On the inconsistency of rigid-body frictional planar mechanics

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June 17, 1987
CMU–CS–87–130

Abstract

This paper reviews the problem of a thin rigid rod sliding on a horizontal surface in the plane, which is commonly cited as an example of the inconsistency of planar rigid–body Newtonian mechanics. We demonstrate the existence of a consistent solution, using Routh's analysis of rigid–body impact.

This work was supported under grants from the System Development Foundation and the National Science Foundation.

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1. Introduction

It is widely accepted in the mechanics literature that Newtonian mechanics of rigid planar bodies with Coulomb friction is sometimes inconsistent, that is, that problems can be posed that have no solution satisfying the axioms of the theory in question. Excellent treatments of this issue are given by Lötstedt (1981), and Erdmann (1984), who independently constructed essentially identical examples demonstrating the inconsistency, and by Kilmister and Reeve (1966) who treat the issues of uniquess and existence in a more general context. Lötstedt attributes his example to Béghin (1923) and Klein (1909), and cites Painlevé (1895) for the first examples of this kind.

The consistency of rigid–body mechanics is an interesting issue in the abstract, but also has some important practical ramifications. Robotics research, in particular, is exploring automatic systems for solving a number of problems in rigid–body mechanics, such as testing whether a structure of objects is stable, where to place kinematic constraints (such as fingertips) so as to move, or immobilize, an object, and determining the evolution of a system for off–line programming and debugging of robotic manipulator systems. Erdmann (1984), and Rajan et al (1987), explore consistency and ambiguity of rigid–body mechanics from a robotics perspective.

The possible existence of inconsistencies in rigid–body mechanics is difficult for some of us to accept. For the reader who has not faced this issue before, we offer the following intuitive explanation, which may make the possibility of inconsistencies more plausible, if not more palatable. To begin, consider the problem of a system of a finite number of particles subject to Newton's laws, and suppose, for concreteness, that the only forces acting among the particles are the result of mutual gravitation. Now, for any given state, i.e. a specification of the instantaneous positions and velocities of the particles, a total force acting on each particle is completely determined. The change in state is obtained by integrating these forces, and is likewise well–defined and completely determined.

The laws of rigid–body mechanics with Coulomb friction are not as simple as the laws of the system described above. In particular, Coulomb's law of sliding friction does not specify the force as a unique function of the state. Rather, it imposes a *constraint* on the force and the resultant states. The law does not suggest an effective means of determining the contact forces, and, in practice, simulations must occasionally search for a set of contact forces satisfying the constraints. Given this state of affairs, it is not too surprising to find that the search might turn up more than one solution (ambiguity) or fail to turn up any solutions (inconsistency).

This argument suggests the plausibility of inconsistency, but to prove inconsistency is a different matter. It suffices to demonstrate a problem with no solution satisfying the axioms of the theory. The example in question is a thin rigid rod sliding along a horizontal surface. In nice cases, the contact produces a force that balances the gravitational force, so that the end of the rod either continues sideways, or accelerates away from the surface. However, with a particular choice of the dynamic parameters, we can arrange for all feasible finite contact forces to generate an angular acceleration of the rod, with the net effect of accelerating the end of the rod into the surface, rather than away from the surface.

The problem is resolved by recognizing that we have an *impact* problem, even though the rod is initially moving horizontally. It is possible to construct impulsive forces, fully in accord with the relevant axioms, that do not accelerate the end of the rod downwards. Small impulsive forces are subject to the same constraint as finite forces, but a large enough impulse can instantaneously halt the rod's sideways motion, after which the constraint imposed by Coulomb on additional impulse is considerably relaxed. The details originate in Routh's (1860) treatment of rigid–body impact, which is further developed by

