VARIATIONAL PRINCIPLES

by

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1. INTRODUCTION

The finite element method is based on variational principles developed for the most part in the last century. While a complete exposition of these principles is beyond the scope of this course, an introduction to the essential elements of variational calculus is required as a foundation for the procedures to follow.

Variational principles for electromagnetic field problems can be derived from two alternative points of view, namely by using Galerkin's method or by using the Rayleigh-Ritz method. Since Galerkin's method is considerably easier to understand than the Rayleigh-Ritz method, and is fact turns out to be more general, this lecture will concentrate on the development of Galerkin's method. Application to electromagnetic field problems will be illustrated by simple examples.

2. A SIMPLE POTENTIAL DISTRIBUTION PROBLEM

A typical electromagnetic field problem which we may use for illustrative purposes is the following: Determine the electrostatic potential distribution ^<x.y) and the capacitance of the coaxial conductor shown in Figure 2-1. In this conductor, the inner conductor is raised, to a potential *f* « 100 volts with respect to the outer conductor.

To solve this problem we may solve Laplace's equation for ^(x,y)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (1)$$

subject to the boundary conditions

$$\phi \Big|_{\text{inner conductor}} = 100 \text{ volts} \qquad (2)$$

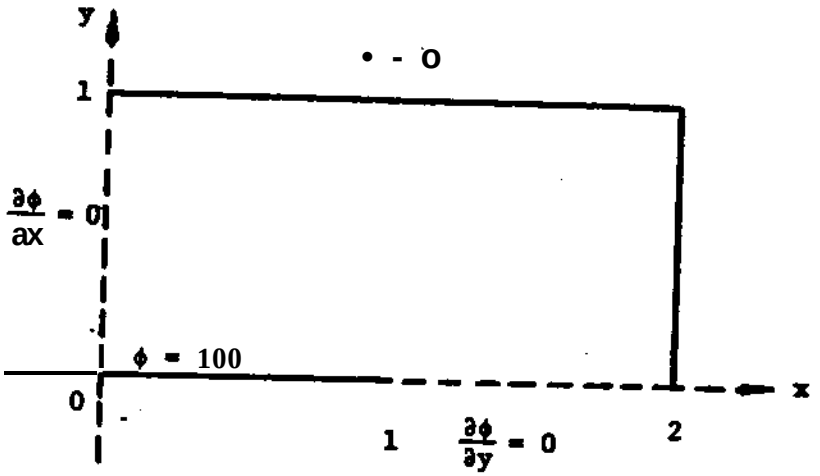

Figure 2-2: One quadrant of the coax of Figure 1 showing the Neuman boundary conditions on the lines of symmetry.

$$3y \Big|_{\substack{y - 0 \\ 1 < x < 2}} - 0 \qquad (5)$$

derived from the fact that the solution must be the same on either side of a symmetry line. In mathematical parlance, boundary conditions specifying the derivative of the unknown, such as is given in equations (4) or (5), are called Neuman boundary conditions, while boundary conditions specifying the potential itself, such as in equation (2) or (3), are called Dirichlet boundary conditions.

Equation (1) is a specific example of the general operator equation

$$A\phi(x,y) - p(x_f y) \tag{6}$$

where A is any linear operator **and** /><x,y) is a given forcing function. The following sections discuss the solution of the general operator form given in equation (6), but the reader may wish for clcanty to keep in mind the more specific problem given in equation (1).

3. GALERKIN'S METHOD

Galerkin's method is simplicity itself. It consists of five easy steps:

1. Write f<x,y) as a linear combination of approximation functions $\{a_j(x,y)\}$

$$\phi^{app}(x,y) = \sum_{j=1}^{n} \phi_j \, a_j(x,y) \tag{7}$$

2. Substitute (7) into (6)

$$\sum_{j=1}^{n} \quad - \quad - \tag{8}$$

3. Multiply both sides of equation* (8) by a_i(x,y> and integrate over the solution region **Q**

$$\sum_{j=1}^{n} \int_{\Omega} a_1(x,y) \, A \, a_j(x,y) d\, 0 \; *j - f \; o(x,y) \; p(x,y) dfl, \; i = 1,\ldots,n \tag{9}$$

4.

