A PRIORI BOUNDS FOR SOLUTIONS OF QUASI-LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

Martin H. Schultz

1967

November, 1967

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1. INTRODUCTION

Let Ω be a region in R^n, $n = 1,2,\ldots,5$, $\partial\Omega$ denote its boundary. We consider quasi-linear elliptic differential equations of the form

$$(1) \quad L[u] \equiv - \sum_{|\alpha|,|\beta|\leq 1} D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}u) = f(x,u) \ , \ x \in \Omega,$$

subject to the boundary conditions

$$(2) \quad u(x) = 0 \ , \ x \in \partial\,\Omega,$$

where we have freely used the standard multi-index notation, cf. [15]. For example, $x = (x_1,\ldots,x_n) \in R^n$ and if $\alpha = (\alpha_1,\ldots,\alpha_n)$ is any index whose components are non-negative integers, $|\alpha| \equiv \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $D^{\alpha} \equiv D_1^{\alpha_1}\ldots D_n^{\alpha_n} \equiv \dfrac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}}$. A basic question, both in proving the existence of a solution of (1), (2) by means of the Schauder Fixed Point Theorem, cf. [1], [8], and in approximating that solution numerically, cf. [5], [6], [7], [11], [12], and [13] is whether or not we can obtain an a priori bounds for classical solutions of (1), (2) in the uniform norm over Ω.

The special case of $n = 2$, $L \equiv -\Delta$ has been studied by many people. In [6] a uniform norm a priori bound was obtained for the case in which $\frac{\partial f}{\partial u} \leq \gamma < \frac{1}{\rho}$, where $\rho \equiv \max_{x \in \Omega} |\psi(x)|$ and $\Delta\,\psi(x) = -1$, $x \in \Omega$, $\psi(x) = 0$, $x \in \partial\,\Omega$. In [9] such an a priori bound was obtained for the case in which

$\liminf_{|u|\to\infty} \frac{f(x,u)}{u} \geq 0$, and in [11] such an a priori bound was obtained for the case in which there exists a positive constant u_0 such that $\frac{f(x,u)}{u} \leq -(\frac{1}{a})^2$ for all $|u| \geq u_0$ where Ω is contained in the strip $|x_1| \leq a$. In this paper, we give new conditions on the problem (1), (2) which guarantee the existence of a uniform norm a priori bound.

2. MAIN RESULTS

We assume throughout this paper that the coefficients, $a_{\alpha\beta}(x)$, $|\alpha|$, $|\beta| \leq 1$, are real-valued, bounded, measurable functions in Ω and that there exists a positive constant C such that if

(3) $a(u,v) \equiv \int_\Omega \sum_{|\alpha|,|\beta|\leq 1} a_{\alpha\beta}(x)D^\alpha u(x)D^\beta v(x)dx$, then

(4) $a(u,u) \geq C^2 \,||u||^2_{W^{1,2}(\Omega)} \equiv C^2 \int_\Omega \sum_{|\alpha|\leq 1} |D^\alpha u(x)|^2 dx$,

for all $u \in W_0^{1,2}(\Omega)$, the completion of the real-valued $C_0^\infty(\Omega)$ functions with respect to $||\cdot||_{W^{1,2}(\Omega)}$. We remark that if L is strongly elliptic, i.e., there exists a positive constant θ such that $\sum_{|\alpha|,|\beta|=1} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta \geq \theta|\xi|^2$ for all $x \in \Omega$ and all real n-vectors ξ, then by Gårding's inequality, cf. [15, pg. 175] there exists a constant S such that the quadratic form associated with the differential operator L + SI satisfies (4) and we may consider the problem

(5) $(L + SI)[u] = f(x,u) + Su\ ,\ x \in \Omega$,

(6) $u(x) = 0,\ x \in \partial\Omega$,

which is equivalent to (1), (2).

We define the real number

$$(7)\quad \Lambda \equiv \operatorname*{Inf}_{\substack{w \in W_0^{1,2}(\Omega) \\ w \neq 0}} \frac{\int_\Omega \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}(x) D^\alpha w(x) D^\beta w(x)\,dx}{\int_\Omega [w(x)]^2\,dx} .$$

Inequality (4) yields the result that $\Lambda > 0$.

