problem is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\ddot{\underline{x}} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\dot{\underline{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \tag{111}$$

The parameters are identical to those of the system described in Example 1 by equation (59). The initial conditions and performance index are also the same. Here, however, there is only one control variable. The problem is to "design" the trajectory u such that the performance index is minimized.

To apply the Fourier-based approach, the equations of motion are modified by introducing an artificial control variable, u^*, such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \ddot{\underline{x}} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dot{\underline{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ u \end{bmatrix} \qquad (112)$$

Optimal trajectories were generated using the algorithms presented at the end of the previous section. The time history of the artificial control variable, u^*, is plotted in Figure 4. Its magnitude is small, and always remains less than 2 x 10^{-4} N. The time history of the control variable, u , is shown in Figure 5. The curves, representing the solution from integrating the Riccati equation and from using the Fourier-based approach, are essentially indistinguishable. The corresponding time histories of the configuration variables are shown in Figure 6. Again, the Riccati-based and Fourier-based solutions appear identical.

3 DISCUSSION

The approach described in this report applies to unconstrained linear optimal control problems with quadratic performance indices. It is applicable to high order systems and to systems with highly penalized terminal configuration variables (and rates). Unlike variational approaches, the approach does not require integration of differential equations. The near optimal solution is obtained by solving a system of linear algebraic equations for free, time-independent parameters. As a result, the approach is integration-free, and thus is computationally very efficient. Ultimately, it is hoped that such an approach can be used for real (or near real) time optimal control of dynamical systems.

3.1 Treatment of Boundary Conditions

By modifying equations (22), (25), and (26), it is possible to apply the method to systems with fixed terminal conditions. For example, if the terminal configuration variable x_{if} is given, then the term $p^\wedge c_{if}$ is known, and equation (22) can be written as:

Xff) = [A-+P2*jf] + tPl P3 P4 <*1 • • • % Pi • • • Ptf]

[Jc,$_0$ $\dot{x}_{if}$ $\ddot{x}_{if}$ a_{iX} . . . a_{iK} b_{iX} . . . b_x7 **(113)**

Note that p_2 and x_{if} have been removed from equations (23) and (24), respectively, since x_{if} is no longer a free variable.

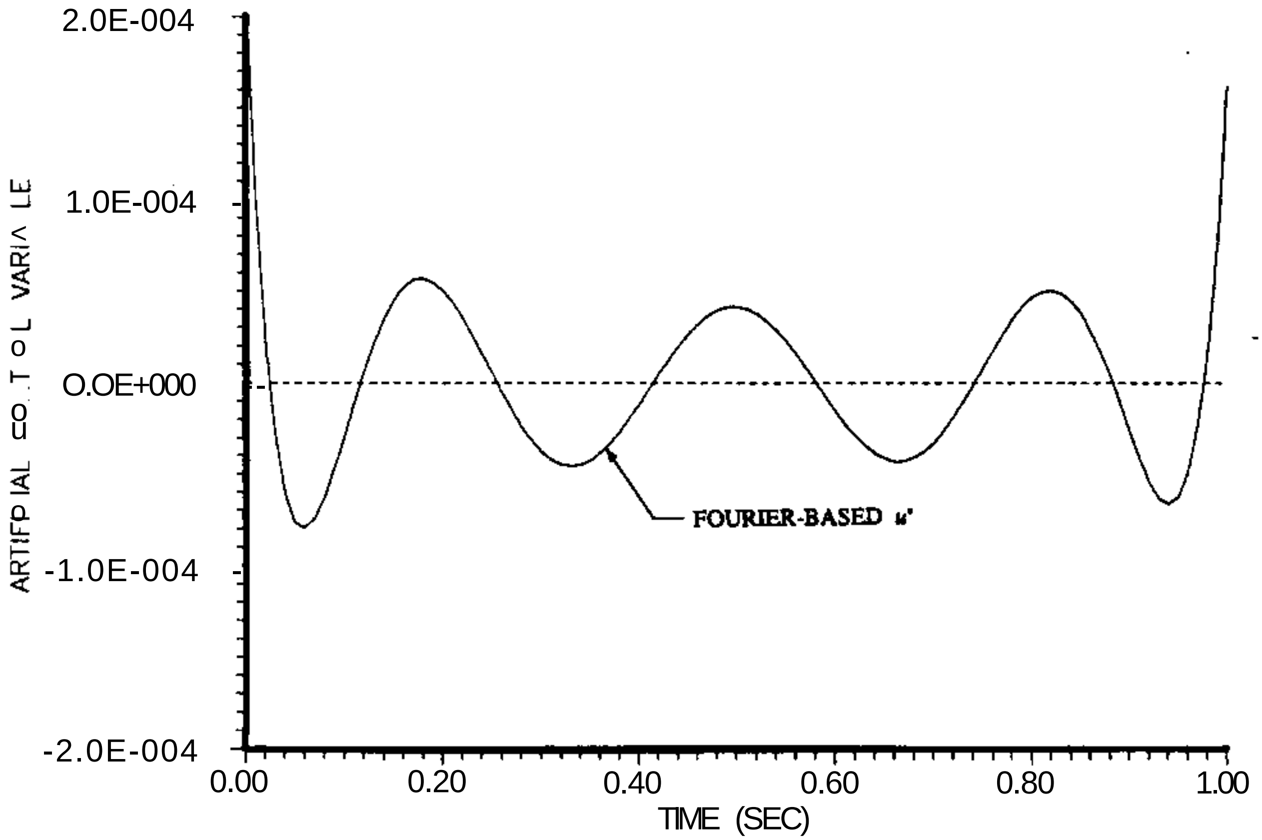

Figure 4. Fourier-based Optimal Solution of the Artificial Control Variable for Example 3.

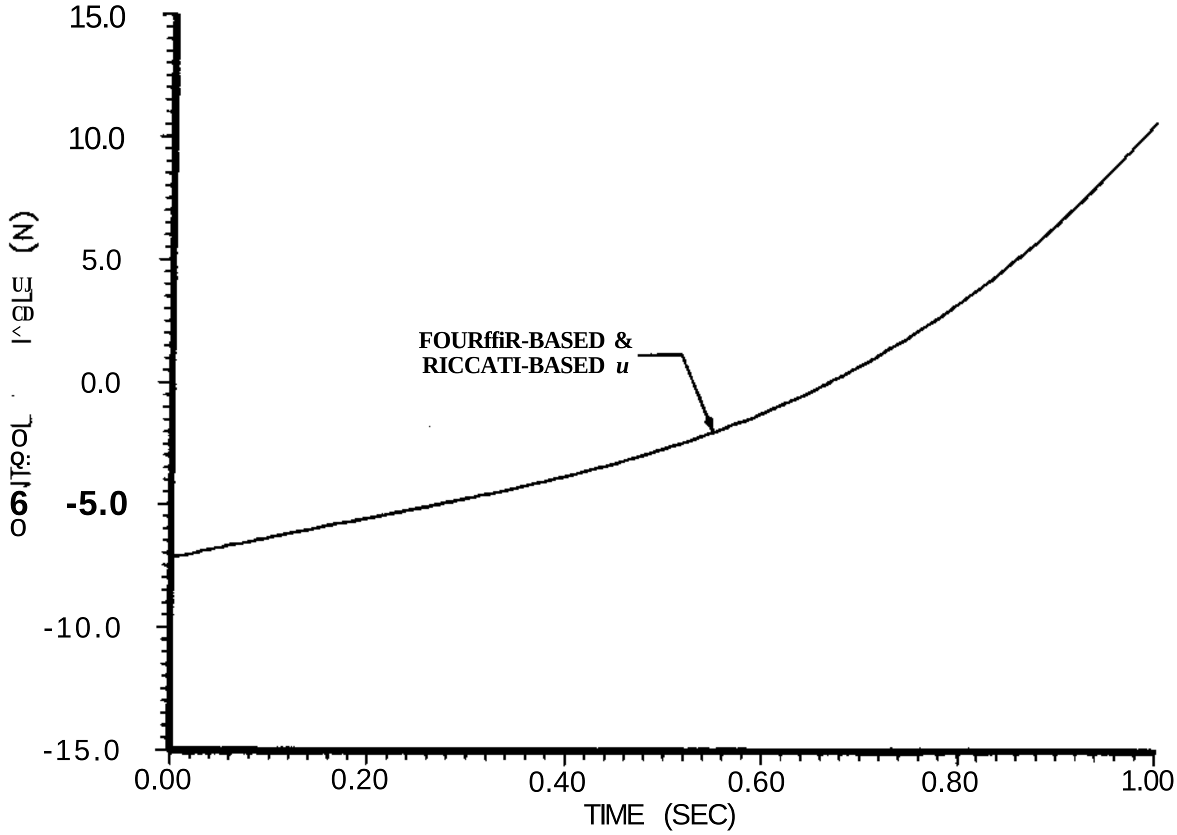

Figure 5. Riccati-Based and Fourier-Based Optimal Solutions of the Control Variable for Example 3.