Notice that equation (9) is actually a matrix equation

$$S \underset{\sim}{\phi} = \underset{\sim}{\rho}$$

(10)

$$S_{ij} = \int_{\Omega} \alpha_i(x,y) \; A \; \alpha_j(x,y) d\Omega$$

$$\rho_i = \int_{\Omega} \alpha_i(x,y) \quad \rho(x,y) d\Omega$$

(11)

and solve this matrix equation for the unknowns $\underset{\sim}{\phi}$.

5. Substitute the solution $\underset{\sim}{\phi}$ of (10) into equation (7) to give the approximate solution $\phi^{app}(x,y)$.

It is a little harder to see why this procedure works as well as it does. One analysis draws heavily on the concept of *linear vector spaces* and goes as follows: Since $\phi(x,y)$ as determined in Galerkin's method is only approximate, it will leave a residual

$$r(x,y) = A \; \phi^{app}(x,y) - \rho(x,y)$$

(12)

If we multiply both sides of equation (12) by $\alpha_i(x,y)$, integrate over Ω, and use equation (9) we find that

$$\int_{\Omega} \alpha_i(x,y) \; r(x,y) \; d\Omega \quad = \quad 0 \quad , \quad i = 1,\ldots,n$$

(13)

Thus, *the residual function in a Galerkin solution is orthogonal to all of the*

approximation functions used to represent +(x,y). If we choose the approximation functions {tf.(x^y)} wisely* they will contain all of the important variations in r(x,y) and leave only a small residual error in the orthogonal subspace. As we shall see in the next lecture, the choice of approximation functions is a major concern in finite element analysis.

4. THE RAYLEIGH-RITZ METHOD

The Galerkin equations (9) can also be derived from an energy point-of-view using a procedure called the Rayleigh-Ritz method. The energy stored in a scalar field *f*(x,y) governed by equation <6> is

$$W_p(\psi) = \frac{1}{2} \int_a \psi(x,y) \; A \; \psi(x,y) d\Omega \tag{14}$$

while the energy associated with the forcing function ^(x,y) is

$$W_k(\psi) = \int \cdot(x,y) \; P(x,y) dfl \tag{15}$$

A basic variational theorem states that for a positive definite system the energy functional

$$F(\psi) = W_k(\psi) - W_p(\psi) \tag{16}$$

is minimized at the true solution equation (6), i.e. when *f*f.*

It is not hard to prove this result Since we are interested in variations in F(f) about the solution point, we may begin by writing the arbitrary function f<x,y) as a sum of the true solution f<x,y) and a second arbitrary function £<x,y> multiplied by a scalar *

$$\psi(x,y) = \phi(x,y) + e\,C(x,y) \tag{17}$$

The first variation of F(f) about the solution of f « • is then given by the derivative of F(p) with respect to e at e - 0

$$\delta F(\psi)\Big|_{\psi - \bullet} = 3e\Big|_{e - 0} \tag{16}$$

Substituting equations (14) - (17) into (18) gives

$$\delta F(\psi)\Big|_{\bullet - \bullet} = \int_\Omega \xi\rho d\Omega - \frac{1}{2}\int_\Omega \xi\, MdQ \quad -\frac{1}{-}\int_Q \phi A\xi d\Omega - e\int_\Omega \xi A\xi d\Omega\Big|_{e - 0} \tag{19}$$

