INTEGRALS WITH A KERNEL IN THE SOLUTION OF NONLINEAR EQUATIONS IN N DIMENSIONS

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ABSTRACT

We consider iterations for solving the nonlinear equation $F(x) = 0$ in the N dimensional Banach space, $1 \le N \le +\infty$, which use "integral information with a kernel". This information consists of the "standard information" $F^{(j)}(x_d)$, $j = 0,1,\ldots,s$ and the integral $\int_0^1 g(t)\, F(x_d + ty_d)dt$ where $s \ge 1$, x_d is an approximation to the solution and y_d depends on the standard information. We show there exists an iteration with order $2s + 1 + \delta_{N,1}$ and prove its optimality.

1. INTRODUCTION

We want to approximate the simple solution α of the nonlinear equation

(1.1) $F(x) = 0$

where

$F: D \to B_2$, D is an open convex subset of B_1, B_1 and B_2 are N-dimensional Banach spaces, $1 \le N \le +\infty$ and $[F'(\alpha)]^{-1}$ is a bounded operator. This problem is often solved by construction of the sequence of successive approximations to α using the standard information on F

$$\mathfrak{N}_s = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d)\},$$

where x_d is a close approximation to α.

In previous papers we investigated another kind of information, namely the integral information

$$\mathfrak{N}_{-1,s} = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 F(x_d + ty_d)dt\},$$

where $s \ge 1$ and y_d depends only on the standard information (see Kacewicz [75a] and [75b]),

We showed there exists an iteration of maximal order $s + 3 - \delta$ (for optimally chosen y_d), where

$$\delta = \begin{cases} 0 \text{ if } N = 1 \text{ or } s \ge 2 \\ 1 \text{ otherwise} \end{cases}$$

Since the maximal order of iterations using the standard information is equal to $s + 1$, the use of the integral increases the maximal order by $2 - \delta$.

In this paper we consider more general kind of integral information, namely integral information with a kernel.

$$(1.2)\quad \mathfrak{N}^{g}_{-1,s} = \mathfrak{N}^{g}_{-1,s}(x_d;\, F) = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d),\ \int_0^1 g(t)F(x_d + ty_d)dt\},$$

where

$s \geq 1$, $y_d = y_d(x_d, F(x_d), F'(x_d), \ldots, F^{(s)}(x_d))$, $g = g(t)$ is a complex function of a complex variable such that $\int_0^1 |g(t)|dt < +\infty$.

Note that if $g(t) \equiv 1$ then $\mathfrak{N}^{g}_{-1,s} = \mathfrak{N}_{-1,s}$. The question is how the maximal order of iteration depends on g.

In Section 2 we define the iteration $I^{g}_{-1,s}$ which uses $\mathfrak{N}^{g}_{-1,s}$ for optimally chosen y_d (see Section 4) and is of order $\min(s+1+m,\ 2s+1+\delta_{N,1})$ (see Section 3 and Corollary 1 in Section 4), where m is an integer depending on g (defined in Section 2) and δ_{ij} is the Kronecker delta. In Section 4 we prove the iteration $I^{g}_{-1,s}$ is maximal. Furthermore we show there exists a polynomial $g = g(t)$ independent on F such that $m = s+\delta_{N,1}$. Since for such g the order is equal to $2s+1+\delta_{N,1}$, the value of the integral with a kernel, which is represented by the vector of size N, increases the maximal order by $s+\delta_{N,1}$.

In Section 5 we show that for N sufficiently large the iteration $I^{g}_{-1,1}$ has smaller complexity index than any interpolatory iteration $I_{0,k}$, which uses the information $\mathfrak{N}_k$, $k \geq 1$, under some assumptions on the cost of computing the value of function, its derivatives, and the integral. In

Section 6 we give examples of the function g and show some connections between information $\mathfrak{N}^{g}_{-1,s}$ and certain two-point information without memory.

2. DEFINITION OF THE ITERATION $I^{g}_{-1,s}$

We shall use the notation

$$(2.1)\quad I_j = \int_0^1 g(t)\, t^{s+j}\, dt, \quad \forall j \geq 1.$$

Let us define

$$(2.2)\quad B_0 = \{g = g(t):\ I_1 = 0\}$$

$$(2.3)\quad B_1 = \{g = g(t):\ I_1 \neq 0,\ I_2 = 0\},$$

$$(2.4)\quad B_m = \left\{g = g(t):\ I_1 \neq 0,\ I_2 \neq 0,\ \frac{I_k}{I_1} = \left(\frac{I_2}{I_1}\right)^{k-1} \quad k = 2,3,\ldots,m, \quad \frac{I_{m+1}}{I_1} \neq \left(\frac{I_2}{I_1}\right)^{m}\right\} \text{ for } m \geq 2.$$

