A Variational Principle for Quasistatic Mechanics

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CMU-RI-TR-86-16

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December 1986

This work was supported by a grant from Xerox Corporation, and by the Robotics Institute, Carnegie Mellon University.

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Abstract

Quasistatic mechanical systems are those in which mass or acceleration are sufficiently small that the inertial term ma in F = *ma* is negligible compared to dissipative forces. In robotics quasistatic mechanics may be used for systems with friction when motions are sufficiently slow. Here we consider a general quasistatic system with constraints and both dissipative and conservative forces. Under some conditions it is possible to replace Newton's law with the simple and intuitive variational principle that the system moves within the space of unconstrained motions, in such a way as to minimize power. For quasistatic systems we find that this *principle of minimum power* is correct if all the *velocity-dependent* forces are parallel to the velocity and have a magnitude independent of velocity, i.e. are essentially equivalent to Coulomb friction. No restriction need be imposed on velocity-independent forces or forces of constraint

1.2. Quasistatic Systems in Robotics

The principle of minimum power has been used by Peshkin and Sanderson [11] to solve a problem similar to the "credit card" example above. Works of Mason [8], Brost [2], Mani and Wilson [6], and Peshkin and Sanderson [12] analyze similar scenarios in which a part free to slide on a tabletop is manipulated by a robot. In all these works the motion of a sliding object must be determined, which is a problem in quasistatic mechanics.

The dynamics of the robot itself or of its effects on the environment cannot be considered within quasistaic mechanics when kinetic effects are important. However many other problems arise in robotics which can be partially or completely analyzed in the quasistatic approximation:

The strength and mode of failure of a grasp as external forces are applied to the grasped part.

The stability or mode of collapse of a partially assembled structure.

Prediction of backlash in a system of gears or tendons (with friction, but at low speeds).

The effect of terrain on the trajectory of a mobile robot with coupled wheels, when wheel slip is an issue.

Rigidity (and deviation from nominal shape) of a robot under load, including frictional coupling of the links of the robot. Similarly, rigidity of a part as it is machined under numerical control, and the shape actually cut.

1.3. Constraints

Constraints are central to the analysis of the above systems. In the "rope" example above, the rope, which is a continuous object, may be approximated by an arbitrarily dense linear collection of point particles, each constrained to be at a fixed (small) distance from its two adjacent neighbors.

The credit card may be considered to be a network of point particles, each constrained to lie at fixed distances from several nearby particles. With enough such constraints the object is rigid.

The credit card and the rope are also affected by an external constraint that keeps them in the plane of the tabletop or of the ground, respectively. And each system is affected by an external, moving constraint: the robot finger, or the hand pulling the rope.

Of course one would not normally analyze a rigid object as a collection of particles and constraints. Simpler specifications of it are possible, having as few as 6 degrees of freedom and no internal constraints. We will use the "collection of particles" specification in discussing the validity of the principle of minimum power, because that specification is completely general. In actually *using* the principle of minimum power, simpler specifications would be employed. This issue is discussed further in section 5.

1.3.1. What is a Constraint?

Real forces exerted on a particle are always continuous functions of the particle's position. The forces of constraint mentioned above are so abrupt, however, that a useful idealization is to consider them to be due to perfectly rigid links, enforcing fixed distances. This idealization is useful because with sufficient rigidity the detailed nature of the forces is unimportant to the motion. However the idealization brings with it difficulties in calculation due to the singularities which may arise.

We therefore segregate the forces which act in a system into two classes. One class, which we will call f_o consists of forces due to the idealized rigid constraints. The second class contains all remaining forces, and will be denoted f_{xc}. ("XC" stands for "except constraints".) $?_{xc}$ may include external fields (e.g. gravitational, electric, magnetic), dissipative forces (e.g. friction, viscosity), and interparticle forces (e.g. spring forces). We have $?_{total} = F_c + ?_{xc}$. Newton's law is simply $F_{tote/} = 0$ in the quasistatic approximation.