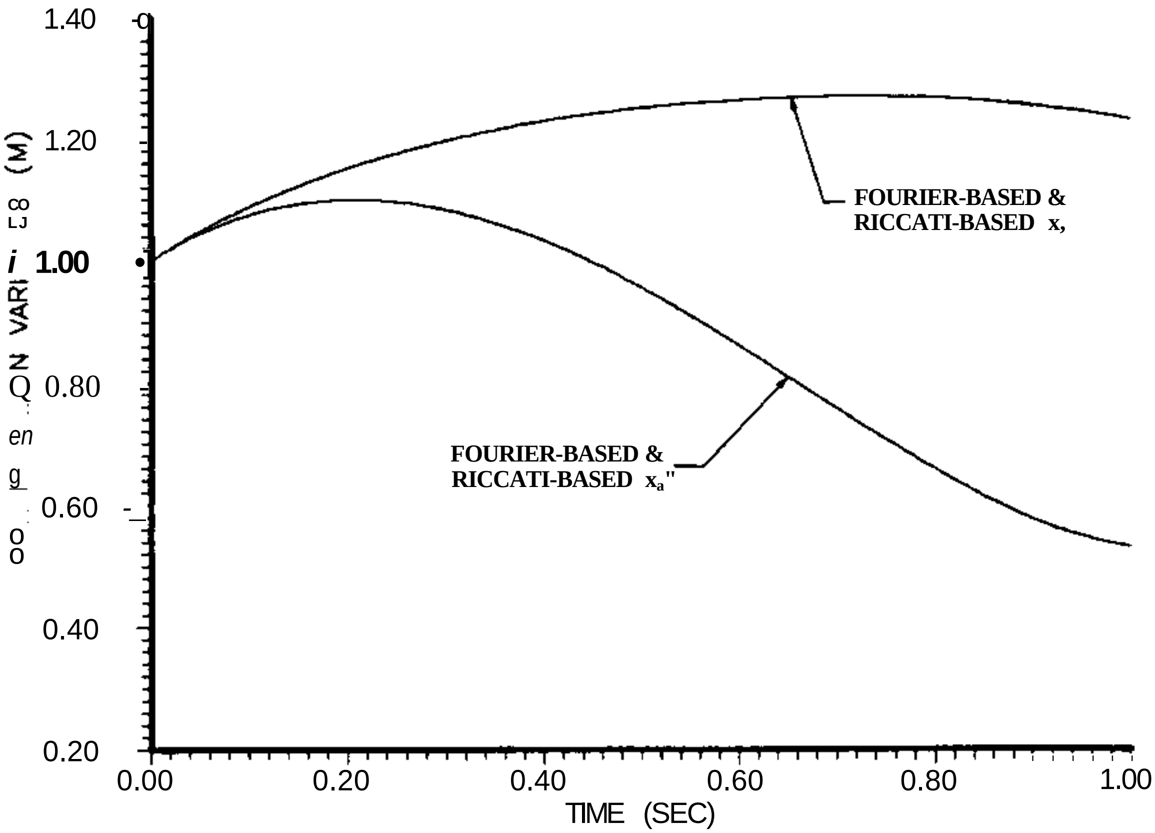

Figure 6. Riccati-Based and Fourier-Based Optimal Solutions of the Configuration Variables for Example 3.

In the same fashion, problems with fixed terminal configuration variable rates *(i.e.,* $\dot{x}_{if}$ and/or $\ddot{x}_{if}$ known) and problems with fixed initial configuration variable "accelerations" *(i.e.,* x_{iQ}) can be handled. In practice, once the fixed boundary conditions are identified, the corresponding rows and columns can be extracted from the coefficient matrix of Y in equation (58) and the contributions of the extracted columns can be subtracted from the corresponding elements of the right-hand side column vector. Using this technique, the same computer routines can be used to handle problems with both fixed and free boundary conditions, eliminating any additional analytical work. Furthermore, problems with linear equality constraints on the boundary conditions *(e.g.,* $x_l(t_f)+x_2(t_f) = l$) can be handled in the same manner.

3.2 Fourier-Based Approach and Optimization

The Fourier-based approach converts the original performance index, which is a quadratic functional, to a quadratic function, as shown in equations (39) and (52). This conversion eliminates the dependence of the performance index on time, a continuous variable, leaving the performance index as a function of a finite number of scalar variables. By using the Fourier-based approach, optimal control problems can be cast as algebraic optimization problems, with an associated rich software-base for their solution. Future research, as described below, is planned to exploit this unique feature of the Fourier-based approach.

33 Admissibility of Traj ectories

The Fourier-based approach does not guarantee the admissibility of the parameterized trajectories. Consequently, trajectories suggested by the Fourier-based approach may not be realizable. In Section 2.2.2 optimization algorithms were proposed to remedy this problem of inadmissibility of trajectories. These algorithms generate the near-optimal trajectories that simultaneously minimize the performance index and the influence of the artificial control variables. It should be noted that the near-optimal trajectories, thus generated, are not necessarily admissible in the strict mathematical sense, since the artificial control variables cannot be driven to zero exactly. Nonetheless, the control variable trajectories generated by the Fourier-based approach are very close to the true optimal solution and thus present no problem in implementation.

4 FUTURE RESEARCH

4.1 Work Statement

The ultimate objective of the research described in this report is to develop a general analytical and computational tool for solving optimal control problems, and hence designing optimal trajectories of dynamical systems. Toward this end, the use of a Fourier-based approximation is suggested since it converts the optimal control problem into an optimization problem which can readily be solved for near optimal solutions. An important finding, as highlighted in Section 2, is that for linear systems the optimization is "integration-free" and thus the near optimal solution can be determined by solving simultaneous linear algebraic equations!

Future work will concentrate on four principal tasks. The first task will involve the application of the linear near optimal control approach to problems with linear constraints on state and/or control variables by utilizing quadratic programming. In the second task, sensitivity functions will be developed to determine the influence of system paramaters on the near optimal control. The third task will involve the extension of the approach to general linear dynamical systems, represented by systems of first order differential equations (such as in state space form). In the fourth task the approach will be generalized to apply to nonlinear dynamical systems where sequential linearization will be employed. By addressing each of these tasks the Fourier-based methodology can be developed as a general tool for optimal system design, and ultimately implemented in a refined, user-oriented computer software package.

4.1.1 Quadratic Programming

In physical systems, constraints on state and control variables commonly occur. For example, state constraints often arise due to safety and/or configurational restrictions. In addition, due to finite-power actuators, control variables generally have magnitude limitations.

In solving for the optimal control of constrained systems, it is typical to formulate the necessary conditions of optimality using Pontryagin's minimum principle. The solution of such constrained optimal control problems, however, is often hindered by numerical difficulties associated with these necessary conditions. Consequently, solutions of constrained optimal control problems of high-order systems are often extremely difficult, if not impossible, to obtain.

State and control constraints have not been addressed explicitly in the formulation of the Fourier-based approach. However, an important advantage of the Fourier-based approach is its potential for solving optimal control problems with linear constraints on state and control variables. These problems can be cast as quadratic programming problems which can be solved by utilizing quadratic programming algorithms [Frank and Wolfe, 1956; Wolfe, 1959; Van De Pann and Whinston, 1964], as described below.

A quadratic programming problem is an optimization problem involving (i) a quadratic objective function, and (ii) linear constraints on the free variables. Mathematically, the problem in general form is to minimize an objective function J_{QP}

$$J_{QP}(\underline{y}) = f\underline{G}a + \underline{G}_{2l} \tag{114}$$

subject to linear constraints

$$2 \geq \underline{0} \tag{115}$$

$$Qsi = g \tag{116}$$

where $\underline{G}_{l}$, $\underline{G}_{2}$, and $\underline{G}_{3}$ are matrices and & and 2 are vectors.

To utilize quadratic programming for solving optimal control problems, it is necessary to modify the constraints on state and control variables such that they are not explicit functions of time. In particular, it is possible to eliminate the dependency of the state and control variables on time by discretizing the trajectory into a finite number of equally-spaced time intervals. The satisfaction of the constraints can then be checked at the discrete time intervals.

For example, consider a problem in which the ith generalized coordinate *x/t)* is always positive, *i.e.,*

$$JC_{£}(O \geq 0 \quad for \quad 0 \leq t \leq t_f \tag{117}$$

By using equation (22), this constraint can be written as

$$Pi + fi^{T}(OXi \wedge \circ \quad for \quad 0 \leq t \leq t_f \tag{118}$$

without losing linearity. To remove time dependency, the constraint can be approximated by

$$Pi \ + \ \pounds^{r}(?AOz \ \wedge \ 0 \quad where \quad <?=1,2,\ldots,\pounds \tag{119}$$

with Ar = $t_f l\ Q$. This equation consists of a system of Q linear constraints, and can be put in the form of equation (115).

