SOME ASPECTS OF THE CYCLIC REDUCTION ALGORITHM
FOR BLOCK TRIDIAGONAL LINEAR SYSTEMS

Don Heller
Department of Computer Science
Carnegie-Mellon University
Pittsburgh, PA   15213

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ABSTRACT

The solution of a general block tridiagonal linear system by a cyclic odd-even reduction algorithm is considered. Under conditions of diagonal dominance, norms describing the off-diagonal blocks relative to the diagonal blocks decrease quadratically with each reduction. This allows early termination of the reduction when an approximate solution is desired. The algorithm is well-suited for parallel computation.

1. Introduction

We propose to solve the block tridiagonal linear system $Ax = v$, i.e.,

$$e_j x_{j-1} + d_j x_j + f_j x_{j+1} = v_j, \quad j = 1, \ldots, N,$$

$$N = 2^{m+1} - 1, \quad m \geq 1, \quad e_1 = f_N = 0.$$

The components are $n \times n$ matrices and n-vectors, so the overall dimension of A is $(Nn) \times (Nn)$. The algorithm considered is related to the cyclic odd-even reduction algorithm developed for the numerical solution of Poisson's equation on a rectangle, [3], [5], and to the more general cyclic reduction technique of Hageman and Varga [4].

The two fundamental operations of cyclic odd-even reduction are the elimination of odd-indexed unknowns and their eventual recovery through back-substitution. If we multiply equation $2j-1$ by $-e_{2j} d_{2j-1}^{-1}$, equation $2j+1$ by $-f_{2j} d_{2j+1}^{-1}$, and add these to equation $2j$, the result is

$$(-e_{2j} d_{2j-1}^{-1} e_{2j-1}) x_{2j-2}$$

$$+ (d_{2j} - e_{2j} d_{2j-1}^{-1} f_{2j-1} - f_{2j} d_{2j+1}^{-1} e_{2j+1}) x_{2j}$$

$$+ (-f_{2j} d_{2j+1}^{-1} f_{2j+1}) x_{2j+2}$$

$$= v_{2j} - e_{2j} d_{2j-1}^{-1} v_{2j-1} - f_{2j} d_{2j+1}^{-1} v_{2j+1}.$$

These new equations, for $j = 1, \ldots, 2^m - 1$, are again a block tridiagonal system, involving only the even indexed unknowns of x. This is the reduction step. Once x_{2j-2} and x_{2j} are computed, they may be substituted into equation $2j-1$ to compute x_{2j-1}. This is the back-substitution step.

Setting $A^{(0)} = A$, $x^{(0)} = x$, $v^{(0)} = v$, we generate a sequence of problems $A^{(i)}x^{(i)} = v^{(i)}$, of block dimension $(2^{m-i+1} - 1) \times (2^{m-i+1} - 1)$. At any stage we can stop the reduction, solve the system at hand and begin the back substitution. Since $A^{(m)}$ is composed of a single block, the reduction stops there. If $A^{(m)}x^{(m)} = v^{(m)}$ is solved exactly, and $x = x^{(0)}$ is computed exactly from the back substitution based on $x^{(m)}$, we have <u>complete cyclic reduction</u>. Now suppose we pick $0 < k \leq m$ and solve $A^{(k)}x^{(k)} = v^{(k)}$ approximately by computing $y^{(k)} \approx x^{(k)}$. If $y = y^{(0)}$ is computed exactly from the back substitution based on $y^{(k)}$, we have <u>incomplete cyclic reduction</u>. The major goal of this paper is an analysis of incomplete cyclic reduction.

If A satisfies certain diagonal dominance conditions, then good approximations $y^{(k)}$ exist, are easily computed, and the errors incurred are not dangerously propagated to the entire solution. It is also possible to pick k based on a few simple a priori calculations. In particular, norms describing the off-diagonal blocks relative to the diagonal blocks decrease quadratically with each reduction. This fact justifies the use of incomplete cyclic reduction.

The cyclic reduction algorithm is well-suited for use on a parallel or pipeline computer, as many of the quantities involved may be computed independently of the others. The special case $n = 1$ has been studied by Lambiotte and Voigt [6], with attention to a pipeline computer, and by Stone [7] who uses a slightly different formulation of the elimination step.

The cyclic reduction algorithm was originally developed by Hockney [5] for the discrete version of Poisson's equation. In this case it is possible to replace the solution of systems by matrix multiplication. It was Hockney's observation that, in the case $n = 1$ and constant diagonals (c.f. remark 7, Section 3), the reduction could be stopped when the ratio of the off-diagonal elements to the diagonal elements fell below the machine precision. Then the tridiagonal system was essentially diagonal and could be solved as such without damage to the solution.

The Buneman algorithm for Poisson's equation [3] also generates approximations to the solution in the course of the reduction. A quadratic convergence result (equations 13.4, 13.6, of [3]) is given but only used in the stability proof. Buzbee [2] developed this to obtain a truncated Buneman algorithm much like our incomplete cyclic reduction, but only applicable to the case $A = L_h + dI$, where L_h is the discrete Laplacian operator. It was then shown how to use this special case in an iterative method to solve the discretization of $u_t = \nabla(a\nabla u) + S$ on a rectangle.