Note that $B_m \neq 0$, $\forall m$. Indeed, the function

$$g(t) = t - \frac{s+2+m}{s+3+m}$$

belongs to B_m for $m = 0,1$. For $m \geq 2$ we can find a function g for which

$$(2.5)\quad I_j = 1,\ j = 1,2,\ldots,m,\ I_{m+1} = 2.$$

Suppose g is of the form

$$(2.6)\quad g(t) = \sum_{i=0}^{m} g_i t^i$$

Then the equalities (2.5) give us the system of linear equations on g_i, $i = 0,1,\ldots,m$

$$(2.7)\quad \sum_{i=0}^{m} \frac{1}{s+j+i+1} g_i = 1 + \delta_{j,m+1} \quad j = 1,2,\ldots,m+1.$$

Since the matrix $\left[\frac{1}{s+j+i+1}\right]_{\substack{i = 0,1,\ldots,m \\ j = 1,\ldots,m+1}}$ is symmetric and positive definite, the coefficients g_i exist and hence $B_m \neq 0$, $\forall m$.

In the remaining part of this paper we often use the notation $h = \varphi(x; F)$ which means that h is the approximation of α obtained by one step of the iteration φ based on x and a certain information on F. Recall that if $z = I_{0,s}(x; F)$, where $I_{0,s}$ means the maximal interpolatory iteration which uses the standard information $\mathfrak{N}_s$ for $s \geq 1$, then

$$\lim_{x\to\alpha} \frac{z-\alpha}{(\alpha-x)^{s+1}} = \frac{F^{(s+1)}(\alpha)}{(s+1)!\, F'(\alpha)} \quad \text{for } N = 1$$

and

$$\lim_{x\to\alpha} \frac{\|z-\alpha\|}{\|\alpha-x\|^{s+1}} \leq \frac{\|[F'(\alpha)]^{-1}F^{(s+1)}(\alpha)\|}{(s+1)!} \quad \text{for } N \geq 2.$$

We define now the iteration $I^g_{-1,s}$ which uses the information $\mathfrak{N}^g_{-1,s}$ for y_d given by

$$y_d = \begin{cases} \text{arbitrary} & \text{if } m = 0 \\ z_d - x_d & \text{if } m = 1, \\ \frac{I_1}{I_2}(z_d - x_d) & \text{if } m \geq 2, \end{cases}$$

where x_d is an approximation to the solution α, $z_d = I_{0,s}(x_d; F)$ and $g \in B_m$.

The next approximation $h_d = I^g_{-1,s}(x_d;\ F)$ in $I^g_{-1,s}$ is defined as a zero of the polynomial $w = w(x) = w(x;\ x_d, F)$,

$$(2.9)\quad w(h_d;\ x_d, F) = 0$$

(with a criterion of its selection, e.g., the nearest zero to x_d), where w is given as follows.

Case I. $N = 1$.

$$(2.10)\quad w(x;\ x_d, F) = F(x_d) + F'(x_d)(x-x_d) + \ldots + \frac{1}{s!} F^{(s)}(x_d)(x-x_d)^s + A(x_d, F)(x-x_d)^{s+1},$$

where

$$(2.11)\quad A(x_d, F) = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{y_d^{s+1} I_1}\left(\displaystyle\int_0^1 g(t)F(x_d+ty_d)dt - \sum_{i=0}^{s} \frac{1}{i!} F^{(i)}(x_d)y_d^i \int_0^1 g(t)t^i dt\right) & \text{otherwise} \end{cases}$$

Case II. $2 \le N \le +\infty$.

$$(2.12)\quad w(x;\ x_d, F) = F(x_d) + F'(x_d)(x-x_d) + \ldots + \frac{1}{s!} F^{(s)}(x_d)(x-x_d)^s + c\left\{\int_0^1 g(t)\ F(x_d+ty_d)dt - \sum_{i=0}^{s} \frac{1}{i!} F^{(i)}(x_d)y_d^i \int_0^1 g(t)t^i dt\right\},$$

where

$$c = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{I_1} & \text{if } m = 1 \\ \dfrac{1}{I_1}\left(\dfrac{I_2}{I_1}\right)^{s+1} & \text{if } m \ge 2 \end{cases}$$

Note that to find a good approximation of h_d in numerical practice it is possible to perform a few Newton steps on the equation (2.9).

We see that for m = 0 $I^g_{-1,s}$ is equal to the well known interpolatory iteration $I_{0,s}$ which uses the standard information $\mathfrak{N}_s$ and is of order s+1. Hence we assume that $m \geq 1$.

