ON THE FACIAL STRUCTURE OF SCHEDULING POLYHEDRA

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Abstract

A well-known job shop scheduling problem can be formulated as follows. Given a graph G with node set N and with directed and undirected arcs, find an orientation of the undirected arcs that minimizes the length of a longest path in G. We treat the problem as a disjunctive program, without recourse to integer variables, and give a partial characterization of the scheduling polyhedron $P(N)$, i.e., the convex hull of feasible schedules. In particular, we derive all the facet inducing inequalities for the scheduling polyhedron $P(K)$ defined on some clique with node set K, and give a sufficient condition for such inequalities to also induce facets of $P(N)$. One of our results is that any inequality that induces a facet of $P(H)$ for some $H \subset K$, also induces a facet of $P(K)$. Another one is a recursive formula for deriving a facet inducing inequality with p positive coefficients from one with $p-1$ positive coefficients. We also address the constraint identification problem, and give a procedure for finding an inequality that cuts off a given solution to a subset of the constraints.

Contents

1. Introduction

We consider the following machine sequencing problem which is a special case of resource-constrained scheduling (for background material see [It 2], [8,...,12]). A number of items have to be processed by performing a sequence of operations on each of them on specified machines. There are n operations to be performed, including a fictitious "stop" (operation n), the objective being to minimize total completion time subject to (i) precedence constraints between the operations, and (ii) the condition that a machine can process only one item at a time, and operations cannot be interrupted. The problem can be stated as

$$\min t_n$$

$$(P) \qquad \begin{array}{ll} t_j - t_i \geq d_{ij} & (i,j) \in A \\ t_i \geq 0, & i \in N \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, & (i,j) \in E \end{array}$$

where t_i is the starting time of operation i, d_{ij} is the minimum required time lapse between starting operation i and starting operation j (for instance, completion time of operation i, plus set-up time for operation j), A indexes the pairs of operations constrained by precedence relations, E the pairs that use the same machine and therefore cannot overlap in time, and "V" is the logical "or". It is useful to represent the problem by a disjunctive graph [1, 2, 10, 12] G » (N°,A°,E), where H° «fcO}U-N is a set of nodes, one for each operation, plus a source node 0; A° * AU[(O,j)!j is not preceded by an operation} is a set of (conjunctive) directed arcs; E is a set of undirected arcs, one for every pair of operations to be performed on the same machine. Solving the problem involves orienting the undirected arcs, i.e., choosing for each of them one of the two possible directions. i_t is therefore convenient to represent each undirected arc by a disjunctive pair of directed arcs, i.e., a pair of which one member needs to be selected: hence the name disjunctive graph. We will use this latter representation, and consider E to consist of pairs of directed arcs (i,j),(j,i), ^ E+ , {(i,j)eE|i<fl •• » Cd.j)«|i > j}, and E - E^+UE-. The arcs of E occur in disjoint maximal cliques (bv a clique we mean a complete digraph), of which there is one for every machine. Thus if M indexes the set of maximal cliques (machines), and for VCN°, < v > denotes the subgraph of G induced by V, then for every rtM, the node set ^ of the r^{th} maximal clique < r_r > corresponds to the set of ' operations to be performed on the same machine (r).

Every directed arc (i,j)c_{AU}E has a positive length d^, while the arcs (O,j)«A°\A have length d_{QJ} - 0. For a pair {(i,j), (j,i)}<E, ^ + d_{ji} t- not only possible, but typical. *We* will assume that the arc lengths are integers satisfying the triable inequality djJ + djfe_ > ^ ¥ ^^ Though this assumption involves some loss of generality, it is realistic for the n-chine sequencing problem. The disjunctive graph G is illustrated in Figure 1

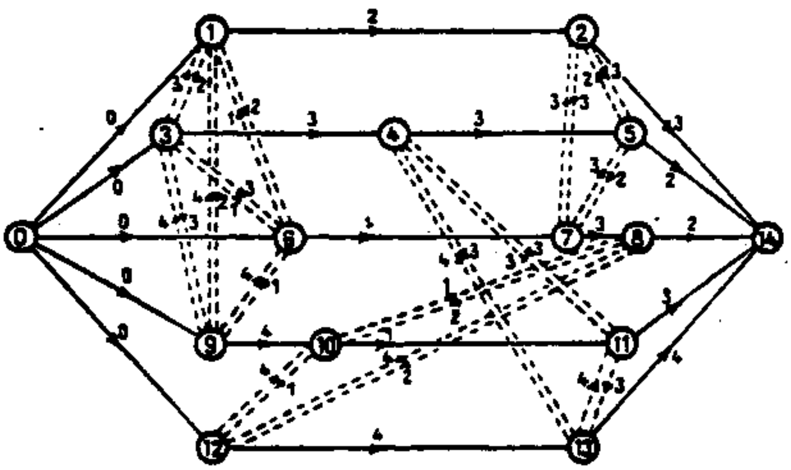


Figure 1

on a problem with 5 items (directed source-sink paths), 4 machines (maximal cliques, whose arcs are shown in dotted lines), and 14 operations (nodes other than the source). The numbers on the arcs are the lengths d_{ij}.

The subgraph obtained from G by deleting the disjunctive arc set E is the ordinary digraph $D = (N^0, A^0)$, in which node 0 has indegree zero and outdegree the number of items, node n has indegree the number of items and outdegree zero, while all remaining nodes have indegree and outdegree one. In fact D is the union of as many disjoint (except for their end nodes) paths from 0 to n, as there are items.

A <u>selection</u> in G consists of exactly one member of each pair of disjunctive arcs in E. Thus, if $\alpha = \frac{1}{2}|E|$, there are 2^{α} possible selections in G. In the undirected representation of E, a selection in G corresponds to an orientation of all the undirected arcs of G.

For every selection S in G, $D_S = (N^0, A^0 \cup S)$ is an ordinary digraph; and the problem obtained from (P) by replacing the set of disjunctive constraints indexed by E with the set of conjunctive constraints indexed by S is the dual of a longest path (critical path) problem in D_S. Thus solving (P) amounts to finding a minimaximal path in the disjunctive graph G, i.e., finding a selection (orientation) S that minimizes the length of a critical path in D_S over the set of all possible selections.

Problem (P) stated at the beginning of this section has a variable t_j associated with every node of G except for 0. One can of course introduce a variable t_0 for node 0, but then the problem does not change if t_0 is constrained by $t_0 = 0$, which leads to the elimination of the variable just introduced. We therefore prefer to work with vectors $t \in R$ that don't have a component t_0 constrained to be 0.

Problem (F) is a disjunctive program. It can also be represented as a mixed integer program by introducing a binary variable for every disjunctive constraint, but there are advantages to <u>not</u> doing that and using instead the disjunctive programming approach (for background see [3, 5]). In this paper we investigate the properties of the scheduling polyhedron P, the closed convex hull of all vectors $t \in \mathbb{R}^n$ satisfying the constraints of (P). Section 2 introduces the polyhedron P, states some of its basic properties, and discusses the relationship of P to polyhedra defined by subsets of the constraint set. Section 3 deals with scheduling polyhedra P(K) defined on a clique with node set K, and characterizes the vertices of P(K). Section 4 gives a complete characterization of the facets of P(K). One of the results is that any inequality that defines a facet of P(H) for some $H \subset K$ also defines a facet of P(K). Another result is a procedure for deriving a facet defining inequality for P(K) with p nonzero coefficients from a facet defining inequality with p-1 nonzero coefficients. This section also lists all the facets of P(K), for K of arbitrary size, having one, two or three nonzero coefficients. Section 5 gives a sufficient condition for an inequality that defines a facet of P(K) to also define a facet of P. The condition is verifiable in $O(|E|)$ time. Finally, section 6 addresses the constraint identification problem and gives a procedure for identifying facet defining inequalities that cut off a given $t \in \mathbb{R}^n$ that violates some of the disjunctions of (P). Some of our results were presented in [4].

2. <u>Some Properties of the Scheduling Polyhedron</u>

Any $t \in \mathbb{R}^n$ satisfying the constraints of (P) will be called a <u>schedule</u> for G. The feasible set of (P), or the set of schedules for G, can be written as

$$T = \left\{ t\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} t_j - t_i \geq d_{ij}, & (i,j)\in A \\ t_i \geq 0, & i\in N \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, & i,j\in K_\ell,\ i\neq j,\ \ell\in M \end{array} \right\}.$$

The closed convex hull of T, clconv T, will be called the <u>scheduling polyhedron</u>, and denoted P(N), or simply P.

T is a disjunctive set, and its convex hull is easiest to describe when T is in disjunctive normal form [3, 4], i.e., in the form $T = \bigcup_{S\in Q} T_S$, where Q is the index set of all selections in G, and T_S is the (polyhedral) set of schedules for the digraph D_S defined by the selection S in G:

$$T_S = \left\{ t\in\mathbb{R}^n \;\middle|\; \begin{array}{ll} t_j - t_i \geq d_{ij} & (i,j)\in A\cup S \\ t_i \geq 0\ , & i\in N \end{array} \right\}.$$

If D_S contains a cycle, $T_S = \emptyset$. So the only selections of interest are those for which the associated digraph D_S has no cycles, i.e., those indexed by $Q^* = \{S\in Q \mid D_S$ is acyclic$\}$, since $T = \bigcup_{S\in Q^*} T_S$. In the sequel we assume that $Q^* \neq \emptyset$. For any $S\in Q^*$, we will denote by $L(i,j)_S$ the length of a longest path from i to j in D^S. The length of the (unique) path* from i to j in D will be denoted by $L(i,j)$.

<u>Theorem</u> 2.1. For every $S\in Q^*$, T_S has dimension n.

<u>Proof</u>. We define n + 1 vectors $t^i\in R^n$, $i=0,1,\dots,n$, as follows. Let t^0 be defined by $t^0_j = L(0,j)_S$, $j=1,\dots,n$; and for $i=1,\dots,n$, let t^i be defined by

$$t_j^i = \begin{cases} 2t_j^o + \epsilon & j = i \\ 2t_j^o & j \neq i, \end{cases}$$

where $0 < \epsilon < 1/2$.

Clearly, $t^o \epsilon T_S$. For $i=1,\ldots,n$, $t_j^i \geq 0$, $\forall j$, and for $(h,j)\epsilon A \cup S$, one can easily check that $t_j^i - t_h^i \geq d_{hj}$. Thus for $i = 0,1,\ldots,n$, $t^i \epsilon T_S$. Also, the n+1 points $t^i \epsilon \mathbb{R}^n$ are affinely independent, since the $n \times n$ matrix whose i^{th} row is $t^i - 2t^o$, $i = 1,\ldots,n$, is ϵ times the identity matrix of order n.||

Corollary 2.2. P is full dimensional.

Next we turn to the extreme points of P. First we characterize the extreme points of T_S for an arbitrary $S\epsilon Q^*$.

Theorem 2.3. A schedule t for D_S is an extreme point of T_S if and only if

(i) $t_n = L(0,n)_S$;

(ii) for $j\epsilon N\backslash\{n\}$, $t_j = L(0,j)_S$ or $t_j = L(0,n)_S - L(j,n)_S$ (or both);

(iii) if $t_j = L(0,j)_S$, then $t_i = L(0,i)_S$ for all i on any longest path from 0 to j; and if $t_j = L(0,n)_S - L(j,n)_S$, then $t_i = L(0,n)_S - L(i,n)_S$ for all i on any longest path from j to n.

Proof. Necessity. Let $t^*\epsilon T_S$ be such that $t_n^* > L(0,n)_S$, and let $N_1 = \{j\epsilon N | t_j^* > L(0,j)_S\}$. Define t^1 and t^2 by $t_j^1 = t_j^* + \epsilon$, $t_j^2 = t_j^* - \epsilon$ for $j\epsilon N_1$, and $t_j^1 = t_j^2 = t_j^*$ for $j\epsilon N\backslash N_1$. Then $t^1, t^2 \epsilon T_S$, $t^1 \neq t^* \neq t^2$, and $t^* = \frac{1}{2}(t^1 + t^2)$, i.e., t^* is not extreme. Thus (i) is necessary.

Now let $t^o \epsilon T_S$ satisfy (i), but violate (ii) for $j\epsilon N^* \subseteq N\backslash\{n\}$; i.e., let $N^* := \{j\epsilon N\backslash\{n\} | L(0,j)_S < t_j^o < L(0,n)_S - L(j,n)_S\}$. Define t' and t'' by $t_j' = t_j^o + \epsilon$, $t_j'' = t_j^o - \epsilon$, $j\epsilon N^*$; and $t_j' = t_j'' = t_j^o$, $j\epsilon N\backslash N^*$. Then for suitably small ϵ, $t', t'' \epsilon T_S$, $t' \neq t^o \neq t''$, and $t^o = \frac{1}{2}(t' + t'')$. Thus (ii) is necessary.

Finally, condition (iii) is implied by (ii), hence it is also necessary.

Sufficiency. Suppose $t \in T_S$ is not extreme. Then t is the convex combination of $t^i \in T_S$, $t^i \neq t$, $i = 1,\dots,n+1$. If $t_n > L(0,n)_S$, condition (i) is violated and we are done. So let $t_n = L(0,n)_S$; then $t_n^i = L(0,n)_S$, $i = 1,\dots,n+1$. Furthermore, for every $j \in N$ such that $t_j = L(0,n)_S - L(j,n)_S$, we have $t_j^i = t_j$, $i = 1,\dots,n+1$. Let N_1° be the set of these indices $j \in N$. Then there exists $j_* \in N \setminus N_1^\circ$ such that $t_{j_*}^i < t_{j_*}$ for some $i \in \{1,\dots,n+1\}$, or else $t^i = t$ for all i. But then $L(0,j^*)_S < t_{j_*}^i < L(0,n)_S - L(j_*,n)_S$ (since $t^i \in T_S$), hence condition (ii) is violated.||

Corollary 2,4. If t is an extreme point of P, then $t_0 = L(0,n)_S$, and $t_j = L(0,j)_S$ or $t_j = L(0,n)_S - L(j,n)_S$ (or both), $\forall j \in N \setminus \{n\}$, for some $S \in Q^*$.

Proof, Every extreme point of P is an extreme point of T_S for some $S \in Q^*$.||

For every (not necessarily maximal) clique < K >, we define a schedule for < K > as a vector $t \in T(K)$, where

$$T(K) = \left\{ t \in R^p \;\middle|\; \begin{array}{ll} t_i \geq L(0,i), & i \in K \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, & i,j \in K,\ i \neq j \end{array} \right\},$$

where $p = |K|$, and L(0,i) is the length of the (unique) path from 0 to i in D = (N°,A°). The closed convex hull of T(K), clconv T(K), will be called the scheduling polyhedron on < K >, and denoted P(K).

For any $V \subseteq N$, we denote by S(V) a selection in < V >, i.e., a set of arcs containing exactly one member of each disjunctive pair of arcs with both ends in V. For $V' \subset V \subseteq N$, we say that the selection S(V) is an extension to < V > of the selection S(V') (the selection S(V') is a restriction to < V' > of the selection S(V)) if the arcs of S(V) with both ends in V' are precisely those of S(V').

A selection S in G (where S is an abbreviation for $S(N)$) is always of the form

$$(2.1) \qquad S = \bigcup_{r \in M} S(K_r)$$

where each $S(K_r)$ is a selection in a maximal clique $\langle K_r \rangle$.