Another formulation of cyclic reduction for the general case that does not require solving systems at each stage is given by Stone [7]: multiply equation 2j-1 by $-e_{2j}d_{2j+1}$, equation 2j by $d_{2j-1}d_{2j+1}$, equation 2j+1 by $-f_{2j}d_{2j-1}$, and add. If $d_{2j-1}d_{2j+1}e_{2j} = e_{2j}d_{2j+1}d_{2j-1}$, and $d_{2j-1}d_{2j+1}f_{2j} = f_{2j}d_{2j-1}d_{2j+1}$, the set of new equations is again tri-

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diagonal. If n = 1 this commutativity condition is no restriction; in fact, our method differs only by a scaling factor. However, for n > 1 the added cost of solving 4n+2 linear systems per new equation may be a small price for systems without commutativity. In addition, Stone's version cannot handle variable block sizes, nor is symmetry preserved except in the case of constant diagonals.

An odd-even reduction for block pentadiagonal systems arising from the discrete biharmonic operator is given in [1]. This problem could also be treated by partitioning into a block tridiagonal form.

2. The Algorithm in Detail and its Relation to Gaussian Elimination

We describe the means of generating a sequence of problems $A^{(i)}x^{(i)} = v^{(i)}$ such that $x_j^{(i+1)} = x_{2j}^{(i)}$, and a back substitution to recover the odd-indexed unknowns of $x^{(i)}$ given $x^{(i+1)}$.

Suppose we have $Ax = v$, an $N \times N$ system with $n \times n$ blocks, $N = 2^{m+1} - 1$,

$$
A = \begin{pmatrix} d_1 & f_1 & & & & \\ e_2 & d_2 & f_2 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & f_{N-1} \\ & & & & e_N & d_N \end{pmatrix} \equiv (e_j, d_j, f_j)_{N \times N},
$$

$e_1 = f_N = 0$. The extension to general values of N and variable block sizes is immediate but will not be considered here. Setting $N_i = 2^{m-i+1} - 1$,

let $A^{(i)} = (e_j^{(i)}, d_j^{(i)}, f_j^{(i)})_{N_i \times N_i}$, $x^{(i)} = (x_j^{(i)})_{N_i}$, $v^{(i)} = (v_j^{(i)})_{N_i}$,

with $A^{(0)} = A$, $x^{(0)} = x$, $v^{(0)} = v$. Then, for $i = 0, \ldots, m - 1$, and $j = 1, \ldots, N_{i+1}$, define

$$
e_j^{(i+1)} = -e_{2j}^{(i)} (d_{2j-1}^{(i)})^{-1} e_{2j-1}^{(i)},
$$

$$
d_j^{(i+1)} = d_{2j}^{(i)} - e_{2j}^{(i)} (d_{2j-1}^{(i)})^{-1} f_{2j-1}^{(i)} - f_{2j}^{(i)} (d_{2j+1}^{(i)})^{-1} e_{2j+1}^{(i)},
$$

$$
f_j^{(i+1)} = -f_{2j}^{(i)} (d_{2j+1}^{(i)})^{-1} f_{2j+1}^{(i)},
$$

$$x_j^{(i+1)} = x_{2j}^{(i)},$$

$$v_j^{(i+1)} = v_{2j}^{(i)} - e_{2j}^{(i)}(d_{2j-1}^{(i)})^{-1}v_{2j-1}^{(i)} - f_{2j}^{(i)}(d_{2j+1}^{(i)})^{-1}v_{2j+1}^{(i)}.$$

Now, suppose $y^{(i+1)}$ is an approximation to $x^{(i+1)}$, perhaps the true value. Then $y^{(i)}$, an approximation to $x^{(i)}$, is found by

$$y_{2j}^{(i)} = y_j^{(i+1)}, \quad j = 1, \ldots, N_{i+1},$$

$$y_1^{(i)} = (d_1^{(i)})^{-1}(v_1^{(i)} - f_1^{(i)}y_2^{(i)}),$$

$$y_{2j-1}^{(i)} = (d_{2j-1}^{(i)})^{-1}(v_{2j-1}^{(i)} - e_{2j-1}^{(i)}y_{2j-2}^{(i)} - f_{2j-1}^{(i)}y_{2j}^{(i)}),$$

$$j = 2, \ldots, N_{i+1},$$

$$y_{N_i}^{(i)} = (d_{N_i}^{(i)})^{-1}(v_{N_i}^{(i)} - e_{N_i}^{(i)}y_{N_i-1}^{(i)}).$$

Of course, it must be shown that $d_j^{(i)}$ is non-singular; this will be done in Section 3 under certain assumptions about A. In addition we show that the factors used in the reduction and back substitution form a quadratically decreasing sequence.

Complete cyclic reduction reduces $Ax = v$ to $A^{(m)}x^{(m)} = v^{(m)}$, solves this exactly and proceeds through the back substitution to an exact solution. Incomplete cyclic reduction stops with $A^{(k)}x^{(k)} = v^{(k)}$, $0 \le k \le m$, solves this approximately and begins the back substitution as described above.

Cyclic reduction is equivalent to block Gaussian elimination without (block) pivoting on a permuted system $(PAP^T)(Px) = (Pv)$. P reorders the vector $(1,2,3,\ldots,N)$ so that the odd multiples of 2^0 come first, followed by the odd multiples of 2^1, the odd multiples of 2^2, etc.