1.4. The Principle of Minimum Power

We define the instantaneous power P_v of a system of particles to be

$$P_v = \sum_i \vec{F}_{XC_i} \cdot \vec{v}_i \tag{1}$$

where i ranges over the particles, $?_{xc_i}$ is all forces acting on particle i except forces of constraint, and $?_i$ is the velocity of particle *L*

Often the principle of minimum power allows a simpler solution for the motion of a system than does Newton's law. Our purpose in this paper is to find out whether the principle of minimum power gives the same result as Newton's law for quasistatic systems. (As P_v is insensitive to mass and acceleration, the principle of minimum power cannot give the correct result for general systems.) We will find that for the principle of minimum power solution to match the Newtonian solution, a restriction must be placed on the form of velocity-dependent forces which act in the system. In particular, we will find that these forces must be parallel to the velocity which gives rise to them and independent of its magnitude. The only such force in common occurrence is Coulomb friction. Therefore the principle of minimum power is only usable for systems with forces of constraint, velocity independent forces, and Coulomb friction.

Because forces of constraint are left out of T_{xc} in equation 1, P_y bears no obvious relation to actual energies of the system. P_v may be more properly called "pseudo-work" in the terminology of [14]. The principle of minimum power is not equivalent to the well-known "principle of virtual work* or to the "principle of least action" of Lagrangian mechanics [5]. These principles apply only when all forces are conservative, i.e. when there are no dissipative forces. (However, see [4] for an extension of Lagrangian mechanics to include radiative and relaxation processes.)

In this paper we will be considering two ways of finding the motion of a system involving constraints without becoming entangled with singularities. The first of these is the familiar method of Lagrange multipliers, using Newton's law. In the second we use the principle of minimum power. By equating the two solutions we determine restrictions on the types of forces for which the principle of minimum power gives the correct (i.e. consistent with **Newton's** law) solution.

We will first consider a single particle system without constraints. A few lines of vector algebra are sufficient to find the restrictions on the types of forces. In section 3 we introduce constraints in terms of "constrained directions" along which the projection of velocity must be zero. In section 4 we generalize the forces from three dimensions to 3n dimensions to represent an n-particle system. The constrained directions generalize easily to 3#i dimensions. The equations derived for the one-particle case retain their form when generalized to n-partfcles. Finally we consider a simple example.

2. One-particle Systems Without Constraints

We will assume that the system has arrived at its present state in accordance with the laws of physics, and ask only what happens in the next moment. The instantaneous velocity alone completely answers that question. We can find the instantaneous velocity by using Newton's law or the principle of minimum power.

2.1. Newton's Law

The Newtonian solution for the instantaneous velocity of a particle in the quasistatic approximation is that velocity which satisfies

$$\vec{F}_{total} = 0 \tag{2}$$

In the absence of constraints, $\vec{F}_{XC} \equiv \vec{F}_{total}$.

2.2. Minimum Power

With P_v as defined in equation 1, and in the absence of constraints, the velocity specified by the principle of minimum power is the one for which

$$\nabla P_v = 0 \tag{3}$$

Note that the gradient is taken with respect to $\vec{v}$, the possible motions.

2.3. Equating the Solutions

We wish to find the conditions under which equations 2 and 3 are satisfied for the same velocity $\vec{v}$, i.e. where the principle of minimum power gives the same solution as Newton's law. Equating equations 2 and 3 and using the definition of P_v from equation 1 we have

$$\vec{F}_{total} = \nabla(\vec{F}_{XC} \cdot \vec{v}) \tag{4}$$

Since $\vec{F}_{XC} \equiv \vec{F}_{total}$ we now drop the subscripts. Equation 4 may be broken into scalar components and transformed:

$$\forall_j \quad F_j = \frac{d}{dv_j}(\vec{F} \cdot \vec{v}) \tag{5}$$

$$\forall_j \quad F_j = \frac{d}{dv_j}\sum_i F_i v_i$$

$$\forall_j \quad F_j = \sum_i v_i \frac{d}{dv_j} F_i + \sum_i F_i \frac{dv_i}{dv_j}$$

$$\forall_j \quad F_j = \sum_i v_i \frac{d}{dv_j} F_i + F_j$$

$$\forall_j \quad 0 = \sum_i v_i \frac{d}{dv_j} F_i \tag{6}$$

The indices i and j run from 1 to 3, as we are dealing with one particle in 3-space. In later sections we will genearlize to n particles in $3n$-space, with i and j running from 1 to $3n$.

Equation 5 (or 6) is the condition, in its most general form, on the types of forces for which the principle of minimum power gives the correct solution.

