

COMPACTNESS IN ALGEBRAIC STRUCTURES

by

Günter H. Wenzel

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If K is a class of relational systems or algebras then $H(K)$, $I(K)$, $S(K)$, $P(K)$ stands for the class of all homomorphic images, isomorphic images, subsystems (resp. subalgebras) and direct products of elements in K .

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Günter H. Wenzel
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Denotations. The following is a list of recurring denotations:

N, N_0, Z, Q, R denotes the set of natural numbers, non-negative integers, integers, rational numbers, real numbers, resp. 2^A is the set of subsets of A , $U(H)$ denotes set-theoretical union (intersection), $V(A)$ denotes lattice-theoretical cup and meet, resp. \wedge and, finally, $\cdot 0$ denotes set-theoretical disjoint union.

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Seminar Notes

by Giinter H* Wenzel

It is the very question which has given birth to the mathematical discipline called « algebra » that (in a modified form) is underlying our topic: The question about the solvability of certain systems of equations in certain pregiven algebraic domains. However, while the original interest was and is directed toward finding solutions of finite systems of equations (or, at least, toward establishing the existence of such solutions) our interest is of a more relative nature: Given a system of equations over some algebraic structure (the terms will, of course, be made precise in the succeeding sections), it is our aim to study the conditions under which we can conclude the existence of a solution of that system provided certain distinguished subsystems are solvable. To clarify our point we will initiate our investigation with the aid of a few simple, informal, illustrating examples:

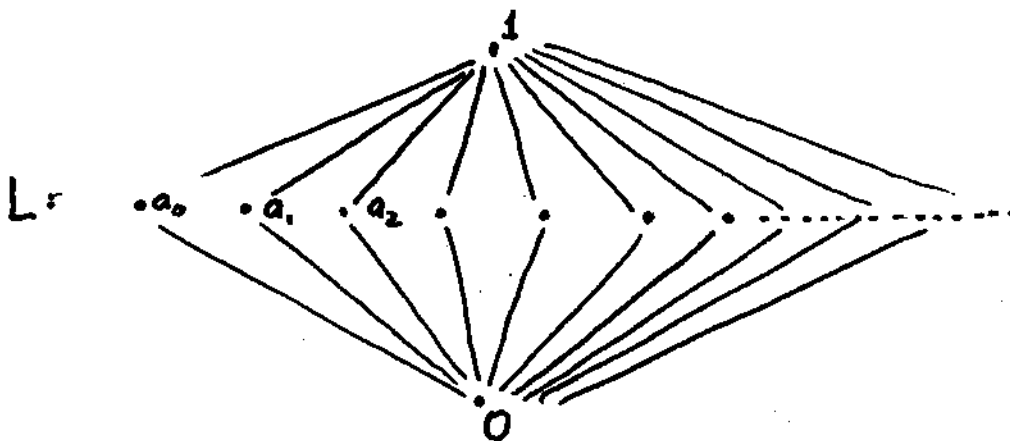
(Ex. 1): If we consider the cyclic group Z of integers with addition $+$ then we narrow our attention to the following system L of equations:

$$\begin{aligned} 3x_0 + x_1 &= 1 \\ x_1 &= 2x_2 \\ x_2 &= 2x_3 \\ &\vdots \\ x_n &= 2x_{n+1} \\ &\vdots \end{aligned}$$

where n runs through the set N of natural numbers. Visibly, if $(x_0, x_1, x_2, \dots, x_n, \dots) \in \mathbb{Z}^{\infty}$ is a solution of Σ then $0 \wedge x_1 = 2^n x_{n+1}$ for every $n \in \mathbb{N}$, i.e. 2^n divides $x_1 \wedge 0$ for every natural number. This being impossible, L has no solution. On the other hand, if L_n denotes the set of the first n equations in L , then we choose x_n and x_{n-1} such that $3x_n + 2^{n-1} = 1$, define $x_{n-1} = 2x_n, x_{n-2} = 2x_{n-1}, \dots, x_1 = 2x_2$ and have, thus, a solution (x_0, x_1, \dots, x_n) of L_n . L is a system of equations which is not solvable, although every finite subsystem is. This example is supplemented by the next one.

(Ex. 2): If B is a complete Boolean algebra with join \vee , meet \wedge , complement $'$, zero 0 and identity 1 , then every system L of equations involving variables, constants of B and the above operations is solvable provided every finite subsystem is. We delay a proof of this fact to a later section.

(Ex. 3): If we replace the Boolean algebra in Ex. 2 by a complete lattice L with join \vee and meet \wedge , then the conclusion is no more true. To see this, let $L = \{0, a_0, a_1, \dots, a_n, \dots, 1\}$, $n < \infty$, where 0 and 1 are respectively the smallest and largest element and the elements a_n are pairwise unrelated:



If S is a set of cardinality \aleph^{ω} then the system $I, = \{x_i \wedge x_j = 0, \vee x_i \vee x_j = 1, i \wedge j \in S\}$ of equations over L is not solvable in L , since i / j implies $x_i \wedge x_j$ for any solution $(x_s)_{s \in S}$ of S would have cardinality $\aleph^{\omega} > |L|$. On the other hand, it is quite evident that every finite subsystem $\mathcal{F} \subset S$ is solvable.

(Ex. 4) It is the same basic technique that together with Dirichlet's prime number theorem yields the following peculiar property of the ring Z of integers: If ω is the initial ordinal of \mathbb{F} , then the system $\mathcal{F} = \{x_{\xi, \eta}^* (m \cdot z_{\xi}^* + n) + Y_{\xi, \eta} (m \cdot z_{\eta} + n) = 1; \xi, \eta < \omega\}$ of equations over the ring Z is not solvable if n, m are relatively prime natural numbers such that $\frac{+1-n}{m} \notin Z$ (Mycielski chose $n = 5, m = 2$). To see this, we realize that for every choice of $z_{\xi} \in Z$ the integer $m \cdot z_{\xi} + n$ is different from $+1$; hence, $\mathcal{F} / r \setminus$ implies $z_{\xi} \in Z$ for every solution $(z_{\xi})_{\xi < \omega}$ of \mathcal{F} which, of course, is impossible in Z . On the other hand, every countable subsystem T of \mathcal{F} is solvable in Z : Let $\xi_0, \xi_1, \xi_2, \dots, \xi_n, \dots$ be the indices ξ, η actually occurring in T and choose z_{ξ} such that $m \cdot z_{\xi} + n = p_{\xi}$ is a prime number for every $\xi \in \mathbb{N}$; moreover we make our choice such that $i \wedge j$ implies $p_i \wedge p_j$. We can do this, since $n + m, n + 2m, n + 3m, \dots$ constitutes an infinite arithmetical progression with initial element n , difference m and $(n, m) = 1$; thus, Dirichlet's theorem assures an infinite number of prime elements in the progression. Since

therefore $(p_i, p_j) = 1$ if $i \neq j$ we can find integers s_j and t_{ij} such that $s_j p_i + t_{ij} p_j = 1$, i.e. E^1 is solvable.

It was Kaplansky who in 1954 observed the impact of this ¹¹ equational behavior¹² on the structure of infinite Abelian groups and he baptized groups with the property that every system of equations whose finite subsystems are solvable is solvable as ¹¹ algebraically compact groups ^M (He used an equivalent, but different, definition). In the sequel the concept has attracted the interest of algebraists and logicians alike and has been studied in both a universal algebraic-logical and classical-algebraic framework. Our attempt in the succeeding sections will be to lay down the foundations of that framework and to give our problem the rigorous setting it visibly needs. Our approach to the main results will be as geodesic as necessary to justify the title, as geodesic as possible without either bypassing some recent results on algebraic constructions (as, e.g., ultra-products) which can be found on nearby side-tracks or neglecting results that serve the aim of a certain degree of self-containedness.

Chapter I* Elements of Universal Algebra.

This chapter will contain a concise account of the basic concepts associated with universal algebras and relational systems. We confine our attention to the algebraic aspects of things delaying the impact on first order-logic to the second **chapter**.

§1. Universal Algebras and Relational Systems.

Universal algebras are the final abstraction **of algebraic** systems as groups, rings, lattices, etc., while relational systems play the same rôle with respect to partially ordered sets, chains, divisibility-domains and so on.

Def. 1: A universal algebra $G = \langle A; F \rangle$ is a pair of sets A, F where A is non-empty, called carrier set of G , and F consists of finitary operations on A (called fundamental operations); i.e. each $f \in F$ is a mapping from some A^{n_f} into A where n_f is a non-negative integer. If one well-orders $F = \{ \langle f_0 \rangle, \langle f_1 \rangle, \dots, \langle f_\alpha \rangle, \dots \}$ and f_α is an n_α -ary operation, then the string $r = \langle n_0, n_1, \dots, n_\alpha, \dots \rangle$ is called the type of G . $K(r)$ stands for the class of all universal **algebras of type T^*** . We will frequently use the same set F to denote **the operations** in different algebras of a fixed type T (thus interpreting F as set of symbols which induce operations on **the universal** algebras $G \in K(r)$). If we want to emphasize **the algebra** G we will also write f^G for f . If $F' \subset F$ then we call $G = \langle A; F' \rangle$ the F' -retract of $G = \langle A; F \rangle$.