Provided that the energy operator A is self-adjoint

$$\int_\Omega \xi A\phi d\Omega = \int_\Omega \phi A\xi d\Omega \tag{20}$$

equation (19) becomes

$$6?(\bullet)\,1 \quad " \int \xi(\rho - A\phi)d\Omega \tag{21}$$

Since $A\phi = \rho$ by definition of ϕ, we conclude that the first variation of $F(\psi)$ about the solution point is zero

$$\delta F(\psi) \Big|_{\psi = \phi} = 0 \tag{22}$$

To show that $F(\phi)$ is a true minimum, we may evaluate the second variation of $F(\psi)$

$$\delta^2 F(\psi) = \frac{\partial^2 F}{\partial \epsilon^2} = \int_\Omega \xi A \xi d\Omega \tag{23}$$

Since the integral $\int_\Omega \xi\ A\ \xi d\Omega$ is always positive with a positive definite operator A, the curvature of $F(\psi)$ is always negative, indicating that $F(\phi)$ is indeed the true, unique minimum value of $F(\psi)$.

The Galerkin equations are obtained from $F(\psi)$ by substituting in equation (21) the expansion for ϕ used before

$$\phi^{app}(x,y) = \sum_{j=1}^{n} \phi_j\ \alpha_j(x,y) \tag{7}$$

In view of equation (22), equation (21) yields

$$\sum_{j=1}^{n} \phi_j \int_\Omega \xi(\rho - A\alpha_j) d\Omega = 0 \tag{24}$$

Setting the arbitrary function ξ in the above equal to each of the approximation functions $\{\alpha_i, i$

(«, • * I* l,.~,n) in torn then yields

$$\sum_{j=1}^{n} \int_{\Omega} \ldots \, d\Omega = \int_{\Omega} a_i \rho \, d\Omega \quad , \quad i = 1, \ldots, n \tag{25}$$

Equation (25) is identical to equation (9) derived using the Galerkin procedure.

Although the Rayleigh-Ritz equations (25) are identical to the Galerkin equations (9), it must be emphasized that the Rayleigh-Ritz procedure may be applied only to systems which possess the self-adjoint property in equation (20), while no such restriction exists in the application of Galerkin's method.

5. A SIMPLE EXAMPLE

We will illustrate the ideas presented in the previous sections with a simple example. Consider the infinite parallel plate capacitor illustrated in Figure 5-1 in which a negative linearly varying charge exists $\rho(x,y) = -x$.

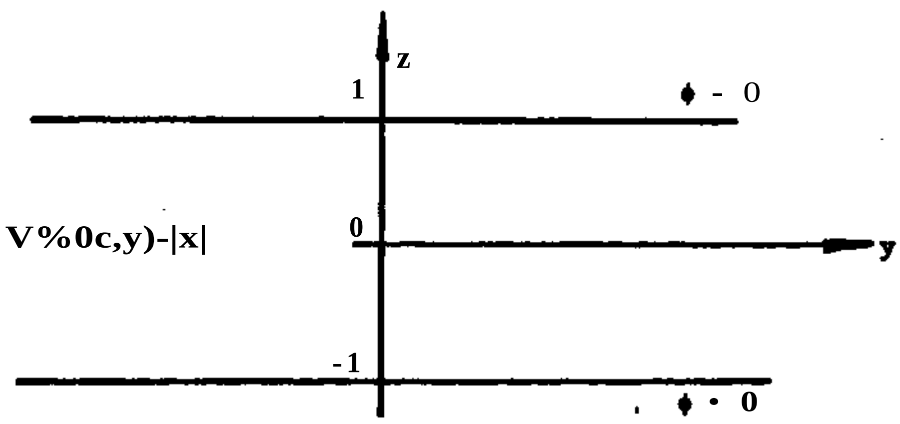

Figure 5-1: An infinite parallel plate capacitor containing a space charge $\rho(x,y) = -|x|$.