The accuracy of this approach can be improved by increasing the number of discretization intervals, although the computational effort grows accordingly. Despite this potential computational cost, the Fourier-based approach represents a systematic approach for handling linearly constrained optimal control problems. Future work will investigate the use of the Fourier-based approach with quadratic programming for solving linearly constrained optimal control problems.

4.1.2 Sensitivity Study

The performance of an optimal control system depends on the trajectories as well as on the values of the system parameters. Systems with identical mathematical models *(i.e,* equations of motion) and identical expressions for their performance indices but different parameter values will generally have different optimal trajectories and different optimal values for their performance indices. Hence, the appropriate selection of the system parameters has the potential to further improve the performance of an optimal control system. The sensitivity of the system parameters on the performance of the system is an important issue to study.

Previously, in optimal control problems sensitivity studies were mathematically difficult to carry out due to the fact that the performance index was a functional involving the continuous variable, time. The Fourier-based approach eliminates the dependence of time on the performance index, making it only a function of system parameters and trajectory variables. A sensitivity analysis is possible since the mathematical complexity has been greatly reduced. A goal of future work will be to develop sensitivity functions relating the system parameters and the performance index.

4.1.3 General Linear Systems

Structural systems differ from general linear systems which can be represented by state equations, *i.e.,* sets of simultaneous first-order ordinary differential equations. In structural system models, the first and second time derivatives of the configuration variables appear explicitly in the equations of motion. It is necessary to ensure the convergence of the near optimal displacement, velocity, and acceleration trajectories. This requirement was the basis for the use of a fifth-order auxiliary polynomial (equation (4)) in the derivation. In linear systems represented by state equations only the first derivative of the state variable appears. Consequently, there is no need to guarantee the convergence of the second derivative of the trajectory. The boundary conditions at the ends of the "acceleration" profile are not required to parameterize the trajectory. It is therefore possible to employ a third-order auxiliary polynomial, in place of the fifth-order polynomial as required for structural systems.

This reduction in the order of the auxiliary polynomial will lead to a change in the relation between the generalized coordinate variables and the variables of the Fourier-based approach. In future work, the necessary and sufficient condition of optimality for the Fourier-based approach will be rederived for this case of general linear systems. Furthermore, the integral tables required to evaluate the performance index will be rederived. This new formulation will include artificial control variables, which are introduced to overcome the problem of inadmissibility of the trajectories, when the order of the system is greater than the number of control variables.

4.1.4 Nonlinear Systems

As indicated in Section 1, Yen and Nagurka [1987a] developed a Fourier-based approach and applied it to generate near-optimal trajectories of general dynamical systems, including nonlinear systems. In this approach, necessary conditions of optimality are not used to solve for the trajectories; rather, a nonlinear programming algorithm is employed to adjust free variables of the parameterized trajectories to minimize a performance index. This method is numerically robust and quite effective in identifying the optimal solution. However, the process of adjusting the free variables, carried out by a nonlinear programming algorithm, is in general a computationally intensive, iterative process. In this process, the performance index must be evaluated at each iteration by a numerical integration routine. The number of iterations required to reach the optimal solution usually becomes significant as the number of free variables increases. As a result, this method may require significant computational effort in finding the optimal trajectories of high-order systems.

In contrast to the method described by Yen and Nagurka [1987a], it is proposed to develop a computationally efficient approach to solve unconstrained nonlinear optimal control problems with quadratic performance indices. To handle these problems it is proposed to generalize algorithms developed for unconstrained linear systems with quadratic performance indices.

For nonlinear optimal control problems the necessary conditions for optimality can be derived by variational calculus and lead to a nonlinear two-point boundary-value problem (2PBVP). As mentioned in the Introduction, initial conditions are generally known for the state variables and terminal conditions are specified for the co-state variables. In addition, for many physical systems (including robotic manipulators) the terminal conditions are specified for the state and control variables. If these are given, the nonlinear system can be linearized about the terminal state. The optimal trajectories of the linearized system can then be determined by the Fourier-based approach for linear systems described in this report.

Future work will explore the efficacy of solving the nonlinear optimal control problem via backward sequential linearization, *i.e.*, it is proposed to "discretize" the continuous variable (time) to set up a series of linear optimal control problems which can readily be solved by the non-iterative Fourier-based approach for linear systems. The corresponding solution would apply for each corresponding time interval.

5 CONCLUSIONS

The approach presented in this report represents a computationally efficient alternative to standard approaches for the design of optimal trajectories of linear and nonlinear dynamical systems. The approach employs a Fourier-based approximation that parameterizes the generalized coordinate trajectories of a dynamic system. The approach avoids solving a 2PBVP, which is typically required in the standard formulation based on the calculus of variations. In addition, the approach is distinct from dynamic programming since it involves significantly fewer free variables. It thereby reduces the large computer storage requirement, a problem that has traditionally hindered the application of dynamic programming to high order systems.

In this report the basic methodology of the Fourier-based near optimal control approach is developed. In particular, the approach is specialized to determine the optimal control of structural systems with quadratic performance indices. The algorithm turns out to be integration-free and in general offers significant computational advantages in comparison to standard approaches (that typically require the integration of differential Riccati equations.) The computational efficiency is due to the fact that the near optimal solution can be found from a system of linear algebraic equations. A further advantage of the approach is that it can handle both free and fixed boundary conditions on the generalized coordinates.

One of the most important features of the Fourier-based approach is that, by parameterizing the trajectories, an optimal control problem is converted into an optimization problem. By using optimization algorithms, such as linear and quadratic programming, values of the trajectory parameters can be found that minimize a performance index. The rich supply of optimization algorithms and software gives the Fourier-based approach significant potential for handling general types of optimal control problems.

Future work is planned to continue investigating the design of Fourier-based optimal trajectories. By further refining the Fourier-based approach and investigating different optimization algorithms, it is hoped that a general tool for optimal trajectory design can be developed that could be used to solve many previously unsolvable optimal control problems.

6 REFERENCES

Bellman, R. E., 1957, *Dynamic Programming,* Princeton University Press, Princeton, NJ.

Canon, M.D., Cullum, CD., and Polak, E., 1970, *Theory of Optimal Control and Mathematical Programming,* McGraw-Hill, Inc., New Yoik, NY.

Frank, M. and Wolfe, P., 1956, "An Algorithm for Quadratic Programming," *Naval Research Logistics Quarterly,* Vol. 3, March and June, pp. 95-110.

Friedland, B., 1986, *Control System Design,* McGraw-Hill, New York.

Hicks, G.A., and Ray, W.H., 1971, "Approximation Methods for Optimal Control Synthesis," *Canadian Journal of Chemical Engineering,* Vol. 49, pp. 522-528.

Johnson, F.T., 1969, "Approximate Finite-Thrust Trajectory Optimization," *AIAA Journal,* Vol. 7, June, pp. 993-997.

Keller, H.B., 1975, "Numerical Solution of Boundary Value Problems for Ordinary Differential Equations: Survey and Some Recent Results on Difference Methods," *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, ed* A.K. Aziz, Academic Press, New York, pp. 27-88.

Kraft, D., 1980, "Comparing Mathematical Programming Algorithms Based on Lagrangian Functions for Solving Optimal Control Problems," *Control Applications of Nonlinear Programming, ed.* H.E. Rauch, Pergamon Press, Oxford.

Kwakernaak, H. **and Sivan,** R., **1972,** *Linear Optimal Control Systems,* Wiley, New York.

Lee, E.B. and Markus, L., 1986, *Foundations of Optimal Control Theory,* Robert Krieger Publishing, Malabar, FL.

Nagurka, MX., Yen, V., and Benaroya, H., 1987, "A Fourier-Based Method for the Suboptimal Control of Nonlinear Dynamical Systems," *Proceedings of the Sixth VPI&SUIAIAA Symposium on Dynamics and Control of Large Structures,* Blacksburg, VA, June 29-July 1.

Nair, G.G., 1978, "Suboptimal Control of Nonlinear Systems," *Automatica,* Vol. 14, pp. 517-519.

Owens, D.H., 1981, Multivariable and Optimal Systems, Academic Press, London.

Patel, R.V. and Munro, N., 1982, *Multivariable System Theory and Design,* Pergamon Press, Oxford.

Pereyra, V., 1984, "A Finite Difference Solution of Boundary Value Problems in Ordinary Differential Equations," *Studies in Numerical Analyses, ed.,* G.H. Golub, The Mathematical Association of America, pp. 243-269.