(Ex.5) : A semilattice S is a universal algebra of type $\langle 2 \rangle$, say $S = \langle S; \vee \rangle$, such that $a \vee a = a$, $a \vee b = b \vee a$, $(a \vee b) \vee c = a \vee (b \vee c)$ holds for all $a, b, c \in S$. A lattice \mathcal{L} is a universal algebra of type $\langle 2, 2 \rangle$, say $\mathcal{L} = \langle L; \vee, \wedge \rangle$ such that both $\langle X; \vee \rangle$ and $\langle L; \wedge \rangle$ are semi-lattices and, in addition, the so-called absorption-laws $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ hold for all $a, b \in L$. In a similar fashion, groups $Q = \langle G; \cdot, ^{-1}, 1 \rangle$ are certain universal algebras of type $\langle 2, 1, 0 \rangle$, rings $\mathcal{R} = \langle R; *, +, ^{-1}, ^{-1}, \sim, 0 \rangle$ are certain universal algebras of type $\langle 2, 2, 1, 1, 0 \rangle$, Boolean algebras $\mathcal{B} = \langle B; \vee, \wedge, ', 1, 0 \rangle$ are certain universal algebras of type $\langle 2, 2, 1, 0, 0 \rangle$, and so we could go on enumerating the known specific algebraic structures. 1_{τ} stands for a 1-element algebra in $K(T)$ and is called n trivial algebra.¹¹ As the reader has doubtless observed, the term M universal algebra¹¹ keeps coming up and is lengthy. So we agree to briefly say n algebraⁿ whenever we mean n universal algebra.¹¹

Def. 2: A relational system $G = \langle A; F, R \rangle$ is a triplet of sets A, F, R where $\langle A; F \rangle$ is an algebra and R is a set of finitary relations on A , i.e. the elements R_i of R are subsets of some A^m where $m \in \mathbb{N}$. Again we well-order $R = \{R_0, R_1, R_2, \dots, R_{fi}, \dots\}_{fi} \langle \beta$ and call $T = \langle T_i \rangle_{i \in \mathbb{N}}$ the type of $\wedge_{i \in \mathbb{N}} r_i$ as the type of $\langle A; F \rangle$ and $r_2 = \langle m_0, m_1, \dots, m_6, \dots \rangle_{fi} \langle \beta$ where $R_{fi} \subseteq A^{m_i}$. $K(T)$ stands for the class of all relational systems of type r .

$R_0 \langle (a_1, \dots, a_{m_0}) \rangle$ is synonymous with $(a_1, \dots, a_{m_0}) \in R_0$. In case $m_5 = 2$ we also write $a_1 R_5 a_2$ or $a_1 \wedge a_2 \wedge R_6$.

(Ex. 6): A partially ordered set, shortly poset, P is a relational system of type $\langle 0; \langle 2 \rangle \rangle$, say $P \bullet \langle \langle P; \langle J, \{f\} \rangle \rangle$, such that (i) $a \neq a$, (ii) $a \leq^{\wedge} b$ and $b \leq^{\vee} a$ implies $a = b$, (iii) $a \leq^{\wedge} b$ and $b \leq^{\wedge} c$ implies $a \leq^{\wedge} c$ holds for all $a, b, c \in P$. In case $a \leq^{\wedge} b$ or $b \leq^{\wedge} a$ holds for any two $a, b \in P$ we call the above poset a chain (or ordered set). In a similar fashion we obtain lattice-ordered groups $Q = \langle G; \{ \cdot, \sim^1, ! \} \rangle \{ \{ \cdot \} \}$ as relational systems of type $r = \langle \langle 2, 1, 0 \rangle; \langle 2 \rangle \rangle$, unique factorization domains $\langle = \langle D; \{ \cdot, +, \sim^1, -, 1, 0 \}; \{ / \} \rangle$ as relational systems of type $T = \langle \langle 2, 2, 1, 1, 0, 0 \rangle; \langle 2 \rangle \rangle$, etc. A relational system $\langle A; \$, R \rangle$ is called a strict relational system.

Again we feel the notation $\langle A; \wedge, R \rangle$ cumbersome and agree that, in case confusion is impossible, the strict relational system $\langle A; (\wedge, R) \rangle$ will be denoted by $\langle A; R \rangle$ with similar adjustment in the type. For a similar reason will we identify $\langle A; F, \phi \rangle$ with the algebra $\langle A; F \rangle$. Finally, we will use the notation R_{γ}^u instead of R_{γ} whenever we feel it necessary to emphasize the underlying relational system G , thus conforming with the corresponding agreement on the operations F .

We should point out that, of course, every algebra induces a strict relational system in a natural manner: Given the algebra $G = \langle A; F \rangle$ we pass to the relational system $G^1 = \langle A; \phi, F^1 \rangle$ where $F^1 = \{ F_{\gamma} \mid f_{\gamma} \in F \}$ is defined by $(a_1, \dots, a_{\gamma+1}) \in F_{\gamma}$ if and only if $f_{\gamma}(a_1, \dots, a_{\gamma+1}) = a_{\gamma+1}$. Thus, why don't we dispense with universal algebras and **non-strict relational systems** in favour of strict relational systems {as, e.g., S^* Kochen

has done in his paper "Ultraproducts in the theory of modules"?) There are two reasons: First of all a good part of our interest is founded in algebra, and dispensing with algebras would, e.g., destroy the subalgebra-lattice in favour of subsystems, a trade that we do not like to accept. Secondly it is our determined aim to stay intuitive whenever possible, devoting ourselves to formalism only where necessary or advisable. Kochen's procedure reflects his prime interest on the model-theoretic aspect of ultraproducts compared with the algebraic implications.

There are other occasions where the use of algebras or relational systems is just a matter of view-point as the following example whose proof we omit illustrates.

(Ex. 7): Let $\mathcal{F} = \langle L; \leq \rangle$ be a poset in which any two elements a_1, a_2 have a greatest lower bound, say $a_1 \wedge a_2$, and a least upper bound, say $a_1 \vee a_2$. Then the algebra $\mathcal{F}' = \langle L; \vee, \wedge \rangle$ is a lattice. Vice versa, if $\mathcal{F}' = \langle L; \vee, \wedge \rangle$ is a lattice and we define the relation \leq on L by $a_1 \leq a_2$ if and only if $a_1 \wedge a_2 = a_1$ (equivalently, $a_1 \vee a_2 = a_2$) then $\mathcal{F} = \langle L; \leq \rangle$ is a relational system with the above properties. Thus, the algebra \mathcal{F}' and the relational system \mathcal{F} just introduced manifest only different approaches to the same underlying object.¹¹

§2. Homomorphisms and Congruences.

We start with two relational systems $G = \langle A; F^*R \rangle$ $P = \langle B; F, R \rangle$ of the same type (if $R = (f)$ we have universal algebras) and a mapping $\langle p : A \rightarrow B$.

Def. 3; (i) $\langle p$ is called a "weak homomorphism"¹¹ if

$$(1) \quad \langle p(f_i(a_1, \dots, a_n)) = f_i(\langle p(a_1), \dots, \langle p(a_n)) \rangle \text{ and}$$

$$(2) \quad R_j(a_1, \dots, a_m) \text{ implies } R_j(\langle p(a_1), \dots, \langle p(a_m) \rangle) \text{ holds for all } f_i \in F, R_j \in R \text{ and } a_i \in A.$$

(ii) $\langle p$ is a "homomorphism"¹¹ if (1),(2) above hold and in addition (3) $R_j(\langle p(a_1), \dots, \langle p(a_m) \rangle)$ implies the existence of $b_1, \dots, b_m \in B$ such that $\langle p(a_i) = \langle p(b_i)$ and $R_j(b_1, \dots, b_m)$.

(iii) $\langle p$ is a "strong homomorphism"¹¹ if (1),(2) above hold and in addition (3) $R_j(a_1, \dots, a_m)$ holds if and only if $R_j(\langle p(a_1), \dots, \langle p(a_m) \rangle)$ is true. We use the notations $\langle p: G_w \rightarrow B$, $\langle p: G \rightarrow H$ and $\langle p: G_s \rightarrow B$ respectively to indicate that $\langle p: A \rightarrow B$ is a homomorphism of type (i), (ii) or (iii). $\text{Hom}_w(G, B)$, $\text{Hom}(G, B)$, $\text{Hom}_s(G, B)$ denote the sets of all homomorphisms of type (i), (ii), (iii), resp.

We note that in case of universal algebras the three notions coincide and agree visibly with the homomorphism-concept used in the diverse specific algebraic structures dealt with in (Ex.5). We reserve the concepts of epi-, mono- and isomorphism for onto, 1-1 and bijective homomorphisms, resp.. A homomorphism $\langle p: G \rightarrow G$ is also called an endomorphism, an isomorphism

$\langle \varphi : G \rightarrow G \rangle$ will be referred to as an automorphism.^{tf} The following remark is quite self-evident and left as exercise:

Remark 1:

- (1) The composition $g \circ f : A \rightarrow C$ of two homomorphisms $f : G \rightarrow H$ and $g : H \rightarrow C$ is again a homomorphism $g \circ f : G \rightarrow C$ if G, B, C are algebras of the same type (To obtain the same result for relational systems we would need strong homomorphisms).
- (2) $\langle \text{End}(G); \circ \rangle$ is a semigroup with identity for all algebras G (where $\text{End}(G) = \text{Hom}(G, G)$).
- (3) $\langle \text{Aut}(G); \circ, 1 \rangle$ is a group for every relational-system G if $\text{Aut}(G)$ is the set of all isomorphisms in $\text{End}(G)$.

Closely related to homomorphisms, congruence relations play the same basic role in the theory of algebras that normal subgroups play in the theory of groups or ideals play in the theory of rings. To lead up to them we recall the concept of an equivalence relation^m on a non-empty set A .

