A NuTE ON PLOW NETWORKS AND STRUCTURES

by

Steven J. Penves

DRC-12-06-82

April, 1982

A NOTE ON FLOW NETWORKS AND STRUCTURES

Steven J. Fenves

1. Introduction

The purpose of this note is to explore certain common features of flow and structural networks, in terms of the effect of the (usually implicit) global equilibrium equations on the range of problems amenable to analysis.

The objective is to extend the standard analysis problem and explore the possibility of "tuning" flow and structural networks by preassigning values of certain inflow and reaction components.

2. Flow Problem

2.1. Standard case

Consider a flow network consisting on n nodes and b branches, governed by:

- **continuity (compatibility)**

$$d = \bar{A}\bar{D} \qquad (1)$$

where d = b-vector of branch head losses (potential differences)
$\bar{D}$ = n-vector of node heads (potentials)
$\bar{A}$ = b x n branch-node augmented incidence matrix

- flow conservation (equilibrium)

$$\bar{Q} = \bar{A}^{*}q \qquad (2)$$

where $\bar{Q}$ = n-vector of nodal flows
q = b-vector of branch flows

- branch constitutive equations

$$q = kd \qquad (3)$$

where k= bxb diagonal matrix of linear[1] branch admittances (flow-headloss relation).

Since the columns of $\bar{A}$ are linearly dependent

[1] **A linear constitutive relation is used for simplicity, so that the symbolic solution can be conveniently** represented.

$$\sum_j a_{ij} = 0 \quad \text{for all } i \tag{4}$$

the n $\bar{D}_j$ in (1) cannot be uniquely determined Conversely, the n conservation equations in (2) are not independent and the $\bar{Q}_j$ cannot be uniquely assigned.

The standard procedure is to designate one node, say, node 0, as the <u>datum</u>, drop the corresponding column of $\bar{A}$ yielding the incidence matrix A, and redefine D as an (n-1) vector of heads (potentials) relative to the head D_0. Then the (n-1) vector Q can be uniquely assigned and

$$Q_0 = -\sum_{j=1}^{n-1} Q_j \tag{5a}$$

is determined from the overall global conservation relation

$$\sum_0^{n-1} Q_j = 0. \tag{5b}$$

With this change, the network can be readily solved Substituting (1) into (3) and the result into (2) yields:

$$\begin{aligned} Q &= A^t k A D \\ &= K D \end{aligned} \tag{6}$$

where K = (n-1)x(n-1) global admittance matrix

Since linear dependence has been removed, K is non-singular and (6) can be solved for the unknowns D:

$$D = K^{-1} Q \tag{7}$$

The remaining unknowns can be obtained by back-substitution into (1) and (3):

$$\begin{aligned} d &= AD = AK^{-1}Q \\ q &= kd = kAK^{-1}Q \end{aligned} \tag{8}$$

2.2. Condensation

It is instructive to recast the above in the notation used for structural condensation. We define

$$\bar{D} = \begin{Bmatrix} D \\ \hline D_0 \end{Bmatrix} \qquad \bar{Q} = \begin{Bmatrix} Q \\ \hline Q_0 \end{Bmatrix} \tag{9}$$

$$\bar{A} = \left[A \;\vdots\; A_0 \right]$$

where A_0 = column of $\bar{A}$ corresponding to the datum node. Substituting as in (6) yields

$$\begin{bmatrix} A^t \\ --- \\ A_0^t \end{bmatrix} [k] \quad [A^t ; A_0] \begin{Bmatrix} D \\ --- \\ D_0 \end{Bmatrix} \tag{10}$$

&

$$= \begin{bmatrix} K & K_{10} \\ K_{01} & K_{00} \end{bmatrix} \begin{Bmatrix} D \\ --- \\ D_0 \end{Bmatrix}$$

The first equation in (10) yields

$$Q = K\,D + K_{10}\,D_0 \tag{11}$$

$$D = K^{-1}\,[Q - K_{10}\,D_0]$$

and the second

$$Q_0 = [K_{01}\,K^{-1}]\,Q + [K_{00} - K_{01}K^{-1}K_{10}]D_0 \tag{12}$$

To establish equivalence with the previous solution, we have to prove that

a) $-K_{01}K^{-1}$ = a row of all 1's; to reproduce (5)

b) $-K^{-1}K_{10}$ = a column of all 1's; physically, the potential at D_0 is additive to all potentials D_j

c) $[K_{00} - K_{01}K^{-1}K_{10}] = 0$; Physically, the flow Q_0 does not depend on the (arbitrary) potential D_0.

