THE USE OF INTEGRALS IN THE SOLUTION OF NONLINEAR EQUATIONS IN N DIMENSIONS*

B. Kacewicz
Department of Computer Science
Carnegie-Mellon University
(On leave from University of Warsaw)

ABSTRACT

We introduce the maximal order iteration $I_{-1,s}$ for solving of the nonlinear equation $F(x) = 0$ in the N dimensional Banach space, $1 \le N \le +\infty$, which uses the "integral information". Integral information consists of the "standard information" $F^{(j)}(x_d)$, $j=0,1,\ldots,s$ and the value of $\int_0^1 F(x_d + ty_d)dt$ where $s \ge 1$, x_d is close to the solution and y_d only depends on the standard information. We show $I_{-1,s}$ is of order $s + 3 - \delta$, where $\delta = 0$ for $N = 1$ or $s \ge 2$ and $\delta = 1$ otherwise. Since the maximal order for the standard information is equal to s+1, the additional value of the integral which is represented by a vector of size N increases the order by $2 - \delta$.

*This work was supported in part by the Office of Naval Research under Contract N0014-67-0314-0010, NR 044-422 and by the National Science Foundation under Grant GJ 32111.

1. INTRODUCTION

We consider the problem of solving the nonlinear equation

(1.1) $F(x) = 0,$

where $F: D \to B_2$, D is an open convex subset of B_1 and B_1, B_2 are Banach spaces, $\dim(B_1) = \dim(B_2) = N$, $1 \le N \le +\infty$. We usually solve (1.1) by iteration and it is often assumed we know the standard information for F, i.e.,

$$\mathfrak{N}_s = \mathfrak{N}_s(x_d;F) = \left\{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d)\right\},$$

where $s \ge 1$ and x_d is an approximation to the solution α. In a previous paper (Kacewicz [75]) we raised the question how other types of information can be used in iteration and what is the maximal order of convergence for such information. We answered this question in the case of "integral information" for scalar equations, $N = 1$. This paper deals with integral information in the multivariate and abstract case.

By integral information we mean

(1.2) $$\mathfrak{N}_{-1,s} = \mathfrak{N}_{-1,s}(x_d;F) = \left\{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 F(x_d + ty_d)dt\right\},$$

where y_d depends on $x_d, F(x_d), \ldots, F^{(s)}(x_d)$ and $s \ge 1$.

Note that $\mathfrak{N}_{-1,s}$ differs from $\mathfrak{N}_s$ by the additional value of an integral which is represented for $N < +\infty$ by a vector of size N.

In Section 2 we define the iteration $I_{-1,s}$ which uses information $\mathfrak{N}_{-1,s}$ and is of order $s + 3 - \delta$, where

$$\delta = \begin{cases} 0 & \text{if } s \ge 2 \text{ or } N = 1 \\ 1 & \text{if } s = 1 \text{ and } N \ge 2 \end{cases}$$

A theorem on the convergence of $I_{-1,s}$ is proved. The main result of this paper is that for optimally chosen y_d the maximal order of iteration is equal to $s + 3 - \delta$, i.e., that $I_{-1,s}$ has order as high as possible. Since the maximal order of the standard information $\mathfrak{N}_s$ is equal to $s + 1$ the additional use of the integral (i.e. N new inputs) increases the maximal order by $2 - \delta$. Note that we can attain the order $s + 3 - \delta$ by using the standard information $\mathfrak{N}_{s+2-\delta}$ which is expressed by $O(N^{s+3-\delta})$ scalar function evaluations whereas the same order is achievable by $\mathfrak{N}_{-1,s}$ using $O(N^{s+1})$ scalar function evaluations. For example, for $2 \le N < +\infty$ using integral information $\mathfrak{N}_{-1,1}$, i.e., $O(N^2)$ scalar function evaluations we have order 3 whereas the same order is achievable by $\mathfrak{N}_2$, i.e., by $O(N^3)$ scalar function evaluations. Informations $\mathfrak{N}_{-1,2}$ and $\mathfrak{N}_4$ have the same maximal order equal to 5 and use $O(N^3)$ and $O(N^5)$ scalar function evaluations, respectively.

In Section 4 we show that the iteration $I_{-1,1}$ has a smaller cost index than any iteration $I_{-1,s}$, $s \ge 2$ and any interpolatory iteration which uses the standard information $\mathfrak{N}_k$, $k = 1,2,3,\ldots$.