For a vector $t \in R^{|N|}$ and a clique $\langle K \rangle$ of G, we will denote by t_K the vector in $R^{|K|}$ whose components are t_j, $j \in K$.

Theorem 2.5. A schedule t for G is an extreme point of P if

(i) t is an extreme point of T_S for some $S \in Q^*$; and

(ii) for every maximal clique $\langle K \rangle$ of G, t_K is an extreme point of $P(K)$.

Proof. Suppose t is a schedule for G that satisfies (i) and (ii), and let t be the convex combination of some $t^i \in P$, $i = 1, \ldots, n+1$. Since t satisfies (ii), $t_K = t^i_K$, $i = 1, \ldots, n+1$, for every clique $\langle K \rangle$ of G. If S is the selection associated with t, this implies that $t^i \in T_S$, $i = 1, \ldots, n+1$. Since t also satisfies (i), $t^i = t$, $i = 1, \ldots, n+1$. Thus t is an extreme point of P. ∎

Given a clique $\langle K \rangle$ in G, we say that a schedule t for G (a vector $t \in T$) is an extension to G of a schedule t' for $\langle K \rangle$ (an extension to T of a vector $t' \in T(K)$) if $t_j = t'_j$, $\forall j \in K$. We say that a schedule t' for $\langle K \rangle$ ~~can~~ be extended to T, if t' has an extension $t \in T$. Conversely, we say that a schedule t' for $\langle K \rangle$ is a restriction to $\langle K \rangle$ of the schedule t for G, if t is an extension of t'. By the choice of the lower bounds $L(0,i)$, $i \in K$, every schedule for G can be restricted to any of the cliques of G. Therefore, for every clique $\langle K \rangle$ of G,

$$(2.2) \qquad P \subseteq P(K) .$$

The more interesting question, of course, is when can a schedule for some clique < K > be extended to a schedule for G. This question is intimately related to the problem of facet lifting, i.e., to the connection between facet inducing inequalities for P(K) and for P. It will be investigated in section 5, where we will give a sufficient condition for an inequality that defines a facet of P(K) to also define a facet of P. This condition is always satisfied for some of the cliques of G, so at least some of the facet inducing inequalities for P(K) are always facet inducing for P itself. This provides the main, though not the only, motivation for focusing in the next 2 sections on the polyhedra P(K).

3. The Scheduling Polyhedron on a Clique

In this section we study the properties of the scheduling polyhedron on a clique, or briefly the clique polyhedron P(K) • clconv T(K). If |K| • p and if we denote 1^ - L(0,i), icK, then

$$T(K) - \left\{ t \in R^p \;\middle|\; \begin{array}{c} t_i \geq L_i\,, \quad i \in K \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, \; \forall\, i,j \in K,\; i \neq j \end{array} \right\}.$$

As before, a vector tcT(K) will be called a schedule for < K >.

Apart from its connection with machine sequencing, and more generally with the resource constrained scheduling problem, the polyhedron T(K) is an interesting object in its own right. A selection S(K) in < K > is the arc set of a tournament in < K >. Every tournament is known to have a directed Hamilton path (i.e., a directed path containing all the vertices), and for an acyclic tournament this path is unique. In fact, every acyclic tournament is the transitive closure of its unique directed Hamilton path. A selection S(K) is therefore uniquely determined by the sequence of the nodes of K in its directed Hamilton path, and conversely, every selection S(K) defines a unique sequence of the nodes of K. Thus the scheduling problem on a clique, namely the problem of finding

$$\min_{t\in T(K)} \max_{i\in K} t_i \qquad (3\text{-}D$$

with $L_\pm$ » 0, $i \in K$, is a "dual" formulation of the problem of finding a shortest Hamilton path in < K >, using node rather than arc variables. The latter problem in turn is polynomially equivalent to the traveling salesman problem (TSP). Indeed, an optimal tour for the TSP yields a shortest Hamilton path by deletion of the largest arc. Conversely, finding for each $i \in K$ a shortest Hamilton path originating in i (which is problem (3.1)) with the extra condition that ^ - 1^ - 0), then adding to each path the unique arc that closes it, and choosing the shortest of the p resulting tours, yields an optimal solution to TSP.

The scheduling polyhedron P(K) on a clique < K > is related to the linear ordering polyhedron P_u on < K > studied recently by Grötschel, Jünger and Reinelt [5]. P_{LQ} is the convex hull of the incidence vectors of acyclic tournaments in < K >. It is a bounded polytope in $*^{p(p"l)}{}_f$ the space spanned by the arcs of the complete digraph < K >, whereas P(K) is an unbounded polyhedron in R^p. When P(K) is specialized to the case where $L_i - 0$, $i \in K$, there is a one to one correspondence between its vertices and acyclic tournaments in < K >, as will be shown later in this section. Hence there is a one to one correspondence between the vertices of P(K) (in the case $L_i = 0$, $i \in K$) and those of PJ^J. One might therefore expect a similarly close relationship between facets of P^ and those of P(K). In fact, however, the facets of P(K) are rather different from, and seemingly unrelated to, those of P_{Tn}. A set of p vertices that lie on a facet of P_{LO} may not lie on a facet of P(K), and vice versa« While the facets of P_{E}^{0} are independent of the arc lengths, the facets of P(K) strongly depend on the arc lengths d_{ij}.

Wheneveir possible without risking confusion, the notation S(K) for a selection in < K > will be abbreviated to S. Every selection S in < K > defines a polyhedron

$$T(K)_S \cdot \left\{ t c \mathbb{R}^{r} \;\middle|\; \begin{array}{ll} t_i \wedge L_i \quad \cdot & i e K \\ t_j - t_i \geq d_{ij}, & (i,j) \in S \end{array} \right\},$$

which is nonempty if and only if S is acyclic. Let Q(K) be the set of selections in < K >, and Q(K)* » (ScQ(K)|s is acyclic}. Then the disjunctive normal form of T(K) becomes

$$T(K) \text{ » } \underset{ScQ(K)*}{\wedge J} T(K)_{q*} .$$

For every ScQ(K)*, the polyhedron $T(K)_S$ is obviously full-dimensional; hence so is P(K).

For ieK and an acyclic selection S in < K >, we define the <u>rank</u> of i in S as the position (rank) of i in the sequence associated with S.

<u>Theorem 3,1</u>. Let S be an acyclic selection in < K > with associated sequence j(l),.•.,j(p). Further, let k_1 s 1, and for i «2,...,s $\leq$ p, let k_i be the smallest integer (if it exists) such that

(3_2) $$L_j < v > L_j\, oc_{iml})^+ < 4 \underset{i-1}{} d_j(k>j <fcfi) \cdot$$

Then t is a vertex of $T(K)_g$ if and only if for i » l,...,s

(3_3) $$'jck^\wedge - {}^L n^\wedge y$$

for k > k_g

(3#4) $$c_{j(k)} \text{ " } c_{j(k-1)} + d_{j(k-1)j(k)}$$

and for kc[k_1,k_s] , k + k_i , 1 * 1,...$_f$s,

$$
(3.5) \qquad t_{j(k)} = \begin{cases} t_{j(k-1)} + d_{j(k-1)j(k)}, & k = k_i+1,\dots,k_i+s_i \\ t_{j(k+1)} - d_{j(k)j(k+1)} & k = k_i+s_i+1,\dots,k_{i+1}-1 \end{cases}
$$

for some «$_1$«(l,...,k$_{1+1}$-k$_1$-l}$_t$ i = l,...,s.

<u>Proof</u>. T(K)$_S$ is a special case of a polyhedron T_S whose vertices are characterized by Theorem 2.3. With the definition of k_i, i » l,...,s, given by (3.2), the conditions (i), (ii), (iii) of Theorem 2.1 specialize to (3.3), (3.4), (3.5) above.||

Note that T(K)$_g$ has exactly one vertex t° of the form

(3.6)

$\overset{o}{C}$jd) * ^{L}j(D'

$\overset{o}{C}$j(k) " ""^jtk)' $\overset{o}{C}$j(k-1) + ^{d}j(k-l)j(k)$^}$» ^{k} ' ^{2} - — P -

namely the vertex for which $s_{\pm}$ » k^-^-1. 1 -l,...,s. This vertex, which we call the <u>main vertex</u> of T(K)$_g$, will be seen to play a special role in the structure of P(K).

<u>Theorem 3.2</u>. The extreme direction vectors of T(K)$_g$ are w^1, i - 1,...,p, defined by

$$
(3.7) \qquad w^{i}_{j(k)} = \begin{cases} 1 & k = p-i+1,\dots,p \\ 0 & \text{otherwise.} \end{cases}
$$

<u>Proof</u>. For any t«T(K)$_g$, t + Xw^lgT(K)$_g$ for all \ > 0 and i » l,...,p, aa one can readily see by substituting t + Xw^1 for t into the constraints defining T(K)$_g$. Thus every w^1 defined by (3.6) is a direction vector of T(K)$_g$. Further, each w^1 satisfies w^l_j» o for j = j (1),..,j (p-i) and w_j* ^ - $w^i_{j(k-1)}$» » 0 for k » p-i+2,...,p. Thus each w^i satisfies with equality

p-1 inequalities whose coefficient matrix has full row rank, and therefore is extreme. ||

Our next theorem gives a necessary and sufficient condition for the main vertex of $T(K)_S$ to be its only vertex, i.e., for $T(K)_S$ to be a cone.

Theorem 3.3. $T(K)_S$ is the displaced polyhedral cone with vertex t^0 defined by (3.6) and extreme direction vectors w^i, $i = 1,\dots,p$, defined by (3.7) if and only if

$$\ell_{j(k-1)} + d_{j(k-1)j(k)} \geq \ell_{j(k)}, \qquad k = 2,\dots,p. \tag{3.8}$$

Proof. If (3.8) holds, then condition (3.2) of Theorem 3.1 is not satisfied for any integer k; hence $s = 1$, (3.4) is vacuous, and (3.3), (3.5) becomes $t^s_{j(1)} = \ell_{j(1)}$, $t_{j(k)} = t_{j(k-1)} + d_{j(k-1)j(k)}$, $k = 2,\dots,p$, which is of the form (3.6). Thus the main vertex is the only vertex of $T(K)_S$. On the other hand, if (3.8) is violated for some $k \in \{2,\dots,p\}$, the definition (3.4), (3.5), (3.6) gives rise to at least two distinct vertices. Thus $T(K)_S$ has exactly one vertex if and only if (3.8) holds. The extreme direction vectors of $T(K)_S$ are given by (3.7), irrespective of the number of vertices. ||

Next we turn to the extreme points and extreme direction vectors of P(K). Naturally, every extreme point of P(K) is an extreme point of $T(K)_S$ for some $S \subset Q(K)^*$; but the converse will be shown to be true only if P(K) satisfies a regularity condition. Also, every extreme direction of P(K) is an extreme direction of $T(K)_S$ for some $S \subset Q(K)^*$, but the converse is never true.

In order to prove some properties of the vertices of P(K) we need a characterization of the extreme direction vectors of P(K), so we start with the latter.

Theorem 3.4. The extreme direction vectors of P(K) are precisely the unit vectors e_i, $i = 1,\dots,p$.

~~Proof~~. For 1 - l,...,n, the unit vector $e_\pm$ is an extreme direction vector of every $T(K)_g$ such that i is the last node of the sequence defined by S. Hence every $*_\pm$ is a direction vector of P(K), and since $\bullet_i$ is a unit vector and T(K) is contained in the positive orthant, each e_t is extreme for P(K). Every other extreme direction vector of $T(K)_{S>}$ for every SeQ(K)*, is the sum of unit vectors; hence none of them is extreme for P(K). Since every extreme direction of P(K) is an extreme direction of $T(K)_g$ for some ScQ(K)*, it follows that P(K) has no extreme direction vectors other than the p unit vectors e_i.||

~~Theorem 3.5~~. Let S be an acyclic selection in < K > with associated sequence j(1).....j(p), p - |K|. Then t° is a vertex of P(K) if and only if t° is the main vertex of $T(K)_g$ for some SsQCK)*, and the conditions

$$(3.9) \qquad L_{j(i)} + d_{j(i)j(1)} > L_{j(1)} \qquad , \qquad i = 2,\ldots,p,$$

and

$$(3.10) \qquad \max[L_{j(k)}, t^o_{j(i-1)} + d_{j(i-1)j(k)}] + d_{j(k)j(i)} > t_{j(i)} \qquad \begin{array}{l} i = 2,\ldots,p-1 \\ k = i+1,\ldots,p \end{array}$$

are satisfied.

~~Proof~~. Sufficiency. Suppose t° is not a vertex of P(K). If t° is not the main vertex of $T(K)_g$ for some SeQ(K)*, we are done. Now assume t° is the main vertex of $T(K)_g$. Since t° is not a vertex of P(K), it is the convex combination of p+1 schedules $t^h \in T(K)$, h - 1.....p+1, such that $t^h \neq t°$, h » l,...,p+1.

Since t° is the main vertex of $T(K)_g$, $t° \leq t$ for all $t \in T(K)_S$ and hence at least one t^h must have a component $t^h_{j(k)}$ such that $t^h_{j(k)} < t°_{j(r)}$ for some k > r. Let r be the smallest integer for which there exists such k.

If r = 1, then

$$L_{j(k)} + d_{j(k)j(1)} \le t^h_{j(k)} + d_{j(k)j(1)}$$
$$\le t^h_{j(1)} = t^o_{j(1)} = L_{j(1)},$$

i.e., condition (3.9) is violated for i = k.

If $r \in \{2,\ldots,p-1\}$, then $t^*_{j(r-1)} < t^h_{j(k)} < t^*_{j(r)}$, and since t^h is a schedule,

$$t^o_{j(r)} = t^h_{j(r)} \ge t^h_{j(k)} + d_{j(k)j(r)}$$
$$\ge \max\{L_{j(k)},\ t^h_{j(r-1)} + d_{j(r-1)j(k)}\} + d_{j(k)j(r)}$$

Therefore, since $t^h_{j(r-1)} = t^o_{j(r-1)}$, (3.10) is violated for i = r.

Necessity. Note first that for any schedule t for < K >, if there exists a schedule t* for < K > such that

(3.11) $t^* \le t$ and $t^*_j < t_j$ for some $j \in K$,

then t is not a vertex of P(K), since it can be expressed as the sum of t* and a positive combination of unit vectors, i.e., direction vectors of P(K). Thus if t° is a vertex of P(K), then t° is the main vertex of $T(K)_S$ for some $S \in Q(K)^*$.