The proof of a) is as follows

Since the columns of $K = A^tA$ are linearly dependent.

$$K_{0j} = -\sum_{i=1}^{n-1} K_{ij} \text{ for all } j \tag{12a}$$

or

$$K_{01} = -[e]K$$

where [e] = row vector of all ones.

Therefore,

$$-K_{01}K^{-1} = [e]\,K\,K^{-1}$$

$$= [e]\,I \tag{12b}$$

$$= [e] \qquad \text{Q.E.D.}$$

The proof of b) follows immediately from symmetry.

The proof of c) follows directly, since

$$K_{00} = -\sum_{i=1}^{n-1} K_{10,i} = -[e]\ K_{10}$$

$$[K_{00} - K_{01}\ K_{11}^{-1}\ K_{10}] = -[e]K_{10} - (-[e]K_{10})$$

$$= 0.$$

Q.E.D.

A sample network and its matrices $\bar{K}$, K, K^{-1}, K_{10} K_{00}, $-K_{01}$ K^{-1} and $[K_{00} - K_{01}\ K^{-1}\ K_{10}]$ are shown in Figure 1.

2.3. Additional Condensation

The analogy with structural condensation can be carried a step further. We partition the nodes into two sets:

a) a set of nodes f (for free) with known applied flows Q_f and unknown potentials D_f; and

b) a set of nodes s (for support) with known D_s and unknown Q_s.[2]

Partitioning (6) accordingly:

$$\begin{Bmatrix} Q_f \\ \hline Q_s \end{Bmatrix} = \begin{bmatrix} A_f^t \\ \hline A_s^t \end{bmatrix} [k] \quad [A_f \mid A_s] \begin{Bmatrix} D_f \\ \hline D_s \end{Bmatrix} \tag{13}$$

$$= \begin{bmatrix} K_{ff} & K_{fs} \\ K_{sf} & K_{ss} \end{bmatrix} \begin{Bmatrix} D_f \\ \hline D_s \end{Bmatrix}.$$

The solution for the unknowns follows:

$$\{D_f\} = K_{ff}^{-1}\ [Q_f - K_{fs}\ D_s] \tag{14a}$$

$$\{Q_s\} = K_{sf}K_{ff}^{-1}Q_f + [K_{ss} - K_{ff}^{-1}K_{fs}]D_s \tag{14b}$$

An interesting degenerate case is when all the nodes are in s. i.e., when we have to

[2] The datum node can be included in set s and we can go back to the augmented form of (1) and (2).

calculate the flows when all the nodal heads (relative to the head at the datum) are specified From the second equation in (13) we get directly

$$Q_s = K_{ss} D_s , \qquad (15)$$

without any simultaneous equations.[3]

3. Structural Problem

3.1. Standard case

With the appropriate redefinition of Q_j, D_j, q_i and d_i as subvectors and of k_{ii} and a_{ij} as submatrices, the linear structural analysis problem is identical to the flow problem.

The conventional approach in structures is that given in Section 2.3: ail reaction (support) nodes *are* included in set s; the displacements of the free nodes are given by (14a) and the reactions at the supports by (14b).

3.2. Dependent and Independent Reactions

It is instructive to recast the structural problem in the notation of Section 2.2. As pointed out above, conventionally the set s includes all reactions, including the "hyperstatic" ones, i.a, reactions beyond those required for rigid-body equilibrium. Unlike the flow network, in which there is one global equilibrium constraint (5b), the structural network is constrained by more than one rigid body equilibrium requirement

The overall rigid-body equilibrium equation for the structure is:

$$0 = [H] \{Q\}$$

$$= [H_0 \; H_1 \; \cdots \; H_j \; \cdots \; H_{n-1}] \begin{Bmatrix} Q_0 \\ \cdots \\ Q_j \\ \cdots \\ Q_{n-1} \end{Bmatrix} \qquad (16)$$

where H_j = nf x nf force translation submatrix from node j to the point where the global equilibrium equation is evaluated
nf = number of rigid-body degrees of freedom (=3 for plane structures; =6 for space structures).