2.4. Forces for which Minimum Power Is Correct

Note that equation 5 is linear. If two types of forces individually satisfy 5, their sum will also.

If a force is independent of velocity, its derivative with respect to any component of velocity will be zero, so it will satisfy 6. Therefore the principle of minimum power is valid for all velocity-independent forces. Most common external forces (electric fields, springs, gravity) are velocity independent. A magnetic field acting on a moving electric charge exerts a velocity-dependent force.

If a force $\vec{F}$ is perpendicular to $\vec{v}$, $(\vec{F}\cdot\vec{v})$ in equation 5 is zero. Therefore equation 5 cannot be satisfied. The principle of minimum power does not find the correct solution for forces which are perpendicular to the velocity which gives rise to them. A magnetic field acting on a moving electric charge is an example of a perpendicular force. This result is not surprising: a perpendicular force can do no work on a particle, and so is invisible in P_v. Yet it does affect the motion.

Finally, consider forces which are parallel to the velocity which gives rise to them. We may write

$$\vec{F} = F\,\overline{v} \qquad (7)$$

where $\vec{F}$ is a scalar and $\overline{v}$ is a unit vector in the direction of $\vec{v}$. Condition 4 becomes

$$F\,\overline{v} = \nabla\,(F\,|v|) \qquad (8)$$

$$F\,\overline{v} = |v|\,\nabla F + F\,\nabla|v|$$

$$F\,\overline{v} = |v|\,\nabla F + F\,\overline{v}$$

$$0 = |v|\,\nabla F \qquad (9)$$

To satisfy equation 9, the gradient of $\vec{F}$ must be zero except at $\vec{v} = 0$. Therefore $\vec{F}$ must be a constant. Such forces are generalized versions of Coulomb friction, where the frictional force is directed opposite to the velocity, but the magnitude of that force is independent of velocity and direction.

We can conclude that the principle of minimum power does not in general give the correct solution for the motion of a one-particle quasistatic system. However the solution is correct if the forces acting can be composed of:

- Forces independent of the velocity of the particles.
- Forces parallel to the velocity of the particles, but whose magnitude is independent of the velocity of the particles.

3. One Particle Systems with Constraints

In this section we include constraints in the Newtonian and minimum power solutions for the motion of a system. By formulating both solutions in terms of the same "constrained directions" along which the projection of the particle's velocity must be zero, the constraint forces in the two solutions are shown to cancel exactly. The question of the equivalence of the Newtonian and minimum Power solutions is thus reduced to the previous case in which no constraints were involved. The constrained directions will be generalized in section 4 to *3n* dimensions.

3.1 * Newtonian Solution by Lagrange Multipliers

When there is a constraint there is a force to maintain the constraint. These "forces of constraint" must be included in f_{total}» 0. Generally the forces of constraint are unknown and cannot be solved directly. The method of *Lagrange multipliers* [3] has been developed to deal with constraints.

In a formulation of the method of Lagrange multipliers well suited to our purposes, each constraint is replaced by a spring which exerts a force proportional to the difference between its length and its "relaxed[1]* length *d.* We denote the proportionality constant *X.* As *X* -»«>, the spring becomes rigid, and therefore acts as a constraint. Recall that rigid constraints were themselves only idealizations of real forces so sharp that their details ceased to be relevant to the motion of a system. Therefore the choice of a very stiff spring to replace the constraint does not reduce the generality of the constraints.

The force exerted by a spring with spring constant *X* constraining a particle to be a distance *d* from the origin, is

$$\vec{f}_s = \lambda\,(d - |\vec{r}|)\,\hat{r} \qquad (10)$$

where *f* is the position of the particle, and *7* indicates a unit vector in the direction of *f.*

We have initially a state of the system (described by the vector *t)* which satisfies the constraints, and ask what happens in the next instant *dt.* We wish to find ?, the vector specifying the instantaneous velocity of the particle. If a particle is constrained to be a distance dfrom the origin, and is presently at that distance, then the constraint may be stated as a restrictton on the instantaneous velocity of the particle: ? must be perpendicular to *t.* The force arising from a violation of this constraint is

$$f_s = -X\$-7)dt7 \qquad (11)$$

t here is a *constrained direction,* the velocity must be perpendicular to this direction. Figure 3-1 illustrates the constrained direction *f.* The velocity of the particle 7, if it is not to violate the constraint, must be perpendicular to the constrained direction. Should it not be perpendicular, the distance from the origin to the particle would increase by *V>7 dt*, and a force of constraint/*, would develop as given by equation 11.