Now suppose t^o and the associated sequence j(1),...,j(p) are such that (3.9) is violated for some $i = q \in \{2,\ldots,p\}$. Let S* be the selection in < K > defined by the sequence $\ell(1),\ldots,\ell(p)$, where $\ell(1) = j(q)$ and

$$\ell(h) = \begin{cases} j(h-1) & h = 2,\ldots,q \\ j(h) & h = q+1,\ldots,p, \end{cases}$$

and let t* be the main vertex of $T(K)_{S^*}$. Then $t^*_{\ell(1)} = L_{\ell(1)} < t^o_{j(q)} = t^o_{\ell(1)}$, since

$$t^o_{j(q)} \geq t^o_{j(1)} + \sum_{h=2}^{q} d_{j(h-1)j(h)}$$

$$\geq L_{j(1)}$$

$$\geq L_{j(q)} + d_{j(q)j(1)}$$

$$\geq t^o_{\ell(1)}$$

where the last two inequalities follow from the assumption that (3.9) is violated for i = q, and $d_{j(q),j(1)} \geq 0$. For h = 2,...,p, we show by induction that $t^*_{\ell(h)} \leq t^o_{\ell(h)}$. We first do this for h∈{2,...,q}. For h = 2,

$$t^*_{\ell(2)} = \max\{L_{\ell(2)},\ t^*_{\ell(1)} + d_{\ell(1),\ell(2)}\}$$
$$= \max\{L_{j(1)},\ L_{j(q)} + d_{j(q),j(1)}\} = L_{j(1)} \leq t^o_{\ell(2)}\ ,$$

since (3.9) is violated for q. Suppose $t^*_{\ell(h)} \leq t^o_{\ell(h)}$ for h = 2,...,r-1, and let h = r ≤ q. Then by the induction hypothesis,

$$t^*_{\ell(r)} = \max\{L_{\ell(r)},\ t^*_{\ell(r-1)} + d_{\ell(r-1)\ell(r)}\}$$
$$\leq \max\{L_{j(r-1)},\ t^o_{j(r-2)} + d_{j(r-2)j(r-1)}\}$$
$$\leq t^o_{j(r-1)} = t^o_{\ell(r)}.$$

Next we proceed to h∈{q+1,...,p}. For h = q+1, we have

$$t^*_{\ell(q+1)} = \max\{L_{\ell(q+1)},\ t^*_{\ell(q)} + d_{\ell(q)\ell(q+1)}\}$$
$$\leq \max\{L_{j(q+1)},\ t^o_{j(q-1)} + d_{j(q-1),j(q+1)}\}$$
$$\leq \max\{L_{j(q+1)},\ t^o_{j(q)} + d_{j(q),j(q+1)}\}$$
$$\leq t^o_{j(q+1)} = t^o_{\ell(q+1)},$$

where we have used the triangle inequality $d_{j(q-1),j(q+1)} \le d_{j(q-1),j(q)} + d_{j(q),j(q+1)}$. If $q+1 = p$ we are done; otherwise suppose $t^*_{\ell(h)} \le t^\circ_{\ell(h)}$ for $h = q+1,\dots,r-1$, and let $h = r \ge q+2$. Then

$$\begin{aligned} t^*_{\ell(r)} &= \max\{L_{\ell(r)}, t^*_{\ell(r-1)} + d_{\ell(r-1)\ell(r)}\} \\ &\le \max\{L_{j(r)}, t^\circ_{j(r-1)} + d_{j(r-1),j(r)}\} \\ &\le t^\circ_{j(r)} = t^\circ_{\ell(r)}. \end{aligned}$$

We have shown that $t^* \le t^\circ$, with $t^*_j < t^\circ_j$ for $j = j(q) \ne \ell(1)$. Thus t^* satisfies (3.11) and hence t° is not a vertex of $P(K)$.

Next suppose t° violates (3.10) for some $i \in \{2,\dots,p-1\}$ and $k \in \{i+1,\dots,p\}$, and let $(i,k) = (q,k)$ be the first such pair. Consider the selection S^* defined by the sequence $\ell(1),\dots,\ell(p)$, where $\ell(q) = j(k)$ and

$$\ell(i) = \begin{cases} j(i) & i = 1,\dots,q-1,\ k+1,\dots,p \\ j(i-1) & i = q+1,\dots,k, \end{cases}$$

and let t^* be the main vertex of $T(K)_{S^*}$. Then $t^*_{\ell(i)} \le t^\circ_{\ell(i)}$ for $i = 1,\dots,q-1$. For $i = q$,

$$\begin{aligned} t^*_{\ell(q)} &= \max\{L_{\ell(q)}, t^*_{\ell(q-1)} + d_{\ell(q-1)\ell(q)}\} \\ &\le \max\{L_{j(k)}, t^\circ_{j(q-1)} + d_{j(q-1)j(k)}\} \qquad (3.12) \\ &< t^\circ_{j(q)} - d_{j(k)j(q)}, \end{aligned}$$

where the last inequality follows from the fact that (3.10) is violated for $(i,k) = (q,k)$. But then from $d_{j(k)j(q)} > 0$ and $t^\circ_{j(q)} < t^\circ_{j(k)} = t^\circ_{\ell(q)}$, we have $t^*_{\ell(q)} < t^\circ_{\ell(q)}$. For $i \in \{q+1,\dots,p\}$ we show by induction that $t^*_{\ell(i)} \le t^\circ_{\ell(i)}$. First let $i \in \{q+1,\dots,k\}$. For $i = q+1$,

$$t^*_{\ell(q+1)} = \max\{L_{\ell(q+1)}, t^*_{\ell(q)} + d_{\ell(q)\ell(q+1)}\}$$
$$= \max\{L_{j(q)}, t^*_{\ell(q)} + d_{j(k)j(q)}\}$$
$$\leq t^o_{j(q)} = t^o_{\ell(q+1)}. \qquad \text{(from 3.12)}$$

Now suppose $t^*_{\ell(i)} \leq t^o_{\ell(i)}$ for $i = q+1,\ldots,r-1$, and let $i = r$, with $q+2 \leq r \leq k$. Then

$$t^*_{\ell(r)} = \max\{L_{\ell(r)}, t^*_{\ell(r-1)} + d_{\ell(r-1)\ell(r)}\}$$
$$\leq \max\{L_{j(r-1)}, t^o_{j(r-2)} + d_{j(r-2)j(r-1)}\}$$
$$\leq t^o_{j(r-1)} = t^o_{\ell(r)}.$$

Next let $i \in \{k+1,\ldots,p\}$. For $i = k+1$,

$$t^*_{\ell(k+1)} = \max\{L_{\ell(k+1)}, t^*_{\ell(k)} + d_{\ell(k)\ell(k+1)}\}$$
$$\leq \max\{L_{j(k+1)}, t^o_{j(k-1)} + d_{j(k-1)j(k+1)}\}$$
$$\leq \max\{L_{j(k+1)}, t^o_{j(k)} + d_{j(k)j(k+1)}\} \text{ (by the triangle inequality)}$$
$$\leq t^o_{j(k+1)} = t^o_{\ell(k+1)}$$

Suppose $t^*_{\ell(i)} \leq t^o_{\ell(i)}$ for $i = k+1,\ldots,r-1$, and let $i = r \geq k+2$. Then

$$t^*_{\ell(r)} = \max\{L_{\ell(r)}, t^*_{\ell(r-1)} + d_{\ell(r-1)\ell(r)}\}$$
$$\leq \max\{L_{j(r)}, t^o_{j(r-1)} + d_{j(r-1)j(r)}\}$$
$$\leq t^o_{j(r)} = t^o_{\ell(i)}.$$

Thus $t^* \leq t^o$ and $t^*_{\ell(q)} < t^o_{\ell(q)}$, hence t^* satisfies (3.10) and t^o is not a vertex of $P(K)$.||

It is of some interest to characterize the situation when every vertex of every polyhedron $T(K)_g$, SeQ(K)*, is also a vertex of P(K). From Theorem 3.5, this is the case if and only if for every acyclic selection S in < K >, $T(K)_S$ is a cone and (3.9), (3.10) holds. But these conditions are not easy to check. Next we give an easy to check necessary and sufficient condition for the vertices of P(K) to be precisely those of the polyhedra $T(K)_C$, in tenns of a regularity condition suggested by (but different from) (3.9), (3.10). We say that a disjunctive set T(K) as well as the polyhedron P(K) and the clique < K > is <u>regular</u> if

(3.13) $\quad L_t + d_{tj} > Lj \quad , \quad$ ¥ i,jeK, i # j

and

(3.14) $\quad d_{ifc} + d_{fej} > d_{\pm j} \quad , \quad$ ¥ i,j,kcK_f i * j * k * i .

As we will presently show, regularity is a necessary and sufficient condition for $T(K)_g$ to be a cone and for (3.9) and (3.10) to hold for every acyclic selection S in < K >. Later we will see that regularity also plays a crucial role in the facial structure of P(K): certain facets exist if and only if T(K) is regular.

<u>Theorem 3.6</u>. The vertices of P(K) are precisely the vertices of the polyhedra $T(K)_g$, ScQ(K)*, if and only if T(K) is regular.

<u>Proof</u>. Sufficiency. Let T(K) be regular. From Theorem 3.3, condition (3.13) implies that each $T(K)_g$ is a cone. We will show that conditions (3.13) and (3*14) imply (3.9) and (3.10) for every acyclic selection S in < K >. Let S be any such selection, with associated sequence j(l),...,j(p), and let t° be the vertex of $T(K)_g$. Then (3.13) clearly implies (3.9), and (3.14) implies

$$(3.15)\qquad t^o_{j(i-1)} + d_{j(i-1)j(k)} + d_{j(k)j(i)} > t^o_{j(i-1)} + d_{j(i-1)j(i)}$$

for all $i = 2,\ldots,p-1$, $k = i+1,\ldots,p$. Further, (3.13) implies

$$t^o_{j(i-1)} + d_{j(i-1)j(i)} \geq L_{j(i-1)} + d_{j(i-1)j(i)} > L_{j(i)},$$

hence $t^o_{j(i-1)} + d_{j(i-1)j(i)} = t^o_{j(i)}$, which together with (3.15) implies (3.10).

Necessity. We show that if any of the conditions (3.13) or (3.14) is violated, there exists some acyclic selection S in < K > such that (3.9) or (3.10) is violated. Suppose (3.13) is violated for some $i,j \in K$. Then (3.9) is violated for every acyclic selection S whose sequence contains j as first node. Now suppose (3.13) holds, but (3.14) is violated for some $i,j,k \in K$. Consider any selection S whose sequence contains i and j as two consecutive nodes, say $i = j(h-1)$ and $j = j(h)$, with $k = j(k)$ such that $k > h$. Then the violation of (3.14) implies

$$(3.16)\qquad t^o_{j(h-1)} + d_{j(h-1)j(k)} + d_{j(k)j(h)} \leq t^o_{j(h-1)} + d_{j(h-1)j(h)}\ .$$

Since (3.13) holds, we have

$$t^o_{j(h-1)} + d_{j(h-1)j(k)} \geq L_{j(h-1)} + d_{j(h-1)j(k)} > L_{j(k)}\ ,$$

which together with (3.16) implies

$$\max\{L_{j(k)},\ t^o_{j(h-1)} + d_{j(h-1)j(k)}\} + d_{j(k)j(h)}$$
$$\leq \max\{L_{j(h)},\ t^o_{j(h-1)} + d_{j(h-1)j(h)}\}$$
$$\geq e_{j(h)}\ \bullet$$

i.e., (3.10) is violated for $i = h$.||

While the regularity conditions (3.13), (3.14) are simpler and much easier to check than the conditions (3.9), (3.10), and while they are necessary and sufficient for (3.9), (3.10) to hold for <u>every</u> acyclic selection, note that they cannot replace (3.9) and (3.10) when it comes to a particular acyclic selection: regularity is a sufficient, but not a necessary condition for some vertex of a particular $T(K)_S$ to be a vertex of $P(K)$.

Example 3.1. Consider the clique $K = \{1,2,3\}$ shown in Fig. 2, with $L_1 = 10$, $L_2 = 8$, $L_3 = 11$; $d_{12} = 1$, $d_{13} = 2$, $d_{21} = 2$, $d_{23} = 4$, $d_{31} = 1$, $d_{32} = 2$.

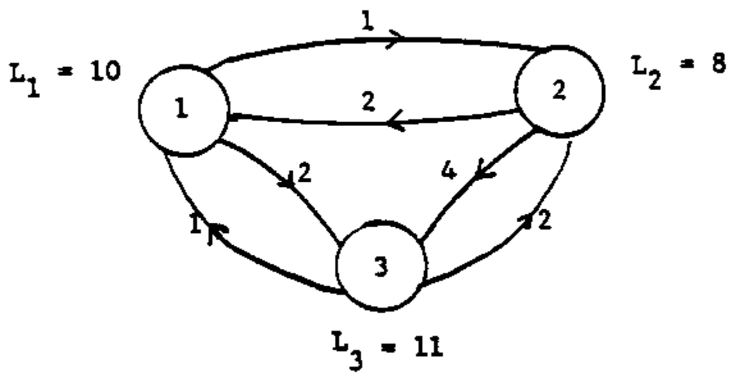


Fig. 2

Condition (3.13) is violated for the ordered pair $\{i,j\} = (2,1)$ and condition (3.14) for the ordered triples $\{2,1,3\}$ and $\{3,1,2\}$. Table 1 lists

Table 1

Sequences associated with S	Main Vertex t° of $T(K)_S$		
	all d_{ij} as specified	$d_{21} = 3$	$d_{21} = 3$, $d_{31} = 2$
1,2,3	(10,11,15)	(10,11,15)*	(10,11,15)*
3,1,2	(13, 8,12)	(13, 8,12)*	(14, 8,12)*
2,3,1	(12,13,11)*	(12,13,11)*	(13,14,11)*
1,3,2	(10,14,12)	(10,14,12)*	(10,14,12)*
2,1,3	(10, 8,12)*	(11, 8,13)*	(11, 8,13)*
3,2,1	(15,13,11)	(16,13,11)	(16,13,11)*

*Vertices of $P(K)$.

the sequences associated with the 3! = 6 acyclic selections $S \in Q(K)^*$ and the main vertex of each of the corresponding polyhedron $T(K)_S$. Because the regularity conditions are violated, only 2 of the 6 main vertices are vertices of $P(K)$: (12,13,11) and (10,8,12). For every other t, there exists some t' such that $t' \leq t$. If we replace $d_{21} = 2$ by $d_{21} = 3$, condition (3.12) is satisfied for all $i,j \in K$, $i \neq j$, and condition (3.13) is violated only for the triplet {3,1,2}. As a result, all but one of the vertices of the polyhedra (now cones) $T(K)_S$ become vertices of $P(K)$, the exception being (16,13,11) (since thetre exists a vertex (12,13,11)). If we also replace $d_{31} = 1$ by $d_{31} = 2$, $T(K)$ becomes regular, and as a result all 6 vertices of the cones $T(K)_{S}$ become vertices of $P(K)$.||

Next we turn to the facets of $P(K)$.

4. Facets of the Clique Polyhedron

Given a convex polyhedron $C \subset R^n$, an inequality $\alpha x \geq \alpha_0$ is said to define (or induce) a k-dimensional face of C, if $\alpha x \geq \alpha_0$ for every $x \in C$ and $\alpha x = \alpha_0$ for k + 1 affinely independent points $x \in C$. Thus the inequality $\alpha x \geq \alpha_0$ defines a ~~facet~~ of C, if $\alpha x \geq \alpha_0$, for all $x \in C$, and $\alpha x = \alpha_0$ for n affinely independent points $x \in C$.

Let $|K| = p$. For $i = 1,\ldots,p!$, let S^i be the i^{th} acyclic selection in $\langle K \rangle$, and let $j_i(1),\ldots,j_i(p)$ be the sequence associated with S^i. Further, let v^i be the main vertex of $T(K)_{S^i}$, i.e., let $v^i \in R^p$ be the vector whose components are defined recursively by

$$(4.1)\qquad v^i_{j_i(k)} = \begin{cases} L_{j_i(1)}, & k = 1 \\ \max\{L_{j_i(k)},\ v^i_{j_i(k-1)} + d_{j_i(k-1),j_i(k)}\}, & k = 2,\ldots,p. \end{cases}$$

Finally, let V be the p! X p matrix whose 1^{th} row is v^{i}, and let

e * $(l_f...fl)^T$ **ave P! components.