Ramesh, A,V., Utku, S., and Garba, J.A., 1987, "A Look into the Computational Complexities and Storage Requirements of Some Riccati Equation Solvers," ARO/AFSOR Conference on Non-Linear Vibrations, Stability, and Dynamics of Structures and

Mechanisms, Virginia Polytechnic Institute and State University, Blacksburg, VA, March 23-25.

Rosenbrock, **H.IL, 1970,** *State-Space and Multivariable Theory,* Thomas Nelson and Sons, Ltd., London.

Schultz, D.G. and Melsa, J.L., **1967,** *State Functions and Linear Control Systems,* McGraw-Hill, New York.

Sirisena, H.R. **and** Tan, K.S., 1974, "Computation of Constrained Optimal Controls Using Parameterization Techniques," *IEEE Transactions on Automatic Control,* Vol. AC-18, August, pp. 431-433.

Tabak, **D., 1970,** "Applications of Mathematical Programming Techniques in Optimal Control: A Survey," *IEEE Transacations on Automatic Control,* Vol. AC-15, December, pp. 688-690.

Tabak, **D. and** Kuo, B.C., **1971,** *Optimal Control by Mathematical Programming,* Prentice-Hall, Inc., Englewood Cliffs, NJ.

Takahashi, Y., Rabins, M.J., **and** Auslander, **D.M., 1970,** *Control and Dynamic Systems,* Addison-Wesley, Reading.

Van De Panne, C **and** Whinston, A., **1964,** "The Simplex and the Dual Method for Quadratic Programming," *Operation Researach Quarterly,* Vol 15, December, pp. 355-388.

Vlassenbroeck and Van Doreen, 1988, "A Chebyshev Technique for Solving Nonlinear Optimal Control Problems," *IEEE Transactions on Automatic Control,* Vol. Ac-33, April, pp. 333-340.

Wolfe, P., 1959, "The Simplex Method for Quadratic Programming," *Econometica,* Vol. 27, No. 3, July, pp. 382-398.

Yen, V. **and** Nagurka, **M.L., 1987a,** "Generating Suboptimal Trajectories of Dynamical Systems by Fourier-Based Approximations," International Symposium on the Mathematical Theory of Networks and Systems (MTNS), Phoenix, AZ, June 15-19.

Yen, V. **and** Nagurka, MX., **1987b,** "A Fourier-Based Optimal Control Approach for Structural Systems," *AIAA Journal of Guidance, Control, and Dynamics,* submitted September.

7 APPENDICES

7.1 Appendix 1: Coefficients of Auxiliary Polynomial

The coefficients of equation (4) are determined from the boundary conditions of x_i, *i.e.*, $x_i(0)$, $\dot{x}_i(0)$, $\ddot{x}_i(0)$, $x_i(t_f)$, $\dot{x}_i(t_f)$, and $\ddot{x}_i(t_f)$, giving rise to six simultaneous algebraic equations. These coefficients are:

$$d_{i0} = x_{i0} - \sum_{k=1}^{K} a_{ik} \qquad (A1-1)$$

$$d_{i1} = \dot{x}_{i0} - \frac{2\pi}{t_f}\sum_{k=1}^{K} k b_{ik} \qquad (A1-2)$$

$$d_{i2} = \frac{1}{2}\ddot{x}_{i0} + \frac{2\pi^2}{t_f^2}\sum_{k=1}^{K} k^2 a_{ik} \qquad (A1-3)$$

$$d_{i3} = 10(x_{if}-x_{i0})t_f^{-3} - \left(6\dot{x}_{i0} + 4\dot{x}_{if} - \frac{20\pi}{t_f}\sum_{k=1}^{K} k b_{ik}\right)t_f^{-2} - \left(\frac{3}{2}\ddot{x}_{i0} - \frac{1}{2}\ddot{x}_{if} + \frac{4\pi^2}{t_f^2}\sum_{k=1}^{K} k^2 a_{ik}\right)t_f^{-1} \qquad (A1-4)$$

$$d_{i4} = 15(x_{i0}-x_{if})t_f^{-4} + \left(8\dot{x}_{i0} + 7\dot{x}_{if} - \frac{30\pi}{t_f}\sum_{k=1}^{K} k b_{ik}\right)t_f^{-3} + \left(\frac{3}{2}\ddot{x}_{i0} - \ddot{x}_{if} + \frac{2\pi^2}{t_f^2}\sum_{k=1}^{K} k^2 a_{ik}\right)t_f^{-2} \qquad (A1-5)$$

$$d_{i5} = 6(x_{if}-x_{i0})t_f^{-5} - 3\left(\dot{x}_{i0} + \dot{x}_{if} - [illegible]\sum_{k=1}^{K} k b_{ik}\right)t_f^{-4} - \frac{1}{2}\left(\ddot{x}_{i0} - \ddot{x}_{if}\right)t_f^{-3} \qquad (A1-6)$$

where $x_{i0} = x_i(0)$, $x_{if} = x_i(t_f)$, and similarly for the corresponding time derivatives.

7.2 Appendix 2: Integral Evaluations

This appendix describes the approach to obtain the closed-form solution of $\underline{A}^*$, $\underline{F}^*$, $\underline{Q}^*$ and $\underline{S}^*$ of equations (53), (54), (55), and (56), respectively, each of which can be represented as the summation of the integrals of six terms, according to equations (42), (43), (44), and (45). Due to the sparseness of £* > H^* > and $\underline{\Phi}^*$, the evaluation of these integrals can be simplified, as demonstrated below.

Consider the first integral of $\underline{A}^*$, *i.e.,*

$$\underline{\Lambda}_1^* = \int_0^{t_f} \underline{\Phi}^{*T} \underline{F}_1 \underline{\Phi}^* dt \qquad \text{(A2-1)}$$

The *i-j th* element of this matrix can be written as

$$\Lambda^*_{1ij} = 01 \sum_{k=1}^{N} \text{F} u/<t>/> \quad . \quad U \; J = h...,mN \qquad \text{(A2-2)}$$

$$= \sum_{k=1}^{N} \sum_{l=1}^{N} F_{1kl} \int_0^{t_f} \phi^*_{ki} \phi^*_{lj} dt \qquad \text{(A2-3)}$$

where F_{lkl} is known from equation (46). Due to the sparseness of $\underline{\Phi}^*$,

$$\Lambda^*_{1ij} = F_{1kl} \int_0^{t_f} \theta_k \% dt \qquad \text{(A2-4)}$$

where k = mod(ż, *m)* (*i.e.,* integer remainder of / divided by *m,* when integer remainder does not equal zero; if remainder equals zero, then k = m), / = mod(/, *m),* *m* is the dimension of *y_i* and

$$[9, \quad e_2 \; . \; . \; . \; ej = \#" \qquad \text{(A2-5)}$$

from equation (28). Closed-form expressions for the integral of equation (A2-4) are provided in Table A2-1.

In a similar manner, the other five integrals of $\underline{A}^*$, the six integrals of $\underline{F}^*$, the six integrals of £T and the six integrals of 2* can be evaluated. These are presented in the accompanying tables (where $v_k = 2nk$). Note that the first three integrals of $\underline{F}^*$ and $\underline{Q}^*$ are identical.

TABLE A2-1: First Integral of $\underline{A}^*$

$(\nu_k=2\pi k)$

Integral Table for $\int_0^{t_f} F_i F_k\, dt$

ω	$\int_0^{t_f}\omega\,dt$	ω	$\int_0^{t_f}\omega\,dt$
ϕ_1^2	[illegible]	$\phi_3\phi_4$	$-\frac{11}{35}$
$\phi_1\phi_2$	[illegible]	$\phi_3\epsilon_k$	$\left(-\frac{\nu_k^2}{5}+12\right)t_f^{-2}$
$\phi_1\phi_3$	$\frac{4}{35}$	$\phi_3\zeta_k$	$\left(-\frac{60}{7}\nu_k+\frac{360}{\nu_k}\right)t_f^{-2}$
$\phi_1\phi_4$	$\frac{t_f}{70}$	ϕ_4^2	$\int_0^{t_f}\phi_1^2\,dt$
$\phi_1\epsilon_k$	$\left(\frac{\nu_k^2}{10}-6\right)t_f^{-1}$	$\phi_4\epsilon_k$	$\int_0^{t_f}\phi_1\epsilon_k\,dt$
$\phi_1\zeta_k$	$\left(-\frac{10}{7}\nu_k+\frac{60}{\nu_k}\right)t_f^{-1}$	$\phi_4\zeta_k$	$-\int_0^{t_f}\phi_1\zeta_k\,dt$
ϕ_2^2	$\frac{120}{7}t_f^{-3}$	$\epsilon_i\epsilon_k \quad i\neq k$	$\left[\frac{1}{5}\nu_i^2\nu_k^2-12(\nu_i^2+\nu_k^2)\right]t_f^{-3}$
$\phi_2\phi_3$	$\frac{60}{7}t_f^{-2}$	e(et t-fc	$\left(\frac{7}{..}-24\nu_k^2\right)t_f^{-3}$
$\phi_2\phi_4$	$\frac{3}{7}t_f^{-1}$	W. «.*	$\left[\frac{120}{7}\nu_i\nu_k-720\left(\frac{\nu_i}{\nu_k}+\frac{\nu_k}{\nu_i}\right)\right]t_f^{-3}$
M.	0	|t«t. t - *	$\left[\frac{120}{7}\nu_k^2-1440\ldots\right]t_f^{-3}$
$\phi_2\zeta_k$	$\left(\frac{720}{7}\nu_k-\frac{720}{\nu_k}\right)t_f^{-3}$	fit.	0
*³	$\frac{192}{35}t_f^{-1}$		