We will require two general forms for forces of constraint The first,

$$/_s = -\lambda \quad \bar{F} \;\; dt\mathbf{F} \qquad \mathbf{(12)}$$

Is used to enforce a fixed distance from a particle to a point in space. (It can be used for fixed Inter-particle distances, too, as we will see in section 4.) By properly selecting constrained directions ? in velocity space, equation 12 is sufficient to represent general distance constraints. Suppose a particle at *t* is constrained to lie a distance *d* from a point *p* fixed in space. Its velocity *S?* must be perpendicular to *(?~P).* The vector ? which represents this constraint is

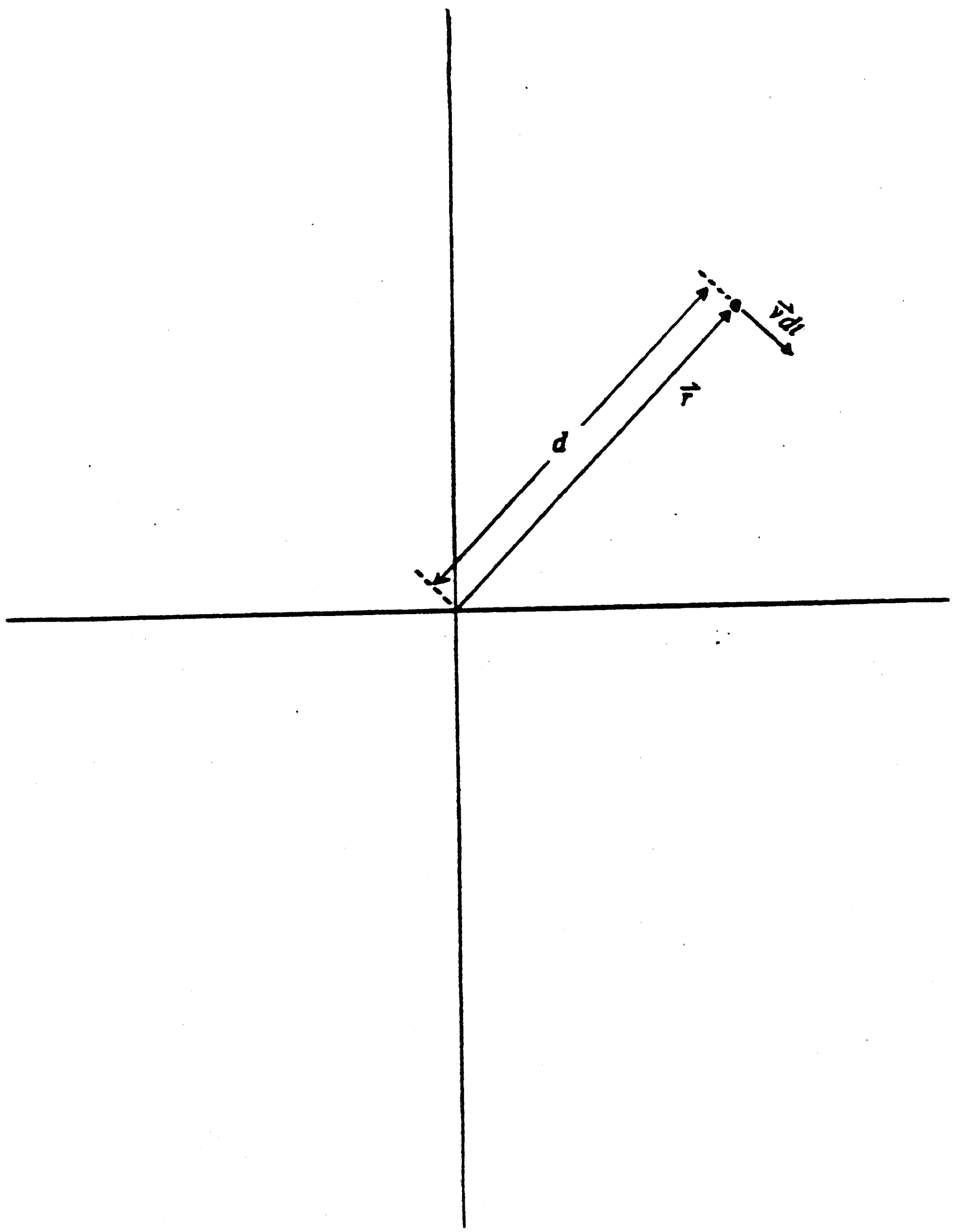

Figure 3-1: $\vec{r}$ is a constrained direction

$$\vec{c}_x = \vec{r}_x - \vec{p}_x$$
$$\vec{c}_y = \vec{r}_y - \vec{p}_y \qquad (13)$$
$$\vec{c}_z = \vec{r}_z - \vec{p}_z$$

The second form of constraint we shall need imposes a velocity $\vec{e}$ on the motion of a particle. We can write a force term to maintain this constraint

$$\vec{f}_s = -\gamma\,((\vec{v} - \vec{e}) \cdot \bar{e})\,\bar{e}\;dt \qquad (14)$$

where γ is another spring constant like χ. This equation may be interpreted to say that if the component of the particle's velocity $\vec{v}$ in the $\bar{e}$ direction differs from $\vec{e}$, we will impose a force in the $\bar{e}$ direction.

Newton's law may now be written as

$$\vec{F}_{XC} - \sum_j \lambda_j\,(\vec{v} \cdot \bar{c}_j)\,\bar{c}_j\,dt - \sum_k \gamma_k((\vec{v} - \vec{e}_k) \cdot \bar{e}_k)\,\bar{e}_k\,dt = 0 \qquad (15)$$

where the second and third terms are the forces of constraint from equations 11 and 13. $\vec{F}_{XC}$ represents all forces other than the constraints.

To solve the system, one must solve for the components of $\vec{v}$ in terms of the multipliers λ_j and γ_k, and then take the limit as all the multipliers go to infinity.

3.2. Minimum Power Solution with Constrained Directions