Def. 4: If A is a non-empty set, then an equivalence relation

\sim on A is a subset of $A \times A$ (i.e. $\sim \subseteq A \times A$) such that (i) $(a, a) \in \sim$, (ii) $(a, b) \in \sim$ implies $(b, a) \in \sim$, (iii) $(a, b), (b, c) \in \sim$ implies $(a, c) \in \sim$ for all $a, b, c \in A$. It has become tradition to write $a \sim b$ or $a \sim b$ for $(a, b) \in \sim$ and to say that ¹¹ a and b are equivalent modulo \sim .^{ft} More generally: $a \sim b \sim c$ stands for $a \sim b$ and $b \sim c$ etc.. The three postulates are known

as reflexivity, symmetry and transitivity, resp..

We assume the reader has learned before that a binary relation on a non-empty set A is a subset of $A \times A$. Thus, equivalence relations are merely reflexive, symmetric and transitive binary relations. Let us, for convenience, also recall that with any two binary relations $p, q \subseteq A^2$ a third binary relation $p \circ q \subseteq A^2$ is defined by $(a, b) \in p \circ q := (a, c) \in p, (c, b) \in q$ for some $c \in A$. This composition obeys the associative law $(p \circ q) \circ r = p \circ (q \circ r)$; hence, if $R_2(A)$ denotes the set of all binary relations on A , then $\langle R_2(A); \circ \rangle$ is a semi-group. We should point out that the fact that θ_1 and θ_2 are equivalence relations on A does not necessarily imply that $\theta_1 \circ \theta_2$ is an equivalence relation as well. As a matter of fact, we suggest the easy exercise to prove that $\theta_1 \circ \theta_2$ is an equivalence relation if and only if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ if and only if θ_1 and θ_2 are permutable. Let us also recall that every $p \in R_0(A)$ determines a so-called inverse relation $p^{-1} \in R_0(A)$ via $(a, b) \in p^{-1} := (b, a) \in p$; both the diagonal-relation $\Delta_A = \{(a, a) \mid a \in A\}$ and the universal relation $\bar{A} = A \times A$ are equal to their respective inverses. Let us not return to our equivalence-relations before agreeing that, in case $p \in R^*(A)$, $(f) \in B \subseteq A$, $p_B^* \in R_2(B)$ is defined by $p_B^* = p \cap B^2$.

Def. 5: If $E(A)$ denotes the set of all equivalence relations on A , $a \in A$ and $\theta \in E(A)$, then $[a]_\theta = \{b \mid b \in A \text{ and } b \theta a\}$ constitutes the so-called equivalence-block of a modulo θ . If $\emptyset \neq S \subseteq A$ then $[S]_\theta = \cup\{[a]_\theta \mid a \in S\}$ is called the closure of

S modulo 9.

A different view-point on equivalence relations is introduced when we speak about ¹ partitions¹¹ of a non-empty set.

Def. 6: If A is a non-empty set and P is a set of non-empty subsets of A (i.e. $p \subseteq A$) then P is a partition of A if (i) $B_1 \cap B_2 \in P$ implies $B_1 = B_2$ or $B_1 \cap B_2 = \emptyset$, (ii) $\cup(B \in P) = A$. The elements of P are called blocks of the partition. Part(A) is the set of all partitions of A .

2

If $P \in \text{Part}(A)$ then we can define the subset P^* of A by the requirement that $(a,b) \in P^*$ holds if and only if there is a block B in P containing both a and b . If $\theta \in E(A)$ then we can define the subset θ' of 2^A as $\theta' = \{[a]\theta; a \in A\}$. It is again an easy matter to check the following remark.

Remark 2: (1) If $P \in E(A)$ then $\theta' \in \text{part}(A)$.

(2) If $P \in \text{Part}(A)$ then $P^* \in E(A)$

(3) $\{(\theta')'\} = \theta$ for every $\theta \in E(A)$

(4) $(P^*)^* = P$ for every $P \in \text{Part}(A)$.

Thus, partitions and equivalence relations on a non-empty set A are in a 1-1 correspondence given by ⁵. This relationship provides an even further ^{f1} identification^{f1} of the two concepts.

Def. 7: Let $\theta \in E(A)$, $P_i \in \text{Part}(A)$, $i = 1, 2$. We say that $\theta \leq \theta_2$ ($P_1 \leq P_2$) if $a \equiv b (\theta)$ always implies $a \equiv b (\theta_2)$ (if every block of P_1 is contained in a block of P_2).

Theorem 1: (1) Both $\langle E(A); \ell \rangle$ and $\langle \text{Part}(A); \ell \rangle$ are lattices as partially ordered sets (see(Ex. 7)). Thus, if $a \vee b$ ($a \wedge b$) denotes the least upper bound (greatest lower bound) of a, b in both lattices, then $E(A) = \langle E(A); V, A \rangle$ and $\langle \text{Part}(A); V, A \rangle$ are lattices. Moreover: $\theta : E(A) \rightarrow \text{Part}(A)$ mapping 0 to 0^5 is a lattice-isomorphism.

(2) $\langle E(A); V, A \rangle$ is a complete lattice, i.e. every set S of equivalence relations in $E(A)$ has both a greatest lower and a least upper bound, denoted by $A(s; s \in S)$ and $V(s; s \in S)$, resp..

Proof: The fact that the mapping $*$ is a homomorphism in both directions is a simple exercise. So we confine ourselves to proving that $\langle E(A); V, A \rangle$ is a complete lattice.

If $S \subseteq E(A)$ and the greatest lower bound (g.l.b.) exists then (following common usage) we denote this g.l.b. by $A(s; s \in S)$ which, in case of finite $S = \{s_1, \dots, s_n\}$, can be replaced by $s_1 A s_2 A \dots A s_n$. Clearly, $A(s; s \in S)$ exists always and $A(s; s \in S) = \bigwedge (s; s \in S)$ if $S \subseteq E(A)$. To show that the least upper bound (l.u.b) $V(s; s \in S)$ always exists we introduce the binary relation θ on A as follows: $a \theta b$ holds if and only if there exist elements $a = a_0, a_1, a_2, \dots, a_n = b$ in A and equivalence relations $s_1, s_2, \dots, s_n \in S$ such that $a \theta_1 a_1 \theta_2 a_2 \dots \theta_{n-1} a_{n-1} \theta_n a_n$. It is straight forward to verify that θ is an equivalence relation which is larger or equal than all $s \in S$. If 0 is an arbitrary equivalence and $0 \geq s$ for all $s \in S$ then $a \theta b$ always implies $a \theta b(0)$, i.e. $a \theta b$ implies $a \theta b(0)$, or equivalently: $0 \leq \bigvee S$. Hence, $0 = V(s; s \in S)$. q.e.d.

Note: We ought to keep in mind the construction of A and V for arbitrary sets $S \subseteq E(A)$ since we use it whenever necessary. For the purpose of studying universal algebras, the concept of \equiv equivalence relation¹¹ is too restricted, for it does not take in account the nature of the fundamental operations F . To take care of this handicap we pass to the more restricted set of congruence relations.

Def. 8.: Let $G = \langle A; F, R \rangle$ be a relational system. Then $\theta = \equiv$ on $E(A)$ is called a congruence relation of G provided that the ¹¹ substitution property (JSP)¹¹ holds; i.e. $a_1 \equiv b_1, a_2 \equiv b_2, \dots, a_n \equiv b_n$ must imply $f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n)$ for all $a_i, b_i \in A$ and $f \in F$. $C(G)$ denotes the set of all congruence relations of G .

We note that we have not made any specifications on the behaviour of the relations in R as far as congruence relations are concerned. This leaves us with the freedom to specify these relations in different situations in a different manner - a freedom that we will make use of in the sequel. By definition, $C(G) \subseteq E(A)$; thus we can apply the binary operations V and \wedge defined on $E(A)$ according to theorem 1 to any two congruence relations θ_1, θ_2 . It can easily be verified that both $\theta_1 \vee \theta_2$ and $\theta_1 \wedge \theta_2$ are not only equivalence relations on A but even congruence relations of G . Thus, $\langle C(G); V, \wedge \rangle$ is itself a lattice if V and \wedge denote the restrictions of the respective operations on $E(A)$. In this sense, we say that $\langle C(G); V, \wedge \rangle$

is a sublattice of $\langle E(A); V, A \rangle$, a concept that will be discussed in the general setting it deserves in the next section. If $0 / S \subseteq C(G)$ then we can form $a = V(s; s; eS)$ and $b = A(s; s; eS)$ in $E(A)$ as discussed before and again we easily verify that $a, b \in C(G)$. Utilizing this stronger property we use the phrase that $\langle C(G); V, A \rangle$ is a complete sublattice of $\langle E(A); V, A \rangle$.
Summing up:

Remark 3; $C(G) = \langle C(G); V, A \rangle$ is a complete lattice and as such a complete sublattice of $t(A) = \langle E(A); V, A \rangle$ for every relational system G .

Before we proceed we illustrate the concept with a few examples:

(Ex, 8) :

- (1) Given the relational system $G = \langle A; F, R \rangle$, then both the identity-relation o (defined by $a \equiv b(o)$ if and only if $a = b$) and the universal relation i (defined by $a \equiv b(i)$ for all $a, b \in A$) are congruence-relations. o is the smallest, i the largest element of $C(G)$. Universal algebras whose only congruences are o and i are known as simple algebras.
- (2) Let $Q = \langle G; *, \sim^{-1}, 1 \rangle$ be a group and $0 \in C(Q)$. Then $a \equiv b(0)$ implies together with $b^{-1} \equiv b^{-1}(Q)$ that $a \cdot b^{-1} \equiv 1(\wedge)$, and vice versa. Hence, two elements a, b are congruent modulo 0 if and only if $a \cdot b^{-1} \equiv 1(0)$. In other words, to know 0 we only need to know $[1]0$. So let $a, b \in [1]Q$, i.e. $a \equiv 1(0)$, $b \equiv 1(0)$, then,

since $i \in S$. Then $\theta = \bigvee_{i \in S} \theta_i$ is the unique smallest congruence relation such that $a_i \theta b_i$ modulo that congruence relation.