<u>Theorem 4,1</u>. The inequality at $\geq$ 1, where a, t e!R^P, defines a facet of P(K) if and only if a is a vertex of the polyhedron

$$F = \left\{ \alpha \in \mathbb{R}^p \;\middle|\; \begin{array}{c} V\alpha \geq e \\ \alpha \geq 0 \end{array} \right\}.$$

<u>Proof</u>, at $\geq$ 1 defines a facet of P(K) if and only if (i) at $\geq$ 1 for all tcP(K), and (ii)at * 1 for p affinely independent points teP(K).

Condition (i) holds if and only if criF. Indeed, every vertex of P(K) is present among die row vectors v^{i} of V; and the extreme direction vectors of P(K) are the rows of the identity matrix associated with the constraint cr $\geq$ 0. Furthermore, every row v^{i} that is not a vertex of P(K), is nevertheless contained in P(K). Hence at $\geq$ 1 is satisfied by all teP(K), if and only if Va $\geq$ e and a $\geq$ 0, i.e., if and only if a€F.

Further, condition (ii) holds if and only if for some integer kc£l,...,p}f P(K) has k extreme points $v^{i(h)}$, h s 1,...,k, and p-k extreme direction vectors $e_{j(h)}$ ' h * k + 1»###»P» such that $v^{i(h}$ *cr • It h s lf...fk and e_{j}va$_{j}$• $\alpha_{j(h)}$ " °» h « k + l,...,p. The "if" part of this statement holds since $v^{i(1)}\alpha$ = 1 and $a_{j(h)}$ * 0 imply $(v^{i(1)} + e_{j(h)})$ a • 1, h - k+l,...,p, and the p points $v^{i(1)},...,v^{i(k)}$, $v^{i(1)} + e_{j(k+1)}$»---»$v^{i(1)} + e_{j(p)}$ «e affinely independent. The "only if" part follows from the fact that any tcT(K) that is not a vertex of P(K) and satisfies at » 1, can be represented as a positive linear combination of extreme points v^{i} of P(K) that satisfy αv^{i} • 1, and extreme direction vectors e_j of P(K) that satisfy e_ja • 0, where the weights of the v^{i} sum to 1. Thus (ii) holds if and only if for some

$k \in \{1,\ldots,p\}$, α satisfies with equality k of the inequalities $v^i\alpha \geq 1$ and p-k of the inequalities $\alpha_j \geq 0$, such that the p inequalities in question form a system of rank p; i.e., if and only if α is a vertex of F.||

Of course, Theorem 4.1 remains true if all redundant inequalities are removed from the system defining F. Because of the large number of constraints that define F, Theorem 4.1 by itself does not seem to offer a practical way of generating facets of P(K). When combined with the next Theorem however, it provides an efficient way of obtaining those facet inducing inequalities with few positive coefficients.

<u>Theorem 4.2</u>. Let < H > and < K > be cliques, with $H \subset K$, $|H| = \ell$ and $|K| = p$, $2 \leq \ell \leq p$. The inequality $\alpha y \geq 1$, where $\alpha, y \in \mathbb{R}^\ell$, defines a facet of P(H), if and only if the inequality $(\alpha,0)t \geq 1$, where $(\alpha,0)$, $t \in \mathbb{R}^p$, de-defines a facet of P(K).

<u>Proof</u>. Necessity. Suppose $\alpha y \geq 1$ defines a facet of P(H). Then there exist ℓ affinely independent points $y^i \in P(H)$, $i = 1,..,\ell$, such that each y^i is a schedule for < H >, and $\alpha y^i = 1$, $i = 1,\ldots,\ell$. Each y^i can be extended to a schedule t^i for < K > as follows. If $S(H)^i$ is the selection in < H > defined by y^i, let $S(K)^i$ be any acyclic extension of $S(H)^i$ to < K > such that the rank in $S(K)^i$ of any $j \in H$ is less than that of any $k \in K\backslash H$. Then let $t^i \in \mathbb{R}^p$ be any vector satisfying $t^i_j = y^i_j$, $j \in H$, $t^i_j \geq L_j$, $j \in K\backslash H$, and $t^i_j - t^i_h \geq d_{hj}$, $\forall\ (h,j) \in S(K)^i$. Extending each y^i in this fashion gives ℓ affinely independent points $t^i \in P(K)$.

The remaining $p-\ell$ schedules are derived from t^1. Assume w.l.o.g. that the nodes of K are numbered in the order defined by $S(K)^1$, i.e., such that $t^1_1 < t^1_2 < \ldots < t^1_p$. For $i = \ell + 1,\ldots,p$, let t^i be defined by $t^i_j = t^1_j$, j - l,...fp - i + 2, and t_j^i • t_j^1 + l_f j - p - i + 2 + $l_{f\#\#.fp}$. Then the (p-l)xp matrix whose rows are the vectors t^i - t^1, i » 2,...,p, is of the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix},$$

where M_u is of rank 2-1, while M_{22} = (m_{ij}) is the (p-X)x(p-X) matrix defined by m^ - 0 if i + j $\leq$ p-X and m_{ij} « 1 if i + j > p-2. Since M_{22} is nonsingular, the rank of M is p-1. Thus (a,0)t $\geq$ 1 defines a facet of P(K).

Sufficiency. If ay $\geq$ 1 does not define a facet of P(H), then it is the consequence of some inequalities $P^i y \geq 1$, i * l,...,k $\leq$ 2, satisfied by every ycT(H). Then the inequalities $(0^i,O)t \geq 1$, i » 1,...,k, where $(0^1 fO)$, teR^P, are satisfied by every tcP(K) (since a restriction to < H > of a schedule for < K > is a schedule for < H >), and imply the inequality (a,O)t $\geq$ 1. Thus (a,0)t $\geq$ 1 does not define a facet of P(K).||

From Theorem 4.2 it follows that the computational effort required to generate a facet inducing inequality for P(K), with positive coefficients restricted to some subset HCK, depends only on the cardinality of H, not that of K. Thus there are large classes of facets of P(K) that can be generated at a fixed computational cost, whatever the size of K. More generally, the work needed to derive a facet inducing inequality for P(K) grows with the number of positive coefficients of the inequality; and facets defined by inequalities with few positive coefficients are easy to generate.

Next we address the question of how one can derive a facet inducing inequality with p positive coefficients from one with p-1 positive coefficients. Let < K > be a clique with |K| * p, let HCK with JH| » p-1, say H » {!,...,p-l}, and let V and W be the matrices whose rows are the vertices of P(K) and P(H), respectively. Note that every row w^i of W corresponds to some row v^k of V, where $v^k = (w^i, v_{kp})$, and the sequence associated with v^k assigns rank p to node p. For all the remaining rows of V, the associated sequence assigns rank p to some node $j \in \{1,\dots,p-1\}$. Let R(V) and R(W) denote the row index sets of V and W, where every row of V that corresponds to a row of W preserves the index of the latter, i.e., the first $|R(W)|$ elements of R(V) are those of R(W).

For any matrix M, let det(M) denote the determinant of M, let M_S denote the matrix whose rows are those rows of M indexed by S, and let M^j be the matrix obtained from M by substituting a column of 1's for the j^{th} column.

~~Theorem 4.3~~. Let W_S be a (p-1)x(p-1) submatrix of W such that the inequality $\alpha x \geq 1$, where the components of α are

$$(4.2) \qquad \alpha_j = \begin{cases} \dfrac{\det(W_S^j)}{\det(W_S)} > 0, & j = 1,\dots,p-1 \\ 0, & j = p, \end{cases}$$

induces a facet of P(K). Further, let

$$(4.3) \qquad \frac{\det(V^p_{S\cup\{k\}})}{\det(V_{S\cup\{k\}})} = \min_{i \in R(V)\setminus S} \frac{\det(V^p_{S\cup\{i\}})}{\det(V_{S\cup\{i\}})}$$

Then the inequality $\beta x \geq 1$, where the components of β are

$$(4.4) \qquad \beta_j = \frac{\det(V^j_{S\cup\{k\}})}{\det(V_{S\cup\{k\}})}, \qquad j = 1,\dots,p,$$

also induces a facet of P(K); and if the minimum in (4.3) is positive and unique, then $\beta_j > 0$, $j = 1,\dots,n$.

__Proof__. Since the inequality $\alpha x \geq 1$ induces a facet of P(K), it also induces a facet of P(H) (Theorem 4.2), hence the vector $\alpha = (\alpha_1,\dots,\alpha_{p-1})$

is a vertex of the polyhedron $F^{W} = \{\alpha \mid W\alpha \geq e,\ \alpha \geq 0\}$ (Theorem 4.1).
We have to show that if (4.3) holds, then θ defined by (4.4) is a vertex of $F^{V} = \{\theta \mid V\theta \geq e,\ \theta \geq 0\}$. Then by Theorem 4.1, the inequality $\theta t \geq 1$ induces a facet of P(K).

Consider the system of equations

$$(4.5) \qquad v^{i}\theta = 1\ ,\quad i \in S$$

where v^{i} is the i^{th} row of V. Since $S \subset R(W)$, each v^{i} is of the form (w^{i}, v_{ip}).
There are two possible cases.

Case 1. There exists no $\theta \in F^{V}$ satisfying (4.5) with $\theta_{p} > 0$. Then there exists some $k \in R(V)\backslash S$ such that (4.5) together with $v^{k}\theta = 1$ implies $\theta_{p} = 0$ and has the unique solution $\theta = \alpha$. Hence the minimum in (4.3) is 0 and $\theta = \alpha$ is a vertex of F^{V}.

Case 2. The minimum in (4.3) is positive, i.e., there exists $\theta \in F^{V}$ satisfying (4.5) with $\theta_{p} > 0$. Then (4.5) defines an edge of F^{V}, one of whose endpoints is $\theta = \alpha$, whereas the other endpoint is given by the smallest value of θ_{p} for which either (i) some inequality $\theta_{j} \geq 0$, $j \in \{1,\ldots,p-1\}$, becomes tight; or (ii) some inequality $v^{i}\theta \geq 1$, $i \in R(V)\backslash S$, becomes tight.
Let θ_{p}^{1} and θ_{p}^{2} be the values of θ_{p} for which (i) and (ii), respectively, occur.
We claim that $\theta_{p}^{1} \geq \theta_{p}^{2}$. For suppose $\theta_{p}^{1} < \theta_{p}^{2}$, i.e., there exists a vector $\theta^{\circ} \in R^{p}$ that satisfies (4.5) and $\theta^{\circ}_{j_*} = 0$ for some $j_* \in \{1,\ldots,p-1\}$, and such that $v^{i}\theta^{\circ} > 1$, $\forall\ i \in R(V)\backslash S$. Then $(\theta^{\circ}_{1},\ldots,\theta^{\circ}_{j_*-1},\theta^{\circ}_{j_*+1},\ldots,\theta^{\circ}_{p})$ is a vertex of $F^{V} \cap \{\theta \mid \theta_{j_*} = 0\}$, hence we have $v^{i}\theta^{\circ} = 1$ for p-1 of those inequalities indexed by $i \in R(V)\backslash S$, for which j_* has rank p in the sequence defined by v^{i}. But this contradicts the assumption that $\theta_{p}^{1} < \theta_{p}^{2}$.

Now θ_{p}^{2} is the value defined by (4.3), namely the p^{th} component of the solution θ, as defined by (4.4), of the system $v^{i}\theta = 1$, $i \in S \cup \{k\}$, where

$k \in R(V) \setminus S$ is the index of the inequality that becomes tight for $\beta_p = \beta_p^2$. Hence β is a vertex of F^V, i.e., $\beta t \geq 1$ induces a facet of P(K).

Further, if the minimum in (4.3) is both positive (as in case 2 above) and unique, then $\beta_j > 0$ for all j, since otherwise, as shown above, the minimum in (4.3) is not unique.||

In the following we will list all facet inducing inequalities for P(K) with 2 or 3 positive coefficients. But first we examine the trivial facet inducing inequalities, i.e., those having a single positive coefficient.

<u>Proposition 4.4</u>. For all $j \in K$, the inequality $t_j \geq L_j$ induces a facet of P(K).

<u>Proof</u>. W.l.o.g, we assume that $L_j > 0$ for all j. This can always be guaranteed by shifting the origin of the coordinate system, which does not affect the facial structure of P(K). Then the vector α defined by $\alpha_j = 1/L_j$, $\alpha_i = 0$, $\forall\, i \neq j$, is a vertex of $F = \{\alpha | V\alpha \geq e,\ \alpha \geq 0\}$, where the rows of V are the vectors v^i defined by (4.1). Hence from Theorem 4.1, the inequality $\alpha t \geq 1$, that is $t_j \geq L_j$, induces a facet of P(K).||

Next we turn to facet defining inequalities with two nonzero coefficients.

<u>Theorem 4.5</u>. Let $< K >$ be a clique. For any $i,j \in K$, $i \neq j$,

$$(4.6) \qquad (d_{ij} + L_i - L_j)t_i + (d_{ji} + L_j - L_i)t_j \geq d_{ij}d_{ji} + L_i d_{ji} + L_j d_{ij}$$

is a nontrivial facet inducing inequality for P(K) if and only if

$$(4.7) \qquad -d_{ji} < L_j - L_i < d_{ij}.$$

<u>Proof</u>. From Theorem 4.2, (4.6) defines a facet of P(K) if and only if it defines a facet of P({i,j}). From Theorem 4.1, this is the case if and only if the point $\alpha^o = (\alpha_i^o, \alpha_j^o)$, where

$$\alpha_i^o = \frac{d_{ij}+L_i-L_j}{d_{ij}d_{ji}+L_i d_{ji}+L_j d_{ij}} \quad , \qquad \alpha_j^o = \frac{{}^{d}1i+V^{L}i}{d_{ij}d_{ji}+L_i d_{ji}+L_j d_{ij}} \quad ,$$

is a vertex of the polyhedron F({i,j}) defined by the inequalities

L^ + maxCLj, $L_\pm$ + d ^ } ^ $\geq$ 1

(4.8) m a x ^ , L_j + d ^ } ^ + L_{jaj} $\geq$ 1

<*$_t$-> °t αj ≥ 0

If (4.7) holds, then the maximum in the first and second inequalities of (4.8) is attained for $L_i + d_{ij}$ and $L_j + d_{ji}$, respectively, and a_o is the unique solution to the system obtained by requiring these two inequalities to be tight. Since *ct°* also satisfies the remaining two inequalities of (4.8), it is a vertex of F({i,j}) and hence the inequality (4.6) defines a facet of P(K). Further, if (4.7) holds, then c£ > 0 and $a_j^?$ > 0, i.e., the facet is nontrivial.

On the other hand, if $L_i - L_j \geq d_{ij}$ or $L_j - L_i \geq d_{ji}$ (both inequalities cannot hold at the same time), then the maximum in the first or second inequality of (4.8) is attained for L_i or L_j, respectively, and the solution to the system of two equations is ^ * 0, a_j » 1/L_j in the first case, a_j * 0, αr^ * 1/L_i in the second; hence in these cases at $\geq$ 1 coincides with one of the two trivial facet defining inequalities associated with the indices i,j, and (4.6) does not induce a facet.||

Note that (4.7) is the regularity condition (3.4) for the clique < U*j} >• Since |[i,j}| * 2, condition (3.5) does not apply. Thus regularity of the clique < {i,j} > is a necessary and sufficient condition for the polyhedron P(K) (where < K > is any clique containing [t,j}) to have a facet inducing inequality at ≥ 1 with $_a$^ > 0, cr.$_J$ > 0 and Q^ * 0, ¥ k # i,j.