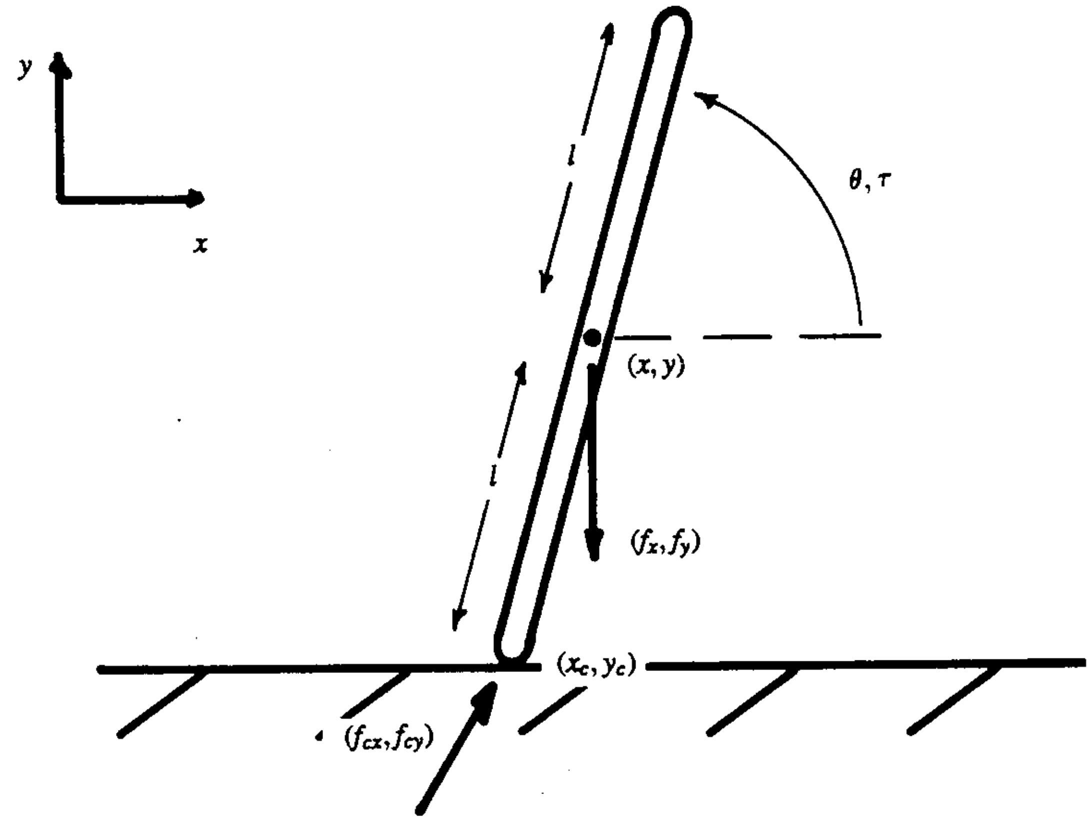

Figure 1: A rigid rod on a frictional surface.

Keller (1986) and Wang and Mason (1987).

2. Finite force analysis of Lötstedt's example

In this section we recapitulate Lötstedt's analysis of the rod sliding on a surface, introducing geometrical methods where Lötstedt relies primarily on algebraic methods. Lötstedt's methods are more suitable than ours when generality is important, but we believe that the geometrical methods are easier to understand.

Consider the rigid rod and horizontal surface of Figure 1. Following Lötstedt, we assume that the mass is distributed symmetrically about the geometrical midpoint of the rod. The ends of the rod are at distance l from the center of gravity.

We have a coordinate system aligned with the surface, and let (x, y) denote the location of the center of mass, (x_c, y_c) the lower end of the rod, θ the angle of the rod with respect to a horizontal reference, (f_{cx}, f_{cy}) the contact force acting on the rod, (f_x, f_y) an applied force acting at the center of mass, and τ an applied torque. The mass of the rod is m, the angular inertia J, and the coefficient of friction at the contact is a constant μ, whether sliding occurs or not.

Purely kinematic considerations provide the following relations for the position, velocity, and acceleration of the end of the rod:

$$
\begin{aligned}
x_c &= x - l\cos\theta && (1)\\
y_c &= y - l\sin\theta && (2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_c &= \dot{x} + \dot{\theta} l \sin\theta && (3)\\
\dot{y}_c &= \dot{y} - \dot{\theta} l \cos\theta && (4)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x}_c &= \ddot{x} + \ddot{\theta} l \sin\theta + \dot{\theta}^2 l \cos\theta && (5)\\
\ddot{y}_c &= \ddot{y} - \ddot{\theta} l \cos\theta + \dot{\theta}^2 l \sin\theta && (6)
\end{aligned}
$$

For simplicity, we will assume the mass and the angular inertia to be 1. (The reader should not assume a uniform distribution of mass, but might imagine a weightless rigid rod with two point masses at unit distance from the center of mass.) Newton's laws give the following equations of motion:

$$
\begin{aligned}
\ddot{x} &= f_x + f_{cx} && (7)\\
\ddot{y} &= f_y + f_{cy} && (8)\\
\ddot{\theta} &= \tau + l f_{cx}\sin\theta - l f_{cy}\cos\theta && (9)
\end{aligned}
$$