We take up the last kind of congruence relations in (Ex.8) in order to describe the lattice $C(G)$ further.

~~Def. 9.~~ Given is a complete lattice $\langle L; \vee, \wedge \rangle$. An element $c \in L$ is compact if $c \leq \bigvee_{s \in S} s, S \subseteq L$, always implies the existence of $s_1, \dots, s_n \in S$ such that $c \leq s_1 \wedge \dots \wedge s_n$. The lattice L is an algebraic lattice if every $t \in L$ can be written as $t = \bigvee \{c \in S \mid c \leq t\}$ where all elements of S are compact.

Theorem 2: $C(G)$ is an algebraic lattice for every relational system G . The compact elements are exactly the congruence relations of the form $\theta = \bigvee_{i=1, \dots, n} \theta_i$.

Proof: To show that the congruence relations $\theta_i = \bigvee_{i=1, \dots, n} \theta_i$ are compact can be left as exercise.

Since $C(G)$ is complete by remark 3 and $\theta = \bigvee_{\theta \in C(G)} \theta$ for all $\theta \in C(G)$ is a triviality, all we need to show is that every compact congruence relation is of the form $\theta = \bigvee_{i=1, \dots, n} \theta_i$. So let θ be compact in $C(G)$.

Since $\theta = \bigvee_{\theta \in C(G)} \theta$ we conclude that $\theta = \bigvee_{i=1, \dots, n} \theta_i$ with $\theta_i = \theta(a_i, b_i)$. Hence, $\theta = \bigvee_{i=1, \dots, n} \theta_i$.

$\bigvee_{i=1, \dots, n} \theta_i = \theta$ q.e.d.

The converse of theorem 2 is true as well. G. Grätzer and E. T. Schmidt proved that every algebraic lattice is representable as congruence lattice of some universal algebra. However, the present proof is deep and involved and must therefore be omitted.

§3. Elementary Algebraic Constructions,

In this section we introduce the main-operations on classes of relational systems or algebras yielding new such systems or algebras which we need in the succeeding discussions. Due to the folklore-character of subject matter we will pursue a slightly sketchy style.

(1) Let $G = \langle A; F, R \rangle$ be a relational system (again $R = \langle f \rangle$ settles implicitly the case of universal algebras) then $H = \langle B; F, R \rangle$ is a subsystem (in case of algebras, subalgebra) of G if (i) $B \subseteq A$, (ii) all operations in H are just the restrictions from the corresponding operations in G and $f_v(k-1, \dots, j_k) \in B$ for all $f_y \in F$, $h_i \in B$, (iii) $R_y^B = R_y^G \cap B^2$ for all $R_y \in R$. If $\emptyset \neq C \subseteq A$ then the set $\mathcal{C} = \{ \langle D; F, R \rangle \mid D \subseteq C \}$ contains G and is therefore non-empty. It is then quite clear that $G(C) = \langle C; F, R \rangle$ is the unique smallest subsystem of G containing C if $[C] = \bigcap \{ D \mid \langle D; F, R \rangle \in \mathcal{C} \}$, $f_y^{G(C)} = f_y^G \upharpoonright [C]^n (= f_y^G \upharpoonright [C]^n)$ restricted from A^n to $[C]^n$, $R_y^{G(C)} = R_y^G \cap [C]^2$. We say that C generates $G(C)$. This definition of a subsystem, though quite adequate for many general purposes, fails to suffice in a number of instances where we are concerned with specific algebraic structures. Thus, e.g., one might like to discuss groups as algebras $Q = \langle G; \cdot \rangle$ of type $\langle 2 \rangle$ satisfying the well-known axiom-system A requiring, a.o., the existence of an element $1 \in G$ such that $1 \cdot g = g \cdot 1 = g$ holds for all $g \in G$. Visibly then $\langle Z^*; \cdot \rangle$ as a subalgebra of the group $\langle Q^*; \cdot \rangle$

(Z^* = non-zero integers, Q^* = non-zero rational numbers), but it is not a subgroup. To cope with such situations we agree on the following terminology: If, within $K(r)$, we single out a certain class ${}^K A(\wedge)$ of relational systems via an axiom-system A , then we define a A -subsystem to be a subsystem which is also an element of ${}^K A(\wedge)$.

(2) Let $G_i, i \in I$, be relational systems of a fixed type r . Then $\text{Tr}(G_i; i \in I) = \langle \text{Tr}(A_i; i \in I); F, R \rangle$ is defined as follows: (i) The elements of $\text{Tr}(A_i; i \in I)$ are all functions $f : I \rightarrow U(A_i; i \in I)$ satisfying $f(i) \in A_i$; (ii) $\forall f, h_1, \dots, h_n \in \text{Tr}(A_i; i \in I)$ then $f \vee (h_1, \dots, h_n)$ (i) = $f \vee (h_1(i), \dots, h_n(i))$ defines the element $f^\vee(h_1, \dots, h_n) \in \text{Tr}(A_i; i \in I)$ for all $f, h_i \in F$; (iii) $R^\vee(h_1, \dots, h_n)$ holds only if $R^\vee(h_1(i), \dots, h_n(i))$ is true for all $i \in I$. The new relational system $\text{Tr}(G_i; i \in I)$ is called the direct (Cartesian) product of the systems G_i . We will use the synonymous notations $h, (h(i))^I, (a_i)^I$ for $h \in \text{Tr}(A_i; i \in I)$ if $a_i = h(i)$. $h(i) = a_i$ is called the i -th component of h , the i -th projection $\text{Tr}(A_i; i \in I) \rightarrow A_i$ is a weak epimorphism for all $i \in I$. If $G_i = G$ for all $i \in I$ then we also write G^I for $\text{Tr}(G_i; i \in I)$.

(3) $B = \langle B; F, R \rangle$ is a subdirect product of the relational systems $G_i = \langle A_i; F, R \rangle, i \in I$, if it is a subsystem of $\text{Tr}(G_i; i \in I)$ such that $\text{Tr}(B) = A_i$ for all i -th projections. In case $G_i = G$ for all $i \in I$ we always have two trivial subdirect products, namely G itself and the diagonal system $A(G) = \langle A; F, R \rangle$ with

$A = \{ \langle a_i \rangle_{i \in I}; a_i = a \in A \}$ which is isomorphic to G . There are instances where these are the only ones: If, e.g., $\mathbb{Z} = \langle \mathbb{Q}; +, \#, -, 0 \rangle$ is the ring of rational numbers viewed as being of type $\langle 2, 2, 1, 0 \rangle$, then we suggest the exercise to prove that indeed \mathbb{Z}^2 and $\mathbb{Z} \times \mathbb{Z}$ are the only subdirect products of two copies of \mathbb{Z} .

(4) If θ is a congruence relation on $G = \langle A; F, R \rangle$ then we define the quotient-system $G/\theta = \langle A/\theta; F, R \rangle$ as follows: (i) $A/\theta = \{ [a]_\theta; a \in A \} \subseteq 2^A$, (ii) $f([a_1]_\theta, \dots, [a_n]_\theta) = [f(a_1, \dots, a_n)]_\theta$ for all $f \in F$, $a_i \in A$, (iii) $R([a_1]_\theta, \dots, [a_m]_\theta)$ holds if and only if there exist $b_1, \dots, b_m \in A$ such that $b_i \theta a_i$ and $R(b_1, \dots, b_m)$. It is an easy exercise to verify that both operations and relations are well-defined and G/θ is a relational system of the type of G . The so-called "canonical projection" $\pi: A \rightarrow A/\theta$ mapping a to $[a]_\theta$ is an epimorphism. As we have seen before, every homomorphism $\langle p \rangle$ determines a congruence relation $\ker(p)$, and every congruence relation θ determines a homomorphism π . As will be stated more precisely in the next but one section, this correspondence is bijective, thus enabling us to deal with homomorphic images of a relational system completely within G itself.

(5) Our last construction introduces the so-called direct (or inductive) limit of relational systems. To do so, let $\langle I; \leq \rangle$ be a directed poset, i.e. a poset in which any two elements have some upper bound, let $G_i = \langle A_i; F, R \rangle$, $i \in I$, be relational systems and assume the $\langle p_{ij} : A_i \rightarrow A_j \rangle$ to be weak

homomorphisms for every pair $i, j \in I$ with $i \leq j$ such that $\langle p_{ij} \dots = \text{identity and } \langle p_{ij} \dots = \langle P_{-v}^\#$. Then we define the equivalence

relation \sim on $\cup_{i \in I} A_i$ by $a_i \sim a_j$ if there exists some $k \in I, k \geq i, j$, such that $\langle P_{ik}^\#(a_i) = \langle P_{jk}^\#(a_j)$ (hence $\langle P_{ij}^\#(a_i) = \langle P_{jk}^\#(a_j)$ for all sufficiently large k).

The resulting set $\cup_{i \in I} A_i / \sim$ of equivalence blocks is called

the direct limit of the directed system of sets $(A_i, I, \langle p)$ and denoted by $\varinjlim (A_i, I, \langle p)$ or, shortly, $\varinjlim A_i$. Finally, $\varinjlim (G_i, I, \langle p)$ is defined as follows:

(i) If $f_r \in F, a_j \in A_j, j = 1, \dots, n, I \geq i_1 \dots i_n$, then $\langle P_{i_1}^\# \dots \langle P_{i_n}^\#(a_j)$ makes, as one easily checks, $\varinjlim A_i$ an operation well-defined on $\varinjlim A_i$.

(ii) If $R_j \in R, b_j \in A_j, j = 1, \dots, m$, then R_j is defined to hold true if and only if there exists $m \geq i_1, \dots, i_m$ such that $R^m(\langle P_{i_1}^\# \dots \langle P_{i_m}^\#(b_j))$ holds true.