One can verify that the polynomial w satisfies the following interpolatory conditions. For N = 1,

$$w^{(j)}(x_d) = F^{(j)}(x_d) \qquad j = 0,1,\dots,s$$

$$\int_0^1 g(t)\ w(x_d+ty_d)dt = \int_0^1 g(t)\ F(x_d+ty_d)dt.$$

For $2 \leq N \leq +\infty$,

$$w(x_d) = F(x_d) + O(\|\alpha-x_d\|^{s+1})$$

$$w^{(j)}(x_d) = F^{(j)}(x_d) \qquad j = 1,2,\dots,s$$

$$\int_0^1 g(t)\ w(x_d+ty_d)dt = \int_0^1 g(t)\ F(x_d+ty_d)dt + O(\|\alpha-x_d\|^{s+1}).$$

3. CONVERGENCE OF THE ITERATION $I^g_{-1,s}$

If the function F is sufficiently smooth in the neighborhood of the zero α, then from (2.10), (2.11), (2.12) and due to the special form of y_d given by (2.8) we have

(3.1) $F(x) - w(x;\ x_d,F) = R(x),$

where

for N = 1

$$
(3.2)\quad R(x) = \begin{cases} \frac{1}{(s+2)!} F^{(s+2)}(x_d)(x-x_d)^{s+2} + O((x-x_d)^{s+3}) + \\ \quad + O((z_d-x_d)^2 (x-x_d)^{s+1}) & \text{if } m = 1 \\ \sum_{k=2}^{m} \frac{1}{(s+k)!} F^{(s+k)}(x_d)(x-x_d)^{s+1}[(x-x_d)^{k-1} - (z_d-x_d)^{k-1}] + \\ + \frac{1}{(s+1+m)!} F^{(s+1+m)}(x_d)(x-x_d)^{s+1}\left[(x-x_d)^m - \frac{I_{m+1}}{I_1}\left(\frac{I_1}{I_2}\right)^m (z_d-x_d)^m\right] + \\ + O((x-x_d)^{s+2+m}) + O((z_d-x_d)^{m+1}(x-x_d)^{s+1}) & \text{if } m \geq 2, \end{cases}
$$

for $2 \leq N \leq +\infty$

$$
(3,3)\quad R(x) = \begin{cases} \frac{1}{(s+1)!}[F^{(s+1)}(x_d)(x-x_d)^{s+1} - F^{(s+1)}(x_d)(z_d-x_d)^{s+1}] + \\ + \frac{1}{(s+2)!} F^{(s+2)}(x_d)(x-x_d)^{s+2} + O(\|x-x_d\|^{s+3}) + \\ + O(\|z_d-x_d\|^{s+3}) & \text{if } m = 1 \\ \sum_{k=0}^{m-1} \frac{1}{(s+1+k)!} [F^{(s+1+k)}(x_d)(x-x_d)^{s+1+k} - F^{(s+1+k)}(x_d)(z_d-x_d)^{s+1+k}] + \\ + \frac{1}{(s+1+m)!}\Big[F^{(s+1+m)}(x_d)(x-x_d)^{s+1+m} - F^{(s+1+m)}(x_d)(z_d-x_d)^{s+1+m} \cdot \\ \quad \frac{I_{m+1}}{I_1}\left(\frac{I_1}{I_2}\right)^m\Big] + O(\|x-x_d\|^{s+2+m}) + O(\|z_d-x_d\|^{s+2+m}) & \text{if } m \geq 2 \end{cases}
$$

From the Brouwer fix point theorem for $N < +\infty$ or the Schauder fix point theorem for $N = +\infty$ (see Ortega and Rheinboldt [70], p.164), from the definition (2.9) of $I^g_{-1,s}$ and (3.1), (3.2) and (3.3) we get the following theorem about convergence of $I^g_{-1,s}$. In Section 4 we shall use the result below to establish the order of $I^g_{-1,s}$.