We can obtain the equations of motion expressed with respect to the coordinates of the contact point, by combining equations 5–9:

$$
\begin{aligned}
\ddot{x}_c &= f_x + \tau l \sin\theta + f_{cx}\left(1 + l^2\sin^2\theta\right) + f_{cy}\left(-l^2\cos\theta\cdot\sin\theta\right) + \dot{\theta}^2 l\cos\theta && (10)\\
\ddot{y}_c &= f_y - \tau l \cos\theta + f_{cy}\left(1 + l^2\cos^2\theta\right) + f_{cx}\left(-l^2\cos\theta\sin\theta\right) + \dot{\theta}^2 l\cos\theta && (11)
\end{aligned}
$$

To obtain Lötstedt's example, we hypothesize a translational motion to the left, with a gravitational applied force:

$$
\begin{aligned}
\dot{x}_c &< 0 && (12)\\
\dot{y}_c &= 0 && (13)\\
\dot{\theta} &= 0 && (14)
\end{aligned}
$$

$$
\begin{aligned}
f_x &= 0 && (15)\\
f_y &= -g && (16)\\
\tau &= 0 && (17)
\end{aligned}
$$

Also substituting

$$f_{cx} = \mu f_{cy} \quad (18)$$

and simplifying, we obtain

$$\ddot{y}_c = -g + a f_{cy} \quad (19)$$

where

$$a = \frac{l^2}{2}\left(\frac{l^2+2}{l^2} + \cos 2\theta - \mu\sin 2\theta\right) \quad (20)$$

The contact conditions dictate that both $\ddot{y}_c$ and f_{cy} be non–negative, which, from inspection of equation 19 implies that a must be positive. To complete the example, then, we choose values for l, μ, and θ to obtain a negative a. In particular, $\mu = \tan 30°$, $\theta = 15°$, and $l = 4$ renders a negative.

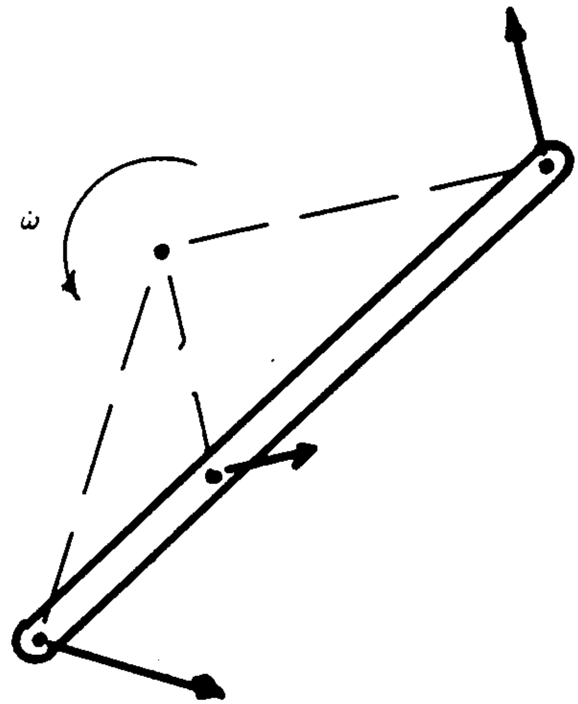

Figure 2: Instantaneous acceleration center.

We can best explain the situation in terms of the *instantaneous acceleration center* (Hall 1961). First, consider a motionless rod subjected to a rotational acceleration about a center (x_a, y_a), as in Figure 2. The acceleration at each point is perpendicular to a line drawn from the acceleration center, and proportional in magnitude to the length of that line. It is apparent that any point to the left of the acceleration center will have a downwards component of acceleration, and, in particular, the bottom of the rod is being accelerated downward.