$\varinjlim G_i = \varinjlim (G_i, I, \langle p)$, thus defined, is the direct limit of the direct system of relational systems $(G_i, I, \langle p)$. If we enlarge I to $I \cup \{I^*\}$ where $i < I^*$ for every $i \in I$ (in case I has a maximal element m we agree on $m = I^*$), denote $\varinjlim (G_i, I, \langle p)$ by G and define $\langle P_{I^*}^\# : A_{I^*} \rightarrow A$ by $\langle P_{I^*}^\#(a_i) = [a_i e^1,$

then $(G_i, I \cup \{1^*\}, \langle p \rangle)$ is still a direct system, and the following remark is evidently true:

Remark 4: $\varinjlim (G^1, \wedge) \cong \varinjlim (G_i, I \cup \{1^*\}, \langle o \rangle)$.

(Ex. 9): Associate with $G = \langle A, F, R \rangle$ the set $I = \{B; \ll = \langle B; F, R \rangle\}$ is a finitely generated subsystem of G . Then $\langle I; \subseteq \rangle$ is a directed poset and $(\ll, 1, \langle p \rangle)$ is a direct system of relational systems if $\ll_B = \ll$ and $\langle \text{PT}_3^* B \rightarrow C \rangle$ is the embedding-map for $B \subseteq C$. One can easily verify that $\varinjlim (\ll_B, I, \langle p \rangle) \cong G$; hence, every relational system is a direct limit of its finitely generated subsystems.

Let us pursue the matter a little further. Many of the results are due to G. Grätzer, although our approach is new at times. If $(G, D, \langle p \rangle)$ is a direct system and $\mathcal{J}, I \subseteq D$ are directed sub-posets of D , then $(G, \mathcal{J}, \langle p \rangle)$, where the $\langle p_{ij} \rangle$ are confined to $i, j \in \mathcal{J}$, is a direct system as well; a similar remark holds for I . Again it is easy to verify the next remark if \mathcal{J} is bounded by I (i.e. for every $j \in \mathcal{J}$ there is some $i \in I$ with $j \ll i$).

Remark 5: $\langle p_{ij} \rangle^* : \varinjlim (A, \mathcal{J}, \langle p \rangle) \rightarrow \varinjlim (A, I, \langle p \rangle)$ defined by $\langle p_{ij} \rangle^* (\{a_i\}_0 = \{a_i\}_1$ where $a_i = \wedge_{j \in \mathcal{J}} (a_j)$, $i \in I$, is a weak homomorphism with kernel 0 defined by $[a_i]_1 \leq [a_j]_2$ (mod Θ), $\exists i \in \mathcal{J} \wedge i \in I$ if and only if $\langle p_{ij} \rangle^* (a_i) = \langle p_{ij} \rangle^* (a_j)$ for some $i \in I$. The image is $([a_i]_1; i \in I, a_i \in GA)$ and there is some $j \in \mathcal{J}$, $a_i \in A$, $\exists k \wedge i, j$ with $\langle p_{ij} \rangle^* (a_i) = \langle p_{ij} \rangle^* (a_j)$

Assuming that, in addition, \mathcal{J} is cofinal with I (i.e. for every $i \in I$ there is some $j \in \mathcal{J}$ with $i \leq j$) we conclude that

$$[a_i]_0 \wedge [a_j]_6 \text{ i.e. } \langle p_i \cdot (a_i) \rangle_{D_1^{5:i}} \wedge \langle p_j \cdot (a_j) \rangle_{D_2, D_2} \text{ for all } j \in \mathcal{J},$$

$$\text{implies } \langle p_i \cdot (a_i) \rangle_{D_1} \wedge \langle p_j \cdot (a_j) \rangle_{D_2} \text{ for all } i \in I, \text{ i.e.}$$

$[a_i]_{D_1} \wedge [a_j]_{D_2}$ (0) (see last remark). Since in this case, as the last remark shows, $(p_i)^\wedge$ is also an onto-mapping, we have the next remark:

Remark 6; If $(G_i, D, \langle p \rangle)$ is a direct system and $\mathcal{J}_1 \subseteq D$ are mutually cofinal we have the isomorphism $\langle p_{\mathcal{J}_1} \cdot \wedge : \text{I}^\wedge \text{m}(G_i, \wedge, \langle p \rangle) \rightarrow \text{I}^\wedge \text{m}(G_i, \mathcal{J}_1, \wedge)$. We recall that G_i^\wedge stands for $\text{I}^\wedge \text{m}(G_i, I, \langle p \rangle)$. It is natural to call two direct systems $(G_i, I, \langle p \rangle)$ and $(B_j, \mathcal{J}, \langle * \rangle)$ equivalent if $G_i^\wedge \cong B_j^\wedge$. In this respect the next result is of some interest. Before starting it we recall McNeille's embedding theorem stating that every poset can be embedded in a complete lattice such that least upper bounds and greatest lower bounds are preserved.

Remark 7; Given the direct system $(G_i, I, \langle p \rangle)$ and the McNeille-embedding of I in a complete lattice $\langle \mathcal{E}, \wedge \rangle$, we get the directed poset $J = I \cup I^*$ with $I^* = \{ \mathcal{J}^{*m} \}$ is a directed subset of I and \mathcal{J}^* its least upper bound in \mathcal{E} ,) and the (by now well-defined) direct system $(G_i, I, \langle p \rangle)$, $(G_i, I, \langle p \rangle)$ and $(G_i, I, \langle p \rangle)$ are equivalent.

The "proof" is rather obvious since $I \cup \{I^*\}$ is a cofinal sub-poset of J . Thus, $\text{I}^\wedge \text{m}(G_i, I, \langle p \rangle) \wedge \text{I}^\wedge \text{m}(G_i, I \cup \{I^*\}, \langle p \rangle)$ by remark 6, while $\text{I}^\wedge \text{m}(G_i, I \cup \{I^*\}, \langle p \rangle) \wedge \text{I}^\wedge \text{m}(G_i, I, \langle p \rangle)$ by remark 4. Still, the remark contains immediately the following corollary

on double-limits a special case of which (Cor. 2) is a theorem due to G. Grätzer*

Corollary 1L: Let $\langle I; \leq \rangle$ be a directed poset with directed subposets $\langle I_p; \leq \rangle, p \in P$. If $I_2 = \bigcup \{I_p; p \in P\} \neq I$ (denotations as in remark 7) is a directed subposet of \tilde{I} cofinal with I , then $\varinjlim (\varinjlim (G_p, I_p, \langle p \rangle), I_2, \langle P \rangle) \cong \varinjlim (G_p, I_p, \langle p \rangle)$ (where, as usual, $\varinjlim (G_p, I_p, \langle p \rangle) = G^{\wedge}$).

We should point out that this double-limit-theorem has been established in a categorical context.

Corollary 2: If $(G, I, \langle p \rangle)$ is a direct system and $I = \bigcup (I_p; p \in P)$ where $\langle I_p; \leq \rangle$ are directed subposets of $\langle I; \leq \rangle, \langle P; \leq \rangle$ is a directed poset and $p_1 \leq p_2$ implies $I_{p_1} \subseteq I_{p_2}$ then $(G', P, \langle p \rangle)$ (with $G'_p = G_{I_p}$ and $\langle p, q \rangle = \langle I_p \wedge I_q \rangle$) is a direct system such that $\varinjlim (G^{\wedge}, I, \langle p \rangle) \cong \varinjlim (G', P, \langle p \rangle)$.

A special instance of a direct limit (called ultra-product) has obtained such outstanding a significance that we decided to give it an extra section, §4.

§4. Introduction of reduced products; ultra-products and set-theoretical properties.

This presentation of reduced products is based on a little remark due to L. Fleischer [1] connecting up direct limits and ultra-products. Assume we are given a non-empty set I , a non-empty set $D \subseteq 2^I$ closed under finite intersection not containing \emptyset and a relational-system $G_i = \langle A_i; F, R \rangle$ of some fixed type r for every $i \in I$. Then $\langle D; \leq \rangle$, defined by $J \leq K \iff J \supseteq K$, is a directed poset which yields a direct system $(B, D, \langle \rho \rangle)$ in the following fashion: (i) If $j \in D$, then $\rho_j = \prod_{i \in j} \rho_i$ (ii) if $J \leq K$, then $\rho_j = \rho_k \circ \rho_{j \setminus k}$ is the projection mapping f to the restriction $f|_{A_j}$.

~~Def. 10~~; Using the notation just introduced, $\lim_{\leftarrow} (f_i, D, \langle \rho \rangle)$ is called the reduced product of the relational systems G_i $i \in I$, modulo D and is denoted by $TT_D(G_i; i \in I)$. (We will slightly modify this after theorem 3).

If we take the above direct system $(f_i, D, \langle \rho \rangle)$ and enlarge D to D^\wedge by adding to D every set $j^\wedge \in 2^I$ containing some $j \in D$, then we can define the direct system $(f_j, D^\wedge, \langle \rho \rangle)$ in exactly the same fashion as we defined $(f_i, D, \langle \rho \rangle)$. Since D is visibly cofinal with D^\wedge , remark 6 of the last section proves the next remark.