A quasistatic system chooses that motion, from among all motions satisfying the constraints, which minimizes the P_v.

In the notation developed above, total instantaneous power may be written

$$P_v = \vec{F}_{XC} \cdot \vec{v} \qquad (16)$$

$\vec{F}_{XC}$ represents all forces other than the constraints. P_v is a scalar quantity, while $\vec{F}_{XC}$ and $\vec{v}$ are vectors. Were it not for the restriction "*among all motions satisfying the constraints*", the motion minimizing total power would satisfy

$$\nabla P_v = 0 \qquad (17)$$

If certain directions of motion $\vec{s}_l$ violate the constraints, we do not care if P_v could be further lowered by moving in those directions. So we only require that P_v is at a minimum when we change $\vec{v}$ in unconstrained directions. In terms of the gradient of P_v, we do not insist that it be zero in all directions, but only in the unconstrained directions. In the constrained directions the gradient of P_v may be non-zero. This requirement may be written

$$\nabla P_v = \sum_l \alpha_l \vec{s}_l \qquad (18)$$

Note that the principle of minimum power is satisfied if equation 18 is true for any set of values of the parameters α_k. Another way of understanding this is that we require P_v to be minimized not over the entire velocity space (of dimension 3 now, but which will be generalized to $3n$), but only on a subspace reduced in dimensionality by the number of constraints. The basis vectors of this subspace are perpendicular to all the constrained directions ?$_r$ P_v is also defined on the complementary subspace whose basis vectors are the constrained directions f_{lt} but it of no interest what the projection of VW onto this space is, because the system is constrained to have zero velocity in this subspace. The principle of minimum power therefore allows VW to be composed of an arbitrary linear combination of the constrained directions.