By symmetry, this problem reduces to solving the one-dimensional boundary value problem

$$\frac{d^2\phi(x)}{dx^2} = - |x| \tag{26}$$

$$\phi(0) = 0 \tag{27}$$

$$\left.\frac{d\phi}{dx}\right|_{x=1} = 0 \tag{28}$$

We shall solve this problem by approximating $\phi(x)$ using quadratic polynomials

$$\phi(x) = a_0 + a_1 x + a_2 x^2 \tag{29}$$

First a word about boundary conditions. As we shall see in the next section, the approximation functions do not always have to satisfy the boundary conditions. For the moment, however, we shall assume that the approximation functions in the Galerkin or the Rayleigb-Ritz methods satisfy the boundary conditions exactly. Imposing the boundary condition (27) on the approximation (29) gives

$$a_0 = 0 \tag{30}$$

Similarly, imposing (28) on (29) gives

$$a_1 = -2a_2 \tag{31}$$

We will therefore make a one-term approximation of the potential distribution *f(x)*

$$\phi^{app}(x) = \phi_1 a_1(x) \tag{32}$$

where the approximation function $a_1(x)$ is

$$a_1(x) = x^2 - 2x \tag{33}$$

The matrix S of equation (10) is one by one. We evaluate it as follows:

$$s_{11} = \int_0^1 (x^2 - 2x) \frac{d^2}{dx^2} (x^2 - 2x)dx$$

$$= 2 \int_0^1 (x^2 - 2x)dx$$

$$= -\frac{1}{3} \tag{34}$$

On the right-hand-side of equation (10) we obtain

$$P_1 = -\int_0^1 (x^2 - 2x)xdx - J2 \tag{35}$$

Therefore, equation (10) yields

$$-\frac{4}{3} *! * \frac{5}{12} \tag{36}$$

or

$$\phi_1 = -\frac{5}{16} \tag{37}$$

The approximate solution is thus

$$^{app}W = \frac{5}{16} (2x - x^2) \tag{38}$$

This should be compared to the exact solution

$$\phi(x) = \frac{1}{2}(x - \frac{1}{3}x^3)$$

(39)

6. NATURAL BOUNDARY CONDITIONS

Up to this point, we have tried to find approximate solutions of the differential equation (6) but have imposed the boundary conditions on the solution exactly. The boundary conditions may also be written in operator form

$$B\phi(x,y)\Big|_\Gamma = g$$

(40)

where Γ is the boundary of the region Ω. We will now show that imposing boundary conditions only approximately in Galerkin's method has some advantages over the exact approach.

Suppose that we have solved a boundary value problem using Galerkin's method and have required that the approximation functions satisfy the boundary conditions exactly, as in the example in the preceding section. Since the solution is approximate, it must bend to match the true solution as well as it can, but it is constrained rigidly to a prescribed behavior on the boundary. If we loosen the condition boundary a little bit, requiring that the solution satisfy the boundary condition. Closely but not exactly, then the approximate solution will be more free to match the true solution in the problem interior. The highest overall accuracy is obtained not with the approximate solution which satisfies the boundary conditions exactly but with that solution which gives the smallest overall error to both the differential equation and to the boundary conditions.

Since equation (40) is an operator equation, we may use Galerkin's method to approximate it. On doing so, we obtain the matrix equation

$$R\,\underset{\sim}{\phi} = \underset{\sim}{g}$$

(41)

$$R_{\pm j} - \int_{\Gamma} a_{\pm}(x_{>y})\, B\alpha_j(x,y)\, d\Gamma$$

$$^{8}i \quad '' \quad /^{a}i^{(X\,t\,y)}\, 8U^{,y)}\, d\Gamma \qquad (42)$$

Note that equations (41) and (42) differ in form from equations (10) and (11) only in that the integrals are evaluated over the boundary *T* and not over the region O.

Equations (10) and (41) represent additional requirements on the approximate solution $\wedge^{arr}(x,y)$, else the boundary conditions would not contribute to the solution. One method of imposing both requirements is to solve the combined system

$$(S + XR)\underline{\alpha} = \underline{p} + X\,\underline{g} \qquad (43)$$

Depending on the value of the parameter X, the boundary conditions on £ are more or less strongly imposed.

Boundary conditions which are imposed only approximately in Galerkin's method are called *natural boundary conditions* This name is derived from the fact that with Laplace's equation, the approximate Neuman boundary condition is obtained in a very natural way.