2. AN INTERPOLATORY-INTEGRAL ITERATION $I_{-1,s}$

Let $\mathfrak{N}_{-1,s}$ be the integral information defined by (1.2). We want to find an iteration which uses $\mathfrak{N}_{-1,s}$ and has order of convergence as high as possible. Such iterations are called maximal.

Wozniakowski [75] has proved that the maximal order is equal to the order of information. So we wish to find y_d such that the order of information is maximized. Let us briefly recall the ideas of the order of information and the order of iteration. Let $\mathfrak{F}$ be a class of functions F,

$$F: D_F \to B_2,\ D_F \subset B_1,\ \dim(B_1) = \dim(B_2) = N$$

which have a simple zero $\alpha = \alpha(F)$ and are analytic in its neighborhood. Let $\{x_d\}$ be a sequence converging to α, $\lim_d x_d = \alpha$. We shall say that $\{F_d\}$ is equal to $F \in \mathfrak{J}$ with respect to $\mathfrak{N}_{-1,s}$ iff

(2a) $\{F_d\} \subset \mathfrak{J}$, $F_d(\alpha_d) = 0$, $\lim_d \alpha_d = \alpha$,

(2b) $\lim_d F_d^{(k)}(\alpha) = G^{(k)}(\alpha)$, $k = 0,1,\dots,$
where $G \in \mathfrak{J}$, $G(\alpha) = 0$,

(2c) $\mathfrak{N}_{-1,s}(x_d;F) = \mathfrak{N}_{-1,s}(x_d;F)\ \ \forall d$, i.e.,
$F^{(k)}(x_d) = F_d^{(k)}(x_d)$, $k = 0,1,\dots,s$,
$\int_0^1 F(x_d + ty_d)dt = \int_0^1 F_d(x_d + ty_d)dt$

The order of information $p = p(\mathfrak{N}_{-1,s})$ is a real number such that

$$p(\mathfrak{N}_{-1,s}) = \begin{cases} \sup A & \text{if } A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where

$$A = \Big\{p \geq 1:\ \forall\{x_d\},\ \lim_d x_d = \alpha,\ \forall\, F \in \mathfrak{J},\ F(\alpha) = 0,\ \forall\{F_d\} \text{ equal to } F \text{ it is true that } \overline{\lim_d} \frac{\|\alpha_d - \alpha\|}{\|x_d - \alpha\|^{p-\epsilon}} = 0,\ \forall \epsilon > 0\Big\}.$$

Let $\varphi_{-1,s}$ be an iteration which uses the information $\mathfrak{N}_{-1,s}$. We shall use the notation $h_d = \varphi_{-1,s}(x_d;F)$ which means that h_d is the approximation of α obtained by one step of $\varphi_{-1,s}$ based on x_d and the information $\mathfrak{N}_{-1,s}$.

The order of iteration $\varphi_{-1,s}$, $p = p(\varphi_{-1,s})$ is a real number such that

$$p(\varphi_{-1,s}) = \begin{cases} \sup B & \text{if } B \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$B = \{p \geq 1: \forall\{x_d\}, \lim_d x_d = \alpha, \forall F \in \mathfrak{F}, F(\alpha) = 0, \forall\{F_d\}$ equal F it is true that

$$\overline{\lim_d} \frac{\|h_d - \alpha_d\|}{\|x_d - \alpha\|^{p-\varepsilon}} = 0, \forall \varepsilon > 0$$

where $h_d = \varphi_{-1,s}(x_d;F)\}$

In this section the maximal iteration $I_{-1,s}$ which uses the information $\mathfrak{N}_{-1,s}$ for the solution of (1.1) is defined and the character of convergence is given.

Let x_d be an approximation to a zero α of F. The next approximation x_{d+1} in $I_{-1,s}$ is defined as a zero of the polynomial $u_d = u_d(y;F)$

(2.1) $\quad u_d(x_{d+1};F) = 0$

(with the criterion of its choice, e.g., the nearest zero to x_d), where u_d is defined as follows.