Theorem 1

Let the iteration $I^{g}_{-1,s}$ be defined by (2.9) and $g \in B_m$. If the function F is sufficiently smooth in the neighborhood of its simple zero α, then the approximation $h_d = I^{g}_{-1,s}(x_d; F)$ is well defined for x_d sufficiently close to α and

(i) For N = 1

$$\lim_{x_d \to \alpha} \frac{h_d - \alpha}{(\alpha - x_d)^{\min(s+1+m,2s+2)}} = \begin{cases} D_{s+1+m} & \text{if } m = 0,1 \\ -\delta_{m,s+1} D_{s+1} \cdot D_{s+2} + \\ + \left(1 - \left(\frac{I_1}{I_2}\right)^m \frac{I_{m+1}}{I_1}\right) \cdot D_{s+1+m} & \text{if } 2 \le m \le s+1 \\ - D_{s+1} \cdot D_{s+2} & \text{if } m > s+1 \end{cases}$$

(ii) For $2 \le N \le +\infty$

$$\lim_{x_d \to \alpha} \frac{\|h_d - \alpha\|}{\|\alpha - x_d\|^{\min(s+1+m,2s+1)}} \le \begin{cases} \|D_{s+1}\| & \text{if } m = 0 \\ \delta_{s,1} \cdot 2\|D_2\|^2 + \|D_{s+2}\| & \text{if } m = 1 \\ \delta_{m,s} \cdot (s+1)\|D_{s+1}\|^2 + \\ + \left|1 - \left(\frac{I_1}{I_2}\right)^m \cdot \frac{I_{m+1}}{I_1}\right| \cdot \|D_{s+1+m}\| & \text{if } 2 \le m \le s \\ (s+1) \cdot \|D_{s+1}\|^2 & \text{if } m > s \end{cases}$$

where $D_k = \frac{1}{k!}[F'(\alpha)]^{-1} F^{(k)}(\alpha)$. ■

Since x_d is an arbitrary point, the theorem above describes the behavior of the function $h = I^{g}_{-1,s}(x; F)$ in the neighborhood of the zero α of F.

4. ORDER OF INFORMATION $\mathfrak{N}^g_{-1,s}$ AND MAXIMALITY OF THE ITERATION $I^g_{-1,s}$

In this section we show that the iteration $I^g_{-1,s}$ has order equal to $\min(s+1+m,\ 2s+1+\delta_{N,1})$ whenever $g \in B_m$. We prove that this order is maximal and y_d given by (2.8) is optimal.

For this purpose we define the order of iteration and the order of information as in Wozniakowski [75b].

Let $\mathfrak{J}$ be a class of functions F,

$$F: D_F \to B_2,\ D_F \subset B_1,\ \dim(B_1) = \dim(B_2) = N$$

which have a simple zero $\alpha = \alpha(F)$ and are analytic in its neighborhood. Let $\{x_d\}$ be a sequence converging to α, $\lim_d x_d = \alpha$. We shall say that $\{F_d\} \subset \mathfrak{J}$ is equal to $F \in \mathfrak{J}$ with respect to $\mathfrak{N}^g_{-1,s}$ iff

(4.1) $F_d(\alpha_d) = 0,\quad \lim_d \alpha_d = \alpha,$

(4.2) $\lim_d F_d^{(k)}(\alpha) = G^{(k)}(\alpha),\quad k = 0,1,\ldots,$

where $G \in \mathfrak{J}$, $G(\alpha) = 0$,

(4.3) $\mathfrak{N}^g_{-1,s}(x_d;F) = \mathfrak{N}^g_{-1,s}(x_d;F_d)\quad \forall d$, i.e.,

$F^{(k)}(x_d) = F_d^{(k)}(x_d),\quad k = 0,1,\ldots,s,$

$\int_0^1 g(t)F(x_d + ty_d)dt = \int_0^1 g(t)F_d(x_d + ty_d)dt$.

The order of information $p = p(\mathfrak{N}^g_{-1,s})$ is a real number such that

$$p(\mathfrak{N}^g_{-1,s}) = \begin{cases} \sup A & \text{if } A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where

$$A = \{p \geq 1:\ \forall\{x_d\},\ \lim_d x_d = \alpha,\ \forall\ F \in \mathfrak{J},\ F(\alpha) = 0,$$

$$\forall\{F_d\} \text{ equal to F it is true that}$$

$$\overline{\lim_d}\ \frac{\|\alpha_d - \alpha\|}{\|x_d - \alpha\|^{p-\epsilon}} = 0, \quad \forall\epsilon > 0\}\ .$$

Let $\varphi^g_{-1,s}$ be an iteration which uses the information $\mathfrak{N}^g_{-1,s}$. The order of iteration $\varphi^g_{-1,s}$, $p = p(\varphi^g_{-1,s})$ is a real number such that

$$p(\varphi^g_{-1,s}) = \begin{cases} \sup B & \text{if } B \neq \emptyset \\ 0 & \text{otherwise}, \end{cases}$$

where

$$B = \{p \geq 1:\ \forall\{x_d\},\ \lim_d x_d = \alpha,\ \forall F \in \mathfrak{J},\ F(\alpha) = 0,\ \forall\{F_d\} \text{ equal to F}$$

it is true that

$$\overline{\lim_d}\ \frac{\|h_d - \alpha_d\|}{\|x_d - \alpha\|^{p-\epsilon}} = 0,\ \forall\epsilon > 0 \text{ where } h_d = \varphi^g_{-1,s}(x_d;F)\}$$

(see Woźniakowski [75b]).