7. THE SIMPLE EXAMPLE REVISITED

Let's return now to the problem considered in Section 5 and relax the requirement that the solution satisfy the Neuman boundary condition at x • 1. Since equation (31) no longer holds, the new approximate solution assumes the form

$$\bullet^{app}(x) - \bullet i\, a_1(x) + *_2\, a_2(x) \qquad (44)$$

where

$$a_1(x) = x$$
$$o_2(x) \quad - x^2$$

(45)

The matrix S of equation (11) is

$$S_{ij} = \int_0^1 a_1(x) \frac{d^2 a_1(x)}{dx^2} \, dx \tag{46}$$

We may integrate this by parts, letting

$$S_{ij} = a_i(x) \frac{da_j(x)}{dx} \Bigg|_{x \, - \, 1} - \int_0^1 \frac{d\alpha_{\pm}(x)}{dx} \frac{da^{\wedge}x)}{dx} \, dx$$

(47)

For a Neuman boundary condition, B • $\backslash_x$ and the matrix R of equation (42) becomes

$$R_{ij} = a_i(x) \frac{dft^{\wedge}x)}{dx} \Bigg|_{x \, = \, 1}$$

(46)

(Note that the integral in equation (42) which is a line integral in two-dimensions reduces to evaluating the integrand at the endpoints of the interval in one-dimensional problems). Equation (43) therefore becomes

$$\sum_{j=1}^{2} (1+X)a_j \left.\frac{da_j}{dx}\right|_{x=1} - \int_0^1 \left[\frac{da_1}{dx}\pm\frac{da_2}{dy}\right] dx \int_{-1}^{1} h_i x dx \,, \quad i = 1,2 \tag{49}$$

(49)

The obvious choice for X in equation (49) is X »-1, • choice which eliminates the first term in equation (49) entirely. For the remaining integrals we obtain

$$s_{12} - s_{21} - \int_{-1}^{1} 2x dx = 1$$

$$s_{22} - \int_{-1}^{1} x^3 dx = \frac{4}{3} \tag{50}$$

$$p_1 = -\int_{-1}^{1} x^2 dx = -\frac{1}{3}$$

$$p_2 = -\int_{-1}^{1} x^3 dx = -\frac{1}{4}$$

Thus equation (49) yields

$$\begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/4 \end{bmatrix} \tag{51}$$

Solving this for $\underline{\phi}$ gives $\phi_1 = 7/12$, $\phi_2 = -1/4$. Consequently, the new approximate solution for $\phi(x)$ is

$$\phi(x) = \frac{7}{12} x - \frac{1}{4} x^2 \tag{52}$$

Numerical values for this solution and for the solutions in equations (38) and (39) are given in Table 1 and plotted in Figure 7-1. Note that although both approximate solutions contain the same polynomial terms, the approximate solution in equation (52) which does not satisfy the Neuman boundary condition exactly is considerably more accurate than the approximate solution in equation (38) which does satisfy it.

Table 7-1: Numerical values for the exact and approximate solutions of equation (26).

x	ϕ^{app} from eq.(38)	ϕ^{app} from eq.(52)	Exact
0.0	0.00000	0.00000	0.00000
0.1	0.05937	0.05583	0.04983
0.2	0.11250	0.10667	0.09867
0.3	0.15938	0.15250	0.14550
0.4	0.20000	0.19333	0.18933
0.5	0.23438	0.22917	0.22917
0.6	0.26250	0.26000	0.26400
0.7	0.28438	0.28583	0.29283
0.8	0.30000	0.30667	0.31467
0.9	0.30938	0.32250	0.32850
1.0	0.31250	0.33333	0.33333

8. SOLUTION OF LAPLACE'S EQUATION

We return now to the solution of Laplace's equation which was introduced in equation (1). For this equation, the Galerkin matrix elements S_{ij} are