Remark 8: If the notations are as above, then

$$TT_D(G^\wedge; i \in I) \cong_{\sim} \prod_{D_1} (G_i; i \in I).$$

Thus, to study reduced products modulo D we can confine our attention to such $D \subseteq 2^I$, $D \neq \langle \emptyset \rangle$, that (i) $J^\wedge \in D$ implies

$J_1 \text{ fl } J_2 \in D$, (ii) $J_1 \in D$ and $J_2 \subseteq J_1$ implies $J_2 \in D$. Such sets $D \subseteq 2^I$ are known as "dual ideals" or "filters" of I . If we take the mere definition of $TT(G_{i \in I})$ as $\lim(H_D, <p)$ we get the carrier-set $U = \bar{D}$ where $U = \cup \{IT(A_j; j \in J); J \in D\}$ and the equivalence relation \equiv_D is defined by $f \equiv_D g$ if and only if $f|_E = g|_E$ for some $E \in D$ and $E \subseteq K \cup J$ (we assume $f \in Tr(A_j; j \in J)$, $g \in Tr(A_k; k \in K)$, $J, K \in D$). In particular fixed some $E \in D$, we conclude that all elements $f \in Tr(A_i; i \in I)$ with a fixed restriction $f|_E$ to $Tr(A_i; i \in E)$ are equivalent among each other and to $f|_E$. Thus, every equivalence class in U has a representative in $Tr(A_i; i \in I)$ and two elements $f, g \in Tr(A_i; i \in I)$ are equivalent if and only if $f|_E = g|_E$ for some $E \in D$. The set-theoretical structure of $Tr(A_i; i \in I) / \bar{D}$ reflects therefore in $Tr(A_i; i \in I) / \bar{D}$, and we end up with a description of $TT(G_{i \in I})$ on the carrier-set $Tr(A_i; i \in I) / \bar{D}$ instead of U provided we can catch the effect of the operations and relations on this simplified carrier-set. This latter task is, of course, a simple one after the foregoing discussion, and we sum it up in the following theorem:

Theorem 3: If $G_i, i \in I$, are relational systems of some fixed type T and D is a filter over I , then $f \equiv_D g = s f|_E = g|_E$ for some $E \in D$ is an equivalence relation on $Tr(A_i; i \in I)$.

Moreover, $B = \langle Tr(A_i; i \in I) / \bar{D}, F, R \rangle$ is (up to isomorphism) the relational system $TT^D(G_{i \in I})$ if (i) $f^y(n^1, \dots, n^y) = n^y$ is equivalent to $[i; f^y(h^1, \dots, h^y), (i) = h(i)], e \in D$ for all $h^j \in Tr(A_i; i \in I)$ and $n_i = (h_i]D$, (ii) $R^y(1^1, \dots, 1^y)$ holds if and only if $f^{isR}_y(h^1(i), \dots, h^y(i))$ holds true) $\in D$.

Def. 11: (i) If we use the notation $\hat{(G^{\wedge}ri^{\epsilon} x)}$ then we will (extending def. 10) assume it has the representation specified in theorem 3, unless stated differently.

(ii) $S(h_1, h_2) = \{i; h_1(i) = h_2(i)\}$ for $\hat{(G^{\wedge}ri^{\epsilon} x)}$ is called the common support of h_1 and h_2

(iii) $S_{\gamma}(h_1, \dots, h_m) = \{i; R_{\gamma}(h_1(i), \dots, h_m(i)) \text{ holds}\}$ is the support of R_{γ} with respect to h_1, \dots, h_m .

We can intuitively say that a filter classifies the subsets of I into two groups: n large ones¹¹ (those in D) and M small ones¹¹ (those in $I-D$). We identify any two elements in $\hat{(A_i; i \in I)}$ whose common support is M large, M thus creating the carrier-set of $IT_D(G_i; i \in I)$. The componentwise application of the fundamental operations f_{γ} is replaced by componentwise application on a M large¹¹ common support, and the validity of a relation is determined by the validity on a n largeⁿ support. This construction (and this is the governing idea which led to its success) will enable us to prove statements that are not necessarily valid for all algebraic structures under discussion but for ¹¹ nearly all¹¹ of them $\dots \dots M$ nearly all^M being a measure to be specified from case to case.

The construction obtains its major significance in the case of an ^{tf} ultra-filter D .^{f!} To establish the necessary tools, we depart to (distributive) lattices.

Def. 12: Let $\mathcal{L} = \langle L; V, A \rangle$ be a lattice. $D \subseteq L$ is called a dual ideal or filter if (1) $a, b \in D$ implies $a \wedge b \in D$, (2) $a \in D$ and $c \geq a$ implies $c \in D$, (3) $D \neq L$ (Thus, in case $L = 2^I$, $V = U$, $A = \text{fl}$, we get the notions introduced above). If D is a filter and $D \subseteq D_1$ ($D_1 = \text{filter}$) implies $D = D_1$, then we call D an ultra-filter. If D is a filter and $a \vee b \in D$ implies always $a \in D$ or $b \in D$ then D is called a prime filter.

(Ex. 10): (i) If $\mathcal{L} = \langle L; V, A \rangle$ is a lattice and $a \in L$ then $[a] = \{b; b \in L \text{ and } b \geq a\}$ is a filter, called the principal filter generated by a .

(ii) If $\mathcal{L} = \langle L; V, A \rangle$ is a lattice and $H \subseteq L$, then $[H] = \{b; b \in L \text{ and } b \wedge h_1 \wedge \dots \wedge h_n \in H \text{ for some } h_1, \dots, h_n \in H\}$ is a filter, called the filter generated by H. It is the smallest filter containing H .

The following is a very important and basic theorem due to M. Stone.

Theorem 4; If $\mathcal{L} = \langle L; V, A \rangle$ is a distributive lattice (i.e. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and, consequently, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$), then every filter I not containing an element a is contained in an ultra-filter of \mathcal{L} not containing a , and all ultra-filters of \mathcal{L} are prime.

proof: Zorn's lemma yields immediately an ultra-filter P containing I and excluding a . Assume $b, c \in P$ but $b \vee c \notin P$.

Then $[I \cup \{b\}] \wedge a$ and $[I \cup \{c\}] \wedge a$, i.e. (see example 10) $a \wedge i_1 \wedge b$ and $a \wedge i_2 \wedge c$ for some $i_1, i_2 \in I$. Thus,

$$a = a \vee a \wedge (i \wedge Ab) \vee (i_2 \wedge Ac) = (ij \wedge Vij) \wedge (j' \wedge vc) \wedge (b \vee i_2) \wedge (b \vee c) \in P.$$

This contradiction proves the theorem* q. e. d.

Thus, in distributive lattices all ultra-filters are prime. The converse is not true as the simple case of chains shows. However, we get the equivalence if we progress to Boolean algebras. Before showing it we derive a little preliminary remark.

Remark 9; If $B = \langle B; V, A, 0, 1 \rangle$ is a Boolean algebra, then $D \subseteq B$ is an ultra-filter (of course, in $\langle B; V, A \rangle$ if and only if for all $a \in B$ the relation $a \in D$ is equivalent to $a' \notin D$).

proof; If D is an ultra-filter and $a, a' \notin D$, then $a \vee a' = 1 \in D$ yields a contradiction against the fact that ultra-filters are prime in distributive lattices, hence Boolean algebras. If $a, a' \in D$ then $a \wedge a' = 0 \in D$ yields $D = B$, another contradiction. Thus, $a \in D$ is equivalent to $a' \notin D$.

Vice versa, if D were not an ultra-filter, then $D \subsetneq D_1$ were true for some ultra-filter D_1 . If $a \in D_1 \setminus D$, then $a' \notin D$ by assumption? thus, $a, a' \in D_1$ implies $a \wedge a' = 0 \notin D_1$ which is impossible. q. e. d.

Corollary: If $B = \langle B; V, A, ', 0, 1 \rangle$ is a Boolean algebra then ultra-filters and primefilters coincide.

proof; We know already that ultra-filters are prime filters. Assume, vice versa, that p is a prime filter and $a \notin p$; then $1 = a \vee a' \in p$ implies that $a' \in p$. Similarly, $a \in p$ implies $a' \notin p$. q. e. d.

Stone's theorem has an immediate, elementary but useful implication whose statement requires our recollection of the following concept:

Def. 13: Let A be a non-empty set and S a subset of 2^A with the property that every finite intersection of elements in S is non-empty. Then S is said to have the "finite intersection property".

Corollary 1: If A is a non-empty set and $S \subseteq 2^A$ has the finite intersection property, then $S_f = \{B; B \subseteq A \text{ and } B \supseteq C \text{ for some finite intersection } C \text{ of elements in } S\}$ is a filter.

Together with Stone's theorem we get the next corollary:

Corollary 2: If $S \subseteq 2^A$ has the finite intersection property, then S is contained in an ultra-filter S_m of the Boolean

algebra $\langle 2^A; \cup, \cap, ', 0, 1 \rangle$.

As remarked before, the reduced products obtain particular significance if the filters involved are ultra-filters. In this case, our intuitive description via $^M \text{large}^M$ and $^M \text{small}^{11}$ sets gets also a much more precise form as expressed in the next remark.

Remark 10: If I is a non-empty set then there is a 1-1 correspondence between the set M of all $\{0,1\}$ -valued, finitely-additive measures on I and the ultra-filters over I .

proof: If $D \subseteq 2^I$ is an ultra-filter, then $\mu_D: 2^I \rightarrow \{0,1\}$, defined by $\mu_D(y) = \begin{cases} 1 & \text{if } y \in D \\ 0 & \text{if } y \notin D \end{cases}$, is an element of M .

If $\mathcal{L} \in M$, then $D_\mu = \{y; y \in 2^I \text{ and } \sum_{j \in I} j(y) = 1\}$ is an ultra-filter over I . Moreover: $D^\wedge = D$ and $\hat{} = \sum_{j \in I} j$ for all ultra-filters D and all elements $\sum_{j \in I} j$ in M . The details are left as exercise. q. e. d.