For $N = 1, u_d$ is the unique interpolatory polynomial which agrees with F with respect to the information $\mathfrak{N}_{-1,s}$ for $y_d = \frac{s+3}{s+2}(z_d - x_d)$, $z_d = x_d - \frac{F(x_d)}{F'(x_d)}$. This case is considered in detail by Kacewicz [75], e.g., it is shown there that $I_{-1,s}$ is of order $s + 3$.

For $N \geq 2$ u_d is given by the formula

(2.2) $$u_d(y;F) = F(x_d) + F'(x_d)(y - x_d) + \ldots + \frac{1}{s!}F^{(s)}(x_d) \cdot (y - x_d)^s + (s + 2)\left(\frac{s+2}{s+3}\right)^{s+1}\left[\int_0^1 F(x_d + ty_d)dt - F(x_d) - \frac{1}{2} F'(x_d)y_d - \ldots - \frac{1}{(s+1)!} F^{(s)}(x_d)y_d^s\right],$$

where

$$(2.3)\quad y_d = \frac{s+3}{s+2}(z_d - x_d), \quad z_d = I_{0,s}(x_d;F),$$

$I_{0,s}$ is the maximal interpolatory iteration which uses the standard information $\mathfrak{N}_s$, thus $\|z_d-\alpha\| = O(\|\alpha-x_d\|^{s+1})$ (see Wozniakowski [74]).

From (2.2) we have

$$(2.4)\quad F(y) - u_d(y;F) = R(y)$$

where

$$(2.5)\quad R(y) = \sum_{k=0}^{2} \frac{1}{(s+1+k)!} F^{(s+1+k)}(x_d)(y - x_d)^{s+1+k} + O(\|y - x_d\|^{s+4}) - (s + 2)\left(\frac{s+2}{s+3}\right)^{s+1}\left[\sum_{k=0}^{2} \frac{1}{(s+2+k)!} F^{(s+1+k)}(x_d)y_d^{s+1+k} + O(\|y_d\|^{s+4})\right].$$

From (2.5) and (2.3) we get

$$(2.6)\quad R(y) = \frac{1}{(s+1)!}\left[F^{(s+1)}(x_d)(y - x_d)^{s+1} - F^{(s+1)}(x_d)\cdot(z_d - x_d)^{s+1}\right] + \frac{1}{(s+2)!}\left[F^{(s+2)}(x_d)(y - x_d)^{s+2} - F^{(s+2)}(x_d)(z_d - x_d)^{s+2}\right] + \frac{1}{(s+3)!} F^{(s+3)}(x_d)(y - x_d)^{s+3} - \frac{(s+3)^2}{(s+4)!(s+2)} F^{(s+3)}(x_d)(z_d - x_d)^{s+3} + O(\|y-x_d\|^{s+4}) + O(\|z_d-x_d\|^{s+4}).$$

From (2.1), (2.4), (2.6) and the Schauder fixed point theorem (see Ortega and Rheinboldt [70]) there follows immediately Theorem 1 which gives the character of convergence of the iteration $I_{-1,s}$ for $N \geq 2$.

As part of this theorem we also state the result for $N = 1$ (Kacewicz [75]).

Theorem 1

Let the iteration $I_{-1,s}$ be defined by (2.1). If the function F is sufficiently smooth in the neighborhood of its simple zero α then the sequence $h_d = I_{-1,s}(x_d;F)$ is well defined for x_d sufficiently close to α and for $2 \le N \le +\infty$

$$\overline{\lim_{x_d\to\alpha}} \frac{\|\alpha-h_d\|}{\|\alpha-x_d\|^{s+3-\delta}} \le \begin{cases} \frac{1}{2}\|[F'(\alpha)]^{-1} F''(\alpha)\|^2 & \text{if } s = 1 \\ \delta_{s,2}\left[\frac{1}{(s+1)!}\|[F'(\alpha)]^{-1}F^{(s+1)}(\alpha)\|\right]^2 (s+1) + \\ + \left|1 - \frac{(s+3)^2}{(s+2)(s+4)}\right| \cdot \frac{\|[F'(\alpha)]^{-1}F^{(s+3)}(\alpha)\|}{(s+3)!} & \text{otherwise} \end{cases}$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise .} \end{cases}$$

For N = 1

$$\lim_{x_d\to\alpha} \frac{h_d-\alpha}{(x_d-\alpha)^{s+3}} = (-1)^{s+2}\left\{\frac{f''(\alpha)\cdot f^{(s+2)}(\alpha)}{2\cdot(f'(\alpha))^2(s+2)!} + \frac{f^{(s+3)}(\alpha)}{(s+4)!}\cdot\frac{1}{(s+2)f'(\alpha)}\right\}$$

■

Note that in the above theorem $\{x_d\}$ is an arbitrary sequence converging to α, hence we can replace x_d, h_d by the continuous variables x, h, respectively.