Now, to apply this observation to Lötstedt's example, consider Figure 3. Note first that the rod is subject to two forces, one of which, the gravitational force, is fixed. The contact force is constrained by Coulomb's law to lie on a ray, making an angle $\tan^{-1}\mu$ from the contact normal. Its magnitude is unconstrained. The contact force and the gravitational force always intersect at the same point, so we can express the total force as a single force acting on a line passing through that intersection point. The family of feasible (finite) forces is illustrated in the figure. Now, without being distracted by the details, for each feasible force, we plot an instantaneous acceleration center. All of the acceleration centers fall on a single horizontal ray, which is delimited on the left by a line perpendicular to the contact force and passing through the center of mass. The distance from the ray to the center of mass varies as ρ^2. (ρ is defined to be the radius of gyration, i.e. $\sqrt{\frac{I}{m}}$.) Now, by decreasing ρ (or, equivalently, leaving ρ at a constant 1 and increasing I) it is easy to see that we can keep the feasible acceleration centers to the right of the bottom end of the rod, which, as we observed earlier, implies that the bottom of the rod accelerates

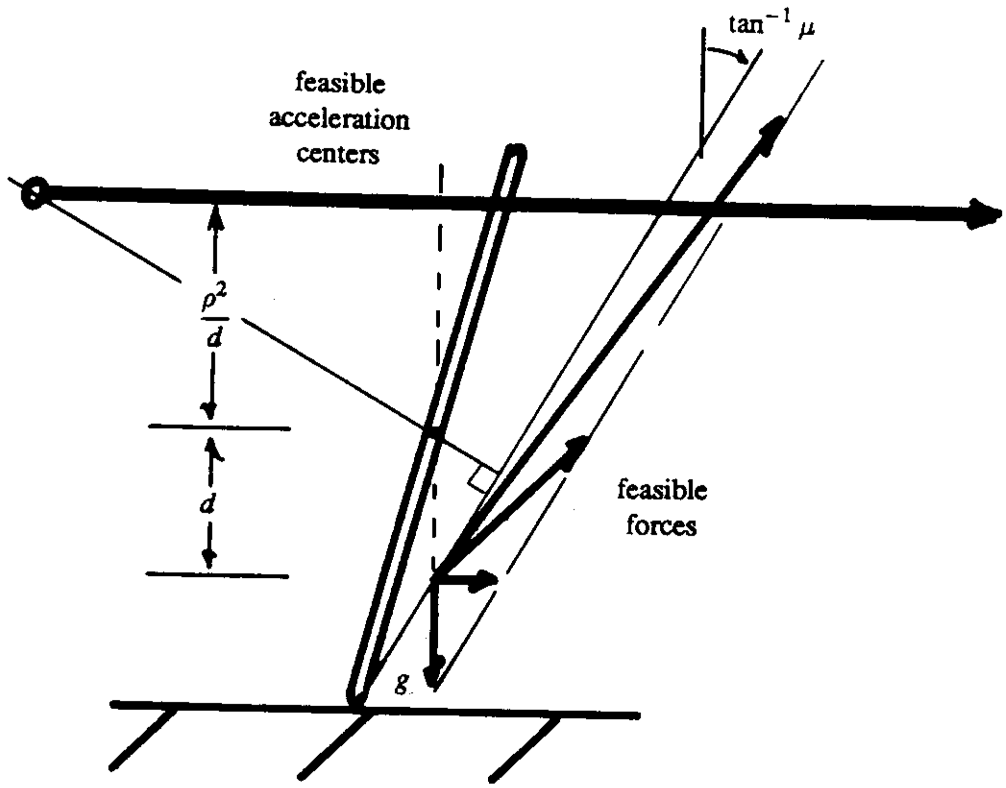

Figure 3: The locus of feasible acceleration centers.

into the table.

3. Impact analysis of Lötstedt's example.

The resolution of Lötstedt's problem lies in viewing the rod/table interaction as a collision, involving impulsive forces. It may seem paradoxical to have a collision between two objects that are not approaching one another (and even, as we shall see, between two objects that are touching but moving away from one another!) but such a collision is surely preferable to an inconsistency in our theory. And, in any case, we have no choice, the collision being admitted by the theory.

In this section, we follow Wang and Mason (1987), which is based on Routh's (1860) analysis. We keep most of the previous section's notation, with some changes to simplify the analysis. The contact point is instantaneously at the origin, and we use, for example, v_{cx} to indicate a velocity rather than our earlier $\dot{x}_c$. P_n and P_t denote the normal and tangential components of impulse, respectively. We do not include any other applied forces. Any finite forces would be negligible relative to the impact forces, although the existence of an external force, such as the gravitational force in the last section, can determine whether the impact *must* occur, or *might* occur.