TABLE A2-2: Second Integral of A*

(vk=2nk)

Integral Table for $\int_0^{t'} q\_'^{T} Eil'\, dt$

UD	$\int_0^{tf} u)\, dt$	ω	$\int_0 \omega\, dt$
	$\frac{t^3_f}{630}$	ff3<74	$-\frac{}{60}$
$o_x o_2$	$-\frac{tf}{84}$	$\sigma_3\epsilon_k$	3 2 . 12 / 140 * vf
	$-\frac{t^2_f}{210}$	O3«t	$-\frac{3}{14}v** \frac{360}{vj}$
$o\backslash° 4$	$\frac{t^3_f}{1260}$		fafdt
$\sigma_1\gamma_k$	$\left(\frac{Vl}{\sqrt{420}} - \frac{6}{v}\right) f'$ 1	$o_A y_k$	$\int_0 o^\wedge t\,dt$
$\sigma_1\delta_k$	I 84 v. vlf'	cr4 6t	rh
ol	7 ^	$\gamma_i\gamma_k \quad i \neq k$	$\left\{\frac{v_i^2 v_k^2}{210} - 12\left[\left(\frac{v_i}{v_k}\right)^2 + \left(\frac{v_k}{v_i}\right)^2\right]\right\} t_f^{-1}$
$o_2 o_3$	$-\frac{3}{14}$	$\gamma_i\gamma_k \quad i = k$	$\left(\frac{v_k^4}{210} + \frac{v_k^2}{2} - 24\right) t_f^{-1}$
$o_2 a_A$	$\frac{h}{84}$	$\delta_i\delta_k \quad i \neq k$	$\left[\frac{3 v_i v_k}{7} - 720\left(\frac{v_i}{v_k^3} + \frac{v_k}{v_i^3}\right)\right] t_f^{-1}$
$o_2 y^\wedge$	0	$\delta_i\delta_k \quad t\text{-}k$	p3 [14 v a * - 1440V.. / v\ } t r
$\sigma_2\delta_k$	$\left(\frac{3}{7}v_k - \frac{720}{v_k^3}\right) t_f^{-1}$	y<<5*	0
σ_3^2	$\frac{8}{35} t_f$		

TABLE A2-3: Third Integral of $\underline{A}$*

(y_k=2nk)

Integral Table for 1 p'T£3£* dt (Jo)

	$\int_{Jo}^{tf}$ uo dt	ω	1 Jo cv dt
p?	ι/ 9240	P3P4	- 2 3 18480 t« r
P1P2	1 8 1 3 I 55440 t '	P_3e^*	\\θ 5040 * vf vj/'
P1P3	- 13 13860 t4 '	P_3P^*	f 5 360>.2 |, -924V*+ vl P
P1P4	*/ 11088	Pi	$\int_0^{t_f} \rho_1^2 \, dt$
Pi<**	V 120 5040 vl) '	P4ttt	Jo
P\Pk	I "1? V 1 Λ V ^55440'* vj vj /'	P4^t	$\int^{tf}$ Jo
P\	181 462*'	$\alpha_i \alpha_k \quad i \neq k$	L 60 2520 U*4 "*JJt,
P2P3	· 3 3 1a 4620(/	$\alpha_i \alpha_k \quad i = k$	^3 - vl t v« 24> \\^2 30 ' 2520 vl) t,
P2P4	$\frac{281}{55440} t_f$	$\beta_i \beta_k \quad i \neq k$	$\left[\frac{5}{462}\nu_i\nu_k - 720\left(\frac{\nu_i}{\nu_k^3} + \frac{\nu_k}{\nu_i^3}\right)\right] t_f$
P2»i	{ 2 120J(/	$\beta_i \beta_k \quad i = k$	$\left(\frac{5}{462}\nu_k^2 - \frac{1440}{\nu_k^4} + \frac{1}{2}\right) t_f$
$P_2£^*$	f 8 1 720>	$\alpha_i \beta_k$	0
P5	$\frac{52}{3465} t_f$		

TABLE A2-4: Fourth Integral of $\underline{A}^*$

Integral Table for $\int_{0}^{t_f} \underline{\phi}^{T} \underline{F}_4 \underline{\sigma}\, dt$

ω	$\int_0^{t_f} \omega\, dt$	ω	$\int_0^{t_f} \omega\, dt$
$\phi_1\sigma_1$	0	$\phi_3\sigma_3$	$\frac{1}{2}$
[illegible]	[illegible]	$\phi_3\sigma_4$	$-\frac{9}{140} t_f$
[illegible]	$\frac{11}{140} t_f$	[illegible]	$\left(-\frac{\nu_k^2}{7} + \frac{360}{\nu_k^2}\right) t_f^{-1}$
[illegible]	$\frac{t_f^2}{140}$	[illegible]	$\left(\frac{2}{7}\nu_k - \frac{12}{\nu_k}\right) t_f^{-1}$
[illegible]	$\frac{-\nu_k^2}{140} + \frac{60}{\nu_k^2} - 1$	$\phi_4\sigma_1$ [illegible]	$-\int_0^{t_f} \phi_1\sigma_4\, dt$
[illegible]	$-\frac{\nu}{7} + \frac{[illegible]}{\nu}$		$-\int_0^{t_f} \phi_1\gamma_k\, dt$
[illegible]	$-\int_0^{t_f} \phi_1\sigma_2\, dt$	$\phi_4\delta_k$	$\int_0^{t_f} \phi_1\delta_k\, dt$
$\phi_2\sigma_2$	0	[illegible]	[illegible]
[illegible]	$-\frac{9}{7} t_f^{-1}$	$\epsilon_k\gamma_l$	0
[illegible]	$\frac{1}{7}$	[illegible]	[illegible]
$\phi_2\gamma_k$	$\left(\frac{2}{7}\nu_k^2 - \frac{720}{\nu_k^2}\right) t_f^{-2}$	[illegible]	$\left(-\frac{11}{14}\nu_k^3 + 12\nu_k + \frac{720}{\nu_k}\right) t_f^{-2}$
$\phi_2\delta_k$	0	[illegible]	[illegible]
[illegible]	$-\int_0^{t_f} \phi_1\sigma_3\, dt$	$\zeta_k\gamma_l$	$-\int_0^{t_f} \epsilon_l\delta_k\, dt$
$\phi_3\sigma_2$	[illegible]	$\zeta_k\delta_l$	0

TABLE A2-5: Fifth Integral of $\underline{A}$*

$(\nu_k = 2\pi k)$

Integral Table for $\int_0^{t_f} \underline{\phi}^{*T} \underline{F}_5 \underline{\rho}^* \, dt$

(V	\ Jo ivdt	ω	$\int_0^{t_f} \omega \, dt$
*Ipl	$-\frac{t^3,}{630}$	$*_3$P4	$\overline{}$ 60
*IP$_2$	$\frac{t_f}{84}$	$\phi_3\alpha_k$	(3 a i !2^
$\phi_1\rho_3$	$\frac{t^1,}{210}$	$\phi_1\beta_k$	/' 3 360 '\
$\phi_1\rho_4$	$-\frac{«?}{1260}$	$\phi_4\rho_i \quad i=1..4$	I *cP4^
*!<**	Vv| 420 J(/	*4<*»	$\int_0^{t_f} \phi_1\alpha_k \, dt$
$\phi_1\beta_k$	$\left(\frac{\nu_k}{84}+\frac{1}{\nu_k}-\frac{60}{\nu_k^3}\right)t_f$	^4^i	-I Jo # i ^ t *
tlPl	$\int_0^{t_f} \phi_1\rho_2 \, dt$	e^p, i-1..4	Jo
*2P2	$-\frac{10}{7}t_f^{-1}$	€$_k$a$_l$ i+k	$\left\{-\nu_i^2\frac{\nu_k^2}{210}+12\left[\left(\frac{\nu_i}{\nu_k}\right)^2+\left(\frac{\nu_k}{\nu_i}\right)^2\right]\right\}t_f^{-1}$
lP	$\frac{3}{14}$	€$_k$a$_t$ i - k	$\left(-\frac{\nu_k^4}{210}+24-\frac{\nu_k^2}{2}\right)t_f^{-1}$
*2P4	$-\frac{«/}{84}$	Ufit	0
$\phi_2\alpha_k$	0	CtPi <-l.-4	Jo *****
$\phi_2\beta_k$	f 3 720V.i	$\zeta_k\alpha_i$	0
*3Pl	J *.p$_3$dt	$\zeta_k\beta_i \quad i\neq k$	$\left[-\frac{3}{7}\nu_i\nu_k+720\left(\frac{\nu_k}{\nu_i^3}+\frac{\nu_i}{\nu_k^3}\right)\right]t_f^{-1}$
**Pl	$\frac{17}{14}$	$\zeta_k\beta_i \quad i=k$	$\left(-\frac{13}{14}\nu_k^2+\frac{1440}{\nu_k^2}\right)t_f^{-1}$
**P*	$-\frac{8}{35}t_f$		

