ERROR BOUNDS FOR BIVARIATE PIECEWISE HERMITE INTERPOLATION

by

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In this paper, we prove, in an important special case, a conjecture of G. Birkhoff, [2], concerning the form of error bounds for bivariate piecewise Hermite interpolation in rectangular polygons.

Let R be the rectangle $[a,b] \times [c,d]$ and consider arbitrary partitions in each coordinate direction of R:

(1) π: $a=x_0 < x_1 < \ldots < x_{N+1}=b$; π': $c=y_0 < y_1 < \ldots < y_{N'+1}=d$,

where N and N' are non-negative integers. We say that $\rho \equiv \pi \times \pi'$ defines a partition on R. For any positive integer m and partition $\rho \equiv \pi \times \pi'$ of R, let $H^{(m)}(\rho;R)$ be the set of all real-valued piecewise-polynomial functions $w(x,y)$ defined on R such that $D^{(i,j)}w \in C^0(R)$ for all $0 \le i, j \le m-1$, and such that $s(x,y)$ is a polynomial of degree $2m-1$ in both x and y in each subrectangle $[x_i,x_{i+1}] \times [y_j,y_{j+1}]$ defined on R by ρ. Given a real-valued function $f(x,y) \in C^{m-1,m-1}(R)$, let its $H^{(m)}(\rho;R)$ - interpolate be the unique element $f_{m,\rho} \in H^{(m)}(\rho;R)$ such that

(2) $$D^{(i,j)} f(x_k,y_\ell) \equiv \frac{\partial^{i+j}f}{\partial x^i \partial y^j}(x_k,y_\ell) = D^{(i,j)} f_{m,\rho}(x_k,y_\ell)$$

for all $0 \le k \le N+1$, $0 \le \;\; \le N'+1$, and all $0 \le i, j \le m-1$.

Let T be any rectangular polygon, i.e., any polygon whose sides are parallel to the coordinate axes in the plane, such as an L-shaped region. We remark that any rectangular polygon can be expressed as a union of rectangles $\bigcup_{i=1}^{k} R_i$ such that $R_i \cap R_j$, $1 \le i, j \le k$ is either void, or a subset of an edge of R_i and an edge of R_j. In this case, we say that the rectangular polygon is <u>composed</u> of the rectangles R_i. Let C be any collection of partitions of T, i.e., each $\rho \equiv \pi \times \pi' \in C$ defines a partition $\pi_i \times \pi_i'$ of each rectangle R_i of T. Then the collection C is said to be <u>regular</u> if and only if there exists three positive constants σ, τ, and η

such that

(3) $\underline{\pi}_i \geq \sigma\, \overline{\pi}_i$ and $\underline{\pi}_i' \geq \sigma\, \overline{\pi}_i$ for all $1 \leq i \leq k$ and all $\rho \in C$ and

(4) $\eta \leq \overline{\pi}_i' \,/\, \overline{\pi}_i \leq \tau$ for all $1 \leq i \leq k$ and all $\rho \in C$ where

$\overline{\pi} \equiv \max\limits_i (x_{i+1} - x_i)$, $\overline{\pi}' \equiv \max\limits_j (y_{j+1} - y_j)$, $\underline{\pi} \equiv \min\limits_i (x_{i+1} - x_i)$, and

$\underline{\pi}' \equiv \min\limits_j (y_{j+1} - y_j)$.

Finally, for any positive integer p let $S^{p,2}(T)$ be the set of all real-valued functions $f(x,y)$ defined on T such that

(5) $D^{(p-i,i)} f \in L^2(T)$ for all $0 \leq i \leq p$, and

(6) $D^{(i,j)} f \in C^0(R)$ for all $0 \leq i+j < p$.

In [3] the following error estimate for bivariate piecewise Hermite interpolation was proved.