3.3. Forces for which Minimum Power Is Correct

We now wish to find the conditions under which equations 15 and 18 are satisfied for the same velocity ? , i.e. where the principle of minimum power gives the same solution as Newton's law. When that occurs we have

$$\vec{F}_{XC} - \sum_j \lambda_j (\vec{v} \cdot \vec{c}_j)\, \overline{c}_j \, dt - \sum_k \gamma_k ((\vec{v} - \vec{e}_k) \cdot \overline{e}_k)\, \overline{e}_k \, dt \\ = \nabla(\vec{F}_{XC} \cdot \vec{v}) - \sum_l \alpha_l \vec{s}_l \tag{19}$$

The constrained directions ^ in the minimum power solution are the directions along which the projection of velocity must be zero to satify the constraints. That is also what the vectors ?$_j$ and t_k are, in the Newtonian solution. The f_t are simply a relabeling of the ?$_j$ and the ?$_t$ The values a, may be chosen arbitrarily, so we choose a_t to be of the form

$$\alpha = \lambda\,(\vec{v} \cdot \vec{c}) < \#, \text{or} \\ \alpha = \gamma\,((\vec{v} - \vec{e}) \cdot \overline{e})\, dt \tag{20}$$

depending on whether ?$_7$ corresponds to a ?$_y$. or a t^ Then the summations in equation 19 cancel leaving only

$$\vec{F}_{XC} = \nabla(\vec{F}_{XC} \cdot \vec{v}) \tag{21}$$

The algebra of equations 5 to 6 applies directly to this equation. The conclusions of section 2.4 therefore apply to one particle systems with constraints, as well.

4. rt-particle Systems with Constraints

We now generalize the above results to n-particle systems. Both the algebra in section 2 and the constrained directions in section 3 generalize to the 3n dimensional velocity space needed for n particles. Henceforth, all vectors will be assumed to be *3n* dimensional, c^ will denote the *x* component of ? for particle *L* If a vector is only three dimensional, it will be indicated as, e.g. ^{3}?.

4.1. Newtonian Solution by Lagrange Multipliers

If a system consists of *n* particles, we can consider a force *T* to be a vector of *3n* components. $t_{mtd} = 0$ then describes the Newtonian solution for the whole system at once.

Equation 12 gave the force required to maintain a constrained direction ?. It is merely a formality to translate equation 12 to a general constrained direction in 3^-space:

$$\vec{f}_s = -\lambda\,(\vec{v} \cdot \overline{c})\, dt \;\; 7 \tag{22}$$

where $\vec{c}$ is a $3n$-vector with components

$$c_{ix} = {}^3c_x \quad c_{iy} = {}^3c_y \quad c_{iz} = {}^3c_z \tag{23}$$

and i is the particle number of the constrained particle. The other $3n$–3 components of $\vec{c}$ are zero.

The force required to impose a velocity ${}^3\vec{e}$ on the motion of a particle (equation 14) also generalizes trivially. We can write a force term to maintain this constraint

$$\vec{f}_s = -\gamma((\vec{v} - \vec{e}) \cdot \bar{e})\,\bar{e}\; dt \text{ where} \tag{24}$$

$$e_{ix} = {}^3e_x \quad e_{iy} = {}^3e_y \quad e_{iz} = {}^3e_z \tag{25}$$

The other $3n$–3 components of $\vec{e}$ are zero.

Many other constraints relating movement of the particles, most of them unphysical, can be expressed in terms of $3n$-vector "constrained directions" $\vec{c}$ or $\vec{e}$. In a n-particle system we need a constraint maintaining inter-particle distances. The $3n$-vector $\vec{c}$ which represents a constrained direction for an inter-particle constraint has six non-zero components. If particle p has position ${}^3\vec{p}$, and particle q has position ${}^3\vec{q}$, then the $3n$-vector $\vec{c}$ which constrains them to maintain their current distance is

$$\begin{aligned} \vec{c}_{px} &= {}^3\vec{p}_x - {}^3\vec{q}_x \\ \vec{c}_{py} &= {}^3\vec{p}_y - {}^3\vec{q}_y \\ \vec{c}_{pz} &= {}^3\vec{p}_z - {}^3\vec{q}_z \\ \vec{c}_{qx} &= {}^3\vec{q}_x - {}^3\vec{p}_x \\ \vec{c}_{qy} &= {}^3\vec{q}_y - {}^3\vec{p}_y \\ \vec{c}_{qz} &= {}^3\vec{q}_z - {}^3\vec{p}_z \end{aligned} \tag{26}$$

The other $3n$–6 components of $\vec{c}$ are zero. The above three types of constraints allow us to tie a particle to a given point in space by a fixed-length link, to impose a velocity on a particle, and to tie two particles of a system to each other by a fixed-length link. Thus rigid bodies may be modeled by specifying three non-coplanar interparticle constraints for each particle. Non-rigid bodies (e.g. a rope) may be modeled by specifying fewer constraints (two per particle in the case of a rope, as discussed in section 1.3).