3. ORDER OF THE INTEGRAL INFORMATION

In this section we prove that the iteration $I_{-1,s}$ has the order $s + 3 - \delta$ which is as high as possible, i.e., $I_{-1,s}$ is maximal and that y_d given by (2.3) is chosen optimally.

This results follow from the theorem concerning the order of the integral information $p(\mathfrak{N}_{-1,s})$.

Theorem 2

Let $1 \le N \le +\infty$, $s \ge 1$ and

$$\mathfrak{N}_{-1,s} = \mathfrak{N}_{-1,s}(x_d;F) = \left\{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 F(x_d+ty_d)dt\right\}$$

where $\quad y_d = y_d(x_d, F(x_d), \ldots, F^{(s)}(x_d))$.

Then

$$p(\mathfrak{N}_{-1,s}) \le s+3-\delta \text{ where } \delta = \begin{cases} 0 \text{ if } N=1 \text{ or } s\ge 2 \\ 1 \text{ otherwise.} \end{cases}$$

Furthermore, if

$$y_d = \frac{s+3}{s+2}(z_d - x_d) \quad \text{for } z_d = I_{0,s}(x_d;F)$$

then

$$p(\mathfrak{N}_{-1,s}) = s+3-\delta.$$

■

Proof

The proof in the case $N = 1$ is omitted since it is given by Kacewicz [75].

Let $N \ge 2$.

We shall prove the first part of Theorem 2. Since the inequality $p(\mathfrak{N}_{-1,s}) \le s+3$ holds for $N = 1$ it also holds for any $2 \le N \le +\infty$. Hence we have to prove that $p(\mathfrak{N}_{-1,1}) \le 3$ and it suffices to consider $N < +\infty$.

Let us define

$$z_d = x_d + \frac{3}{4}y_d, \ \forall d.$$

Case I. $\overline{\lim_d} \frac{\|z_d-\alpha\|}{\|x_d-\alpha\|} > 0$ for certain $F \in \mathfrak{F}$, $F(\alpha) = 0$,

$$\{x_d\}, \ \lim_d x_d = \alpha.$$

Without loss of generality we can assume that $\overline{\lim_d} \frac{\|\alpha-z_d\|}{|\alpha_2-z_{2d}|} > 0$ and $\overline{\lim_d} \frac{\|\alpha-x_d\|}{|\alpha_1-x_{1d}|} > 0$.

Define

$$F_d(x) = F(x) + [(x_1 - x_{1d})^2 (x_2 - z_{2d}), \underbrace{0, \ldots, 0}_{N-1}]^T, \quad \forall d$$

where, in general, m_i and m_{id} denote the i-th components of the vectors m, m_d, respectively.

One can verify that $\langle F_d \rangle$ is equal to F with respect to $\mathfrak{N}_{-1,s}$ and

$$(3.1) \quad \overline{\lim_d} \frac{\|\alpha - \alpha_d\|}{\|\alpha - x_d\|^3} > 0,$$

where $F_d(\alpha_d) = 0$.

<u>Case II</u>. $\overline{\lim_d} \frac{\|z_d - \alpha\|}{\|x_d - \alpha\|} = 0$ for any F and $\langle x_d \rangle$.

For any $F \in \mathfrak{F}$, $F(\alpha) = 0$ let

(i) $\lim_d x_d = \alpha$, $x_{1d} \neq \alpha_1, x_{2d} \neq \alpha_2, \lim_d \frac{\alpha_1 - x_{1d}}{\alpha_2 - x_{2d}} = 1$, $x_{id} = \alpha_i$ for $i = 3,4,\ldots,N$.

From here it follows that y_{2d} can be equal 0 only for finite number of d.

Hence without loss of generality we can assume that $y_{2d} \neq 0$, $\forall d$.