For $N = 7$,

$$
PAP^T = \begin{pmatrix}
d_1 & & & & f_1 & & \\
& d_3 & & & e_3 & & f_3 \\
& & d_5 & & & f_5 & e_5 \\
& & & d_7 & & e_7 & \\
e_2 & f_2 & & & d_2 & & \\
& & e_6 & f_6 & & d_6 & \\
& e_4 & f_4 & & & & d_4
\end{pmatrix}
$$

The block factorization of $PAP^T = LU$ is

$$
L = \begin{pmatrix}
I & & & & & & \\
& I & & & & & \\
& & I & & & & \\
& & & I & & & \\
e_2 d_1^{-1} & f_2 d_3^{-1} & & & I & & \\
& & e_6 d_5^{-1} & f_6 d_7^{-1} & & I & \\
& e_4 d_3^{-1} & f_4 d_5^{-1} & & e_2^{(1)}(d_1^{(1)})^{-1} & f_2^{(1)}(d_3^{(1)})^{-1} & I
\end{pmatrix}
$$

$$U = \begin{pmatrix}
d_1 & & & & f_1 & & \\
& d_3 & & & e_3 & & f_3 \\
& & d_5 & & & f_5 & e_5 \\
& & & d_7 & e_7 & & \\
& & & & d_1^{(1)} & & f_1^{(1)} \\
& & & & & d_3^{(1)} & e_3^{(1)} \\
& & & & & & d_1^{(2)}
\end{pmatrix}$$

If we let $d_j^{(i)} = \ell_j^{(i)} u_j^{(i)}$ be an LU factorization, $\ell_j^{(i)}$ with unit diagonal, then the ordinary LU factorization of $PAP^T = L'U'$ is

$$L' = \begin{pmatrix}
\ell_1 & & & & & & \\
& \ell_3 & & & & & \\
& & \ell_5 & & & & \\
& & & \ell_7 & & & \\
e_2 u_1^{-1} & f_2 u_3^{-1} & & & \ell_1^{(1)} & & \\
& & e_6 u_5^{-1} & f_6 u_7^{-1} & & \ell_3^{(1)} & \\
& e_4 u_3^{-1} & f_4 u_5^{-1} & & e_2^{(1)}(u_1^{(1)})^{-1} & f_2^{(1)}(u_3^{(1)})^{-1} & \ell_1^{(2)}
\end{pmatrix}$$

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$$U' = \begin{pmatrix} u_1 & & & & \ell_1^{-1}f_1 & & \\ & u_3 & & & \ell_3^{-1}e_3 & \ell_3^{-1}f_3 & \\ & & u_5 & & \ell_5^{-1}f_5 & \ell_5^{-1}e_5 & \\ & & & u_7 & \ell_7^{-1}e_7 & & \\ & & & & u_1^{(1)} & (\ell_1^{(1)})^{-1}f_1^{(1)} & \\ & & & & & u_3^{(1)} & (\ell_3^{(1)})^{-1}e_3^{(1)} \\ & & & & & & u_1^{(2)} \end{pmatrix}$$

This factorization gives rise to a related method, first solving L'z = v, then U'x - z. It is easily seen that $z_j^{(i)} = (\ell_j^{(i)})^{-1}v_j^{(i)}$.

We may now apply any of the theorems about Gaussian elimination to PAP^T and obtain a corresponding result about this modification of cyclic reduction for A. In particular, if A is strictly diagonally dominant or positive definite, so is PAP^T and thus Gaussian elimination is well-defined and stable [10].

Finally, note that a Cholesky factorization of $d_j^{(i)} = \ell_j^{(i)}\ell_j^{(i)T}$ may be used if A is positive definite. The result is a Cholesky factorization $PAP^T = LL^T$.

We now turn to the question of storage requirements. If we use block Gaussian elimination in the natural ordering of A (see, for instance, [8]), then no additional block storage is required since there is no block fill-in. However, it has already been seen that cyclic reduction does generate fill-in. The amount of fill-in is the number of off-diagonal blocks of $A^{(1)}, \ldots, A^{(k)}$, which is $\Sigma_{i=1}^{k} 2N_i - 2 = 2N + 2 - 2^{m-k+2} - 4k$. Since the

matrix A requires 3N - 2 storage blocks, this is not excessive. Of course, we have not considered the internal structure of the n×n blocks, as this is highly dependent on the specific system. Also note that $v_j^{(i)}$ may overwrite $v_2 i_j$, and that $x_j^{(i)} = x_2 i_j$ may overwrite $v_j^{(i)}$. Thus no additional storage is required for these vectors.

A time estimate for cyclic reduction is somewhat more difficult. Suppose we are going to make the reduction from $A^{(i)}$ to $A^{(i+1)}$. Each of the N_{i+1} new equations may be formed independently of the others, so cyclic reduction may be used effectively on a parallel or pipeline computer. For a particular computer, the execution time may or may not depend on the number of arithmetic operations actually performed, as many of them would be done simultaneously. A first approximation shows that $19/3 \, n^3 N$ multiplications are used in complete cyclic reduction. This compares with $7/3 \, n^3 N$ for block Gaussian elimination in the given ordering [8].

The use of synchronous parallelism, as with the Illiac IV and CDC Star, creates some constraints which are not so important on a sequential computer. First, pivoting causes inefficiencies when the pivot selections differ between the diagonal blocks. If A is strictly diagonally dominant or positive definite, then no pivoting is required. Secondly, the choice of data structures to represent the matrices and vectors must conform to the machine's requirements for the parallel or vector operations. In the case of the Illiac IV, the problem of communication of data between processing elements must also be considered. It may be that the construction of a program is more important to the success of the whole than the

construction of a numerical method.