TABLE A2-6: Sixth Integral of A*

$(v_k=2nk)$

Integral Table for I a*¹ E₀ρ' dt Jo '"

(V	r^h I Jo cvdt	ω	I CD Gft
<7,P1	0	^3P3	0
σ₁ρ₂	- t) / 84	^3P4	- / 1008
OiP*	13 / 5040 t³,	^3«i	r vj 360\ V 280 + vj / r
Olp*	- t⁴r / 5040	σ₁β₁	T 53 1 12\
$o_x a_k$	V 5040 v\ v⁴j '	$a_4 p_c$ i- 1..4	-J r'/ «/0 ff,P«^
OiPt	V 280 vl)^tf	$a_4 a_t$	Jo
OlPl	-1 r^tf Jo $o_x p_2 dt$	°*Pk	∫h Jo $o_x p_k dt$
O2P2	1/2	V*P, i-1..4	-\ rh Jo $o_t a_t dt$
O2P2	- 11/84 t,	γ₁α₁	0
G*PA	84	γ₁β₁ i≠k	v 140 v? vj
$a_2 a_t$	vl/42 - 1 - 720/vj	$y_g p_t$ i-/c	-vj/140 ^ 12 * v / f 720/vj - v4/2
o*Pt	0	fi^p, i - 1..4	-J c;^tdt
°*P\	-1 rh Jo OxPzdt	$6_t a_t$ itk,	«/o
O*P2	-1 rh $a_2 p_3 dt$	δ₁β₁	0

TABLE A2-7: First Integral of $\underline{V}$ and $\underline{QT}$
(v_k=2nk)

Integral Table for $\int_0^{t_f} \omega\, dt$

ω	$\int_0^{t_f} \omega\, dt$
$r_t\phi_1$	$\left(\frac{1}{60}r_{t2}+\frac{3}{140}r_{t3}\right)t_f$
$r_t\phi_2$	$-\left(r_{t1}+r_{t2}+\frac{6}{7}r_{t3}\right)t_f^{-1}$
$r_t\phi_3$	$r_{t1}+\frac{4}{5}r_{t2}+\frac{22}{35}r_{t3}$
$r_t\phi_4$	$\left(\frac{r_{t2}}{60}+\frac{r_{t3}}{35}\right)t_f$
$r_t\epsilon_k$	$\left[r_{t2}\left(\frac{v_k^2}{30}-2\right)+r_{t3}\left(\frac{v_k^2}{20}-3\right)\right]t_f^{-1}$
$r_t\zeta_k$	$r_{t3}\left(\frac{v_k}{7}-\frac{6}{v_k}\right)t_f^{-1}$

TABLE A2-8: Second Integral of $\underline{V}$ and $\underline{Q}^*$

($v_k=2nk$)

Integral Table for I $\underline{a}'^T F\_2 g\_dt$
Jo

ω	$\int_0^{t_f} \omega \, dt$
$q_t\sigma_1$	$\left(-\frac{1}{140}q_{t2}-\frac{3}{560}q_{t3}-\frac{1}{252}q_{t4}\right)t_f^2$
$q_t\sigma_2$	$q_{t0}+\frac{2}{7}q_{t2}+\frac{5}{28}q_{t3}+\frac{5}{42}q_{t4}$
$q_t\sigma_3$	$\left(\frac{13}{105}q_{t2}+\frac{1}{8}q_{t3}+\frac{5}{42}q_{t4}\right)t_f$
$q_t\sigma_4$	−(105 112 126;' '
$q_t\gamma_k$	$q_{t2}\left(1-\frac{v_k^2}{60}\right)+q_{t3}\left(1-\frac{v_k^2}{70}-\frac{6}{v_k^2}\right)+q_{t4}\left(1-\frac{v_k^2}{84}-\frac{12}{v_k^2}\right)$
$q_t\delta_k$	$q_{t2}\left(\frac{2}{v_k}-\frac{v_k}{21}\right)+q_{t3}\left(\frac{3}{v_k}-\frac{v_k}{14}\right)+q_{t4}\left(\frac{4}{v_k}-\frac{24}{v_k^3}-\frac{17}{210}v_k\right)$

TABLE A2-9: Third Integral of $\underline{P}$ and $\underline{Q}^*$
$(y_k = 2\pi k)$

Integral Table for $\int_0^{t_f} \underline{P}^{*T} \underline{P}_3 \underline{p}\, dt$

ω	$\int_0^{t_f} \omega\, dt$
$p_t p_{t1}$	$\left(\frac{p_{t0}}{120}+\frac{p_{t1}}{280}+\frac{p_{t2}}{1008}+\frac{p_{t3}}{1680}+\frac{p_{t4}}{2640}\right)t_f^3$
$p_t p_{t2}$	$\left(\frac{1}{2}\ \frac{5}{14}\ \frac{37}{168}p_{t3}+\frac{11}{60}p_{t4}+\frac{31}{198}p_{t5}\right)t_f^3$
$p_t p_{t3}$	$-\left(\frac{1}{10}p_{t0}+\frac{13}{210}p_{t1}+\frac{5}{168}p_{t3}\ \frac{1}{45}p_{t4}\ \frac{17}{990}p_{t5}\right)t_f^2$
$p_t p_{t4}$	$\left(\frac{p_{t0}}{120}+\frac{p_{t1}}{210}+\frac{p_{t3}}{504}+\frac{p_{t4}}{720}+\frac{p_{t5}}{990}\right)t_f^3$
$p_t \alpha_k$	$\left[\left(-1+\frac{v_k^2}{60}\right)p_{t0}+\left(-\frac{1}{2}+\frac{v_k^2}{120}\right)p_{t1}+\left(-\frac{1}{4}+\frac{v_k^2}{336}+\frac{3}{v_k^2}\right)p_{t3}+\left(-\frac{1}{5}+\frac{v_k^2}{504}+\frac{4}{v_k^2}-\frac{24}{v_k^4}\right)p_{t4}+\left(-\frac{1}{6}+\frac{v_k^2}{720}+\frac{5}{v_k^2}-\frac{60}{v_k^4}\right)p_{t5}\right]t_f$
$p_t \beta_k$	$\left[\left(\frac{v_k}{42}-\frac{1}{v_k}\right)p_{t1}+\left(\frac{17}{840}v_k+\frac{6}{v_k^3}-\frac{1}{v_k}\right)p_{t3}+\left(\frac{v_k}{60}-\frac{1}{v_k}+\frac{12}{v_k^3}\right)p_{t4}+\left(\frac{19}{1386}v_k-\frac{1}{v_k}+\frac{20}{v_k^3}-\frac{120}{v_k^5}\right)p_{t5}\right]t_f$

TABLE A2-10: Fourth Integral of _P

(y_k=2Kk)

Integral Table for /"t I £'T E_A g_dt Jo

U)	$\int_0^{t_f}$ (vdt
$q_t \phi_1$	$\left[\frac{1}{60} q_{t2} + \frac{3}{140}(q_{t3} + q_{t4})\right] t_f$
$q_t \phi_2$	$-\left(q_{t2} + \frac{6}{7} q_{t3} \quad \frac{5}{7} q_{t4}\right) t_f^{-1}$
$q_t \phi_3$	$q_{t0} + \frac{4}{5} q_{t2} \quad + \frac{22}{35} q_{t3} \quad \frac{1}{2} q_{t4}$
$q_t \phi_4$	^60 35 28 J '
$q_t \epsilon_k$	$\left[\left(\frac{v_k^2}{30} - 2\right) q_{t2} + \left(\frac{v_k^2}{20} - 3\right) q_{t3} + \left(\frac{2}{35} v_k^2 - 4 + \frac{24}{v_k^2}\right) q_{t4}\right] t_f^{-1}$
Qtt t	$\left[\left(\frac{v_k}{7} - \frac{6}{v_k}\right) q_{t3} + \left(\frac{2}{7} v_k - \frac{12}{v_k}\right) q_{t4}\right] t_f^{-1}$