The following kinematic relations must hold:

$$v_{cx} = v_x + y\omega \tag{21}$$
$$v_{cy} = v_x - x\omega \tag{22}$$

$$\Delta v_{cx} = \Delta v_x + y\Delta\omega \tag{23}$$
$$\Delta v_{cy} = \Delta v_y - x\Delta\omega \tag{24}$$

where $\Delta v_{cx} = v_{cx} - v_{cx0}$ etc., and the following impulse–momentum laws relate the effect of impulse:

$$m\Delta v_x = P_t \tag{25}$$
$$m\Delta v_y = P_n \tag{26}$$
$$m\rho^2\Delta\omega = P_t y - P_n x \tag{27}$$

Substituting into the kinematic equations, we obtain:

$$\Delta v_{cx} = \frac{P_t}{m} + y\frac{P_t y - P_n x}{m\rho^2} \tag{28}$$
$$\Delta v_{cy} = \frac{P_n}{m} - x\frac{P_t y - P_n x}{m\rho^2} \tag{29}$$

Now, the reason that impact works is that we can obtain enough zorch to instantaneously cancel the tangential motion at the contact point. We will call the condition of zero tangential relative motion *sticking*. We also define a condition called *maximum compression*, occuring at zero normal relative motion. Each of these conditions defines a linear relation between P_n and P_t. To find the sticking condition, we set $v_{cx} = 0$, obtaining:

$$0 = v_{cx0} + P_t\frac{\rho^2 + y^2}{m\rho^2} - P_n\frac{xy}{m\rho^2} \tag{30}$$

To find the maximum compression condition, we set $v_{cy} = 0$, obtaining:

$$0 = v_{cy0} + P_n\frac{\rho^2 + x^2}{m\rho^2} - P_t\frac{xy}{m\rho^2} \tag{31}$$

These two linear relations define lines in *impulse space*, which are plotted in Figure 4, using the same parameter values as the previous section. We have also plotted the line $P_t = \mu P_n$ through the origin, making an angle $\tan^{-1}\mu$ with the vertical. Using these three lines we can construct an impulse that satisfies the laws of Newton and Coulomb and preserves the rigidity of the rod. Although the impact is assumed to be instantaneous, it is convenient to think about the impulse accumulating from zero. Using Figure 4, the *characteristic point* representing the cumulative impulse begins at the origin, and moves along the line $P_t = \mu P_n$. The reason is that differential impulse is force, and since the rod is sliding leftwards, the differential force must obey Coulomb's law. Note, however, that eventually the characteristic point reaches the line of sticking. Now, Coulomb's law allows $dP_t \leq \mu dP_n$, resisting any impending resumption of sliding. In this case, the characteristic point can satisfy this constraint by moving along the sticking line.

To complete the construction of the total impulse, we push the characteristic point to the line of maximum compression. A perfectly plastic collision would terminate at this point, with the rod's bottom instantaneously at rest with respect to the surface. A perfectly elastic collision would continue until the normal component of impulse is doubled, and would pop away from the surface, with the path of the rod's bottom end perpendicular at the surface. Intermediate cases, with coefficients of restitution between 0 and 1, terminate between these two extremes, and bounce away from the surface with varying amounts of energy.