Thorough discussions of the use of cyclic reduction on parallel and pipeline computers for the case $n = 1$ may be found in [6] and [7]. The basic method given here is a matrix analogue of the method examined in the former paper.

3. Convergence Theorems

In this section we examine the errors due to approximation in incomplete cyclic reduction. The off-diagonal blocks of $A^{(i)}$ are measured by considering norms of

$$B^{(i)} \equiv (-(d_j^{(i)})^{-1} e_j^{(i)}, \ 0, \ -(d_j^{(i)})^{-1} f_j^{(i)}),$$

and the multipliers used in the reduction from $A^{(i)}$ to $A^{(i+1)}$ are measured by considering norms of

$$C^{(i)} \equiv (-e_j^{(i)}(d_{j-1}^{(1)})^{-1}, \ 0, \ -f_j^{(i)}(d_{j+1}^{(i)})^{-1}).$$

We use the vector and matrix infinity norms

$$||v||_\infty = \max_{j=1,\ldots,p} |v_j|,$$

$$||M||_\infty = \max_{i=1,\ldots,p} \Sigma_{j=1}^q |M_{ij}|,$$

and the matrix one norm

$$||M||_1 = \max_{j=1,\ldots,q} \Sigma_{i=1}^p |M_{ij}|,$$

where v is a p-vector and M is p×q. It will be convenient to define

$$\rho_i(M) = \Sigma_{j=1}^q |M_{ij}|, \quad i = 1, \ldots, p.$$

Recall that the incomplete cyclic reduction algorithm reduces $Ax = v$ to $A^{(k)} x^{(k)} = v^{(k)}$, $0 \leq k \leq m$, computes an approximation

$y^{(k)} \approx x^{(k)}$, finally arriving at an approximation $y = y^{(0)} \approx x^{(0)} = x$.

__Theorem.__ Suppose d_j^{-1} exists, $j = 1, \ldots, N$, and $||B^{(0)}||_\infty < 1$. Then

1. Cyclic reduction is well-defined; i.e., $(d_j^{(i)})^{-1}$ exists,

    $i = 0, \ldots, m; \; j = 1, \ldots, N_i$.

2. In incomplete cyclic reduction, $||x - y||_\infty = ||x^{(k)} - y^{(k)}||_\infty$.

3. If $y_j^{(k)} = (d_j^{(k)})^{-1} v_j^{(k)}$, then $\dfrac{||x^{(k)} - y^{(k)}||_\infty}{||x^{(k)}||_\infty} \leq ||B^{(k)}||_\infty$.

4. $||B^{(i+1)}||_\infty \leq ||(B^{(i)})^2||_\infty < 1, \; i = 0, \ldots, m - 1$.

5. If $||C^{(0)}||_1 < 1$, then $||C^{(i+1)}||_1 \leq ||(C^{(i)})^2||_1 < 1$,

    $i = 0, \ldots, m - 1$.

__Proof.__ We first show parts 1, 4, and 5, then 2 and 3. Assume that $A^{(i)}$ has invertible diagonal blocks, $||B^{(i)}||_\infty < 1$, $||C^{(i)}||_1 < 1$. The induction hypothesis $||C^{(i)}||_1 < 1$ will be used only in the proof of part 5. Superscripts will be deleted for convenience, and are assumed to be (i) unless otherwise stated. Define

$$\sigma_1 = d_1^{-1} f_1 d_2^{-1} e_2,$$

$$\sigma_j = d_j^{-1} e_j d_{j-1}^{-1} f_{j-1} + d_j^{-1} f_j d_{j+1}^{-1} e_{j+1}, \; j = 2, \ldots, N_i - 1,$$

$$\sigma_{N_i} = d_{N_i}^{-1} e_{N_i} d_{N_i-1}^{-1} f_{N_i-1},$$

$$\tau_j = d_j \sigma_j d_j^{-1}, \; j = 1, \ldots, N_i,$$

$$S = \text{diag}(\sigma_j),$$

$$T = \text{diag}(\tau_j),$$

$$E_1 = 0, \quad E_j = -e_j d_{j-1}^{-1} e_{j-1}, \quad j = 2, \ldots, N_i,$$

$$D_j = d_j(I - \sigma_j) = (I - \tau_j)d_j, \quad j = 1, \ldots, N_i,$$

$$F_j = -f_j d_{j+1}^{-1} f_{j+1}, \quad j = 1, \ldots, N_i-1, \quad F_{N_i} = 0.$$

Now, σ_j is the diagonal block of the jth block row of B^2, so we have $||\sigma_j||_\infty \leq ||B^2||_\infty \leq ||B||_\infty^2 < 1$. Similarly, τ_j is the diagonal block of the jth block row of C^2, so $||\tau_j||_1 \leq ||C^2||_1 \leq ||C||_1^2 < 1$. Thus, from either condition, $(I - \sigma_j)^{-1}$ and $(I - \tau_j)^{-1}$ exist, as does $D_j^{-1} = (I - \sigma_j)^{-1}d_j^{-1} = d_j^{-1}(I - \tau_j)^{-1}$. But $d_j^{(i+1)} = D_{2j}$, so part 1 is proven.