We set

$$(3.2) \quad F_d(x) = F(x) + \left[(x_1 - x_{1d})^2 - \frac{y_{1d}^2}{y_{2d}^2}(x_2 - x_{2d})^2, \underbrace{0, \ldots, 0}_{N-1}\right]^T, \quad \forall x \in D_F, \forall d.$$

One can verify that $\langle F_d \rangle$ is equal F.

Let $a_d = \frac{\alpha_1 - x_{1d}}{\alpha_2 - x_{2d}}$ $(\lim_d a_d = 1)$.

From (3.2) it follows that

$$(3.3)\quad \|\alpha-\alpha_d\| = c_d\|F_d(\alpha)\| = c_d|a_d(z_{2d}-\alpha_2) - (z_{1d}-\alpha_1)|\cdot$$

$$\cdot\ |a_d(z_{2d}-\alpha_2) + (z_{1d}-\alpha_1) + 2(\alpha_1-x_{1d})|,$$

where $\overline{\lim_d}\, c_d = c > 0$, $F_d(\alpha_d) = 0$.

One can verify that there exists a function F and $\{x_d\}$ satisfying the (i) condition such that

$$(3.4)\quad \overline{\lim_d} \frac{|a_d(z_{2d}-\alpha_2) - (z_{1d}-\alpha_1)|}{\|\alpha-x_d\|^2} > 0.$$

Indeed, otherwise the iteration φ for the solution of scalar equations $f(x) = 0$ defined as follows

$$\beta_{d+1} = \varphi(\beta_d;f) = z_{2d}(x_d,F(x_d),F'(x_d))-z_{1d}(x_d,F(x_d),F'(x_d))$$

where $F(x) = [x_1,f(x),x_3,\ldots,x_N]^T$

and $x_d = \left[\frac{f(\beta_d)}{f'(\beta_d)},\ \beta_d,\ \underbrace{0,\ldots,0}_{N-2}\right]^T$

has order of convergence greater than the order of used information, which is a contradiction.

Hence we proved that for arbitrary $y_d=y_d(x_d,F(x_d),F'(x_d))$ there exist $F \in \mathfrak{J}$, $F(\alpha) = 0$, $\{x_d\}$, $\lim_d x_d = \alpha$, $\{F_d\}$ equal to F such that

$$\overline{\lim_d} \frac{\|\alpha-\alpha_d\|}{|\ \|x_d-\alpha\|^3} > 0,$$

which means that $p(\mathfrak{N}_{-1,1}) \leq 3$. This proves the first part of Theorem 2. We shall prove the second part of Theorem 2.

Let $y_d = \frac{s+3}{s+2}(z_d - x_d)$, $z_d = I_{0,s}(x_d;F)$,

where $\lim_d x_d = \alpha$, $F(\alpha) = 0$, $F \in \mathfrak{I}$.

For any $\langle F_d \rangle$ equal to F we have

$$F(y) - F_d(y) = F(y) - u_d(y;F) + u_d(y;F_d) - F_d(y),$$

where u_d is given by (2.2).

From the above, (2.4) and (2.6) for $y = \alpha$ we get

$$\|\alpha - \alpha_d\| = O(\|\alpha - x_d\|^{s+3-\delta}),$$

which completes the proof of Theorem 2. ∎

Let $h_d = I_{-1,s}(x_d;F)$, where $F \in \mathfrak{I}$, $F(\alpha) = 0$. From Theorems 1 and 2 it follows that

$$\|h_d - \alpha_d\| = O(\|x_d - \alpha\|^{s+3-\delta})$$

for any $\langle x_d \rangle$, $\lim_d x_d = \alpha$ and $\langle F_d \rangle$ equal to F, $F_d(\alpha_d) = 0$.

Hence the following corollary holds

<u>Corollary</u>

The iteration $I_{-1,s}$ is of order $s+3-\delta$.

Let $\psi_{-1,s}$ be a class of iterations which use information $\mathfrak{N}_{-1,s}$. The iteration $I_{-1,s}$ has the maximal order in the class $\psi_{-1,s}$, i.e.,

$$p(I_{-1,s}) = \sup_{\varphi_{-1,s} \in \psi_{-1,s}} p(\varphi_{-1,s}), \quad s \geq 1.$$ ∎

4. COMPLEXITY INDEX

The complexity index of an iteration φ of the order p is a measure of the total cost of estimation of the solution α of (1.1). It is defined (see Traub and Wozniakowski [75]) by

$$z(\varphi;F) = \frac{c(\mathfrak{N};F) + c(\varphi)}{\log p}$$

where $\mathfrak{N}$ is the used information,

$c(\mathfrak{N};F)$ is the information cost,

$c(\varphi)$ is the combinatory cost.