The columns of [H] can always be permuted so as to partition (16) into

[3] For n=2 and actual non-linear headloss-flow relations, this is the two-reservoir problem.

$$0 = [H_0' \,|\, H_1'] \begin{Bmatrix} Q_0' \\ \hline Q_1' \end{Bmatrix} \tag{17}$$

where H_Q' is square and non-singular. This permits us to solve for the dependent reactions Q_Q' as a function of the independent loads (which may include reaction components):

$$Q_0' = -[H_0']^{-1} H_1' \, Q_1' \tag{18}$$

An example is worked out in Figure 2.[4]

Restrictinig set s to contain only the support nodes corresponding to the dependent reactions Q_Q', and comparing (18) to (14), we get the interesting results:

a) $K_{sf}K_{ff}^{-1} = -[H_0']^{-1} H_1'$ to reproduce (18);

(19a)

b) $K_{ff}^{-1}K_{fs} = -H_1'^{t}[H_0'^{t}]^{-1}$;

(19b)

physically, the free displacements D_f depend kinematically on the rigid-body support displacements D_s.[5]

c) $[K_{ss} - K_{sf} K_{ff}^{-1} K_{fs}] = 0$;

(19c)

physically, the statically dependent reactions Q_s are independent of the rigid-body support displacements D_s.

These three relations correspond exactly to scalar relations in Section 2.2.

The proof follows essentially the same reasoning as in Section 2. Without loss of generality, assume that the rigid-body reactions are applied at node 0 and that the global equilibrium equations are evaluated at that point Then substitute into (16) the definitions from (13):

$$\begin{aligned} 0 &= H_0 Q_s + H_1 Q_f \\ &= H_0[K_{sf}D_f + K_{ss}D_s] + H_1[K_{ff}D_f + K_{fs}D_s] \end{aligned} \tag{20}$$

---

[4] Note that this is not a "pure" truss network, in that only a few of the 2n possible load components Q_i *are* shown.

[5] Known *as* the Mohr correction to the Williot diagram in graphic statics.

$$- [H_0 K_{sf} + H_1 K_{ff}] D_f + [H_0 K_{ss} + H_1 K_{fs}] D_s$$

This must be true for any D_f and D_s. Therefore, from the first matrix, we can prove a):

$$\begin{aligned} H_0 K_{sf} + H_1 K_{ff} &= 0 \\ H_0 K_{sf} &= -H_1 K_{ff} \\ K_{sf} K_{ff}^{-1} &= -H_0^{-1} H_1 \qquad \text{Q.E.D.} \end{aligned} \tag{21}$$

The proof of b) follows from symmetry. From the second matrix in (20), we obtain the proof of c) using the result of (21):

$$\begin{aligned} H_0 K_{ss} + H_1 K_{fs} &= 0 \\ K_{ss} + H_0^{-1} H_1 K_{fs} &= 0 \\ K_{ss} - K_{sf} K_{ff}^{-1} K_{fs} &= 0 \qquad \text{Q.E.D.} \end{aligned} \tag{22}$$

3.3. "Adjustable" reactions

We have shown that reactions can be treated as assignable (known) quantities subject only to the global equilibrium constraints (16). This generalization of the standard problem raises the possibility of using analysis to adjust reactions to achieve some design objective (e.g., limit a reaction at a given support or equalize reactions among supports).

The general problem is thus as follows:

Solve

$$\bar{K} \bar{u} = \bar{Q} \tag{23a}$$

subject to

$$H^* \bar{Q} = 0 \tag{23b}$$

where H^* includes both the rigid-body equilibrium constraints of (1b) and the additional imposed constraints on the reactions.

To solve, we factor the constraint equation (23b) so that

$$[H^*_0 \;\; H^*_1] \begin{Bmatrix} Q_s \\ Q_f \end{Bmatrix} = 0 \tag{24}$$

then $Q_s = [-H^{*-1}_0 H^*_1] \; Q_f = H^* \, Q_f$. (25)

An example of the use of (25) is shown in Figure 3. Since the four reactions Q_1

through Q_4 in Figure 2 *are* constrained by only three global equilibrium equations, an additional constraint $Q_1 = Q$ is introduced and the reactions solved for. Note that this is an equilibrium solution only. An elastic analysis must be performed to calculate the relative support displacements D_2 and D_3 relative to D_4 necessary to provide this distribution of reactions.