<u>Theorem 1</u>. Let T be a rectangular polygon composed of the rectangles $R_i \equiv [a_i,b_i] \times [c_i,d_i]$, $1 \leq i \leq k$, in the (x,y)-plane, and let C be a regular collection of partitions of T. If $f \in S^{p,2}(T)$ where $p \geq 2m$, and $f_{m,\rho}$ is the $H^{(m)}(\rho;R_i)$-interpolate of f on each R_i, $1 \leq i \leq k$, then setting $\nu \equiv \max\limits_{1\leq i\leq k} \overline{\pi}_i$, there exists a positive constant M such that

(7) $\| D^{(k,\ell)}(f-f_{m,\rho}) \|_{L^2} \leq M(\nu)^{2m-k-\ell} \left(\sum\limits_{j=0}^{2m} \| D^{(j,2m-j)} f \|_{L^2}^2 \right)^{1/2}$, for all $0 \leq k,\ \ell \leq m$ with $0 \leq k+\ell \leq 2m-1$.

G. Birkhoff has conjectured, [2], that there exists a positive constant K such that the interpolation error, bounded in (7), has a bound of the form

$$(8)\quad \| D^{(k,\ell)}(f-f_{m,\rho}) \|_{L^2} \leq K(\nu)^{2m-k-\ell}(\| D^{(2m,0)}f \|^2_{L^2} + \| D^{(0,2m)}f \|^2_{L^2})^{1/2} .$$

The purpose of this paper is to prove not only (8) but a whole class of such error bounds for a restricted class of functions f, cf. Theorem 3. We remark that such error bounds for this restricted class of functions is of particular importance, as they have applications to the study of error bounds for the Galerkin procedure for approximating the solution of Dirichlet boundary value problems for elliptic differential equations, cf. [3].

We first introduce some multi-index notation. Let R^n denote real n-space, $x = (x_1, \ldots, x_n) \in R^n$, and $|x| \equiv (x_1^2 + \ldots + x_n^2)^{1/2}$. For an index or exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$, whose components are integers, $|x| \equiv \alpha_1 + \ldots + \alpha_n$. For any $x \in R^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $x^\alpha \equiv x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $D^\alpha \equiv D_1^{\alpha_1} \ldots D_n^{\alpha_n} \equiv \dfrac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$. Let Ω be any open set in R^n, $C_0^m(\Omega)$ be the set of real-valued $C^m(\Omega)$ functions which have compact support in Ω,

$$(9)\quad \|u\|_m \equiv \Big(\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^2\, dx\Big)^{1/2}$$ for all $u \in C_0^m(\Omega)$, and $W_0^m(\Omega)$ be the completion of $C_0^m(\Omega)$ with respect to the norm (9). We remark that $\|u\|_{L^2} = \|u\|_0$ for all $u \in L^2(\Omega)$.

For all $\phi \in W_0^m(\Omega)$ define the form

$$(10)\quad B(\phi) \equiv \sum_{|\alpha|=|\beta|=m} \int_\Omega a_{\alpha\beta} D^\alpha\phi D^\beta\phi dx,$$

where the coefficients $a_{\alpha\beta}$ are real constants. Our main result gives a necessary and sufficient condition that there exists a positive constant C such that $CB(\phi) \geq ||\phi||_m^2$ for all $\phi \in W_0^m(\Omega)$.

Theorem 2. There exists a positive constant C such that

(11) $CB(\phi) \geq ||\phi||_m^2$ for all $\phi \in W_0^m(\Omega)$ if and only if there exists a positive constant γ such that

(12) $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^{\alpha+\beta} \geq \gamma|\xi|^{2m}$ for all $\xi \in R^n$, i.e., $B(\phi)$ is a strongly elliptic form.

Proof. If: Let $|\phi|_j = (\int_\Omega \sum_{|\alpha|=j} |D^\alpha u|^2 dx)^{1/2}$ for all $\phi \in W_0^j(\Omega)$. By Lemma 7.7 of [1], there exists a positive constant K_1 such that

(13) $B(\phi) \geq K_1 \gamma \; |\phi|_m^2$ for all $\phi \in W_0^m(\Omega)$.

By the Poincare inequality, cf. Lemma 7.3 of [1], there exist two positive constants K_2 and d such that

(14) $||\phi||_m^2 \equiv \sum_{j=0}^{m} |\phi|_j^2 \leq K_2 \sum_{j=0}^{m} d^{m-j} \; |\phi|_m^2$

for all $\phi \in W_0^m(\Omega)$. Combining (13) and (14) we obtain

(15) $||\phi||_m^2 \leq \dfrac{K_2 \sum_{j=0}^{m} d^{m-j}}{K_1\gamma} B(\phi)$, for all $\phi \in W_0^m(\Omega)$, which yields (11) with

$C \equiv \dfrac{K_2 \sum_{j=0}^{m} d^{m-j}}{K_1\gamma}$.