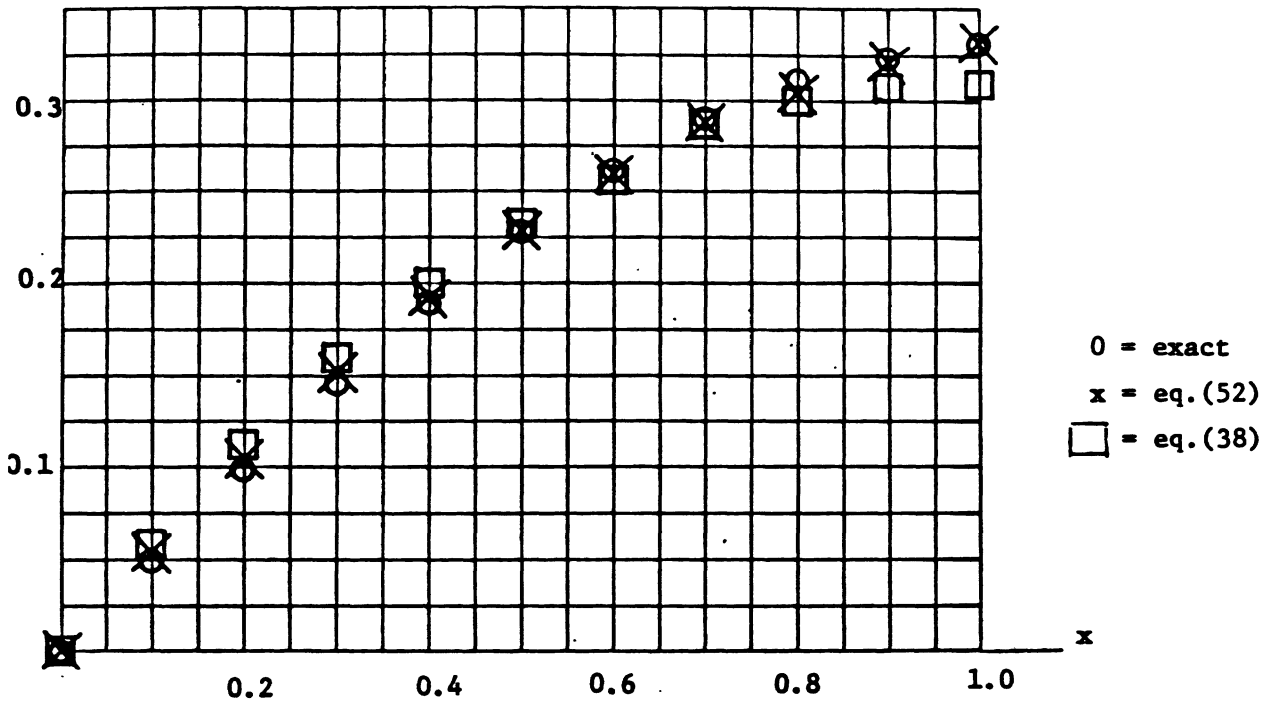

Figure 7-1: Graph of the exact and approximate solutions of equation (26).

$$S_{ij} = \int_{\Omega} \alpha_i \nabla^2 \alpha_j \, d\Omega \tag{53}$$

and the elements of the matrix R corresponding to the Neuman boundary condition are

$$R_{ij} = \int_{\Gamma} \alpha_i \frac{\partial \alpha_j}{\partial n} d\Gamma \tag{54}$$

Green's theorem states that

$$\int_{\Omega} \alpha_i \nabla^2 \alpha_j \, d\Omega = \int_{\Gamma} \alpha_i \frac{\partial \alpha_j}{\partial n} d\Gamma - \int_{\Omega} \nabla \alpha_i \cdot \nabla \alpha_j d\Omega \tag{55}$$

Consequently, the Galerkin equation (43) for Laplace's equation with natural Neuman boundary conditions is

$$\sum_{j=1}^{n} \left[(1+X) \int_{\Omega_r} \nabla a_i \cdot \nabla a_j \, d\Omega - \int_{\Gamma_Q} a_i \, a_j \, d\Gamma \right] \phi_j = 0$$