Woźniakowski [75a] proved that the order of information is equal to the maximal order of convergence. We shall use this property to show $I^g_{-1,s}$ is maximal.

We now prove the theorem about order of information $\mathfrak{N}^g_{-1,s}$.

Theorem 2

Let $\mathfrak{N}^g_{-1,s}$ be the integral information with a kernel

$$\mathfrak{N}^g_{-1,s} = \mathfrak{N}^g_{-1,s}(x_d;F) = \{F(x_d),\ F'(x_d),\ldots,F^{(s)}(x_d),\ \int_0^1 g(t)F(x_d + ty_d)dt\}$$

where

$$s \geq 1,\ y_d = y_d(x_d,\ F(x_d),\ldots,F^{(s)}(x_d)),\ g = g(t)$$

is a complex function of a complex variable such that $\int_0^1 |g(t)|dt < +\infty$ and $g \in B_m$. Then

$$p(\mathfrak{N}^g_{-1,s}) \le \min(s+1+m,\ 2\hat{s}+1+\delta_{N,1}) .$$

Furthermore, if

$$y_d = \begin{cases} \text{arbitrary} & \text{if } m = 0, \\ z_d - x_d & \text{if } m = 1, \\ \frac{I_1}{I_2}(z_d - x_d) & \text{if } m \ge 2, \end{cases}$$

where $z_d = I_{0,s}(x_d;F)$

then

$$p(\mathfrak{N}^g_{-1,s}) = \min(s+1+m,\ 2s+1+\delta_{N,1}).$$ ∎

Proof

We shall prove the first part of Theorem 2, i.e., we shall show that there exist $F \in \mathfrak{F}$, $F(\alpha) = 0$, $\{x_d\}$, $\lim_d x_d = \alpha$ and $\{F_d\}$ equal to F, $F_d(\alpha_d) = 0$ such that

$$(4.4) \quad \overline{\lim_d} \frac{\|\alpha-\alpha_d\|}{\|x_d-\alpha\|^{\min(s+1+m,2s+1+\delta_{N,1})}} > 0.$$

We consider two cases.

Case I. $N = 1$

Let $F \in \mathfrak{F}$, $F(\alpha) = 0$ and $e_d = \alpha - x_d$, where $\lim_d e_d = 0$. We set

$$(4.5) \quad F_d(x) = F(x) + (x-x_d)^{s+1}[(x-x_d)^{m-\gamma}-b_d], \ \forall d,$$

where

$\gamma = b_d = 0$ for $m = 0$ and

$$\gamma = \begin{cases} 0 & \text{if } \overline{\lim_d} \left|1 - \frac{y_d}{e_d} \cdot \frac{I_2}{I_1}\right| = 0 \\ m-1 & \text{otherwise,} \end{cases}$$

$$b_d = y_d^{m-\gamma} \frac{I_{m+1-\gamma}}{I_1} \quad \text{for } m \geq 1.$$

One can verify that $\{F_d\}$ is equal to F. Moreover,

$$|\alpha-\alpha_d| = c_d|F_d(\alpha)| = c_d\begin{cases}|e_d|^{s+1} & \text{if } m = 0\\ |e_d|^{s+1}.|e_d^{m-\gamma}-y_d^{m-\gamma}.\frac{I_{m+1-\gamma}}{I_1}| & \text{otherwise,}\end{cases}$$

where

$$F_d(\alpha_d) = 0 \text{ and } \overline{\lim_d}\, c_d = c > 0.$$

From above we have

$$(4.6) \quad \overline{\lim_d} \frac{|\alpha-\alpha_d|}{|e_d|^{s+1+m}} > 0, \quad \forall m.$$

This proves (4.4) for $m = 0$ or 1. Hence assume $m \geq 2$. Let us now consider the functions $\{F_d\}$ given by (4.5) with $\gamma = m-1$. This means that

$$F_d(x) = F(x) + (x-x_d)^{s+1}(x-x_d-y_d \frac{I_2}{I_1}).$$

Let z_d be defined by

$$z_d = x_d + \frac{I_2}{I_1} y_d, \ \forall d$$

and let the function F and the sequence $\{x_d\}$ be such that

$$\overline{\lim_d} \frac{|z_d-\alpha|}{|e_d|^{s+1}} > 0.$$

Then $\{F_d\}$ is equal to F and

$$(4.7) \quad \overline{\lim_d} \frac{|\alpha-\alpha_d|}{|e_d|^{2s+2}} > 0.$$

Hence, (4.6) and (4.7) prove (4.4) for $N = 1$.