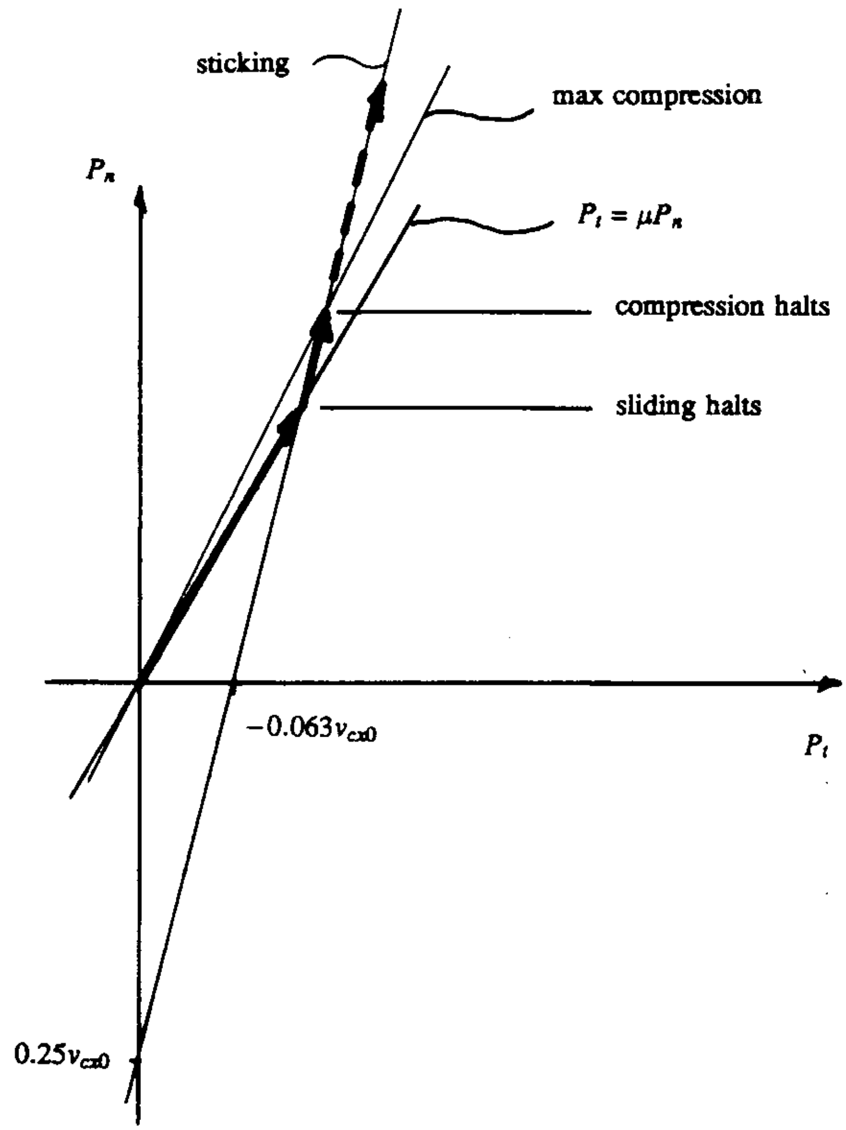

Figure 4: Impact analysis.

4. Remarks

The next step is to review the literature for other examples of inconsistency. While we have just begun this process, we cannot help but wonder whether the consistency of rigid–body mechanics might be an open question.

We note in parting that this problem has important ramifications in the analysis of impact, besides its obvious relevance to the foundations of rigid–body mechanics. In our earlier work, (Wang and Mason 1987) we neglected the possibility of zero approach velocity, and we should also note that the impact might occur for small negative approach velocities. We are also rethinking our use of the coefficient of restitution. It is particularly noteworthy that Newton's use of the coefficient of restitution is inapplicable to this problem, being defined as the ratio of the initial and final normal velocities. Poisson's definition of coefficient of restitution, which relates the normal impulses during compression and restitution, is applicable, although we are disturbed by the possibility of cases in which two distinct phases of compression and restitution might not be present.

References

1. H. Béghin. 1923. Sur certain problèmes de frottement. *Nouvelles Annales, 5e série* 2 (1923–24), pp. 305–312.

2. M. A. Erdmann. 1984. On motion planning with uncertainty. M.S. thesis, Electrical Engineering and Computer Science, Massachusetts Institute of Technology.

3. A. S. Hall. 1961. *Kinematics and Linkage Design*. Waveland Press, Prospect Heights, Illinois.

4. J. B. Keller. 1986. Impact with friction. *Journal of Applied Mechanics* 53, pp. 1–4.

5. C. W. Kilmister and J. E. Reeve. 1966. *Rational Mechanics*, American Elsevier, New York.

6. F. Klein. 1909. Zu Painlevés Kritik der Coulombschen Rebungsgesetze. *Z. f. Math. und Phys.*, 58, S. 186–191.

7. P. Lötstedt. 1981. Coulomb friction in two–dimensional rigid body systems. *Zeitschrift für Angewandte Mathematik un Mechanik*, 61, pp. 605–615.

8. P. Painlevé. 1895. Sur les lois du frottement de glissement. *Comptes Rendus de l'Académie des Sciences*, 121, pp. 112–115, 140 (1905), pp. 702–707, 141 (1905), pp. 401–405 and 546–552.

9. V. T. Rajan, R. Burridge, and J. T. Schwartz. 1987. Dynamics of a rigid body in frictional contact with rigid walls: motion in two dimensions. *Proceedings, 1987 IEEE Int. Conf. on Robotics and Automation*. Raleigh, North Carolina.

10. E. J. Routh. 1860. *Dynamics of a system of rigid bodies*. MacMillan, London.

11. Y. Wang and M. T. Mason. 1987. Modeling impact dynamics for robotic operations. *Proceedings, 1987 IEEE Int. Conf. on Robotics and Automation*. Raleigh, North Carolina.