The elastic solution can be obtained by substituting the now known values of Q_s on the left-hand side of (13); with D_s now treated as an unknown:

$$\left\{ \begin{array}{c} Q_f \\ \hline H^*Q_s \end{array} \right\} = \begin{bmatrix} K_{ff} & K_{fs} \\ K_{sf} & K_{ss} \end{bmatrix} \left\{ \begin{array}{c} D_f \\ \hline D_s \end{array} \right\} \qquad (26)$$

from the second equation, we get D_s

$$D_s = K_{ss}^{-1} [H^* Q_s - K_{sf} D_f]. \qquad (27)$$

Substituting into the first equation

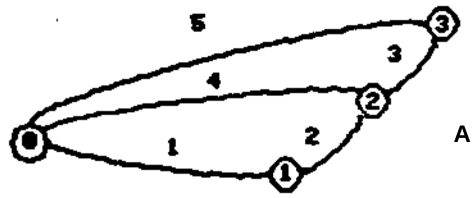
$$Q_f = K_{ff} D_f + K_{fs} K_{ss}^{-1} [H^* Q_s - K_{sf} D_f]$$

after rearrangement we get

$$D_f = [K_{ff} - K_{fs} K_{ss}^{-1} K_{sf}]^{-1} [I - K_{fs} K_{ss}^{-1} H^*]Q_f . \qquad (28)$$

Further use of (25) - (28) remains to be explored

It should be noted that the approach presented does not appear to be useful for the flow network, because of the scalar nature of the overall flow equilibrium equation For example, if in a water distribution network it is desired to provide equal inflow at two nodes, say nodes 0 and i, the magnitude of $Q_i = -\ 1/2 \sum_{j=1}^{n-1} Q_j$ is easily calculated, and Q_i is applied as a negative outflow. The D_i obtained from the solution (7) represents the head at i (relative to D_0) necessary to achieve the constraint of $Q_i = Q_0$.



All K_i * 1

$$\bar{K} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 0 \end{array} \\ \left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ \hline -1 & -1 & -1 & 3 \end{array}\right] \end{array}$$

$$K = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$K_{01} = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}$$