We assume that $f(x,u)$ is a measurable function with respect to $x \in \Omega$, continuous with respect to u for almost all $x \in \Omega$ and there exists a constant $\gamma < \Lambda$ such that

(8) $\dfrac{f(x,u) - f(x,0)}{u} \le \gamma < \Lambda$ for almost all $x \in \Omega$ and all $u \neq 0$. Clearly (8) is satisfied if $f(x,u)$ is continuously differentiable with respect to u for almost all $x \in \Omega$ and $\frac{\partial f}{\partial u}(x,u) \le \gamma < \Lambda$ for almost all $x \in \Omega$. Our first result gives an a priori bound in the $||\cdot||_{W^{1,2}(\Omega)}$ - norm for generalized solutions of (1), (2). It improves and extends Lemma 4 of [7], which considers the case of $n = 1$.

Theorem 1. Let $u(x)$ be a generalized solution of (1), (2), i.e.,

$$(9)\quad \int_\Omega \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}(x) D^\alpha u D^\beta \phi\,dx = \int_\Omega f(x,u)\phi\,dx$$

for all $\phi \in W_0^{1,2}(\Omega)$. Then

$$(10)\quad ||u||_{W^{1,2}(\Omega)} \le \frac{1}{C}\left(\int_\Omega \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}(x) D^\alpha u D^\beta u\,dx\right)^{\frac{1}{2}} \equiv \frac{1}{C}[a(u,u)]^{\frac{1}{2}} \le$$

$$\le \frac{1}{C(\Lambda-\gamma)^{1/2}} \left(\int_\Omega [f(x,0)]^2\,dx\right)^{1/2} \equiv M.$$

Proof. Setting $\phi = u$ in (9) and using inequality (8) we obtain

$$a(u,u) = \int_\Omega f(x,u)u\,dx \le \int_\Omega f(x,0)\,u\,dx + \gamma \int_\Omega u^2(x)\,dx \le$$

$$\le \left(\int_\Omega u^2(x)\,dx\right)^{1/2} \left(\int_\Omega f^2(x,0)\,dx\right)^{1/2} + \frac{\gamma}{\Lambda}\, a(u,u).$$

Inequality (10) follows from the fact that $\gamma < \Lambda$. QED.

The next result follows directly from Theorem 1 and the Sobolev Imbedding Theorem cf. [15].

Corollary. If $n = 1$ and $u(x)$ is a generalized solution of (1), (2) then $u(x) \in C^0[0,1]$ and $||u||_{C^0[0,1]} \equiv \max_{x \in [0,1]} |u(x)| \le \frac{1}{2} M$, where M is the positive constant defined in (8).

The situation for $n > 1$ is not quite as simple. We say that the differential operator L given in (1) is regular in Ω if and only if the following condition is satisfied. If $g \in L^p(\Omega)$ $(1 < p < \infty)$ and $u \in W_0^{1,2}(\Omega) \cap L^p(\Omega)$ $(1 < p < \infty)$ is such that $a(u,v) = \int_\Omega g(x)\, v(x)\, dx$ for all $v \in W_0^{1,2}(\Omega)$, then $u \in W^{2,p}(\Omega)$ and

(11) $||u||_{W^{2,p}(\Omega)} \le k\, \{ ||g||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} \}$ where k is a positive constant independent of u. We remark that any differential operator, L, satisfying (4) is regular in Ω if Ω and the coefficients $a_{\alpha\beta}(x)$ are sufficiently smooth, cf. [2, Theorem 8.2].

Theorem 2. Let $n = 2,\ldots,5$ and L be regular in Ω. If there exists positive constants A, k, t, and ε such that

(12) $|f(x,u)| \le A + k\, |u|^t$, if $n = 2$, for all $x \in \Omega_1$ $-\infty < u < \infty$, or

(13) $|f(x,u)| \le A + k\, |u|^{4n/(n-2)(n+2\varepsilon)}$, if $n = 3,\ldots,5$, for all $x \in \Omega$, $-\infty < u < \infty$, and

(14) $\frac{n}{2} + \varepsilon \le \frac{2n}{n-2}$

and $u(x)$ is a generalized solution of (1), (2), then $u(x) \in C^0(\Omega)$ and satisfies an a priori bound in the uniform norm.

Proof. We first consider the case $n = 2$. By Sobolev's Imbedding Theorem, it suffices to show that any generalized solution, $u(x)$, satisfies an a priori bound in $W^{2,2}(\Omega)$. Hence, by inequality (11) it suffices to show that u satisfies an a priori bound in $L^2(\Omega)$ and $g(x,u)$ satisfies an a priori bound in $L^2(\Omega)$. By Sobolev's Imbedding Theorem u satisfies such an a priori bound and by a result of Vainberg, cf. [14] $g(x,u)$ satisfies such an a priori bound if and only if inequality (12) holds.