TABLE A2-11: Fifth Integral of $\underline{V}$

(v_k=2nk)

Integral Table for $\int_0^{t_f} {}_5\underline{p}\, dt$

ω	$\int_0^{t_f} \omega \, dt$
$p_t\phi_1$	$\left[\frac{3}{140}(p_{t3}+p_{t4})+\frac{5}{252}p_{t5}\right]t_f$
$p_t\phi_2$	$-\left(p_{t1}+\frac{6}{7}p_{t3}+\frac{5}{7}p_{t4}+\frac{25}{42}p_{t5}\right)t_f^{-1}$
$p_t\phi_3$	$p_{t0}+p_{t1}+\frac{22}{35}p_{t3}+\frac{1}{2}p_{t4}+\frac{17}{42}p_{t5}$
$p_t\phi_4$	$\left(\frac{1}{35}p_{t3}+\frac{1}{28}p_{t4}+\frac{5}{126}p_{t5}\right)t_f$
$p_t\epsilon_k$	$\left[\left(\frac{\nu_k^2}{20}-3\right)p_{t3}+\left(\frac{2}{35}\nu_k^2-4+\frac{24}{\nu_k^2}\right)p_{t4}+\left(\frac{5}{84}\nu_k^2-5+\frac{60}{\nu_k^2}\right)p_{t5}\right]t_f^{-1}$
$p_t\zeta_k$	$\left[\left(\frac{\nu_k}{7}-\frac{6}{\nu_k}\right)p_{t3}+\left(\frac{2}{7}\nu_k-\frac{12}{\nu_k}\right)p_{t4}+\left(\frac{17}{42}\nu_k-\frac{20}{\nu_k}+\frac{120}{\nu_k^3}\right)p_{t5}\right]t_f^{-1}$

TABLE A2-12: Sixth Integral of $\underline{V}$
($v_k=2nk$)

Integral Table for $\int_0^{t_f} \omega^T \underline{E}_6 \underline{\xi}\, dt$

ω	$\int_0^{t_f} \omega\, dt$
$p_t\sigma_1$	$-\left(\frac{1}{120}p_{t1}+\frac{3}{560}p_{t3}+\frac{1}{252}p_{t4}+\frac{1}{336}p_{t5}\right)t_f^2$
$p_t\sigma_2$	$p_{t0}+\frac{1}{2}p_{t1}+\frac{5}{28}p_{t3}+\frac{5}{42}p_{t4}+\frac{p_{t5}}{12}$
$p_t\sigma_3$	$\left(\frac{1}{10}p_{t1}+\frac{1}{8}p_{t3}+\frac{5}{42}p_{t4}+\frac{1}{9}p_{t5}\right)t_f$
$p_t\sigma_4$	$-\left(\frac{p_{t1}}{120}+\frac{p_{t3}}{112}+\frac{p_{t4}}{126}+\frac{p_{t5}}{144}\right)t_f^2$
$p_t\gamma_k$	$\left(1-\frac{v_k^2}{60}\right)p_{t1}+\left(1-\frac{v_k^2}{70}-\frac{6}{v_k^2}\right)p_{t3}$ $+\left(1-\frac{v_k^2}{84}-\frac{12}{v_k^2}\right)p_{t4}+\left(1-\frac{5v_k^2}{504}-\frac{20}{v_k^2}+\frac{120}{v_k^4}\right)p_{t5}$
$p_t\delta_k$	$\left(\frac{3}{v_k}-\frac{v_k}{14}\right)p_{t3}+\left(\frac{4}{v_k}-\frac{17}{210}v_k-\frac{24}{v_k^3}\right)p_{t4}+\left(\frac{5}{v_k}-\frac{v_k}{12}-\frac{60}{v_k^3}\right)p_{t5}$

TABLE A2-13: Fourth Integral of Q*

($\nu_k = 2\pi k$)

Integral Table for $\int_0^{t_f} r_t \Omega_4 \, dt$

ω	$\int_0^{t_f} \omega \, dt$
$r_t\sigma_1$	$-\left(\frac{1}{120} r_{t1} + \frac{1}{140} r_{t2} + \frac{3}{560} r_{t3}\right) t_f^2$
$r_t\sigma_2$	$\frac{1}{2} r_{t1} \quad \frac{2}{7} r_{t2} \quad \frac{5}{28} r_{t3}$
$r_t\sigma_3$	$\left(\frac{1}{10} r_{t1} + \frac{13}{105} r_{t2} + \frac{1}{8} r_{t3}\right) t_f$
$r_t\sigma_4$	$-\left(\frac{r_{t1}}{120} + \frac{r_{t2}}{105} + \frac{r_{t3}}{112}\right) t_f^2$
$r_t\gamma_k$	$\left(1 - \frac{\nu_k^2}{60}\right)(r_{t1} + r_{t2}) + \left(1 - \frac{\nu_k^2}{70} - \frac{6}{\nu_k^2}\right) r_{t3}$
$r_t\delta_k$	$\left(\frac{2}{\nu_k} - \frac{\nu_k}{21}\right) r_{t2} + \left(\frac{3}{\nu_k} - \frac{\nu_k}{14}\right) r_{t3}$

TABLE A2-14: Fifth Integral of $\underline{Q}^*$

(v_k=2%k)

Integral Table for $\int_0^{t_f} \mathbf{r}^T \pounds_s \pounds^* \, dt$

a>	$\int_0^{t_f} \alpha \, dt$
$r_t \rho_1$	$\left(\frac{\ }{280} \quad \frac{\ }{560} \quad \frac{\ }{1008} \right)$
$r_t \rho_2$	$\left(\frac{5}{14} r_{t1} + \frac{23}{84} r_{t2} \cdot \frac{37}{168} r_{t3} \right) t_f$
$r_t \rho_3$	$-\left(\frac{13}{210} r_{t1} \cdot \frac{1}{24} r_{t2} + \frac{5}{168} r_{t3} \right) t_f^2$
$r_t \rho_4$	$\left(\frac{r_{t1}}{210} \quad \frac{r_{t2}}{336} \cdot \frac{r_{t3}}{504} \right) t_f^3$
$r_t \gamma_k$	$\left[\left(\frac{v_k^2}{120} - \frac{1}{2} \right) r_{t1} + \left(\frac{v_k}{210} \cdot \frac{2}{v_k} \right) r_{t2} + \left(\frac{v_k^2}{336} \cdot \frac{3}{v_k} \right) r_{t3} \right] t_f$
$r_t \delta_t$	$\left[\left(\frac{v_k}{\ } \quad \frac{1}{\ } \right) (r_{t1} + r_{t2}) + \left(\frac{17}{840} v_k - \frac{1}{v_k} + \frac{6}{v_k^3} \right) r_{t3} \right] t_f$

TABLE A2-15: Sixth Integral of Q^*

$(\nu_k = 2\pi k)$

Integral Table for $\int_0^{t_f}$ [illegible] dt

[illegible]	$\int_0^{t_f} v_j \, dt$
$q_t \rho_1$	$\left(\frac{q_{t0}}{120} + \frac{q_{t2}}{560} + \frac{q_{t3}}{1008} + \frac{q_{t4}}{1680}\right) t_f^3$
$q_t \rho_2$	$\left(\frac{1}{2} q_{t0} + \frac{23}{84} q_{t2} + \frac{37}{168} q_{t3} + \frac{11}{60} q_{t4}\right) t_f$
$q_t \rho_3$	$-\left(\frac{1}{10} q_{t0} + \frac{1}{24} q_{t2} + \frac{5}{168} q_{t3} + \frac{1}{45} q_{t4}\right) t_f^2$
$q_t \rho_4$	$\left(\frac{q_{t0}}{120} + \frac{q_{t2}}{336} + \frac{q_{t3}}{504} + \frac{q_{t4}}{720}\right) t_f^3$
$q_t \alpha_k$	$\left[\left(-1 + \frac{\nu_k^2}{60}\right) q_{t0} + \left(\frac{\nu_k^2}{210} - \frac{1}{3} + \frac{2}{\nu_k^2}\right) q_{t2}$ $+ \left(\frac{\nu_k^2}{336}$ [illegible] $\right) q_{t4}\right] t_f$
$q_t \beta_k$	$\left[\left(\frac{\nu_k}{42} - \frac{1}{\nu_k}\right) q_{t2} + \left(\frac{17}{840}\nu_k - \frac{1}{\nu_k} + \frac{6}{\nu_k^3}\right) q_{t3} + \left(\frac{\nu_k}{60} - \frac{1}{\nu_k} + \frac{12}{\nu_k^3}\right) q_{t4}\right] t_f$