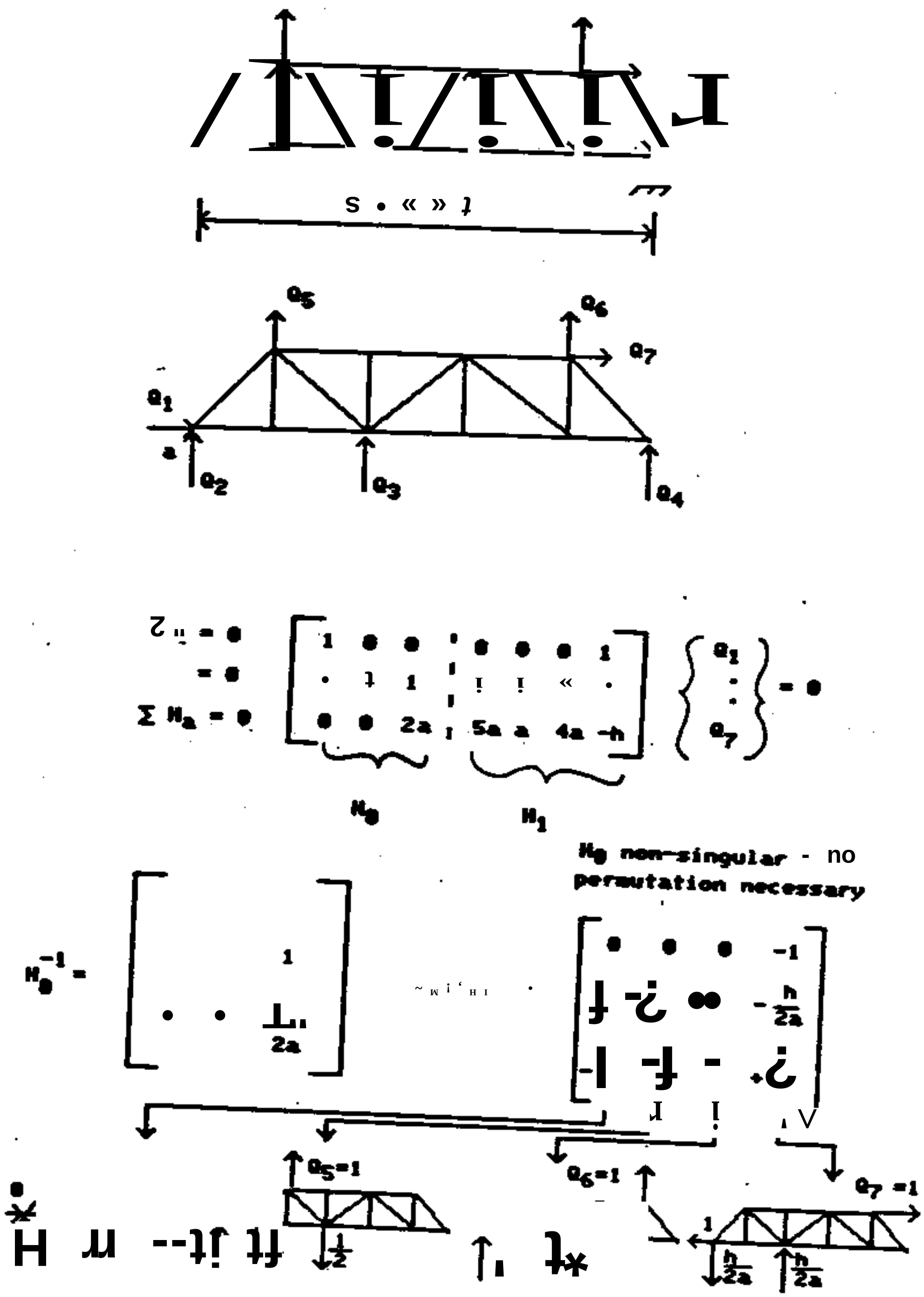
K_{00} = CO

$$K^{-1} = \frac{1}{8} \begin{bmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

$-K_{01}K^{-1}$ • Q i 0

$$K_{00} - K_{01}K^{-1}K_{10} = 3 - 3 = [0]$$

Figure 1.



Q1
Q2
Q3
Q4
Q5
Q6
Q7
a
$\Sigma M_a = 0$
M_0
M_1
M_0 non-singular - no permutation necessary
$M_0^{-1} =$
$Q_5=1$
$Q_6=1$
$Q_7=1$

Figure 2.

Sot $Q_a = Q_1$

$$
\begin{array}{r}
\text{IV * 8} \\
\Sigma H_a = 0 \\
Q_a = Q_1
\end{array}
\quad
H^* \rightarrow
\left[\begin{array}{ccc|ccc}
1 & 1 & 1 & I & 1 & \bullet \\
6 & 2a & Sa & a & 4a & -h \\
-1 & \bullet & 1 & \bullet & \bullet & 6
\end{array}\right]
\begin{Bmatrix} Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 8 \\ 8 \end{Bmatrix}
$$

$$\underbrace{\quad\quad}_{H_0^*} \qquad \underbrace{\quad\quad}_{H_1^*}$$

•l * IH • 8 ignorod • follows *from* statics

$$Q_3 = -H_0^{*-1} H_1^* Q_f$$

$$
\begin{Bmatrix} Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}
= -
\begin{bmatrix}
-1 & 2 & -\frac{h}{a} \\
3 & -3 & -\frac{2h}{a} \\
-1 & 2 & -\frac{h}{a}
\end{bmatrix}
\begin{Bmatrix} Q_5 \\ Q_6 \\ Q_7 \end{Bmatrix}
$$

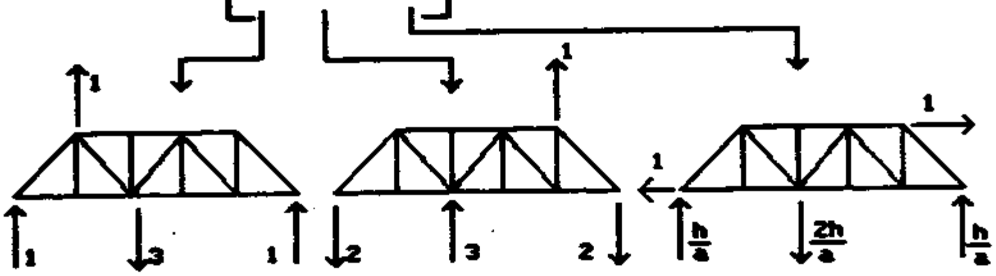


Figure 3.