<u>Case II</u>. $2 \le N \le +\infty$

Since the inequality $p(\mathfrak{N}^g_{-1,s}) \le \min(s+1+m, 2s+2)$ holds for $N = 1$ it also holds for any $2 \le N \le +\infty$. Hence, we want to now show that for $2 \le N \le +\infty$ $p(\mathfrak{N}^g_{-1,s}) \le 2s+1$, i.e., that (4.4) holds for $m > s$. It suffices to consider the case $N < +\infty$. Let $z_d = x_d + \frac{I_2}{I_1} y_d$, $\forall d$, $z_d = z_d(x_d, F(x_d), \ldots, F^{(s)}(x_d))$. If there exist $F \in \mathfrak{F}$, $F(\alpha) = 0$ and $\{x_d\}$, $\lim\limits_d x_d = \alpha$ such that

$$\overline{\lim_d} \frac{\|z_d - \alpha\|}{\|x_d - \alpha\|} > 0,$$

then the family of functions

$$F_d(x) = F(x) + [(x_1 - x_{1d})^{s+1}(x_2 - z_{2d}), \underbrace{0, \ldots, 0}_{N-1}]^T, \ \forall d$$

is equal to F with respect to $\mathfrak{N}^g_{-1,s}$ and (4.4) holds for zeros α_d of F_d.

In the formula above, x_{1d}, z_{2d} denote the components of vectors x_d, z_d respectively such that

$$\overline{\lim_d} \frac{\|\alpha - x_d\|}{|\alpha_1 - x_{1d}|} > 0 \text{ and } \overline{\lim_d} \frac{\|\alpha - z_d\|}{|\alpha_2 - z_{2d}|} > 0,$$

and

$$x = [x_1, \ldots, x_N]^T.$$

Hence assume

$$(4.8) \quad \overline{\lim_d} \frac{\|z_d - \alpha\|}{\|x_d - \alpha\|} = 0 \text{ for any } F \text{ and } \{x_d\}.$$

Let the sequence $\{x_d\}$ satisfy the conditions

(i) $\lim\limits_d x_d = \alpha$, $x_{1d} \ne \alpha_1$, $x_{2d} \ne \alpha_2$, $\lim\limits_d \frac{\alpha_1 - x_{1d}}{\alpha_2 - x_{2d}} = 1$, $x_{id} = \alpha_i$ for $i = 3,4,\ldots,N$, where $F(\alpha) = 0$.

From the assumptions above, it follows that y_{2d} can be equal zero only for a finite number of d, hence without loss of generality we can assume that $y_{2d} \neq 0$ $\forall d$.

Let us define

$$(4.9)\quad F_d(x) = F(x) + \left[(x_1-x_{1d})^{s+1} - \frac{y_{1d}^{s+1}}{y_{2d}^{s+1}}(x_2-x_{2d})^{s+1},\ \underbrace{0,\ldots,0}_{N-1}\right]^T,$$

One can verify that $\{F_d\}$ is equal to F. From (4.9) it follows that

$$(4.10)\quad \|\alpha-\alpha_d\| = h_d\,\|F_d(\alpha)\| = \tilde{h}_d\,|a_d(z_{2d}-\alpha_2) - (z_{1d}-\alpha_1)|\,\cdot$$

$$\cdot\,|\,a_d^s(z_{2d}-x_{2d})^s + a_d^{s-1}(z_{2d}-x_{2d})^{s-1}(z_{1d}-x_{1d}) + \ldots$$

$$\ldots + a_d(z_{2d}-x_{2d})\cdot(z_{1d}-x_{1d})^{s-1} + (z_{1d}-x_{1d})^s|,$$

where

$$\overline{\lim_d}\,\tilde{h}_d = h > 0,\ F_d(\alpha_d) = 0,\ a_d = \frac{\alpha_1-x_{1d}}{\alpha_2-x_{2d}}\ (\lim_d a_d = 1).$$

It can be verified that there exists a function F and $\{x_d\}$ satisfying the (i) condition such that

$$(4.11)\quad \overline{\lim_d}\,\frac{|a_d(z_{2d}-\alpha_2) - (z_{1d}-\alpha_1)|}{\|\alpha-x_d\|^{s+1}} > 0.$$