We also have

$$||S||_\infty = \max_{j=1,\ldots,N_i} ||\sigma_j||_\infty \leq ||B^2||_\infty < 1,$$

$$||T||_1 = \max_{j=1,\ldots,N_i} ||\tau_j||_1 \leq ||C^2|| < 1.$$

Now, define

$$J = (-D_j^{-1}E_j, 0, 0, 0, -D_j^{-1}F_j),$$

$$K = (-E_j D_{j-2}^{-1}, 0, 0, 0, -F_j D_{j+2}^{-1}).$$

Since $e_j^{(i+1)} = E_{2j}$, $d_j^{(i+1)} = D_{2j}$, $f_j^{(i+1)} = F_{2j}$, it is easily shown that $||B^{(i+1)}||_\infty \leq ||J||_\infty$ and $||C^{(i+1)}||_1 \leq ||K||_1$. We also have $B^2 = (I - S)J + S$, and $C^2 = K(I - T) + T$. Thus, for $\ell = 1, \ldots, N_i n$,

$$||B^2||_\infty \geq \rho_\ell(B^2) = \rho_\ell(S) + \rho_\ell((I - S)J)$$

$$\geq \rho_\ell(S) + \rho_\ell(J) - \rho_\ell(S)||J||_\infty$$

$$= \rho_\ell(J) + \rho_\ell(S)(1 - ||J||_\infty).$$

We now show by contradiction that $1 > ||J||_\infty$. Suppose $||J||_\infty = 1$. For

some ℓ, we have $||J||_\infty = \rho_\ell(J) = 1$, and for this ℓ, $1 > ||B^2||_\infty \geq \rho_\ell(J) = 1$, a contradiction. Suppose $||J||_\infty > 1$. For some ℓ, $||J||_\infty = \rho_\ell(J) > 1$; using $1 > \rho_\ell(J) + \rho_\ell(S)(1 - ||J||_\infty)$, we have $1 < \rho_\ell(S)$. But we showed earlier that $\rho_\ell(S) \leq ||S||_\infty < 1$, so this is also a contradiction. Thus $||J||_\infty < 1$ and $||B^2||_\infty \geq \rho_\ell(J) + \rho_\ell(S)(1 - ||J||_\infty) \geq \rho_\ell(J)$. Since this holds for each ℓ, $||B^2||_\infty \geq ||J||_\infty$, establishing part 4. A similar proof holds for C, establishing part 5.

Consider the vector $x^{(i)} - y^{(i)}$. We have

$$x_1 - y_1 = -d_1^{-1}f_1(x_1^{(i+1)} - y_1^{(i+1)}),$$

$$x_{2j} - y_{2j} = x_j^{(i+1)} - y_j^{(i+1)}, \quad j = 1, \ldots, N_{i+1}$$

$$x_{2j+1} - y_{2j+1} = -d_{2j+1}^{-1}e_{2j+1}(x_j^{(i+1)} - y_j^{(i+1)})$$

$$- d_{2j+1}^{-1}f_{2j+1}(x_{j+1}^{(i+1)} - y_{j+1}^{(i+1)}),$$

$$j = 1, \ldots, N_{i+1} - 1,$$

$$x_{N_i} - y_{N_i} = -d_{N_i}^{-1}e_{N_i}(x_{N_{i+1}}^{(i+1)} - y_{N_{i+1}}^{(i+1)}).$$

Define z by

$$z = (0, x_1^{(i+1)} - y_1^{(i+1)}, 0, x_2^{(i+1)} - y_2^{(i+1)}, 0, \ldots,$$

$$x_{N_{i+1}}^{(i+1)} - y_{N_{i+1}}^{(i+1)}, 0).$$

We have $||x^{(i+1)} - y^{(i+1)}||_\infty = ||z||_\infty$ by construction. Also by construction,

$$||x^{(i)} - y^{(i)}||_\infty = \max(||x^{(i+1)} - y^{(i+1)}||_\infty, ||Bz||_\infty).$$

But $||Bz||_\infty \leq ||B||_\infty ||z||_\infty < ||z||_\infty = ||x^{(i+1)} - y^{(i+1)}||_\infty$, so
$||x^{(i)} - y^{(i)}||_\infty = ||x^{(i+1)} - y^{(i+1)}||_\infty$. This proves part 2. Suppose
our approximation for incomplete cyclic reduction is $y_j^{(k)} = (d_j^{(k)})^{-1} v_j^{(k)}$. Then
$x^{(k)} - y^{(k)} = B^{(k)} x^{(k)}$, so $||x^{(k)} - y^{(k)}||_\infty \leq ||B^{(k)}||_\infty ||x^{(k)}||_\infty$.

<div align="center">Q.E.D.</div>

Remarks and Corollaries.

1. For $n = 1$ and A irreducibly diagonally dominant, there is a
similar result, but now $||B^{(i+1)}||_\infty \leq ||(B^{(i)})^2||_\infty \leq 1$. The example A
$= (1,2,1)$, for which $||B^{(i)}||_\infty = 1$, $i = 0, \ldots, m-1$, $||B^{(m)}||_\infty = 0$,
shows that strict inequality is necessary for incomplete cyclic reduction
to be effective. Note that $||B^{(m)}||_\infty = 0$ for any A, since $A^{(m)}$ is only a
single block.

2. Let $PAP^T = L\Delta U$ be the unique factorization with Δ block diagonal,
L and U lower and upper unit triangular. (c.f. Section 2.) Then
$||L||_1 \leq 1 + ||c^{(0)}||_1$, $||U||_\infty \leq 1 + ||B^{(0)}||_\infty$. In L there are at most 2m blocks
per row off the diagonal, so $||L||_\infty \leq 1 + 2nm ||c^{(0)}||_1$. Similarly,
$||U||_1 \leq 1 + 2nm ||B^{(0)}||_\infty$.