All of the equations in preceding sections apply to $3n$ dimensional velocities as well as to the 3 dimensional velocities for which they were explained. Therefore we can generalize the conclusion of section 2 to apply to n-particle systems as well: The principle of minimum power does not in general give the correct solution for the motion of a quasistatic system. However the solution is correct if the forces acting can be composed of:

- Forces of constraint.
- Forces independent of the velocity of the particles.
- Forces parallel to the velocity of the particles, but whose magnitude is independent of the velocity of the particles. Such forces are essentially Coulomb friction.

5. Examples

Note that in using the principle of minimum power, it is not necessary to model the problem as a collection of particles and constraints. That was done only for purposes of generality in the sections above. Any set of parameters which includes all the degrees of freedom of the system may be used. The required constraints are only those which impose restrictions on the parameters chosen.

For instance, in section 1.3, we mentioned a system in which a credit card slides on a tabletop. The card can be considered to be a network of point particles connected by so many constraints that the network becomes rigid. But the principle of minimum power can also be applied to a much simpler specification of the card: we may consider only the 3-space coordinates of 3 non-colinear points of the card. In that case the only constraints which are needed are those which constrain the three points to lie in the plane of the tabletop. A still simpler specification of the card is one in which only the x and y coordinates of one point of the card are used, with the z coordinate understood to be that of the tabletop. One angle describing the orientation of the card must also be given. In this specification no constraints are needed. The motion of the card is solved in [9] and [10].

As an simple example system, consider the two-dimensional one-particle system shown in figure 5-1. A moving constraint imposes a velocity v_x on the particle, in the $+x$ direction. (The constraint could be a frictionless vertical fence.) The constraint applies a force only in the $+x$ direction. An external constant force (e.g. gravity) acts in the $-y$ direction with magnitude mg. A dissipative force ηv^n opposes the velocity v of the particle. (η should not be interpreted as a coefficient of friction, as we have not defined any normal force which gives rise to it. In particular, note that "gravity" acts in the y direction, rather than perpendicular to the plane of motion.)

Coulomb friction corresponds to $n=0$, viscous friction to $n=1$. If $n=0$ and $\eta<mg$, the particle will accelerate in the $-y$ direction violating the quasistatic approximation, so we will assume $\eta>mg$. After motion begins, the particle will approach a terminal velocity. Until the terminal velocity is achieved, the motion of the particle is sensitive to its mass, so the quasistatic approximation is not appropriate. We will consider only the time period after inertial effects have been damped out. Motion will then be uniform with time. We wish to find the velocity v_y of the particle as a function of v_x and the dissipative parameters η and n.

5.1. Newtonian Solution

The external force mg must be equal to the y component of the dissipative force:

$$mg = f_y = \eta \, v^n \frac{v_y}{v} \tag{27}$$

The constraint moving at velocity v_x determines the x component of the particle's velocity. Using

$$v^2 = (v_x^2 + v_y^2) \tag{28}$$

we obtain an implicit solution for v_y:

$$\frac{mg}{\eta} = v_y \, (v_x^2 + v_y^2)^{(n-1)/2} \tag{29}$$

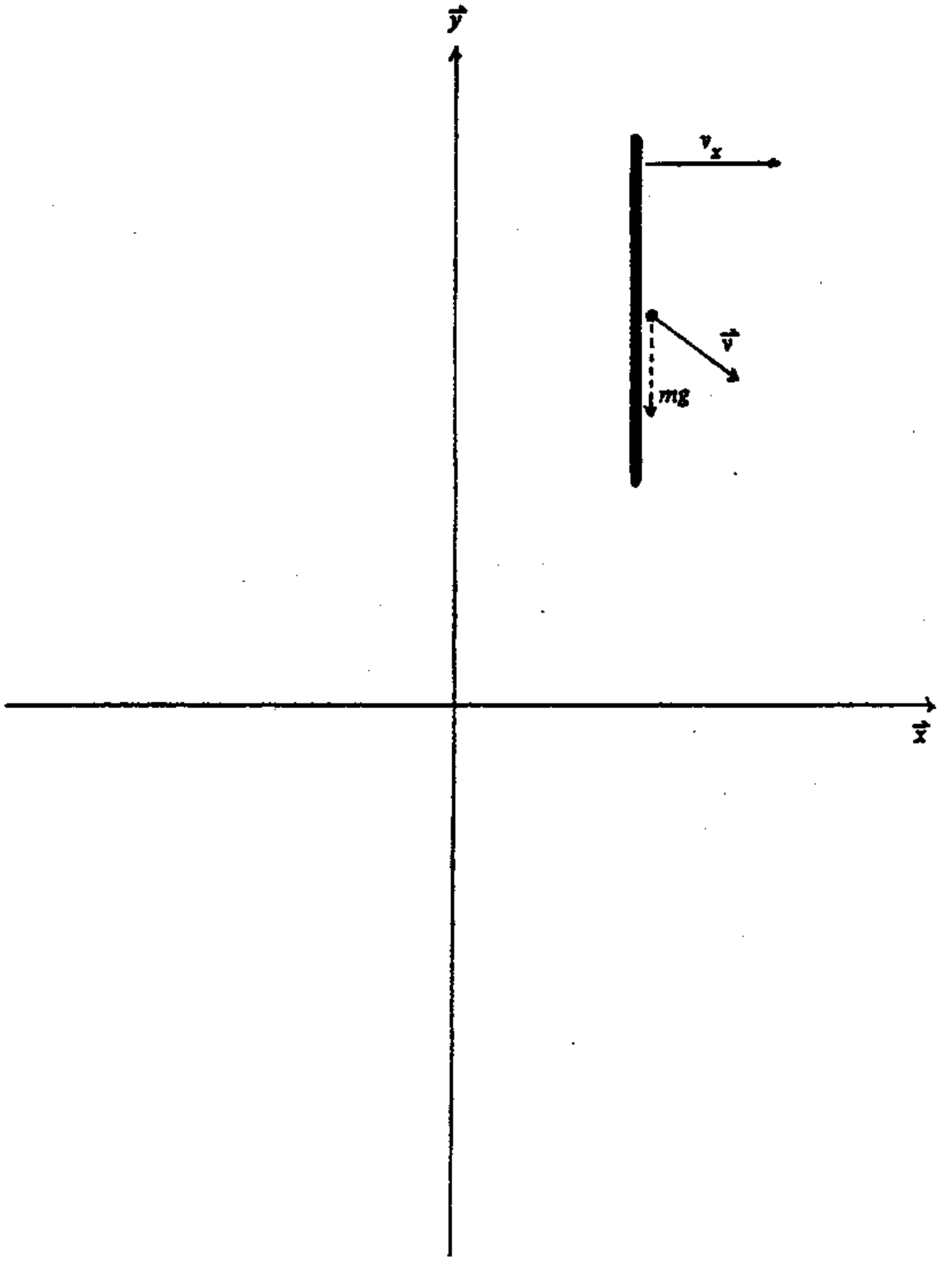

Figure 5-1:. Example erf a quasistatic system

5.2. Minimum Power Solution

Instantaneous power due to the external force is $-mgv_y$. The dissipative force is *riv",* so power is ηv^{n+1}. Total power is then

$$P_v = -mg\,v_y + \eta\,(v_y^2 + v_x^2)^{(n+1)/2} \tag{30}$$

v^ is constrained; v_y unconstrained. We minimize P_v with respect to v^:

$$0 = \frac{dP}{dv_y} *= -mg + {}^{\wedge}\,iti(v/+vflp-W^{1}2v_y \quad {}_{w}{+}i \tag{31}$$

Solving we find

$$\frac{\wedge}{\eta}S\text{-}VyCvZ + v/{}^{\wedge}Wdi+l) \tag{32}$$

which is equivalent to the correct answer (equation 29) only when $n = 0$. As concluded in section 2.4, (p.5), for the principle of minimum power to be correct dissipative forces must be velocity independent, i.e. $n = 0$.

6. Acknowledgements

We wish to acknowledge the careful readings and useful suggestions of Matt Mason and Robert Schumacher. This work was supported by a grant from Xerox Corporation, and by the Robotics Institute, Carnegie-Mellon University.