Next we characterize the facet inducing inequalities with 3 nonzero coefficients for an arbitrary clique < K > with JK| « p. From Theorem 4.2, an inequality of the form $a_{j_1} t_{j_1} + a_{j_2} t_{j_2} + a_{j_3} t_{j_3} \geq 1$ induces a facet of P(K) if and only if it induces a facet of P(£j$_1$, j$_2$> J$_3$}), the clique polyhedron defined on the vertex set (j$_1$, j$_2$, j$_3$}. From Theorem 4.1, this is the case if and only if a * (a_{j_1}, a_{j_2}, a_{j_3}) is a vertex of the polyhedron

F - (aelR^{3}jV<* $\geq$ e, $_a$ $\geq$ 0)$_f$

where ec $\mathbb{R}^6$ and V is the 6 x 3 matrix whose rows are defined by (4.1) for p » 3. To simplify the notation, we assume that (j^ J$_2$, J$_3$) • [1,2,3}. Denoting by p_i the sequence (permutation) associated with row v^i of V, we will assume that the rows of V, indexed by R(V), are ordered so that

p_1 = (1,2,3) $\qquad$ P$_4$ - d,3,2)

p_2 = (2,3,1) $\qquad$ P$_5$ - (2,1,3)

p_3 = (3,1,2) $\qquad$ P* * (3,2,1).

Further, we will assume that < {1,2,3} > is regular; which implies that the matrix V is of the form

$$V = \begin{pmatrix} L_1 & L_1 + d_{12} & L_1 + d_{12} + d_{23} \\ L_2 + d_{23} + d_{31} & L_2 & L_2 + d_{23} \\ L_3 + d_{31} & L_3 + d_{31} + d_{12} & L_3 \\ h & L_1 + d_{13} + d_{32} & L_1 + d_{13} \\ L_2 + d_{21} & L_2 & L_2 + d_{21} + d_{13} \\ L_3 \quad d_{32} + d_{21} & L_3 + d_{32} & h \end{pmatrix}$$

As in Theorem 4.3, we let $V_{\{i,k,\ell\}}$ denote the 3 X 3 matrix consisting of rows i,k,ℓ of V, and let $V^j_{\{i,k,\ell\}}$ be the matrix obtained from $V_{\{i,k,\ell\}}$ by substituing 1 for every entry of column j.

<u>Theorem 4.6</u>. Let $K = \{1,\ldots,p\}$, let $P(\{1,2,3\})$ be regular, and let every 4 x 4 submatrix of (V,e) be nonsingular. Then $P(K)$ has exactly four facets induced by inequalities $\alpha t \geq 1$ with $\alpha_j > 0$ for $j = 1,2,3$, $\alpha_j = 0$ for $j = 4,5,\ldots,p$. In particular the coefficients of the four inequalities are defined by

$$(4.9) \qquad \alpha_j = \det(V^j_{\{i,k,\ell\}})/\det(V_{\{i,k,\ell\}}), \quad j = 1,2,3,$$

$\alpha_j = 0$, $j = 4,5,\ldots,p$, where the four triplets $i,k,\ell \subset R(V)$ are $\{1,5,r\}$, $\{2,6,s\}$, $\{3,4,t\}$ and $\{r,s,t\}$, with $\{r,s,t\} = \{2,3,1\}$ or $\{4,5,6\}$.

<u>Proof</u>. From Theorem 4.2, an inequality $\alpha t \geq 1$ with $\alpha_j = 0$, $j = 4,5,\ldots,p$, induces a facet of $P(K)$, if and only if the inequality $\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 \geq 1$ induces a facet of $P(\{1,2,3\})$. From Theorem 4.1, this is the case if and only if α is a vertex of the polyhedron $F = \{\alpha \in R^3 \mid V\alpha \geq e,\ \alpha \geq 0\}$.

According to a classical result of Steinitz, the number of vertices of a polytope (bounded polyhedron) in R^3 is bounded by $2f - 4$, where f is the number of facets; and this bound is attained when the polytope is simple (totally nondegenerate), i.e., when each vertex lies on exactly 3 facets, or, equivalently, on exactly 3 edges (see for instance Grünbaum [5], p. 190). Now F is never simple, since $v_{11} = v_{41}$, $v_{22} = v_{52}$ and $v_{33} = v_{63}$; as a result each of the 3 vertices having a single positive component (namely: $\alpha_1 = 1/L_1$, $\alpha_2 = \alpha_3 = 0$; $\alpha_2 = 1/L_2$, $\alpha_1 = \alpha_3 = 0$; and $\alpha_3 = 1/L_3$, $\alpha_1 = \alpha_2 = 0$) lies on 4 facets, i.e., is degenerate, if it exists at all (i.e., if $L_j > 0$). Furthermore, F is unbounded. We therefore define a polytope (bounded polyhedron) F^*, obtained from F by

(i) assuming $L_j > 0$, $j = 1,2,3$ (this guarantees the existence of the 3 vertices with one positive component);

(ii) replacing L_j by $L_j + \epsilon > L_j$, $j = 1,2,3$, in rows 4,5,6 (this makes those same 3 vertices nondegenerate); and

(iii) adding the inequality $\alpha_1 + \alpha_2 + \alpha_3 \le M$, where $M > 1/L_j$, $j = 1,2,3$ (this makes F^* bounded).

Given the regularity of < {1,2,3} > and the assumption that every 4 x 4 submatrix of (V,e) is nonsingular, F^* is simple; and listing its vertices allows us to list those of F.

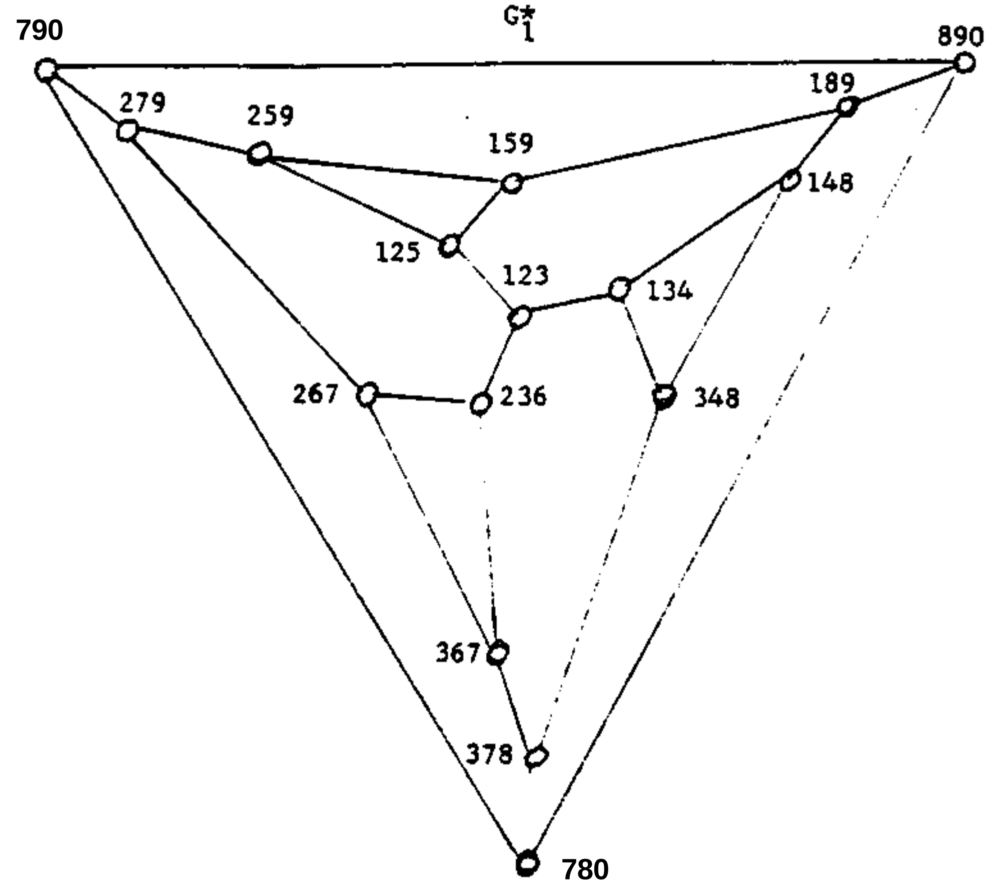
Since F^* has 10 facets (defined by the 6 inequalities $v^i\alpha \ge 1$, the 3 inequalities $\alpha_j \ge 0$, and the inequality introduced in (iii)), it has $2f - 4 = 16$ vertices. Of these, 3 lie on the plane $\alpha_1 + \alpha_2 + \alpha_3 = M$ and are therefore not vertices of F. Another triplet consists of the 3 vertices with exactly one positive component; these are also vertices of F. A third triplet of vertices of F^*, also shared with F, are those with exactly two positive components, that give rise to the facet defining inequalities (4.6) for the corresponding 2-clique polyhedra. A fourth triplet consists of those vertices of F^* having two positive components, whose counterparts in F have a single positive component (because of the degeneracy caused by $v_{11} = v_{41}$, $v_{22} = v_{52}$, $v_{33} = v_{63}$). This is a total of 12 vertices of F^* (6 vertices of F) with one or two positive components (see Table 2, in which the facets are numbered from 1 to 6 for $v^i\alpha \ge 1$, $i = 1,\ldots,6$; 7,8,9 for $\alpha_j \ge 0$, $j = 1,2,3$; and 0 for $\alpha_1 + \alpha_2 + \alpha_3 \le M$). Thus there are 4 facets left, each with 3 positive components.

From Theorem 4.3, there is a vertex with 3 positive components adjacent to every vertex with 2 positive components. Two vertices (of a 3-dimensiooal polytope) are of course adjacent if and only if they share

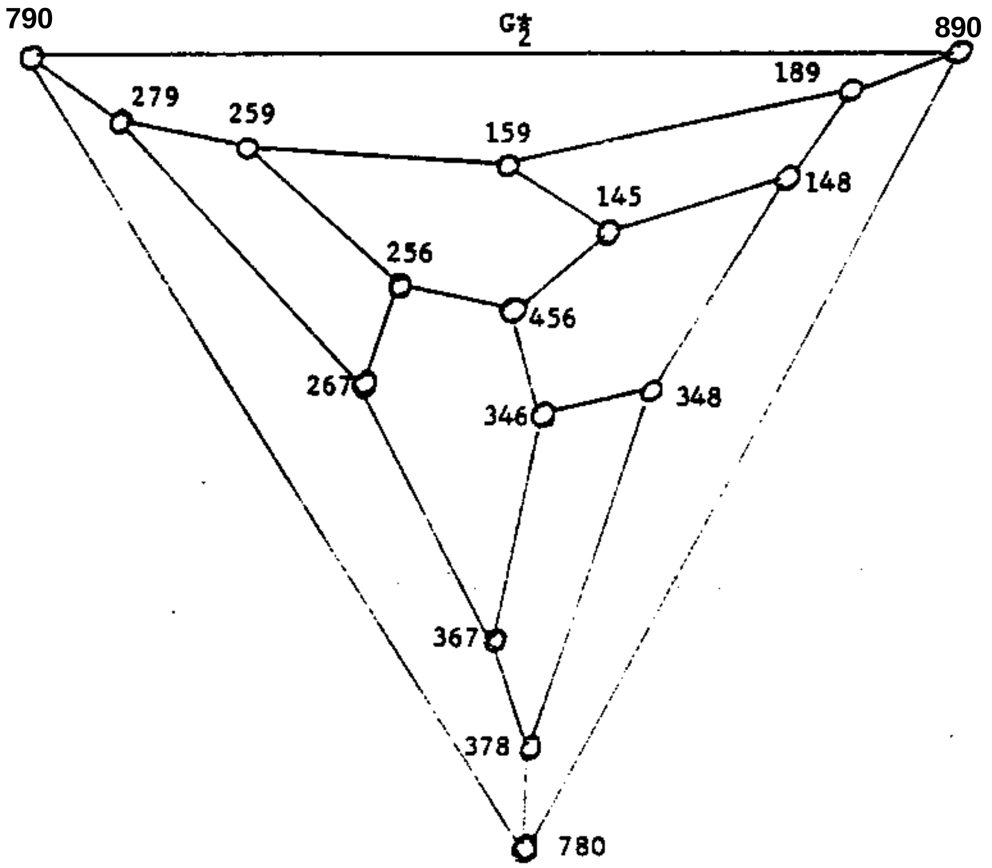
Table 2

Vertex of F*	Positive compoaents	Lies on facets	Vertex of F	Positive components	Lies on facets
1	j - 1	0,8,9	-	j - 1	1,4,8,9
2	j » 2	0,7,9	-		
3	j » 3	0,7,8	-		
4	j - 1	1,8,9	1	j - 1	1,4,8,9
5	j - 2	2,7,9	2	j » 2	2,5,7,9
6	j - 3	3,7,8	3	j - 3	3,6,7,8
7	j - 1,2	1,5,9	4	j » 1,2	1,5,9
8	j • 2,3	2,6,7	5	j - 2.3	2,6,7
9	j - 1.3	3.4,8	6	j » 1,3	3,4,8
10	j - 1,2	2.5,9	2	j - 2	2,5,7,9
11	j - 2,3	3,6.7	3	j '- 3	3,6,7,8
12	j - 1,3	1,4,8	1	j - 1	1,4,8,9

two facets. Thus the vertices with 3 positive components adjacent to {1,5,9}, {2,6,7} and {3,4,8} are of the fora {1,5,r}, {2,6,s} and {3,4,t}, respectively; whereas those adjacent to {2,5,9}, {3,6,7} and {1,4,8} are of the form {2,5,u}, {3,6,w} and {1,4,2}, respectively. Clearly, at least 3 of these 6 potential vertices are distinct, and we know there exists a 4^{th} vertex with 3 positive components. Finally, every vertex is adjacent to exactly 3 other vertices. Checking all possible combinations shows that there are only two ways of satisfying these requirements, namely if