For the integral information

$$c(\mathfrak{N}_{-1,s};F) = c(I) + c(\mathfrak{N}_s;F)$$

where $c(I)$ and $c(\mathfrak{N}_s;F)$ are the costs of the computed integral $\int_0^1 F(x_d + ty_d)dt$ and the standard information $\mathfrak{N}_s$, respectively. We want to compare the cost of $I_{-1,s}$ with the cost of the interpolatory iteration $I_{0,k}$ which uses the standard information $\mathfrak{N}_k$ and has order $k + 1$. $I_{-1,s}$ is better than $I_{0,k}$, iff

$$z(I_{-1,s};F) < z(I_{0,k};F), \text{ i.e.,}$$

$$\text{(4.1)} \quad c(I) < \frac{\log(s+3-\delta)}{\log(k+1)} c(\mathfrak{N}_k;F) - c(\mathfrak{N}_s;F) + \frac{\log(s+3-\delta)}{\log(k+1)} c(I_{0,k}) - c(I_{-1,s}).$$

Let $c(F^{(i)})$ denote the cost of computing $F^{(i)}(x)$. This cost can be measured by the total number of arithmetical operations needed to compute $F^{(i)}(x)$ as well as by the cost of data access. Note that in most recent computers the cost of data access of indexed variables exceeds the cost of a single arithmetical operation. Let $2 \le N < +\infty$. Let $c(F) = N$. Then it is reasonable to assume that $c(I) = O(N)$. Since $F^{(i)}(x)$ can be represented by $O(N^{i+1})$ scalar function evaluations it seems natural to assume that $c(F^{(i)})$ is comparable with the cost of $O(N^{i+1})$ scalar function evaluations. Thus, let $c(F^{(i)}) = O(N^{i+1})$, $\forall i \ge 1$. It is easy to see that the combinatory costs $c(I_{0,k})$ and $c(I_{-1,s})$ are increasing functions of k and s, respectively.

Then we have

$$\min_{k\ge 1} z(I_{0,k};F) = \min_{k\ge 1} \frac{c(\mathfrak{N}_k;F)+c(I_{0,k})}{\log(k+1)} = z(I_{0,1};F),$$

and

$$\min_{s\ge 1} z(I_{-1,s};F) = \min_{s\ge 1} \frac{c(\mathfrak{N}_s;F)+c(I)+c(I_{-1,s})}{\log(s+3-\delta)} = z(I_{-1,1};F).$$

This implies that the Newton iteration $I_{0,1}$ and the iteration $I_{-1,1}$ are optimal for the problem $F(x) = 0$ in the classes $\{I_{0,k}\}_{k=1,2,\dots}$ and $\{I_{-1,s}\}_{s=1,2,\dots}$ respectively. Since $(\log 3-1)c(\mathfrak{N}_1,F) + \log 3\ c(I_{0,1}) - c(I_{-1,1}) = O(N^2)$, (4.1) holds for large N which means that $I_{-1,1}$ is better than the Newton iteration, and hence better than any iteration $I_{0,k}$, $k \ge 1$.

5. INTEGRAL INFORMATION WITH KERNELS

Finally we shall discuss a more general type of integral information, i.e., integral information with kernels

$$\mathfrak{N}_{-1,s}^{g} = \left\{F(x_d), F'(x_d), \dots, F^{(s)}(x_d), \int_0^1 g(t)F(x_d+ty_d)dt\right\}$$

where

$g = g(t)$ is a complex function of a complex variable such that $\int_0^1 |g(t)|dt < +\infty$, $y_d = y_d(x_d, F(x_d), \ldots, F^{(s)}(x_d))$, $s \geq 1$.