3. If $\beta \equiv ||B^{(0)}||_\infty < 1$, $0 < \varepsilon < 1$, then

$$k = \max(0, \min(m, \left\lceil \log_2 \left(\frac{\log_2 \varepsilon}{\log_2 \beta} \right) \right\rceil))$$

guarantees $||B^{(k)}||_\infty \leq \varepsilon$. If $\beta = \frac{1}{2}$, $\varepsilon = 2^{-20} \doteq 10^{-6}$, then $k = 5$ and we

should have N > 32 for incomplete cyclic reduction to be applied.
In fact, $||B^{(5)}||_\infty \leq 2^{-32} \doteq 10^{-9.5}$, so the results of incomplete
cyclic reduction would be much better than required. In the case
$n = 1$, $e_j = f_j = e$, $d_j = d$, some improvements may be made in es-
timating the $B^{(i)}$ norms. Since $||B^{(i)}||_\infty = 2e/d$, $E = F = -e^2/d$,
$D = d - 2e^2/d$, we have $||B^{(i+1)}||_\infty = 2E/D = ||B^{(i)}||^2/(2 - ||B^{(i)}||^2)$
$< ||B^{(i)}||^2$. This observation has been made by Stone [7]. Since
$x^2/2 < x^2/(2 - x^2) < x^2$ for $0 < x < 1$, k must be larger than

$$\log_2\left(\frac{\log_2(\epsilon/2)}{\log_2(\beta/2)}\right)$$

in order to have $||B^{(k)}||_\infty \leq \epsilon$.

4. Superconvergence does occur, as indicated in the proof. In
figure 1 we illustrate the errors by component for $N = 31$, $n = 1$, $k = 3$, $A = (-1,4,-1)$, $x_j = 1$. If the linear system arises from discretiz-
ing a continuous problem with the function values specified on the
boundary, e.g., x_0 and x_{N+1}, the choice $N = 2^{m+1} - 1$ is very desirable.
This is because the spikes in the error curve are as far from the
boundary as possible. The choice $N = 2^{m+1}$ is perhaps the least desirable,
as it places a spike next to the boundary. See also remark 8, below.

5. Since $B^{(k)}$ is the matrix associated with the block Jacobi
iteration for $A^{(k)}$, [9], and $||B^{(k)}||_\infty$ will be small, we may use this iter-
ation to improve the estimate $y^{(k)}$. In fact, the choice $y_j^{(k)} = (d_j^{(k)})^{-1}v_j^{(k)}$
is the iterate following the initial choice $y_j^{(k)} = 0$. If $y^{(k,0)}$ is an
approximation to $x^{(k)}$, and $y_j^{(k,i+1)} = (d_j^{(k)})^{-1}(v_j^{(k)} - e_j^{(k)}y_{j-1}^{(k,i)} - f_j^{(k)}y_{j+1}^{(k,i)})$,

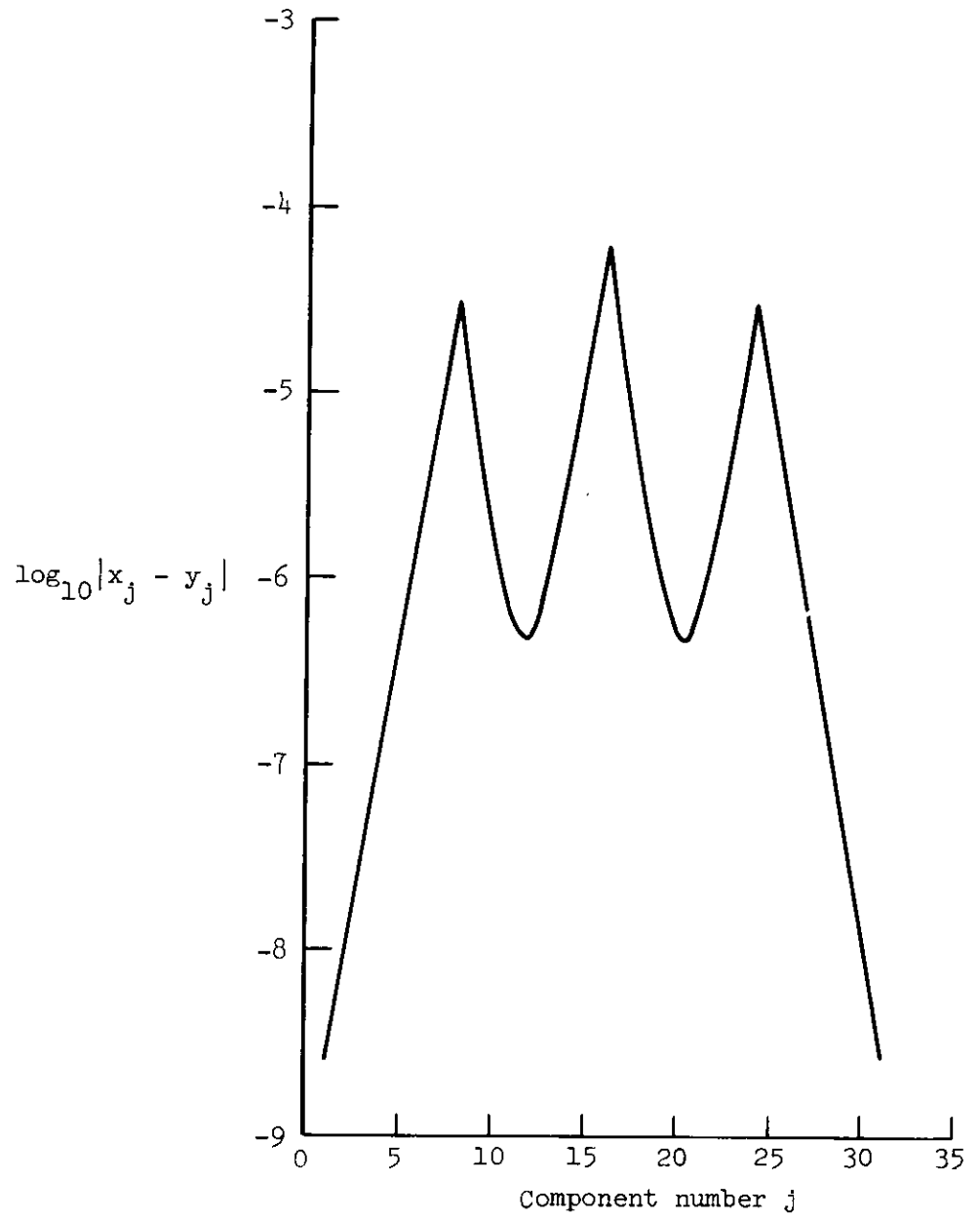Figure 1.- $\log_{10} |x_j - y_j|$ vs. j.