Indeed, otherwise (due to the similar argument which was used by Kacewicz [75b]) the iteration φ for the solution of the nonlinear scalar equation $f(y) = 0$ defined as follows

$$\beta_{d+1} = \varphi(\beta_d;f) = z_{2d}(x_d,F(x_d),\ldots,F^{(s)}(x_d)) - z_{1d}(x_d,F(x_d),\ldots,F^{(s)}(x_d))$$

where β_d is close to the solution (but not equal),

$$F(x) = [x_1, f(x_2), x_3, \ldots, x_N]^T$$

and

$$x_d = [\beta_d - I_{0,s}(\beta_d;f), \beta_d, \underbrace{0,\ldots,0}_{N-2}]^T$$

has the order of convergence greater than s+1, i.e., greater than the order of used information, which is a contradiction.

Finally, from (4.11) and (4.10) follows the inequality (4.4) for $m > s$, which means that $p(\mathfrak{N}^g_{-1,s}) \le 2s+1$. This proves Case II and also the first part of Theorem 2.

We shall prove the second part of Theorem 2. We want to show that for arbitrary $F \in \mathfrak{F}$, $F(\alpha) = 0$, $\{x_d\}$, $\lim_d x_d = \alpha$, $\{F_d\}$ equal to F, $F_d(\alpha_d) = 0$ we have

$$(4.12) \quad \overline{\lim_d} \frac{\|\alpha-\alpha_d\|}{\|x_d-\alpha\|^{\min(s+1+m, 2s+1+\delta_{N,1})}} < +\infty.$$

Since $\|\alpha-\alpha_d\|$ is at least of order s+1, (4.12) holds for $m = 0$. Assume $m \ge 1$. Since $\{F_d\}$ is equal to F we have

$$(4.13) \quad \|F_d(\alpha)\| \le \|w(\alpha;x_d,F)\| + \|F_d(\alpha) - w(\alpha;x_d,F_d)\|$$

where the polynomial $w = w(x;x_d,F)$ is given by (2.10) for $N = 1$ and (2.12) for $2 \le N \le +\infty$.

From (3.2) for $N = 1$ and (3.3) for $2 \le N \le +\infty$ we get

$$(4.14) \quad \|\alpha-\alpha_d\| = O(\|F_d(\alpha)\|) = O(\|x_d-\alpha\|^{\min(s+1+m, 2s+1+\delta_{N,1})}).$$

Hence (4.12) holds which completes the proof of the Theorem 2. ∎

Since

$$\|I^g_{-1,s}(x_d;F)-\alpha_d\| \le \|I^g_{-1,s}(x_d;F)-\alpha\| + \|\alpha-\alpha_d\|$$

we get from Theorem 1 and (4.14)

$$\overline{\lim_d} \frac{\|I^g_{-1,s}(x_d;F)-\alpha_d\|}{\|x_d-\alpha\|^{\min(s+1+m,2s+1+\delta_{N,1})}} < +\infty$$

for any $F \in \mathfrak{J}$, $F(\alpha) = 0$, $\{x_d\}$, $\lim_d x_d = \alpha$, and $\{F_d\}$ equal to F, $F_d(\alpha_d) = 0$. Hence, from the definition of the order of iteration and Theorem 2 we have

Corollary 1

Let $g \in B_m$. Then

$$p(I^g_{-1,s}) = \min(s+1+m,2s+1+\delta_{N,1}).$$ ■

From Corollary 1 and Theorem 2 there follows immediately

Corollary 2

Let $\psi^g_{-1,s}$ be the class of iterations which use information $\mathfrak{N}^g_{-1,s}$. Then

$$p(I^g_{-1,s}) = \sup_{\varphi^g_{-1,s}\in\psi^g_{-1,s}} p(\varphi^g_{-1,s}),$$

i.e., the iteration $I^g_{-1,s}$ is maximal. ■

Note that the order of information and at the same time order of iteration $I^g_{-1,s}$ is maximized and equal to $2s+1+\delta_{N,1}$ iff $m \ge s+\delta_{N,1}$. Thus, for the function g chosen such that $m = s+\delta_{N,1}$ (see (2.6) and (2.7)) one additional value of the integral which is represented by N new data increases the order by $s+\delta_{N,1}$.