Def. 14: If $D \subset 2^I$ is an ultra-filter over I , then $\hat{} (G \wedge i \in I)$ is called an ultra-product. If all $G \wedge i = G$, we write $T_D(\sum_{j \in I} j) = G_D^I$ and call the resulting relational system an ultra-power. The remaining results in this section can be found in Kochen's paper on ultra-products [1].

Theorem 5: Every relational system G can be embedded in every of its ultra-powers. More precisely: If G , the set I and the ultra-filter D over I are given, then

$$j = j_{G,D,I} : G \longrightarrow G_D^I,$$

defined by $j(a) = (a)_{\sum_{j \in I} j}$ is a monomorphism.

proof: If $a = b$ then, clearly, $j(a) = j(b)$; so the mapping is well-defined. If $a \neq b$ then $S(j(a), j(b)) = \{f \in \sum_{j \in I} j \mid f(a) \neq f(b)\}$, i.e. $j(a) \neq j(b)$; so the mapping is 1-1. The homomorphism-properties are verified in a similar fashion. q. e. d.

We can use this theorem to find a first instance in which our new construction yields no new result.

Corollary: If G is a finite relational system (i.e. $\aleph_0 < |A| < \aleph_1$) then $G_D^I \wedge G$ holds for every non-empty I and ultra-filter D over I .

proof: If $f \in A^\wedge$, $A = \{a^1, \dots, a_n\}$ and $A_i = \{i; f(i) = a_i\}$ for every $i \in 1, \dots, n$, then $I = A_1 \cup \dots \cup A_n$; thus, since D is prime and $I \in D$, $A_{i_0} \in D$ for some $1 \leq i_0 \leq n$. We conclude that

$$S(f, (a_i)_{i \in I}) = A_{i_0} \in D, \text{ i.e. } f = (a_i)_{i \in I} = j_{D, I} \circ \pi_{i_0}^{-1}(a_{i_0}).$$

Hence, $j_{D, I}$ (as introduced in theorem 5) is onto and there-

fore an isomorphism. q. e. d.

The following remarks are results in the same direction - showing the limits of the ultraproduct-construction.

Remark 11: If D is a principal ultra-filter over I then $\prod_D (G_i; i \in I) \cong G_{i_0}$ for some $i_0 \in I$.

proof: Since, as one easily verifies, the principal ultra-filters over I are exactly the filters $[i_0)$ with $i_0 \in I$, we conclude that $D = [i_0)$ for some $i_0 \in I$. Thus $r = g$ is equivalent to $f(i_0) = g(i_0)$, and $\text{cpz } \prod_D (A_i; i \in I) \rightarrow A_{i_0}$ mapping f to $f(i_0)$ is an isomorphism. q. e. d.

Corollary 1: $G_D^I \cong 1$ for every principal ultra-filter over I .

Corollary 2: If I is a finite non-empty set and D an ultra-filter then $\prod_D (G_i; i \in I) \cong G_{i_0}$ for some $i_0 \in I$.

proof: If I is finite, then every ultra-filter is principal, q. e. d.

The next remark (whose proof is essentially due to Halmos) is a cardinal number-theoretical counterpart to the corollary to theorem 5.

Remark 12: If $I = \{1, 2, 3, \dots, n, \dots\}$ is the set of natural numbers and G_n a finite relational system of fixed type r for every n such that the set $\{n; |A_n| \leq m\}$ is finite for every natural number m , then $\{n \in I \mid \text{for every } D \text{ non-principal ultra-filter over } I, \dots\}$

proof: Clearly $|\{n \in I \mid \dots\}| \leq |\{n \in I \mid \dots\}|$. Thus, we are done if we can construct an injection $\langle p \rangle: \{0,1\}^{\omega} \rightarrow \prod_{n \in I} A_n$ where $\{0,1\}^{\omega}$ is the set of all countable $\{0,1\}$ -valued sequences s (or equivalently, $\{0,1\}^{\omega}$ consists of all functions $s: \omega \rightarrow \{0,1\}$) which, of course, has cardinality 2^{ω} .

Due to our assumptions we can assume that $|A_1| \leq |A_2| \leq \dots \leq |A_n| \leq \dots$ and $\lim_{n \rightarrow \omega} |A_n| = \aleph_0$. Thus, if $g(n)$ is the unique natural number satisfying $g(n) = 1$ in case $|A_n| = 1$ and $2^{g(n)} \leq |A_n| < 2^{g(n)+1}$ in case $|A_n| \geq 2$, we conclude that (i) $g(n)$ is increasing and $\lim_{n \rightarrow \omega} g(n) = \omega$ and (ii) there exists an injection $F_n: \{0,1\}^{g(n)} \rightarrow A_n$

for every natural number n (if $\{0,1\}^{g(n)}$ is the set of all functions from $\{1, 2, \dots, g(n)\}$ to $\{0,1\}$). Let us agree to denote

by F_n the restriction of $s \in \{0,1\}^{\omega}$ to $\{1, \dots, g(n)\}$. Then $F_n(s) = F_n(s \upharpoonright g(n))$ is well-defined by $F_n(s) = F_n(s \upharpoonright g(n))$,

and we claim that $\langle p \rangle = \prod_{n \in I} F_n: \{0,1\}^{\omega} \rightarrow \prod_{n \in I} A_n$ defined by

$\langle p \rangle(s) = \prod_{n \in I} F_n(s \upharpoonright g(n))$ is an injection. To see this, let $s \neq t$, $s, t \in \{0,1\}^{\omega}$

Then, for large enough n , $s \upharpoonright g(n) \neq t \upharpoonright g(n)$ hence, for large enough n

(say, for all $n \geq n_0$), $F_n(s \upharpoonright g(n)) \neq F_n(t \upharpoonright g(n))$. Consequently $\langle p \rangle(s) \neq \langle p \rangle(t)$

holds for all $n \geq n_0$, i.e. $\langle p \rangle(s) \neq \langle p \rangle(t)$. Since

D is a non-principal ultra-filter, it cannot contain finite sets;

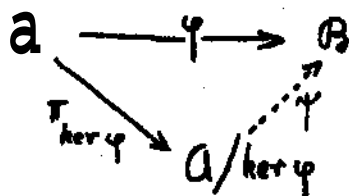
thus, $\langle p \rangle(s) \neq \langle p \rangle(t) \in D$, i.e. $\langle p \rangle(s) \neq \langle p \rangle(t)$ or $\langle p \rangle(s) \neq \langle p \rangle(t)$. q. e. d.

§5. The Homomorphism - Theorem and the Isomorphism - Theorems.

The following is a fundamental theorem which ties up congruences and homomorphisms in universal algebras.

Theorem 6 (Homomorphism-Theorem):

Let G, B be algebras of type r and $\langle p: G \rightarrow B \rangle$ a homomorphism then there exists a unique monomorphism $\theta: G/\ker \langle p \rangle \rightarrow B$ such that the diagram

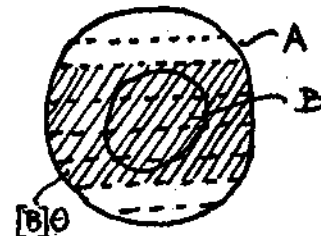


is commutative. In particular: $\langle p(G) \rangle \cong G/\ker p$.

proof: Of course, we have to define $\theta([a]_{\ker \langle p \rangle}) = \langle p(a) \rangle$. Since $a \equiv b \pmod{\ker \langle p \rangle}$ is equivalent to $\langle p(a) \rangle = \langle p(b) \rangle$, the mapping θ is well-defined and 1-1. The homomorphism-property is easily verified, q. e. d.

Corollary: Every homomorphism is a product of first an epimorphism and then a monomorphism.

The homomorphism-theorem occurs in an overwhelming number of situations; in particular, it is underlying the succeeding isomorphism-theorems sometimes referred to as ^H Lasker-Noether isomorphism theorems.ⁿ We choose a subalgebra $B = \langle B; F \rangle$ of G a $\langle A; F \rangle$, a congruence relation θ on G and the closure $[B]_{\theta}$ of B modulo θ in A (see diagram). The dotted lines in the diagram indicate θ . Evidently there is a 1-1 correspondence between the blocks of θ



θ
J3

(the restriction of θ to B) and $\theta|_B$ is the whole and only content of the so-called first isomorphism-theorem is the fact that that 1-1 correspondence constitutes indeed an isomorphism of the relevant algebras.

Theorem 7 (1st isomorphism theorem):

If $B = \langle B; F \rangle$ is a subalgebra of $G = \langle A; F \rangle$ and θ is a congruence relation of G , then $B/\theta = \langle B/\theta; F \rangle$ is a subalgebra of G and $B/\theta \cong B/\theta|_B$.

proof: The fact that B/θ is closed under the operations of F is easily verified; so B/θ is a subalgebra. If $\varphi: B \rightarrow B/\theta|_B$ is defined by $\varphi(b) = [b]_{\theta|_B}$ then it is trivially an epimorphism. What about $\ker \varphi$? $b_1 \sim b_2 \pmod{\ker \varphi}$ is equivalent to $(\varphi(b_1) = \varphi(b_2))$, i.e. to $b_1 \sim b_2 \pmod{B/\theta}$ hence to $b_1 \sim b_2 \pmod{B/\theta}$. Thus, $\ker \varphi = B/\theta$, and the homomorphism-theorem shows that $B/\theta|_B \cong B/\theta$ q. e. d.

Corollary: If $M = \langle H; \cdot, \sim^1, 1 \rangle$ is a subgroup and $N = \langle N; \cdot, \sim^1, 1 \rangle$ is a normal subgroup of the group $Q = \langle G; *, \sim^1, 1 \rangle$, then $M \cdot N/N; \cdot, \sim^1 \cong M/(M \cap N) \cdot N; \cdot, \sim^1$.