For the cases $n = 3,4$ and 5, it suffices by Sobolev's Imbedding Theorem, to show that u satisfies an a priori bound in $W^{2,\frac{n}{2}+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. By inequality (11), it suffices to show that u satisifes an a priori bound in $L^{\frac{n}{2}+\varepsilon}(\Omega)$ and $g(x,u)$ satisfies an a priori bound in $L^{\frac{n}{2}+\varepsilon}(\Omega)$. u satisfies such an a priori bound since by Sobolev's Imbedding Theorem it satisfies such an a priori bound in $L^{2n/n-2}(\Omega)$ and $\frac{n}{2} + \varepsilon \leq \frac{2n}{n-2}$. $g(x,u)$ satisfies such an a priori bound by a result of Vainberg, cf. [14]. QED.

We remark that for the case $n = 2$, $L = -\Delta$, Theorem 2 extends the result of [6] since $\frac{1}{\rho} \leq \Lambda$, cf. [3].

3. AN APPLICATION

Theorem 3. If L and $f(x,u)$ satisfy the hypotheses of Theorem 2, then (1), (2) has a generalized solution.

Proof. By Theorem 2, if u is a generalized solution of (1), (2) then $u \in C^0(\Omega)$ and $||u||_{C^0(\Omega)} \leq B$ for some positive constant B. Consider the modified boundary value problem

$$\text{(15)}\quad L[u] = \tilde{f}(x,u) = \begin{cases} f(x,B+1) & \text{if } u \geq B+1 \\ f(x,u) & \text{if } |u| \leq B+1 \\ f(x,-B-1) & \text{if } u \leq -B-1 \end{cases} \quad , \; x \in \Omega,$$

$$\text{(16)}\quad u(x) = 0 \;,\; x \in \partial\,\Omega.$$

It is easy to see the generalized solutions of (15), (16) satisfy the same a priori bound as those of (1), (2), and hence it suffices to show that (15), (16) has a generlaized solution. However, the existence of such a solution follows by applying the Schauder Fixed Point Theorem to the mapping $L^{-1}\,\tilde{f}$ of $W_0^{1,2}\,(\Omega)$ into itself, cf. [1], [8]. QED.

References

[1] Adams, R. A., "A quasi-linear elliptic boundary value problem", Canad. J. Math. 18 (1966), 1105-1112.

[2] Agmon, S., "The Lp approach to the Dirichlet problem", Ann. Scuola Norm. Sup. Pisa 13 (1959), 405-448.

[3] Barta, J., "Sur la vibration fondamentale d'une membrane", C. R. Acad. Sci. Paris 204 (1937), 472-473.

[4] Berger, M. S., "An eigenvalue problem for nonlinear partial differential equations", Trans. Amer. Math. Soc. 120 (1965), 145-184.

[5] Bers, L., "On mildly non-linear partial differential equations of elliptic type", J. Res. Nat. Bur. Standards Sect. B 51 (1953), 229-236.

[6] Ciarlet, P.G., "Variational methods for non-linear boundary-value problems" (103 pp). Doctoral Thesis, Case Institute of Technology (1966).

[7] Ciarlet, P. G., M. H. Schultz, and R. S. Varga, "Numerical methods of high-order accuracy for non-linear boundary value problems. I. One dimensional problem", Numer. Math. 9 (1967), 394-430.

[8] Cronin, J., Fixed points and topological degree in non-linear analysis, Amer. Math. Soc. Math. Survey No. 11 (1964).

[9] Greenspan, D., and S. V. Porter, "Numerical methods for mildly non-linear elliptic partial differential equations II", Numer. Math. 7 (1965), 129-146.

[10] Levinson, N., "The Dirichlet problem for $\Delta u = f(P,u)$", J. Math. Mech. 12 (1963), 567-576.

[11] Parter, S. V., "Numerical methods for mildly non-linear elliptic partial differential equations I", Numer. Math. 7 (1965), 113-128.

[12] Parter, S. V., "Remarks on the numerical computation of solutions of $\Delta u = f(P,u)$", Numerical Solution of Partial Differential Equations, 73-83, edited by J. Bramble, Academic Press, New York (1966).

[13] Parter, S. V., "Maximal solutions of mildly non-linear elliptic equations", Numerical Solutions of Non-Linear Differential Equations, 213-238, edited by D. Greenspan, J. Wiley and Sons, Inc., New York (1966).

[14] Vainberg, M., "On the continuity of some operators of special type", Dokl. Akad. Nauk SSSR 73 (1950), 253-255.

[15] Yosida, K., Functional analysis (458 pp), Academic Press, New York (1965).