5. COMPLEXITY INDEX

We want to compare the complexity indices of the iterations $I^g_{-1,s}$ and $I_{0,k}$. The complexity index z is defined by

$$z = z(\varphi;F) = \frac{c(\mathfrak{N};F)+c(\varphi)}{\log p}$$

where φ is an iteration of order p which use the information $\mathfrak{N}$, $c(\mathfrak{N};F)$ is the information cost and $c(\varphi)$ is the combinatory cost (see Traub and Woźniakowski [75]). For the integral information with a kernel the cost $c(\mathfrak{N}^g_{-1,s};F)$ consists of the costs of the standard information $c(\mathfrak{N}_s;F)$ and the computed integral $c(I)$. Let us assume that $m = s+\delta_{N,1}$. Then $p(I^g_{-1,s}) = 2s+1+\delta_{N,1}$ and one can verify that $z(I^g_{-1,s};F) < z(I_{0,k};)$ iff

$$(5.1)\quad c(I) < \frac{\log(2s+1+\delta_{N,1})}{\log(k+1)} c(\mathfrak{N}_k;F) - c(\mathfrak{N}_s;F) + \frac{\log(2s+1+\delta_{N,1})}{\log(k+1)} c(I_{0,k}) - c(I^g_{-1,s}).$$

Let $2 \le N < +\infty$ and $c(F^{(i)})$ denote the cost of computing $F^{(i)}(x)$. $c(F^{(i)})$ depends on the total number of arithmetical operations as well as on the cost of data access (which is usually greater than the cost of single arithmetical operation). Let $c(F) = N$. Then we assume that $c(I) = O(N)$ and since $F^{(i)}(x)$ can be represented in general by $O(N^{i+1})$ scalar function evaluations, assume that $c(F^{(i)}) = O(N^{i+1})$. Since the information costs $c(\mathfrak{N}_k;F)$ and $c(\mathfrak{N}^g_{-1,s};F)$ are of order N^{k+1} and N^{s+1} respectively and the combinatory costs $c(I_{0,k})$ and $c(I^g_{-1,s})$ are increasing functions of k and s respectively, we have for large N

$$(5.2)\quad \min_{k\ge1} z(I_{0,k};F) = z(I_{0,1};F) \text{ and}$$

$$(5.3)\quad \min_{s\geq 1} z(I^{g}_{-1,s};F) = z(I_{-1,1};F).$$

However, it should be stressed that if $c(F^{(i)})$ is essentially less than N^{i+1} then (5.2) and (5.3) are not necessarily true. Under our assumptions

$$(\log 3-1)c(\mathfrak{N}_1;F) + \log 3\ c(I_{0,1}) - c(I^{g}_{-1,1}) = O(N^2)$$

which means that (5.1) holds for large N. From here, (5.2) and (5.3), it follows that $I^{g}_{-1,1}$ has smaller complexity index than any iteration $I_{0,k}$, $k \geq 1$ and any $I^{g}_{-1,s}$, $s \geq 2$.

6. EXAMPLES

1. Let $g(t) \equiv 1$. Then $m = 2$ and order $p(I^{g}_{-1,s}) = \min(s+3, 2s+1+\delta_{N,1}) = s+3-\delta$ where

$$\delta = \begin{cases} 0 & \text{if } N = 1 \text{ or } s \geq 2 \\ 1 & \text{otherwise,} \end{cases}$$

which agrees with Kacewicz's [75b] result.

2. Let $N = 1$ and $g(t) = \delta(t-1)$, where δ is a generalized function such that

$$\int_{-\infty}^{+\infty} \delta(t-1)F(t)dt = F(1)$$

for any function F with bounded support (see Gel'fand and Shilov [64]). Then the information is of the form

$$\mathfrak{N}^{g}_{-1,s} = \{F(x_d),\ldots,F^{(s)}(x_d),F(x_d+y_d)\}.$$

Note that $I_j = 1$, $\forall j$ and hence $\frac{I_k}{I_1} = \left(\frac{I_2}{I_1}\right)^{k-1}$, $\forall k$. Then formally we can set

$m = +\infty$ and the order of information $p(\mathfrak{N}^g_{-1,s})$ is equal to $\min(s+1+\infty, 2s+2) = 2s+2$, which agrees with the optimal order of this special Hermitian information (see Woźniakowski [75b]).

3. Let $N = 1$ and $g(t) = \delta_k(t-1)$, where $\int_{-\infty}^{+\infty} \delta_k(t-1)f(t)dt = F^{(k)}(1)$ for any sufficiently smooth F with bounded support.

Then the information is of the form

$$\mathfrak{N}^g_{-1,s} = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), F^{(k)}(x_d+y_d)\}$$

and it was considered by Brent [74]. It is easy to see that if $k > s+1$ then $I_1 = 0$, hence $m = 0$ and the order is equal to $s+1$. If $k \le s+1$ then $I_j = \frac{(s+j)!}{(s+j-k)!}$ hence $m = 2$ and order is equal to $s+3$ which agrees with Brent's result.

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