(56)

Again, the simplest choice is to take X « -1, in which case equation (56) reduces to

$$\sum_{j=1}^{n} \left[\int_{\Omega} \nabla a_i \cdot \nabla a_j \, d\Omega \right] \phi_j = 0$$

(57)

Equation (57) is usually derived in the literature in the following way: The energy functional corresponding to equation (1) is

$$F(\phi) = -\int_{\Omega} f \cdot \nabla^2 \phi \, d\Omega$$

(58)

Using Green's theorem and assuming Neuman boundary conditions, (58) is converted to

$$F(\phi) = \frac{1}{2} \int_{\Omega} \nabla \phi \cdot \nabla \phi \, d\Omega$$

(59)

Substituting for f from equation (7), and minimizing with respect to the coefficients $f_{\%}$ then gives equation (57).

While the tradition derivation of the Calerkin equation *is* admirably short, it confuses the issue of natural boundary conditions. The step from equation (58) to equation (59) makes it appear that the boundary integral is simply "thrown away". In view of the derivation in Section 6, however, we see that this boundary integral is not thrown away but in fact cancels with the addition of the boundary approximation term.

Finally, in conclusion, Figure 8-1 presents the solution of the electrostatics field problem gives in Figure 2-1, using equation (57) and fourth order polynomials for the approximation functions. Details of the specific polynomials used to obtain this solution are left to the next lecture.

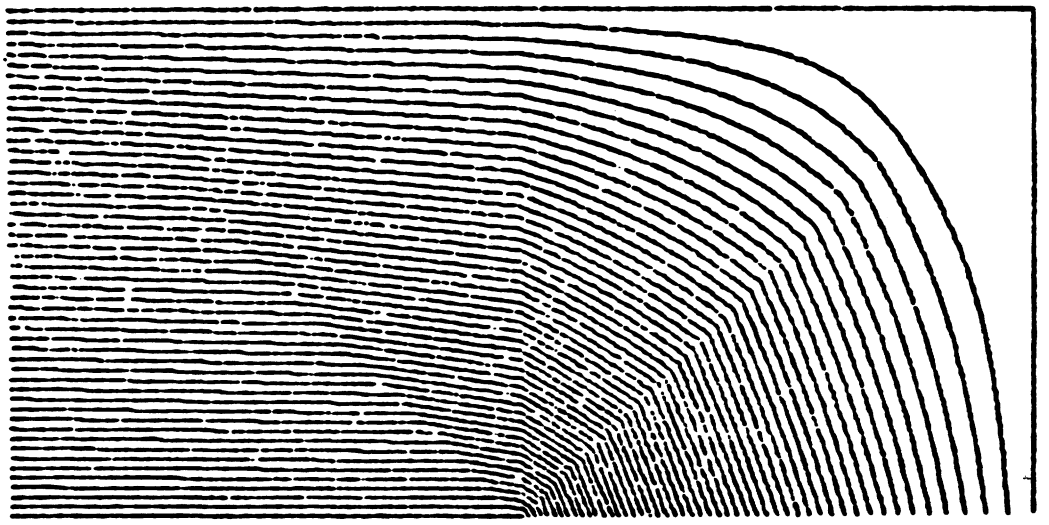

Figure 8-1: Equipotential contours in the two-conductor electrostatics field problem given in Figure 1

REFERENCES

Of the many books describing variational calculus, the best is a text by Ivar Stakgold. This book presents difficult mathematical concepts in a manner which makes them seem elementary and is quite often highly entertaining. Introductory treatments of variational methods may also be found in books by Prenter and by Heubner. Readers wishing to study a more rigorous analysis are recommended the books by Courant and Hilbert and by Mikhlin and Smolitskiy.

1. R. Courant and D. Hilbert, Methods of Mathematical Physics. New York: Interscience, 1953.

2. K. H. Heubner, The Finite Element Method for Engineers. New York: John Wiley, 1975.