then $||x^{(k)} - y^{(k,\ell)}||_\infty \leq ||B^{(k)\ell}||_\infty \, ||x^{(k)} - y^{(k,0)}||_\infty.$ k and ℓ

should be chosen to minimize computation time subject to the constraint

$||B^{(0)}||_\infty^{2^k\ell} \leq \varepsilon.$ Any of the other standard iterative methods may be

used in place of the Jacobi iteration. For related results using the

theory of non-negative matrices and regular splittings, see [4].

6. Varah [8] uses the condition $||d_j^{-1}||(||e_j|| + ||f_j||) \leq 1,$

$||\cdot||$ an n×n matrix norm, to show that block Gaussian elimination in the

natural ordering is well-defined and stable. Let $\delta_j^{(i)} = ||(d_j^{(i)})^{-1}||$

$(||e_j^{(i)}|| + ||f_j^{(i)}||),$ $\delta^{(i)} = \max_j \delta_j^{(i)}.$ Using a proof similar

to the one given here, it may be shown that, if $\delta^{(0)} < 1,$ then $\delta^{(i+1)}$

$< \delta^{(i)2}.$ Again, an irreducibility condition may replace strict in-

equality, but for this case incomplete cyclic reduction might not be

effective.

7. If A is symmetric, then $C^{(i)} = B^{(i)T},$ so that $||C^{(i)}||_1 =$

$||B^{(i)}||_\infty.$ If A is positive definite, then since $A^{(i)}$ is also positive

definite, the spectral radius of $B^{(i)}$ is less than 1, [9].

8. Consider the matrix

$$A = \begin{pmatrix} d_1 & f_1 & & & & & e_1 \\ e_2 & d_2 & f_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & e_{N-1} & d_{N-1} & f_{N-1} \\ f_N & & & & & e_N & d_N \end{pmatrix}, \quad N = 2^{m+1}$$

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and the linear system $Ax = v$. This matrix form typically arises from discretizing an elliptic equation with periodic boundary conditions. If we modify the row operations of cyclic reduction to include

$$\text{ROW}(N) - e_N d_{N-1}^{-1} \text{ROW}(N - 1) - f_N d_1^{-1} \text{ROW}(1),$$

it is seen that the reduced system has the same form as A, and again only the even-indexed unknowns appear. The reduction may be continued until we have $A^{(m)} x^{(m)} = v^{(m)}$, a block 2×2 system. Moreover, if $B^{(i)}$ and $C^{(i)}$ are appropriately redefined all the conclusions of our main result hold. The proof is nearly identical. The superconvergence effects are only slightly less dramatic.

9. In view of the previous remark, there ought to be a more general underpinning to the quadratic convergence phenomenon.

<u>Theorem.</u> Let $H^{(0)}$ be any matrix with invertible diagonal blocks. Define

$$D^{(i)} = \text{the block diagonal part of } H^{(i)},$$
$$J^{(i)} = I - D^{(i)-1} H^{(i)},$$
$$K^{(i)} = I - H^{(i)} D^{(i)-1},$$
$$H^{(i+1)} = (I + K^{(i)}) H^{(i)} = H^{(i)} (I + J^{(i)}).$$

If $||J^{(0)}||_\infty < 1$ then $||J^{(i+1)}||_\infty \leq ||J^{(i)2}||_\infty$.
If $||K^{(0)}||_1 < 1$ then $||K^{(i+1)}||_1 \leq ||K^{(i)2}||_1$.

<u>Proof.</u> Note that $H^{(i)} = D^{(i)}(I - J^{(i)}) = (I - K^{(i)})D^{(i)}$, so $H^{(i+1)} = D^{(i)}(I - J^{(i)2}) = (I - K^{(i)2})D^{(i)}$. Let $S^{(i)} =$ the block diagonal part of $J^{(i)2}$, $T^{(i)} =$ the block diagonal part of $K^{(i)2}$, so $D^{(i+1)} = D^{(i)}(I - S^{(i)}) = (I - T^{(i)})D^{(i)}$. Now, $||S^{(i)}||_\infty \leq ||J^{(i)2}||_\infty < 1$, $||T^{(i)}||_1 \leq ||K^{(i)2}||_1 < 1$, so $D^{(i+1)}$ is invertible. We have

$$J^{(i)2} = (I - S^{(i)})J^{(i+1)} + S^{(i)}$$

$$K^{(i)2} = K^{(i+1)}(I - T^{(i)}) + T^{(i)}$$

The rest of the proof is as before.