Only if: By Theorem 7.12 of [1], there exists a positive constant γ such that

(16) $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^{\alpha+\beta} \geq \gamma \sum_{|\alpha|=m} \xi^{2\alpha}$.

Inequality (12) follows from the fact that

(17) $\sum_{|\alpha|=m} \xi^{2\alpha} = |\xi|^{2m}$ for all $\xi \in R^n$.

This latter equality can be proved by induction on m. In fact, for m=1 we have $\sum_{|\alpha|=1} \xi^{2\alpha} = \xi_1^2 + \ldots + \xi_n^2 = |\xi|^2$. If we assume that $\sum_{|\alpha|=m-1} \xi^{2\alpha} = |\xi|^{2(m-1)}$, then $\sum_{|\alpha|=m} \xi^{2\alpha} = (\xi_1^2 + \ldots + \xi_n^2) \sum_{|\alpha|=m-1} \xi^{2\alpha} = (\xi_1^2 + \ldots + \xi_n^2 \ |\xi|^{2m}$. QED.

Corollary. There exists a positive constant N such that

(18) $||\phi||_m^2 \leq N \sum_{i=1}^{n} || \frac{\partial^m \phi}{\partial x_i^m} ||_0^2$ for all $\phi \in W_0^m(\Omega)$.

Proof. By Theorem 2, it suffices to show that the form $B(\phi) \equiv \sum_{i=1}^{n} \int_\Omega \left(\frac{\partial^m \phi}{\partial x_i^m}\right)^2 dx$ is strongly elliptic. Thus, we must show that there exists a positive constant γ such that

(19) $\sum_{i=1}^{n} \xi_i^{2m} \geq \gamma \ (\sum_{i=1} \xi_i^2)^m$, for all $\xi \in R^n$. However, since all norms on R^n are equivalent, there exists a positive constant Q such that

(20) $(\sum_{i=1}^{n} \xi_i^{2m})^{\frac{1}{2m}} \geq Q \ (\sum_{i=1}^{n} \xi_i^2)^{1/2}$.

Inequality (19) follows from (29) by raising both sides of the inequality (20) to the 2m-th power and setting $\gamma = Q^{2m}$. QED.

Using the result of Theorem 1, we have

Theorem 3. Let T be a rectangular polygon composed of the rectangles $R_i \equiv [a_i,b_i] \times [c_i,d_i]$, $1 \leq i \leq k$, in the (x,y)-plane, and let C be a regular collection of partitions of T. If $f \in S^{p,2}$ (T), where $p \geq 2m$, $D^{(i,j)} f(x,y) = 0$ for all (x,y) in the boundary of T, for all $0 \leq i+j \leq m$, $B(\phi)$ is any strongly elliptic form, and $f_{m,\rho}$ is the $H^{(m)}(\rho;R_i)$-interpolate of f on each R_i, $1 \le i \le k$, then setting $\nu \equiv \max_{1\le i\le k} \bar{\pi}_i$, there exists a constant positive K such that

$$|| D^{(k,\ell)}(f-f_{m,\rho}) ||_0 \le K(\nu)^{2m-k-\ell}(B(f))^{1/2}$$

for all $0 \le k,\ell \le m$ with $0 \le k+\ell \le 2m-1$.

Using Theorem 3 and the previous Corollary we have the

Corollary. If the hypotheses of Theorem 3 hold, then there exists a positive constant K such that

$$||D^{(k,\ell)}(f-f_{m,\rho}) ||_0 \le K (\nu)^{2m-k-\ell}(|| D^{(2m,0)} f ||_0^2 + || D^{(0,2m)} f ||_0^2)^{1/2}.$$

for all $0 \le k,\ell \le m$ with $0 \le k+\ell \le 2m-1$.

References

[1] Agmon, S., "Lectures on elliptic boundary value problems," Princeton; D. Van Nostrand Co., Inc., 1965,pp. 291.

[2] Birkhoff, G., private communication.

[3] Birkhoff, G., M. H. Schultz, and R. S. Varga, "Piecewise Hermite interpolation in one and two variables with applications to partial differential equations;"to appear in Numerische Mathematik.