TABLE A2-16: Integrals of $\underline{Z}^*$

Integral Table for $\underline{I}^* = \frac{1}{t_f}\int_0^{t_f} \underline{L}\,dt$

ω	$\frac{1}{t_f}\int_0^{t_f} \omega\,dt$
$r_t r_k$	$\frac{1}{3}(r_{t1}r_{k1})+\frac{1}{4}(r_{t1}r_{k2}+r_{t2}r_{k1})$ $+\frac{1}{5}(r_{t1}r_{k3}+r_{t2}r_{k2}+r_{t3}r_{k1})+\frac{1}{6}(r_{t2}r_{k3}+r_{t3}r_{k2})+\frac{1}{7}(r_{t3}r_{k3})$
$q_t q_k$	$q_{t0}q_{k0}+\frac{1}{3}(q_{t0}q_{k2}+q_{t2}q_{k0})+\frac{1}{4}(q_{t0}q_{k3}+q_{t3}q_{k0})$ $+\frac{1}{5}(q_{t0}q_{k4}+q_{t2}q_{k2}+q_{t4}q_{k0})+\frac{1}{6}(q_{t2}q_{k3}+q_{t3}q_{k2})$ $+\frac{1}{7}(q_{t2}q_{k4}+q_{t3}q_{k3}+q_{t4}q_{k2})+\frac{1}{8}(q_{t3}q_{k4}+q_{t4}q_{k3})+\frac{1}{9}(q_{t4}q_{k4})$
$p_t p_k$	$p_{t0}p_{k0}+\frac{1}{2}(p_{t0}p_{k1}+p_{t1}p_{k0})+\frac{1}{3}(p_{t1}p_{k1})$ $+\frac{1}{4}(p_{t0}p_{k3}+p_{t3}p_{k0})+\frac{1}{5}(p_{t0}p_{k4}+p_{t1}p_{k3}+p_{t3}p_{k1}+p_{t4}p_{k0})$ $+\frac{1}{6}(p_{t0}p_{k5}+p_{t1}p_{k4}+p_{t4}p_{k1}+p_{t5}p_{k0})$ $+\frac{1}{7}(p_{t1}p_{k5}+p_{t3}p_{k3}+p_{t5}p_{k1})+\frac{1}{8}(p_{t3}p_{k4}+p_{t4}p_{k3})$ $+\frac{1}{9}(p_{t3}p_{k5}+p_{t4}p_{k4}+p_{t5}p_{k3})+\frac{1}{10}(p_{t4}p_{k5}+p_{t5}p_{k4})+\frac{1}{11}(p_{t5}p_{k5})$
$r_t q_k$	$\frac{1}{2}r_{t1}q_{k0}+\frac{1}{3}r_{t2}q_{k0}+\frac{1}{4}(r_{t1}q_{k2}+r_{t3}q_{k0})$ $+\frac{1}{5}(r_{t1}q_{k3}+r_{t2}q_{k2})+\frac{1}{6}(r_{t1}q_{k4}+r_{t2}q_{k3}+r_{t3}q_{k2})$ $+\frac{1}{7}(r_{t2}q_{k4}+r_{t3}q_{k3})+\frac{1}{8}(r_{t3}q_{k4})$
$r_t p_k$	$\frac{1}{2}r_{t1}p_{k0}+\frac{1}{3}(r_{t1}p_{k1}+r_{t2}p_{k0})+\frac{1}{4}(r_{t2}p_{k1}+r_{t3}p_{k0})$ $+\frac{1}{5}(r_{t1}p_{k3}+r_{t3}p_{k1})+\frac{1}{6}(r_{t1}p_{k4}+r_{t2}p_{k3})$ $+\frac{1}{7}(r_{t1}p_{k5}+r_{t2}p_{k4}+r_{t3}p_{k3})+\frac{1}{8}(r_{t2}p_{k5}+r_{t3}p_{k4})+\frac{1}{9}(r_{t3}p_{k5})$
$q_t p_k$	$q_{t0}p_{k0}+\frac{1}{2}q_{t0}p_{k1}+\frac{1}{3}q_{t2}p_{k0}$ $+\frac{1}{4}(q_{t0}p_{k3}+q_{t2}p_{k1}+q_{t3}p_{k0})+\frac{1}{5}(q_{t0}p_{k4}+q_{t3}p_{k1}+q_{t4}p_{k0})$ $+\frac{1}{6}(q_{t0}p_{k5}+q_{t2}p_{k3}+q_{t4}p_{k1})+\frac{1}{7}(q_{t2}p_{k4}+q_{t3}p_{k3})$ $+\frac{1}{8}(q_{t2}p_{k5}+q_{t3}p_{k4}+q_{t4}p_{k3})+\frac{1}{9}(q_{t3}p_{k5}+q_{t4}p_{k4})+\frac{1}{10}q_{t4}p_{k5}$

7.3 Appendix 3: Coupling in Performance Index

This appendix shows how the coupling between y_A and y_B is used to obtain an expression of the performance index, *J*, which is solely a function of y_B.

From equation (103), the performance index can be written as:

$$/ = i^T(A* + \underline{z}^T \underline{HZ})y\_ + \underline{fr} + \underline{n}*^T z + \underline{r} \qquad \text{(A3-i)}$$

Substituting from equations (91), (108), and (109), the performance index can written as:

$$J = ij\underline{\Delta}_{11Z4} + i\,j\_A\,\hat{} + \hat{}(A\underline{2}!+A\underline{L}_2)Z4 + *Ili + \&E2 + ?* \qquad \text{(A3-2)}$$

The first term of equation (A3-2) can be rewritten using equation (90) as

$$\begin{aligned} \underline{y}_A^T \underline{\Delta}_{11} \underline{y}_A &= (\underline{D}_1 \underline{y}_B + \underline{D}_2)^T \underline{\Delta}_{11} (\underline{D}_1 \underline{y}_B + \underline{D}_2) \\ &= \underline{y}_B^T \underline{D}_1^T \underline{\Delta}_{11} \underline{D}_1 \underline{y}_B + \underline{y}_B^T (\underline{D}_1^T \underline{\Delta}_{11} \underline{D}_2 + \underline{D}_1^T \underline{\Delta}_{11}^T \underline{D}_2) + \underline{D}_2^T \underline{\Delta}_{11} \underline{D}_2 \end{aligned} \qquad \text{(A3-3)}$$

The third term of equation (A3-2) can be rewritten using equation (90) as

$$\begin{aligned} \underline{y}_B^T (\underline{\Delta}_{21} + \underline{\Delta}_{12}^T) \underline{y}_A &= \underline{y}_B^T (\underline{\Delta}_{21} + \underline{\Delta}_{12}^T)(\underline{D}_1 \underline{y}_B + \underline{D}_2) \\ &= \underline{y}_B^T (\underline{\Delta}_{21} + \underline{\Delta}_{12}^T) \underline{D}_1 \underline{y}_B + \underline{y}_B^T (\underline{\Delta}_{21} + \underline{\Delta}_{12}^T) \underline{D}_2 \end{aligned} \qquad \text{(A3-4)}$$

The fifth term of equation (A3-2) can be rewritten using equation (90) as

$$\begin{aligned} \underline{y}_A^T \underline{E}_1 &= (\underline{D}_1 \underline{y}_B + \underline{D}_2)^T \underline{E}_1 \\ &= \underline{y}_B^T \underline{D}_1^T li + ejl, \end{aligned} \qquad \text{(A3-5)}$$

Substituting equations (A3-3), (A3-4), and (A3-5) into equation (A3-2) gives equation (104), *i.e.*,

$$J = iL\hat{}lB + ijr* + ?** \qquad \text{(A3-6)}$$

where

$$A^{**} = \underline{D}_1^T(\underline{\Lambda}_{11}\underline{D}_1 + \underline{\Lambda}_{12}) + \underline{\Lambda}_{21}\underline{D}_1 + \underline{\Lambda}_{22} \qquad \text{(A3-7)}$$

$$\underline{\Gamma}^{**} = [\underline{D}_1^T(\underline{\Lambda}_{11} + \underline{\Lambda}_{11}^T) + \underline{\Lambda}_{12} + \underline{\Lambda}_{21}]\underline{D}_2 + \underline{D}_1^T\underline{E}_1 + \underline{E}_2 \qquad \text{(A3-8)}$$

$$\underline{\Sigma}^{**} = \underline{D}_2^T\underline{\Lambda}_{11}\underline{D}_2 + \underline{D}_2^T\underline{E}_1 + \underline{\Sigma}^{*} \qquad \text{(A3-9)}$$