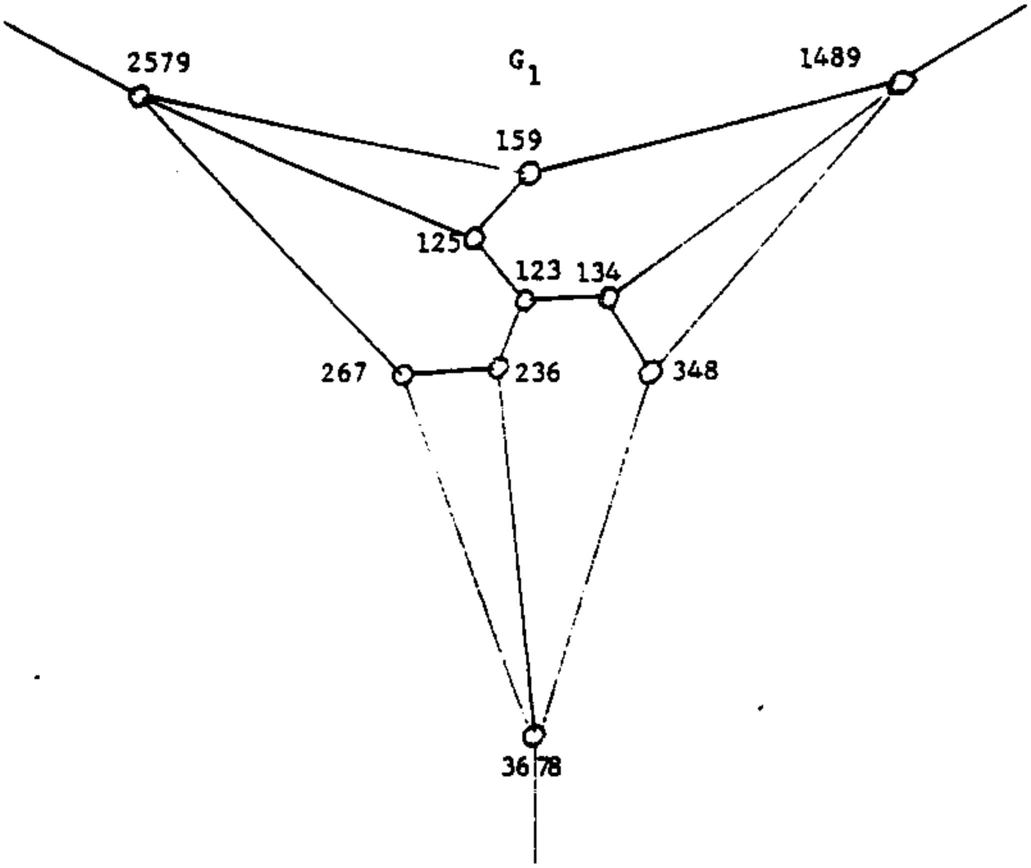


G_1^*
790
890
279
259
159
189
148
125
123
134
267
236
348
367
378
780

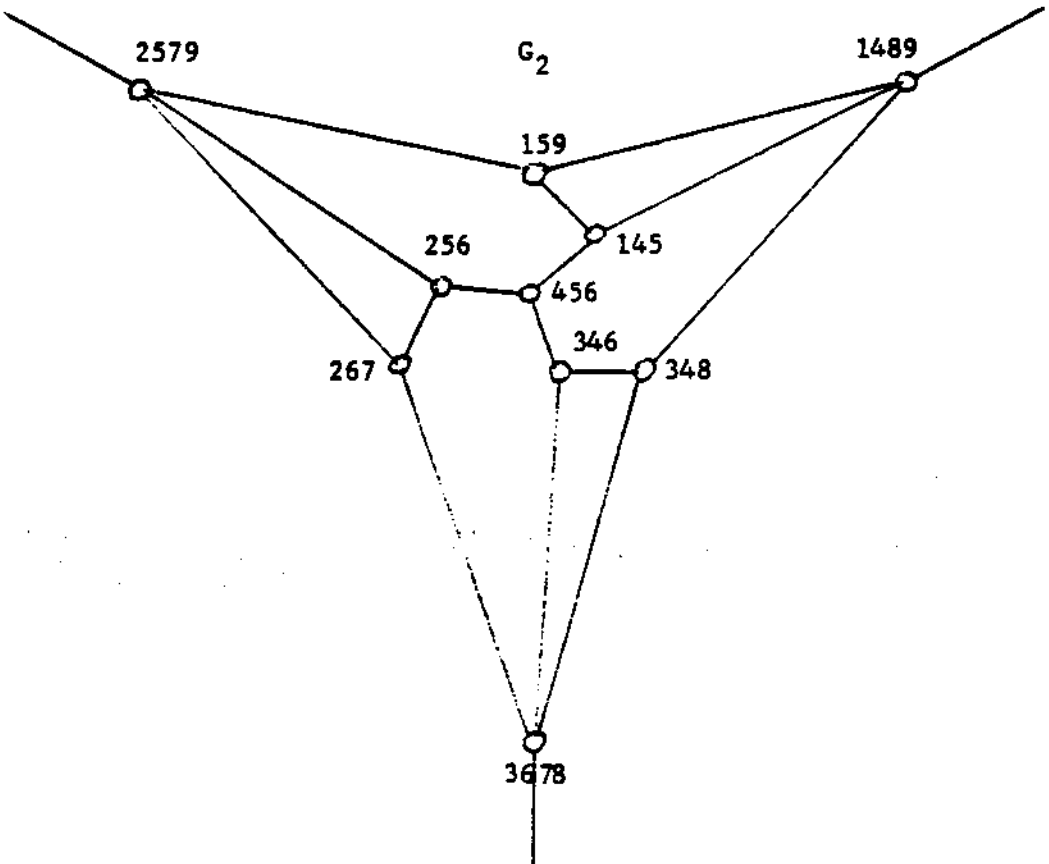


G_2^*
790
890
279
259
159
189
145
148
256
456
267
346
348
367
378
780

Fig. 3



G1
2579
1489
159
125
123
134
267
236
348
3678



G2
2579
1489
159
145
256
456
346
267
348
3678

Fig. 4

(r,s,t} - {2,3,1} and {u,w,z} - (1,2,3}, or if {r,s,t} » £4,5,6} and {u,w,z} «■ (6,4,5}. In the first case, there exists a vertex {1,2,5}, adjacent to [1,5,9} and to [2,5,9}; a vertex [2,3,6}, adjacent to [2,6,7} and [3,6,7}; and a vertex [1,3,4}, adjacent to (3,4,8} and to (1,4,8}. The 4^{th} vertex with 3 positive components is in this case (1,2,3}, adjacent to (1,2,5}, (2,3,6} and (1,3,4}. In the second case, there exists a vertex (1,4,5}, adjacent to (1,5,9} and (1,4,8}; a vertex (2,5,6}, adjacent to (2,6,7} and {2,5,9}; and a vertex (3,4,6}, adjacent to (3,4,8} and {3,6,7}. The fourth vertex in this case is {4,5,6}, adjacent to {1,4,5}, {2,5,6} and {3,4,6}.

Thus the only two possible facial structures of F* are those represented by the graphs G_1^* and G_2^* of Fig. 3.||

Note that the polytope F*, which is bounded and totally nondegenerate (simple), has 16 vertices and 24 edges. The (unbounded) polyhedron F has at most (i.e., when the only degenerate vertices are those with 1 positive component) 10 vertices and 18 edges, as shown in Fig. 4, where G_1 and G_2 are the "graphs" of F (the 3 unbounded edges of F being represented by "half-edges[11] pf G_x and *G^*.

Thus P(K) has at most 4 facets induced by inequalities at $\geq$ 1 with $a_j > 0$ for j - 1,2,3. The regularity of < {1,2,3} > is a necessary condition for the existence of 4 distinct facets of this type, but is not by itself sufficient. For sufficiency we need, besides regularity, the absence of any singular 4 x 4 submatrices of (V,e), as assumed in the Theorem.

<u>Example 4.1</u>. Let G be the disjunctive graph shown in Fig. 5.

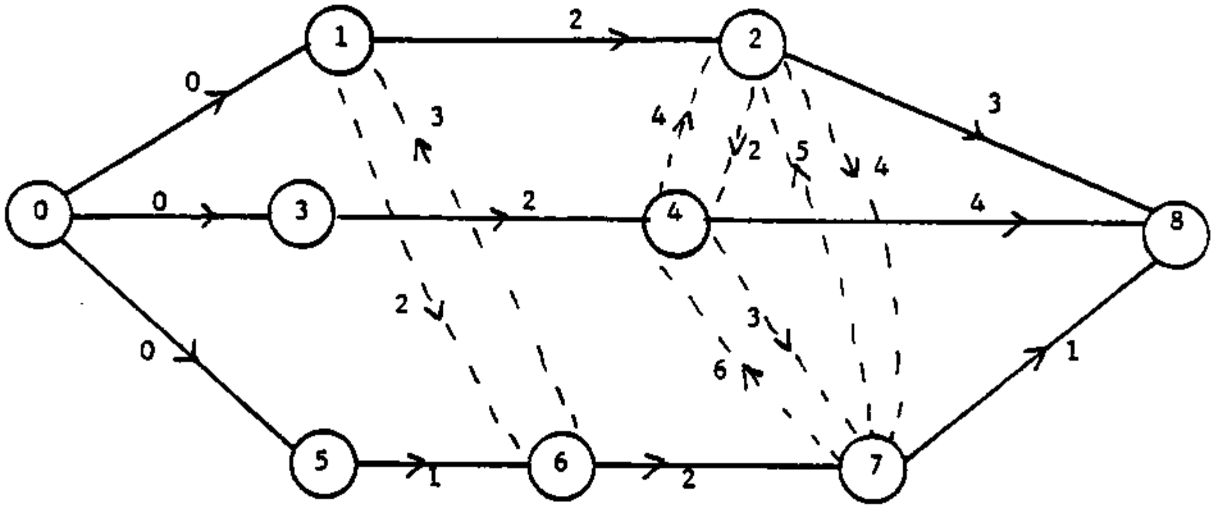


Fig. 5.

6 **has** two disjunctive cliques, induced by the node sets K«i * {1,6} **and** K_2 - {2,4,7}, respectively. For < 1^ > we have L_1 » L(0,l) * 0, L_g - L(0,6) - 1, and d_{16} « 2, d_{gl} * 3. PO^) has 3 facets, defined by the inequalities t, > 0, t. > 1 (Proposition 4.4), and t- + 4t. > 8 (Theorem 4.5).

For < Kj >, w« **have** L_2 - L(0,2) » 2, L_4 - L(0,4) - 2, 1^ - L(0,7) - 3, and d_{24} - 2, $d_{2?}$ » 4, d_{42} - 4, d_{47} - 3, $d_{?2}$ » 5, $d_{?4}$ - 6. We see that < K_2 > is regular, **and the matrix** defining the polyhedron F is

$$
V = \begin{array}{c}
\begin{array}{ccc} 2 & 4 & 7 \end{array} \\
\begin{pmatrix}
10 & 2 & 5 \\
8 & 10 & 3 \\
2 & 12 & 6 \\
6 & 2 & 10 \\
13 & 9 & 3
\end{pmatrix}
\end{array}
$$

P(K_2) has 10 facets: 3 of them are defined by the trivial inequalities t_2 ^ 2> C4 ^ 2» C7 - 3 (Proposition 4.4); another 3 by the inequalities

$$t_2 + 2t_4 \geq 10$$
$$t_2 + 2t_7 \geq 14$$
$$2t_4 + 7t_7 \geq 39$$

with 2 positive coefficients (Theorem 4.5); and, finally, 4 facets are defined by inequalities with 3 positive coefficients (Theorem 4.6):

$$5t_2 + 16t_4 + 4t_7 \geq 102$$
$$t_2 + 5t_4 + 19t_7 \geq 115$$
$$13t_2 + 3t_4 + 24t_7 \geq 206$$
$$t_2 + t_4 + 3t_7 \geq 27.$$

These 4 inequalities correspond, in the notation of Theorem 4.6, to the vertices {1,2,5}, {1,3,6}, {1,3,4} and {1,2,3}, respectively, of F. Here we have multiplied each inequality with the determinant in the denominator of the expression (4.9) in order to express them in integers.||

5. Lifting the Facets of the Clique Polyhedron

In this section we address the question as to how the results of the previous sections can be used to derive facet inducing inequalities for the general scheduling polyhedron $P = \text{clconv } T$ introduced in section 1. In particular, we give a sufficient condition for a facet inducing inequality for one of the clique polyhedra $P(K)$ to also be facet inducing for P.

We introduce some additional notation. For any $i \in N$, let $B(i)$ and $A(i)$ be the set of nodes $j \in N$ "before i" and "after i," respectively, in the digraph $D = (N^0, A^0)$; that is,

$$B(i) = \{j \in N \setminus \{i\} \mid \text{there exists a directed path } P(j,i) \text{ in } D\}$$

$$A(i) = \{j \in N \setminus \{i\} \mid \text{there exists a directed path } P(i,j) \text{ in } D\}.$$

Further, for any VCN, let

$$B(V) \gg \bigcup_{icV} B(i) \quad , \quad A(V) * \bigcup_{icV} A(i).$$

Before addressing the issue of lifting the facets of the clique polyhedron, we examine the role of the trivial inequalities $t_i \geq L(0,i)$ in determining the facial structure of P.

<u>Theorem 5.1</u>. For every icN, the inequality $t_i \geq L(0,i)$ defines a (n-q)-dimensional face of P, where q « |(i}UB(i)|.

<u>Proof</u>. Every tcP that satisfies t_i » L(0,i) also satisfies t_j » L(O,j) for every jcB(i). Hence the face Pn{teH^a|t_i » L(0,i)} of P is at most (n-q)-dimensional, where q * |[i}UB(i)j. To show that it is exactly (n-q)-dimensional, we will construct n-q+1 affinely independent schedules in G that satisfy t_i * L(0,i). Let S be any acyclic selection in G such that for all rcM, if K_r has a node jcB(i), then j has rank 1 in S(K_r); and let 4(h,j)$_s$ denote the length of a longest path from h to j in D_g » (N°$_f$ A^{0}US). Further, let the nodes of D_S be numbered such that (h,j)cAUS implies h < j, and in addition, (i}UB(i) * {l,-..,q}. Such a numbering exists, since D_g is an acyclic digraph and every hcB(i) (IK$_r$ has rank 1 in S(K)$_r$. We then define t° by t°_j « *(0,j)$_S$. Clearly, t° is a schedule in G. Further, by the definition of S, 4(0,j)$_g$ - L(O,j) for all jc(i}UB(i), hence t° satisfies t_i » L(0,i). The next n-q schedules t^h, h » l,...,n-q, are defined recursively by fcj* « t^*^{1} for jcN\{n-h+l] and t^1? » t^1?"^{1} + 1 for j - n-h+1. Each of these vectors is a schedule that satisfies t_i = L(0,i). Then the (n-q) x n matrix whose rows are the vectors $t^h - t^o$, h * l,...,n-q, is of the form M » (M^ M$_2$), where M.$_1$ is (n-q) X q, while M$_2$ is the (t>q) X (n-q) nonsingular matrix

$$M_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 1 \\ \hline l & \cdots & l & l \end{pmatrix}$$

Thus M has rank n-q, and the n-q+1 schedules t^h are affinely independent.||

<u>Corollary 5.2</u>. The inequality $t_i \geq L(O,i)$ defines a facet of P if and only if B(i) « 0.

Next we address the question of lifting the facets of clique polyhedra. We need a couple of definitions and some auxiliary results.

Let < K > be a clique, S(K) an arbitrary acyclic selection in K, and < K_f > the maximal clique containing < K > . As before, let M be the index set of the maximal cliques of G. We will'say that the selection

$$S - \bigcup_{rcM} S(K_r)$$

is a <u>conformal</u> extension of S(K) to G, if it satisfies the following requirements:

(i) S(K^) is any acyclic extension of S(K) to < K_i > , such that, if ieK and jeKAK, the rank of i in S(KJ is less than that of j.

(ii) For rcM\[i} such that $K_r \cap B(K) = 0$, S(K_r> is any acyclic selection in < K_r > .

(iii) For r«M\{X} such that K_rnB(K) + 0, S(K_r) is any acyclic selection in < K_r > such that

(a) if icK$_r$HB(K) and jeK$_r$\B(K), the rank of i in S(K_r) is less than that of j;

O) if jcK$_r$P.B(i) for some icK, the rank of j in S(K_r) is no greater than the rank of i in S(K); and

(Y) if i,hcK, j(i)eK_rnB(i), j(h)cK_r f! B(h), and the rank in S(K) of i is less than that of h, then the rank in S(K_r) of j(i) is less than that of j(h).