<div align="right">Q.E.D</div>

In figure 2 we show the $H^{(i)}$ matrices for $m = 3$, starting with $H^{(0)} = A$, for both the block tridiagonal and periodic forms. In each case $H^{(m+1)}$ is block diagonal. For the first matrix, the $(8,8)$ block of $H^{(3)}$ has separated from the others in that there are no other blocks in its row and column. This block, in fact, is $A^{(3)}$. For the second matrix, $A^{(3)}$ is composed of the $(8,8)$, $(8,16)$, $(16,8)$ and $(16,16)$ blocks of $H^{(3)}$. Cyclic reduction, it is seen, computes only selected rows and columns of the $H^{(i)}$.
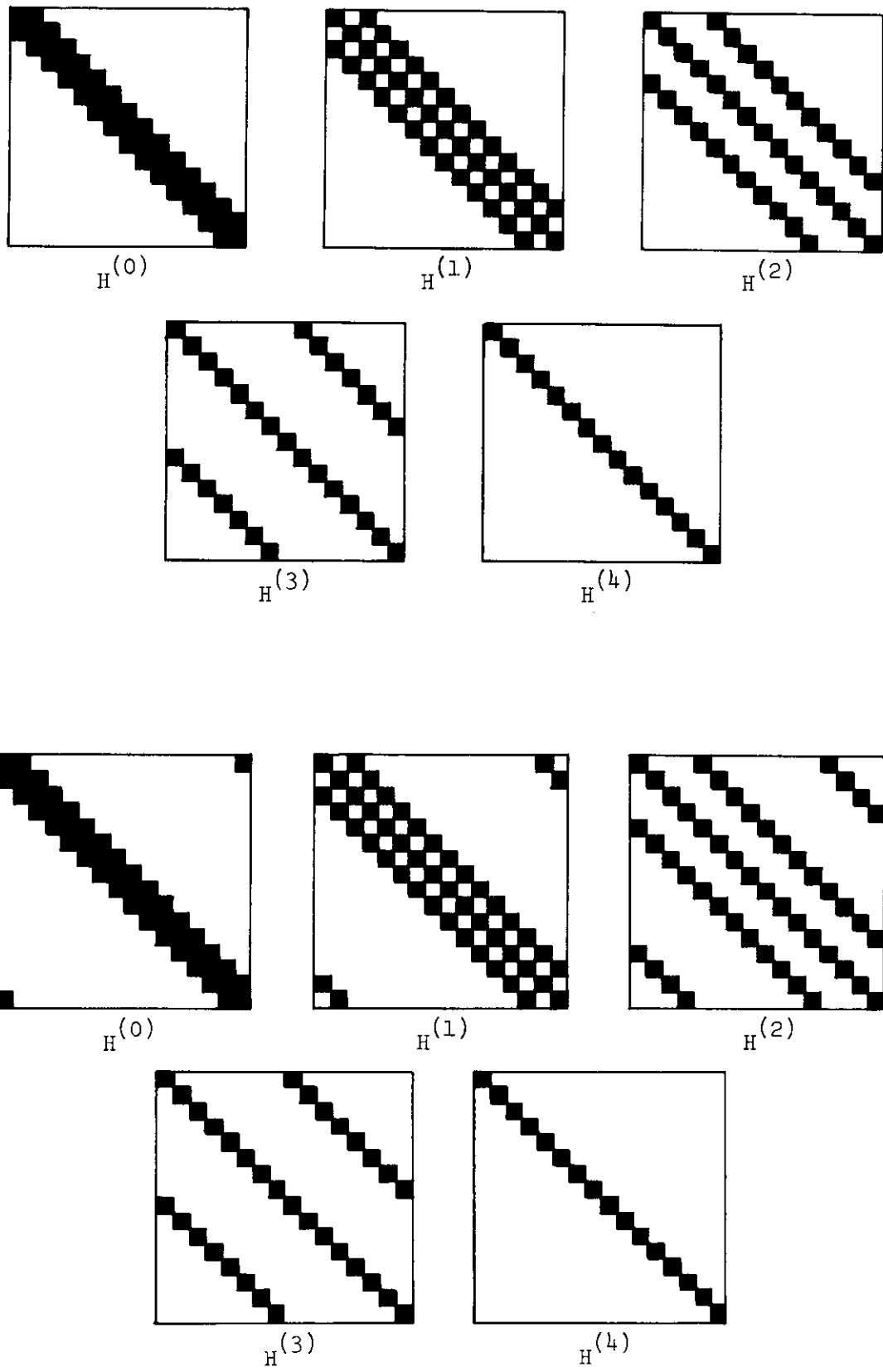
ACKNOWLEDGMENTS

Figure 2.- $H^{(i)}$, i = 0, ..., m+1; m = 3.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*   The Solution of a general block tridiagonal linear system by a cyclic odd-even reduction algorithm is considered. Under conditions of diagonal dominance, norms describing the off-diagonal blocks relative to the diagonal blocks decrease quadratically with each reduction. This allows early termination of the reduction when an approximate solution is desired. The algorithm is well-suited for parallel computation.

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