Let $I_j = \int_0^1 g(t)t^{s+j}dt$ and let $m = m(g)$ be the integer defined as follows.

$$m = \begin{cases} 0 \text{ if } I_1 = 0 \\ 1 \text{ if } I_1 \neq 0,\ I_2 = 0 \\ k \text{ if } I_1 \neq 0,\ I_2 \neq 0,\ \dfrac{I_i}{I_1} = \left(\dfrac{I_2}{I_1}\right)^{i-1} \quad \text{for } i = 2,3,\ldots,k \\ \qquad\qquad \text{and } \dfrac{I_{k+1}}{I_1} \neq \left(\dfrac{I_2}{I_1}\right)^{k},\ k \geq 2. \end{cases}$$

There exists an iteration $I_{-1,s}^{\ g}$ which uses information $\mathfrak{N}_{-1,s}^{\ g}$ for the suitable chosen y_d,

$$y_d = \begin{cases} \dfrac{I_1}{I_2}(z_d - x_d) & \text{if } m \geq 2 \\ z_d - x_d & \text{otherwise, } z_d = I_{0,s}(x_d;F) \end{cases}$$

such that

1. y_d is optimal
2. $p(I_{-1,s}^{\ g}) = \begin{cases} \min(s+1+m, 2s+2) & \text{if } N = 1 \\ \min(s+1+m, 2s+1) & \text{if } 2 \leq N \leq +\infty \end{cases}$
3. $I_{-1,s}^{\ g}$ is maximal
4. There exists $g = g(t)$ such that

$$p(I_{-1,s}^{\ g}) = \begin{cases} 2s+2 & \text{if } N = 1 \\ 2s+1 & \text{if } 2 \leq N \leq +\infty \end{cases}$$

The proof is based on techniques similar to those used here. These results will be reported in a future paper.

ACKNOWLEDGMENT

I wish to express my appreciation to J. F. Traub and H. Wozniakowski for their helpful comments during the preparation of this paper.

6. REFERENCES

Kacewicz [75] Kacewicz, B., "An Integral-Interpolatory Iterative Method for the Solution of Non-Linear Scalar Equations," Computer Science Department Report, Carnegie-Mellon University, Pittsburgh, Pa., 1975.

Ortega and Rheinboldt [70] Ortega, J. M. and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York and London, 1970.

Traub [64] Traub, J. F., Iterative Methods for the Solution of Equations, Prentice Hall, 1964.

Traub and Wozniakowski [75] Traub, J. F. and H. Wozniakowski, "Strict Lower and Upper Bounds on Iterative Complexity," to appear in Analytic Computational Complexity edited by J. F. Traub, Academic Press, 1975. (Also available as a Computer Science Department Report, Carnegie-Mellon University, Pittsburgh, Pa., 1975.)

Wozniakowski [74] Wozniakowski, H., "Maximal Stationary Iterative Methods for the Solution of Operator Equations," SIAM J. Numer. Anal., Vol. 11, No. 5, October 1974, 934-949.

Wozniakowski [75] Wozniakowski, H., "Generalized Information and Maximal Order of Iteration for Operator Equations," SIAM J. Numer. Anal., Vol. 12, No. 1, March 1975, 121-135.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE USE OF INTEGRALS IN THE SOLUTION OF NONLINEAR EQUATIONS IN N DIMENSIONS		5. TYPE OF REPORT & PERIOD COVERED Interim 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) B. Kacewicz		8. CONTRACT OR GRANT NUMBER(s) N00014-67-A-0314-0010 NR 044-422; CMU 1-51039
9. PERFORMING ORGANIZATION NAME AND ADDRESS Carnegie-Mellon University Computer Science Dept. Pittsburgh, PA 15213		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE June 1975 13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS (of this report) 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We introduce the maximal order iteration $I_{-1,s}$ for solving of the nonlinear equation $F(x) = 0$ in the N dimensional Banach space, $1 \le N \le +\infty$, which uses the "integral information". Integral information consists of the "standard information" $F^{(j)}(x_d)$, $j = 0, 1, \ldots, s$ and the value of $\int_0^1 F(x_d + ty_d)dt$ where $s \ge 1$, x_d is close to the solution and y_d only depends on the standard information. We show $I_{-1,s}$ is of order $s + 3 - \delta$, where $\delta = 0$ for $N = 1$ or $s \ge 2$ and $\delta = 1$ otherwise. Since the maximal order for the standard information is equal to $s + 1$, the attitional value of the integral which is represented by a vector of size N increases the order by $2 - \delta$.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)