For any ieN, B(i) is the set of nodes jcN\Ci) lying on the (unique) path P(0,i) from 0 to i in D. Therefore every clique has at most one node in B(i), Let M(i) be the index set of cliques that have such a node, i.e., M(i) » {reM(K_r H B(i)#J}, and let [j_r (i)} » K_rnB(i).

A (not necessarily maximal) clique < K > of G will be called <u>dominant</u>, if for every i.heK such that M(i)riM(h)*J, and every reM(i)nM(h),

$$d_{j_r(i)j_r(h)} + L(j_r(h),h) < L(j_r(i),i) + d_{ih} \cdot \tag{5,1}$$

The term "dominant" seems justified by the properties of these cliques.

<u>Lemna 5.3</u>. Let < K > be a dominant clique in G, and S(K) an acyclic selection in < K > . Then every conformal extension S of S(K) to G has the property that, if icK, jc{O}UB(K) and i is reachable from j in the digraph D_S » (N^o, A^oUS), every longest path from j to i in D_S contains only arcs of A^oUS(K).

~~Proof~~. Let S be a conformal extension of S(K) to G, and for some icK, jc{0}llB(K), let $P(j,i)_s$ be a longest path from j to i in D_g. Suppose now that $P(j,i)_s$ contains an arc of S\S(K); in particular, let (J_1,J_2) be the last such arc encountered when $P(j,i)_S$ is traversed in the direction of its arcs, and let (J_1»J_2)«$S(K_r)$• Then from property (iii) of S, for k * 1,2, J_kcB(K); in particular, j_k lies on the unique path P(0,i_k) in D for some

$i_k \in K$, and $(i_1, i_2) \in S(K)$. Further, if < K > is dominant, $d_{j_1 j_2} + L(j_2, i_2) < L(j_1, i_1) + d_{i_1 i_2}$ and replacing the segment of $P(j,i)_s$ from j_1 to i_1 by the path $P(j_1, i_1) \cup \{(i_1, i_2)\}$ yields a path from j to i in D_c longer than $P(j,i)_s$. This proves that $P(j,i)_s$ cannot contain any arc of $S \setminus S(K)$.||

Theorem 5.4. Let < K > be a dominant clique in G, y° a schedule for < K > with associated selection S(K), and S a conformal extension of S(K) to G. Then the vector $t^\circ \in K^a$ defined by

$$(5.2) \qquad t^\circ_j = \begin{cases} y^\circ_j & j \in K \\ L(0,j)_S & j \in B(K) \\ U - L(j,n)_S & j \in N \setminus K \cup B(K) \end{cases}$$

is a schedule for G if U is sufficiently large to satisfy, for any selection V in G, the condition

$$(5.3) \qquad U > \max \{L(0,n)_V, \max_{j \in K} \{y^\circ_j + L(j,n)_V\}\} .$$

Proof. We show that t° is a schedule for G by showing that it is a schedule for D_S. For this purpose we examine all the arcs of D_S and show that t° satisfies the associated inequalities. All pairs i,j considered below are such that $(i,j) \in A' \cup S$.

If both i and j belong to any one of the three sets K, B(K) or $N \setminus K \cup B(K)$, then substituting the values of t° given by (5.2) into the inequality $t^\circ_j - t^\circ_i \geq d_{ij}$ shows the latter to be satisfied.

For $i \in B(K)$, $j \in N \setminus K \cup B(K)$, $t_j - t_i = U - L(j,n)_S - L(0,i)_S \geq d_{ij}$, since $U > L(0,n)_S \geq L(0,i)_S + d_{ij} + L(j,n)_S$.

For $i \in C$, $j \in N \setminus K \cup B(K)$, $t_j^\circ - t_i^\circ = U - L(j,n)_S - y_i^\circ \geq d_{ij}$, since $U \geq y_i^\circ + L(i,n)_S \geq y_i^\circ + d_{ij} + L(j,n)_S$.

It remains to be shown that the constraints are also satisfied for $i \in B(K)$, $j \in K$; for all remaining ordered pairings of the three index sets used in the definition of t°, the corresponding arc sets are empty.

Now for $i \in B(K)$ and $j \in K$, $t_j^\circ - t_i^\circ = y_j^\circ - L(0,i)_S$. Let the rank of node j in $S(K)$ be k. The schedule y° satisfies the inequalities $y_{j(h)}^\circ \geq L(0,j(h))$, $h = 1,\ldots,p$, and $y_{j(h)}^\circ - y_{j(h-1)}^\circ \geq d_{j(h-1),j(h)}$, $h = 2,\ldots,p$, where $p = |K|$ and h is the rank of $j(h)$ in $S(K)$. It is not hard to see that these inequalities, plus the fact that $j = j(k)$, imply

$$(5.4) \qquad y_j^\circ \geq \max\{L(0,j(k)),\ L(0,j(k-1)) + d_{j(k-1),j(k)}, \ldots, L(0,j(1)) + \sum_{h=2}^{k} d_{j(h-1),j(h)}\}.$$

The expression on the righthand side of (5.4) represents the length of a longest among those paths from 0 to j in D_S, which use only arcs in $A^\circ \cup S(K)$. Since $\langle K \rangle$ is a dominant clique, from Lemma 5.3 this is equal to $L(0,j)_S$, the length of any longest path from 0 to j in D_S. Hence we have

$$t_j^\circ - t_i^\circ = y_j^\circ - L(0,i)_S \geq L(0,j)_S - L(0,i)_S \geq d_{ij}.$$

Since t° satisfies all the inequalities associated with the arcs of D_S, it is a schedule for D_S, hence for G.||

We are now ready to state the main result of this section.

<u>Theorem 5.5</u>. Let $\langle K \rangle$ be a (not necessarily maximal) dominant clique of G, with $|K| = p > 1$. If the inequality $ay \geq 1$, where $a, y \in R^p$, de-

fines a facet of P(K), then the inequality $(\alpha,0)t \geq 1$, where $(\alpha,0),t\in \mathbb{R}^n$, defines a facet of P.

<u>Outline of proof</u> If the inequality $\alpha y \geq 1$ defines a facet of P(K), there exists a set of p extreme points y^i, $i = 1,\ldots,p$ of P(K), such that $\alpha y^i = 1$, $i = 1,\ldots,p$.

Since $< K >$ is dominant, from Theorem 5.4 every y^i has at least one conformal extension t^i to G. From each such schedule t^i for G, additional schedules can be constructed by adding a small positive scalar to certain components. Using this approach one can in fact construct n affinely independent schedules t^i for G, each of which is an extension of some schedule for $< K >$ and therefore satisfies $\alpha t^i = 1$. This proves that the inequality $(\alpha,0)t \geq 1$ induces a facet of P. Details are given in an Appendix.||

6. Identifying Violated Inequalities

For every clique $< K >$ of G, let $\mathcal{F}(K)$ be the set of all facet inducing inequalities for P(K) = clconv T(K), and let $\mathcal{F} = \cup\mathcal{F}(K)$, where the union is taken over all cliques of G. In order to be able to use the inequalities of $\mathcal{F}$ as cutting planes in an algorithm for solving (P), one needs a way to solve the following

<u>Constraint Identification Problem</u> (CIP). Given some $t^o\in \mathbb{R}^n$ that satisfies $t^o_j - t^o_i \geq d_{ij}$, $(i,j)\in A$, $t^o_i \geq 0$, $i\in N$, but violates some of the disjunctions defining T, find an inequality in $\mathcal{F}$ violated by t^o, or show that none exists.

Let $t^o\in \mathbb{R}^n$ be as defined in CIP, let $< K >$ be a clique at least one of whose disjunctions is violated by t^o, let F(K) be the polyhedron defined in Theorem 4.1 relative to $< K >$, and denote by t_K the vector whose components are t_j, $j\in K$. Further, let α^o be defined by

(6.1) $$t^o_K\alpha^o = \min\{t^o_K\alpha \mid \alpha \in F(K)\}.$$

Then if $t^o_K\alpha^o < 1$, the inequality $\alpha^o t_K \geq 1$ obviously cuts off t^o and CIP is solved. Otherwise we have

<u>Proposition 6.1</u>. If $t^o_K\alpha^o \geq 1$, $t^o_K \in P(K)$, i.e., t^o satisfies all the inequalities of $\mathcal{S}(K)$.

<u>Proof</u>. If $t^o_K\alpha^o \geq 1$, then from the definition of α^o, $\alpha t^o_K \geq 1$ for every vertex α of $F(K)$.||

Thus the procedure that suggests itself for solving CIP is to choose some clique $< K >$ at least one of whose disjunctions is violated by t^o_K, and solve (6.1). However, in the absence of additional information we may well choose a clique $< K >$ for which $t^o_K\alpha^o \geq 1$. Also, if $< K >$ is large, solving (6.1) is expensive.

The next Theorem gives a sufficient condition for $\mathcal{S}(K)$ to contain an inequality violated by t^o. The condition occurs frequently and is easy to check. Furthermore, the Theorem restricts the size of $< K >$ to the minimum subject to the above condition.

<u>Theorem 6.2</u>. Let t^o be as defined in CIP. Let $< K >$ be a (not necessarily maximal) clique, with $|K| \geq p$ and $t^o_{j(1)} \leq \ldots \leq t^o_{j(p)}$, such that t^o satisfies

(6.2) $$t^o_{j(1)} = L(0,j(1)),$$

(6.3) $$t^o_{j(p)} < t^o_{j(p-1)} + d_{j(p-1),j(p)},$$

and, if $p \geq 3$,

(6.4) $$t^o_{j(k)} = t^o_{j(k-1)} + d_{j(k-1),j(k)}, \quad k = 2,\ldots,p-1.$$

Further, let $a°$ be defined by (6.1). Then the inequality $a°t_K \geq 1$ cuts off t^o.

<u>Proof</u>. We prove by contradiction that t^R tf(K). It then follows that $\mathcal{F}(K)$ contains an inequality that cuts off t°, and from (6.1), cr°t^ $\geq$ 1 is such an inequality.

Suppose t_K^oeP(K). Then there exist vectors t_K^ieT(K) and scalars $\lambda_i \geq 0$, i » l,...,p+l, such that

$$t_K^o = \sum_{i-1}^{p+1} t_K^i \lambda_i \quad , \quad \sum_{l-l}^{9+1} \lambda_i = 1.$$

Since $t_{j(1)} \geq L(O,j(l))$ for any t^TOO and $t^o_{j(1)}$ - L(O,j(l)), we have c^i_jd) * c^o_J(D^f * " 1ı*"ı(ri"1* SlalUrly, since t $_{(fc)}$ > max[L(0,j (k)), ^(k-l) + ^{d}j(k-l),$_{j(}$k)^ for aU ^W and ^{e}Jck) " ^{c}jCk-l) + ^{d}j(k-l),j(k)' k » 2,...,p-l$_f$ it follows that $t^i_{j(k)}$ * ^f^) * ^{k} * Zf-fP^if whenever p $\geq$ 3. But then from (6.3), for at least one icf1,... ,p+1}$_f$ we have t^, $_N$ < t° $_N$ < ^(P-D +^{d}J(p-D J(P) " $c^i_{j(p-1)}$ + $d_{j(p-1),j(p)}$, contrary to the assumption that t^€T(K), i » l$_f$...$_f$p+l. Thus $t_K^o \notin P(K)$.||

Condition (6.2) of Theorem 6.2 requires that the smallest component of t_g^o be equal to the lower bound on its value in any schedule. This condition is always met by a basic schedule t° for those cliques < K > such that no node of B(K) is contained in any disjunctive clique. For other cliques, the condition may or may not be satisfied, but it is of course easy to check. The remaining conditions simply state that a minimum size clique to be considered is the one with node set K » (j(1),...,j(p)}, where j(l) is the node for which $t^o_{j(1)}$ » L(O,j(l)), and j(p) is the first node in the sequence defined by t° for which the condition $t_{j(p)} - t_{j(p-1)} \geq d_{j(p-1),j(p)}$ (and hence the corresponding disjunction) is violated.

When there is no clique for which the conditions of Theorem 6.2 are satisfied, there is no guarantee that a° defined by (6.1) cuts off t°. In such cases it is a reasonable heuristic to choose a clique for which (6.3) and (6.4) are satisfied, while $t_{j(1)} - L(0,j(1))$ is small (in comparison with other cliques), and which has not yet been used to derive a cut.

Example 6.1. Consider the disjunctive graph G of Example 4.1. Minimizing t_g subject to $t_j - t_i \geq d_{ij}$ (i,j)εA and $t_i \geq 0$, iεN, yields t° = (0,2,0,2,0,1,3,6). Since tj » L(0,l) = 0 and t£ » 1 < tj + d_{16} « 2, the clique induced by {1,6} satisfies the conditions of Theorem 6.2. Thus we solve

$$\min 0a_1^* + 1a_6^*$$
$$\text{s.t. } 0\alpha_1 + 2\alpha_6 \geq 1$$
$$4\alpha_1 + 1\alpha_6 \geq 1$$
$$\alpha_1, \alpha_6 \geq 0$$

and find (α_1, α_6) - (1/8, 1/2), which yields the inequality

violated by t°. Since < {1,6} > is a dominant clique, this inequality induces a facet of P. Minimizing t_7 subject to the same constraints as before, plus ^ + $4t_g \geq 8$, yields t^1 » (0,2,0,2,0,2,4,6).

Since t_2^1 » L(0,2) » 2 **and** t£ < tj + d^ * 4, the clique induced by {2,4} satisfies the conditions of Theorem 6.2. Solving

$$\min 2a^\wedge + 2a\#$$

$$6\alpha_2 + 2\alpha_4 \geq 1$$
$$\alpha_2, \alpha_4 \geq 0$$

yields (a_2^*, a£) - d/10, 1/5), and the inequality

$$t_2 + 2t_4 \geq 10$$

violated by t^1. Again, < {2,4} > is a dominant clique and hence the inequality induces a facet of P. Adding this inequality to the earlier constraint set on t and minimizing $t_?$ yields t^2 » (0,4,0,3,0,2,4,7).

The conditions of Theorem 6.2 are no longer satisfied, since t_j° > L(O,j) for j » 2,4,7. However, each of the cliques not yet used to derive a cut, i.e., (4,7}, (2,7} or {2,4,7}, provides an inequality that cuts off t^2 (this can be seen by checking the list of facet-inducing inequalities for P(K_2) in Example 4.1). In particular, if we take the clique {2,4,7}, then solving

$$\begin{array}{rrrrl}
\text{oin} & 2a_2 + & 4«_4 + & 4«_7 & \\
\text{s.t.} & 2«_2 + & 4«_4 + & 7_{a?} & \geq 1 \\
 & 10\alpha_2 + & 2\alpha_4 + & 5\alpha_7 & \geq 1 \\
 & 8\alpha_2 + & 10\alpha_4 + & 3\alpha_7 & \geq 1 \\
 & 2\alpha_2 + & 12\alpha_4 + & 6\alpha_7 & \geq 1 \\
 & 6\alpha_2 + & 2\alpha_4 + & 10a_? & \geq 1 \\
 & 13a_2 + & 9\alpha_4 + & 3a_? & \geq I \\
 & & & \alpha_2, \alpha_4, a_7 & \geq 0,
\end{array}$$

yields (a_2^3, $a_{4>}^3$ c^) - (13/206, 3/206, 24/206) (with $s_\pm$ » 0 for i - 1,3,4), and the (facet inducing) inequality

$$13t_2 + 3t_4' + 24t_y \geq 206,$$

which cuts off t^2.

References

[1] E. Balas, "Finding a Minimaximal Path in a Disjunctive PERT Network." Theorie des Graphes, Journées Internationales d'Etude, Dunod, 1967, p. 21-30.

[2] E. Balas, "Machine Sequencing via Disjunctive Graphs: An Implicit Enumeration Algorithm," Operations Research, 17, 1969, p. 941-957.

[3] E. Balas, "Disjunctive Programming." Annals of Discrete Mathematics, 5, 1979, p. 3-51.

[4] E. Balas, "Disjunctive Programming and Combinatorial Optimization." Paper presented at the Symposium on the Application of Discrete Mathematics, Cambridge, Mass., June 27-30, 1983.

[5] M. Grötschel, M. Jünger and G. Reinelt, "Facets of the Linear Ordering Polytope." Report No. 82217-OR, Institut für Ökonometrie und Operations Research, University of Bonn, May 1982.

[6] B. Grünbaum, Convex Polytopes, Wiley, 1967,

[7] R. G. Jeroslow, "Cutting Plane Theory: Disjunctive Methods." Annals of Discrete Mathematics, 1, 1977, p. 293-330.

[8] J. K. Lenstra, Sequencing by Enumerative Methods, Mathematical Centre Tracts 69, Mathematisch Centrum, Amsterdam, 1977

[9] J. F. Muth and G. L. Thompson, Industrial Scheduling, Prentice-Hall, 1963.

[10] L. Németi, "Das Reihenfolgeproblem in der Fertigungs-programmierung und Linearplanning mit logischen Bedingungen." Mathematica (Cluj), £, 1964, p. 87-99.

[11] A. H. G. Rinnooy Kan, Machine Scheduling Problems: Classification, Complexity and Computations, Nijhoff, The Hague, 1976.

[12] B. Roy and B. Sussman, "Les Problèmes d'Ordonnancement avec Contraintes Disjonctives." SEMA, Note DS 9 bis, Decembre 1964.

Appendix: Proof of Theorem 5.5.

We will make use of the following auxiliary result:

Lemma 5.6. Let < K > be a dominant clique of G, y^o an extreme point of P(K) with associated selection S(K), and S a conformal extension of S(K) to G. Further, let t^o be the extension to G of y^o defined by (5.2), and let k∈K be such that $y^o_k > L(0,k)$. Then every path $P(i,j)_S$ in D_S originating with some i∈B(k) and such that $t^o_s - t^o_r = d_{rs}$ for all $(r,s)\in P(i,j)_S$, terminates in some j∈B(K).

Proof. Let $P(i,j)_S$ be a path in D_S originating with some i∈B(k) and such that $t^o_s - t^o_r = d_{rs}$ for all $(r,s)\in P(i,j)_S$. Since $t^o_i = L(0,i)_S$, there exists a (longest) path $P(0,i)_S$ from 0 to i in D_S such that $t^o_s - t^o_r = d_{rs}$ for all $(r,s)\in P(0,i)_S$. It then follows that the path $P(0,j)_S := P(0,i)_S \cup P(i,j)_S$ is a longest path from 0 to j in D_S, since $t^o_s - t^o_r = d_{rs}$ for all $(r,s)\in P(0,j)_S$. Now suppose j∈K. Since < K > is a dominant clique of G, it then follows from Lemma 5.3 that $P(0,j)_S$ contains only arcs of $A^o \cup S(K)$, i.e., is of the form $P(0,k) \cup P(k,j)_S$, where P(0,k) is the (unique) path from 0 to k in D. But then P(0,k) is a longest path from 0 to k in D_S and $t^o_k = L(0,k)$, contrary to $t^o_k = y^o_k > L(0,k)$, as assumed in the Lemma. Thus j∉K.

Suppose next that j∈N\K∪B(K), and let (r,s) be the (unique) arc of $P(i,j)_S$ such that r∈B(K), s∈N\K∪B(K). Then from the definition of t^o, $t^o_s - t^o_r > d_{rs}$, contrary to our assumption about $P(i,j)_S$. This proves that j∉N\K∪B(K). Consequently j∈B(K). ||

Proof of the Theorem. Let y^i, i = 1,...,p, be extreme points of P(K), each of which satisfies $\langle y^i \rangle = 1$. We will construct n schedules t^i for G, each of which is an extension of one of the p schedules y^i for < K >, and therefore satisfies $(a,0)t^i = 1$. We will then prove that these n vectors $t^i \in R^n$ are affinely independent, by showing that the (n-1) x n matrix whose rows are the vectors $t^i - t^1$, i = 2,...,n, is of full row rank.

W.l.o.g., we assume that the numbering of the nodes of G is such that K = {1,...,p}, B(K) = {p+1,...,q}, and N\K∪B(K) = {q+1,...,n}.

(i) First, we extend to G the p affinely independent schedules y^i, i = 1,...,p, for < K >. To this end for i = 1,...,p we let $S(K)_i$ be the selection in < K > associated with y^i, and S_i a conformal extension of $S(K)_i$ to G, with the proviso that the arcs of S_i chosen freely under rule (ii) of the definition of a conformal extension (see section 5) are the same for all icfl,...,p}. Next, for i « 1,...,P, we let t^1 be the extension of y^1 to G defined by (5.2) for S » S , with the proviso that the scalar U used in the definition of t be the same for all kcfl,...,?}• The fact that the vectors t defined in this way are schedules for G follows from Theorem 5.4. Note that our specific* tions for S. and t^1 imply that L(j,n) S * L(j,n) S and t* * t^1, jeN\K'JB(K), i - 2,...,n.

Subtracting the vector t from each of the p-1 vectors t , i - 2,...,p, yields the (p-1) X n matrix M. whose rows are t - t , i«2,...,p, and which is of the form ^ » 0**,** M_{2}, 0). Here M-- is the (p-1) X p full row rank matrix whose rows are the p-1 linearly independent vectors y - y , i = 2,...,p, M_{9} is (p-1) x (q-p), and 0 is the (p-1) X (n-q) zero matrix.

(ii) The next q-p schedules t , i * p+l,...,q, are generated as follows. For every node kcK there exists at least one among the p vectors y chosen at the beginning of this proof, say y , such that y > L(0,k). To see why this is true, notice that if y * L(0,k) for i * l,...,p, then the p vectors y lie in the (p-2)-dimensional subspace of $\mathbb{R}^P$ defined by the two equations ay * 1 and y ^{s} L(0,k), hence they cannot be affinely independent.

Now let S(K) be the selection in < K > associated with y , S a conformal extension to G of S(K)., and t the extension to G of y defined by (5.2) for S - $S_{1(fc)}$. For icB(k), let $A(i)_{i(fc)}$ be the set of nodes jcN reachable from i (including i itself) by a path $P(i,j)_{S_{i(k)}}$ in D_g such that for every (r,s)cP(i,j) , $t_s^{i(k)} - t_r^{i(k)}$ « d_{rs} , and let

$$A(B(k))_{i(k)} = \bigcup_{i \in B(k)} A(i)_{i(k)}.$$

Then from Lemma 5.6, $A(B(k))_{i(k)}$ CB(K), k = l,...,p, and since for each ke{l,...,p} by definition $A(B(k))_{i(fc)}$ contains B(k),

$$\bigcup_{k=1}^{P} A(B(k))_{i(k)} = B(K).$$

W.l.o.g., let the q-p nodes of B(K) be numbered in such a way that

$$A(B(1))_{i(1)} = \{p+1,\ldots,p+\beta_1\}$$

$$A(B(2))_{i(2)} \setminus A(B(1))_{i(1)} - \{p+\beta_1+1,\ldots,p+\beta_2\}$$

- -

$$A(B(p))_{i(p)} \setminus \bigcup_{r=1}^{p-1} A(B(r))_{i(r)} = \{p+\beta_{p-1}+1,\ldots,p+3_p\},$$

with $p + 9_p$ * q; and, ia addition, if i,je(p + 9_{k-1} + l,...,p + 9_k} for some ke{l,...,p} (where we define $9_o = 0$) and (i,j)cA US(K) , then i < j.

We then define the vectors t*"" for h » I,...,$?_1$ by

$$t_j^{p+h} = \begin{cases} t^{i(1)} + \epsilon_h & , \quad j = p+\beta_1 - h+1,\ldots,p+\beta_1 \\ t^{i(l)} & , \quad \text{otherwise} \end{cases}$$

with 0 < ^ < 1, h » !,..•,3^, and for h » P_{fe-1} + 1,...,^, k » 2,...,p, by

$$t_j^{p+h} = \begin{cases} t_j \;\top\; \epsilon_h , & j \in \{p+\beta_{k-1}+\beta_k \,\text{"}\, h+1,\ldots,p+\beta_k\} \cup \hat{A}(B(k))_{i(k)} \\ t_j^{i(k)} & \text{otherwise}, \end{cases}$$

where $0 < c_h < 1$, ¥ h, and

$$\hat{A}(B(k))_{i(fe)} - A(B(k))_{1(k)} \cap \left(\bigcup_{r=1}^{k-1} A(B(r))_{i(r)}\right).$$

From Lemma 5.6 and the definition of $A(B(k))_{i(k)}$ each of the vectors t^{p+h} defined above is a schedule for $D_{S_{i(k)}}$, hence for *G*.

Renumbering the schedules t^{p+h}, h • 1,...,0 (^{s}q-p) as t^i, i = p+1,...,q, and subtracting from each t^i the vector t^1, we obtain the (q-p) X n matrix M^2 whose rows are $t^i - t^1$, i - p+l,...,q, and which is of the form M^ » $(M_{21}, M_{22}, 0)$. Here $K^{\wedge\wedge}$ is (q-p) X p, 0 is the (q-p) X (n-q) zero matrix, and M_{22} is a (q-p) X (q-p) lower block triangular matrix of the form

$$M_{22} = \begin{pmatrix} T_1 & 0 & . & . & . & 0 \\ X_{21} & T_2 & . & . & . & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ X_{s1} & X_{s2} & . & . & . & T_s \end{pmatrix}$$

where the i^{th} diagonal block is

$$T_i = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & \epsilon_{\beta_{i-1}}+1 \\ 0 & \cdot & \cdot & \cdot & \epsilon_{\beta_{i-1}}+2 & \epsilon_{\beta_{i-1}}+2 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \epsilon_{\beta_i} & \cdot & \cdot & \cdot & \epsilon_{\beta_i} & \epsilon_{\beta_i} \end{pmatrix}$$

(iii) Finally, we construct the last n-q schedules in G from the schedule t^1 as follows. W.l.o.g. we let the nodes of $N \setminus K \cup B(K) = \{q+1,\ldots,n\}$ be numbered in such a way that, if $i,j \in \{q+1,\ldots,n\}$ and $(i,j) \in A^o \cup S_1$, then $i < j$. Then for $i = q+1,\ldots,n$, we define t^i by $t^i_j = t^1_j$ for $j = 1,\ldots,n - i + q$, and $t^i_j = t^1_j + 1$ for $j = n - i + q + 1,\ldots,n$. The resulting vectors t^i are obviously schedules for G, since t^1 is a schedule, no component was decreased, and if a component j was increased, all components corresponding to nodes reachable from j by a directed path in D_{S_1} were increased by the same amount. Furthermore, since the first p components of t^1 were not changed, clearly these schedules t^i also satisfy $\alpha t^i = 1$.

Subtracting t^1 from each t^i, and letting the vectors $t^i - t^1$, $i = q+1,\ldots,n$ be the rows of a matrix M_3, we obtain $M_3 = (0, 0, M_{33})$, where the two zero matrices are $(n-q) \times p$ and $(n-q) \times (q-p)$, respectively, while M_{33} is the $(n-q) \times (n-q)$ nonsingular matrix whose element in position (i,j) is 0 if $i + j < n-q$, and 1 if $i + j \geq n-q$.

(iv) It remains to be shown that the n schedules t^i that we have constructed are affinely independent. We will do this by showing that the $(n-1) \times n$ matrix whose rows are the n-1 vectors $t^i - t^1$, $i = 2,\ldots,n$, is of full row rank. From parts (i), (ii) and (iii) of the proof. this matrix is of the form

$$M = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & M_{33} \end{pmatrix},$$

where $M^\wedge$ is (p-1) x p and has full row rank. Let $\hat{M}_{11}$ be a (p-1) x (p-1) nonsingular submatrix of $M^\wedge$, and let $\hat{M}_{21}$ be the matrix obtained from M_{21} by removing the column corresponding to the one that was removed from M_{11}. Further, let us permute the blocks of columns of M_{22} and the corresponding blocks of M_{12} by reversing the order of the s blocks, and let $\hat{M}_{22}$ and $\hat{M}_{12}$ be the resulting matrices.

Then M is of full row rank if and only if the (n-1) X (n-1) matrix

$$\hat{M} = \begin{pmatrix} \hat{M}_{11} & \hat{M}_{12} & 0 \\ \hat{M}_{21} & \hat{M}_{22} & 0 \\ 0 & 0 & M_{33} \end{pmatrix}$$

is nonsingular. Since $\hat{M}_{11}$ and M_{33} are nonsingular, $\hat{M}$ is nonsingular if and only if the matrix $\hat{M}_o := \hat{M}_{22} - \hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}$ is nonsingular. It is not hard to see that the numbers e_j used in the construction of $\hat{M}_{22}$ can always be chosen in a way that makes $\hat{M}_o$ nonsingular. We show this by induction on q-p. For q-p = 1, the condition is c_{ft} £ $m^\wedge$, where m_* is the first element of the last row of $M_{21}M_{11}^{-1}M_{12}$. Such c_B obviously exists. Suppose the condition can be satisfied for q-p = 1,2,...,t-1, and let q-p = t. Let A be the matrix consisting of the last t rows and first t columns of $\hat{M}_o$. Denoting by a_{ij} the elements of A and by A_{ij} the cofactor of a_{ij}, and using expansion by the last column of A, we have

$$\det(A) = a_{1t}A_{1t} + \sum_{i=2}^{t} a_{it}A_{it} .$$

By the induction hypothesis, there exist numbers $0 < e_j < 1$, $j = \beta_s$, $\beta_s - 1, \ldots, \beta_s - t + 1$, such that $A_{1t} \neq 0$. Since $a_{1t} = e_{\beta_s - t} - m_{1t}$, where m_{1t} is the element of $\hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}$ in the position corresponding to a_{1t}, we have that det (A) $\neq$ 0 if and only if

a condition which can obviously be satisfied. This completes the induction.

Thus the n schedules t^i for G, i = 1,...,n, are affinely independent. In addition, each one of them is an extension of a schedule for < K >, hence satisfies $(c^*,0)t = 1$. Therefore the inequality $(a,0)t \geq 1$ defines a facet of P.||

to also induce facets of P(N). One of our results is that any inequality that induces a facet of P(H) for some $H \subset K$, also induces a facet of P(K). Another one is a recursive formula for deriving a facet inducing inequality with p positive coefficients from one with p-1 positive coefficients. We also address the constraint identification problem, and give a procedure for finding an inequality that cuts off a given solution to a subset of the constraints.

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A well-known job shop scheduling problem can be formulated as follows. Given a graph G with node set N and with directed and undirected arcs, find an orientation of the undirected arcs that minimizes the length of a longest path in G. We treat the problem as a disjunctive program, without recourse to integer variables, and give a partial characterization of the scheduling polyhedron P(N), i.e., the convex hull of feasible schedules. In particular, we derive all the facet inducing inequalities for the scheduling polyhedron P(K) defined on some clique with node set K, and give a sufficient condition for such inequalities (cont'd)

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