Similarly simple is the second isomorphism-theorem. To prepare it we again set out with a universal algebra $G = \langle A; F \rangle$, but choose two congruence relations $\theta, \phi \in C(G)$. Clearly, $\theta|_A \cdot \phi|_A$ mapping $[a]_{\theta}$ to $[a]_{\phi}$ is an epimorphism provided it is well-defined. To be well-defined, $[a]_{\theta} = [b]_{\theta}$ must imply that $[a]_{\phi} = [b]_{\phi}$, i.e. $\theta \cap \phi$ is a necessary and sufficient condition.

Remark 13: If G is a universal algebra and $\theta, \phi \in C(G)$, then $\tau_{\theta, \phi} : A/\theta \rightarrow A/\phi$, defined by $\tau_{\theta, \phi}(t^a 19) = [a]_\phi$, is an epimorphism if and only if $\theta \leq \phi$.

Def. 15: The kernel of $\tau_{\theta, \phi}$ in remark 13 is denoted by θ/ϕ and is a congruence relation on G/θ .

Remark 14: If $\theta \leq \phi$ are congruence relations on G , then $[a]_\theta \in [b]_\phi$ (θ/ϕ) holds if and only if $a \equiv b(\phi)$.

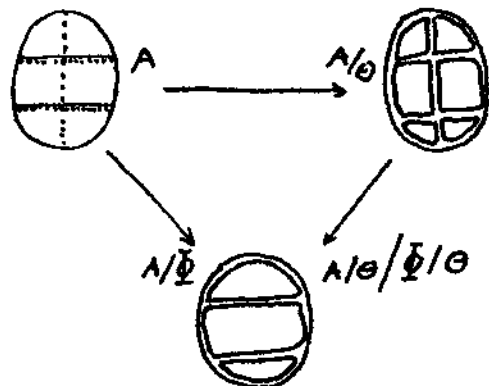
The last remark is, of course, an immediate corollary to the definition of θ/ϕ . The so-called second isomorphism-theorem is now not more than a re-statement of the homomorphism-theorem for the special epimorphism $\tau_{\theta, \phi}$.

Theorem 8 (2nd isomorphism-theorem).

If θ, ϕ are congruence relations on the algebra G and $\theta \leq \phi$, then $G/\theta \sim G/\theta A/\phi$.

Corollary: If $J_{N_i} = \langle N_i, \cdot, \cdot^{-1}, 1 \rangle$, $i = 1, 2$, are normal subgroups of $Q = \langle G; \cdot, \cdot^{-1}, 1 \rangle$ and $N_2 \subset N_1$, then $\langle G/N_2; \cdot, \cdot^{-1}, 1 \rangle \sim \langle G/N_1/N_2/N_1; \cdot, \cdot^{-1}, 1 \rangle$.

To realize the simple content of the second isomorphism-theorem, we visualize it via diagrams: θ is represented by the dotted, ϕ by the solid lines.



Thus, vaguely speaking, the 2nd isomorphism theorem states that it does not matter whether we pass from G to G/θ immediately, or whether we pass to it via any algebra G/θ with $\theta \leq \phi$.

Thus, if we go down to the bare essentials, theorem 8 conveys the "surprising" information that, if $G \triangleleft \theta_1 \triangleleft \theta_2 \triangleleft \dots \triangleleft \theta_n$ are congruence relations on G , then, in

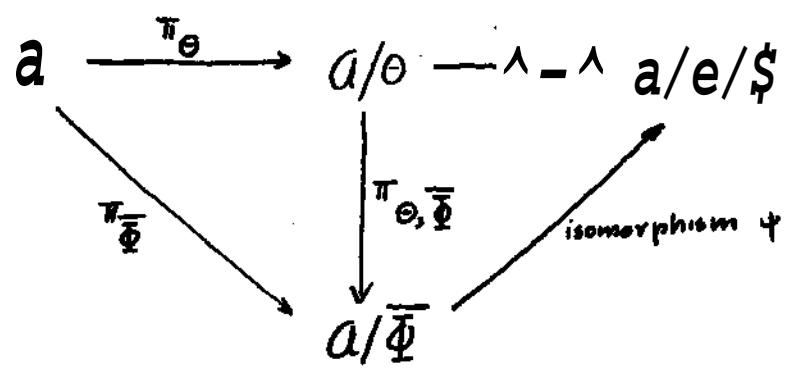
$$\begin{array}{c} \pi_{\theta_1} \circ \pi_{\theta_2} \circ \dots \circ \pi_{\theta_n} \\ G \xrightarrow{\pi_{\theta_1}} G/\theta_1 \xrightarrow{\pi_{\theta_2}} G/\theta_2 \xrightarrow{\pi_{\theta_3}} \dots \xrightarrow{\pi_{\theta_n}} G/\theta_n \end{array}$$

the mappings π_{θ_n} and $\pi_{\theta_{n-1}, \theta_n} \circ \pi_{\theta_{n-2}, \theta_{n-1}} \circ \dots \circ \pi_{\theta_1, \theta_2} \circ \pi_{\theta_1}$ are equal.

Before we proceed we ought to mention the following little result showing the role of the congruence relations θ/θ'

Remark 15: If θ is a congruence relation on G , then $\pi_{\theta}: [G] \rightarrow [G/\theta] = \{[x] \mid x \in G \text{ and } x \sim \theta\} \rightarrow C(G/\theta)$, defined by $\pi_{\theta}([x]) = [x]_{\theta}$ is a lattice-isomorphism $\pi_{\theta}: \langle [G]; V, A \rangle \rightarrow \langle C(G/\theta); V, A \rangle$.

proof: We leave the verification that π_{θ} is a homomorphism as exercise. To see that π_{θ} is onto we select $[x]_{\theta} \in C(G/\theta)$ and combine the two canonical projections $\pi_{\theta}: G \rightarrow G/\theta$ and $\pi_{\theta'}: G/\theta \rightarrow G/\theta'$ to $\pi_{\theta'} \circ \pi_{\theta}: G \rightarrow G/\theta'$ an epimorphism with kernel, say, θ' . Since $\ker(\pi_{\theta'} \circ \pi_{\theta})$ is evidently larger or equal to $\ker \pi_{\theta} = \theta$, we conclude that $\theta \subseteq \theta'$, i.e. $\theta' \in [0]$. Moreover we have the following commutative diagram:



The isomorphism θ (assured by the homomorphism-theorem) maps $[a]\bar{\theta}$ to $[[a]G] \langle \theta \rangle$. Hence, $\ker(\pi \circ \theta) = \ker(\theta \circ \pi) = \ker \pi \circ \theta^{-1}$ i.e. $\bar{\theta}/\theta = \langle \theta \rangle$ thus, $\pi \circ \theta^{-1} = \pi$ which establishes the onto-part. π is 1-1, for $a \neq b \in G$ and $a \neq b$ implies, say, $a \in G_2 \setminus G_1$; i.e. $\ker(\pi \circ \theta^{-1}) \cap \ker \pi = \emptyset$. Thus, $\pi \circ \theta^{-1} = \pi \circ \theta^{-1} \circ \theta = \pi$. In particular, $\pi \circ \theta^{-1} \circ \theta = \pi$.

Corollary; If $Q = \langle G; -, \sim^1, 1 \rangle$ is a group with normal subgroup $T1 = \langle N; *, \sim^1, 1 \rangle$, then there is a lattice-isomorphism from the lattice of all normal subgroups of G containing N to the lattice of all normal subgroups of G/N .

Slightly more complex, though equally elementary, the final isomorphism-theorem is alternately referred to as 3rd isomorphism-theorem, respectively Zassenhaus¹ lemma. In case of groups, the result is due to Zassenhaus; its formulation for universal algebras seems part of the 'folklore'¹¹ on the subject matter (see Cohn [1], Grätzer[1], Wyler [1]). To state it we need to consider one more congruence-relation.

Def. 16: If B is a subset of the non-empty set A and ρ, τ are binary relations on A, B , resp., i.e. $\rho \in R_2(A)$, $\tau \in R_2(B)$, then $\rho \circ \theta \circ \tau \in R_2(A)$ is defined by $(a, b) \in \rho \circ \theta \circ \tau = : (a, c) \in \rho$, $(c, d) \in \tau$, $(d, b) \in \theta$ for some $c, d \in B$ (thus, compared with [2], the applicability of θ has been slightly widened).

Remark 16: If IB is a subalgebra of G , $a \in C(B)$, $\theta \in C(G)$ and $\#_B \langle \tau \rangle$, then $\rho \circ \theta \circ \tau \in R_2(A)$ is a congruence relation on $[B]A$.

proof: Reflexivity and symmetry are immediately clear. So let

us assume $a \equiv b \pmod{[B]_0}$ and $b \equiv c \pmod{[B]_0}$,
 i.e. $a - b \in [B]_0$ and $b - c \in [B]_0$ with $a, b, c \in B$, $c_i \wedge d_i \in B$.

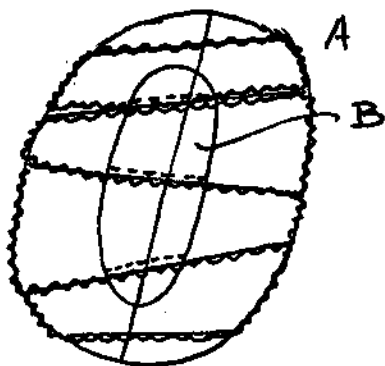
In particular, $c_1 - d_1 \in [B]_0$, i.e. $c_1 \wedge d_1 \in [B]_0$; since $J \geq 0_B$,
 we conclude that $c_1 \equiv d_1 \pmod{[B]_0}$. Thus, $a - c \in [B]_0$ shows $a \equiv c$

(transitivity is established. To check the

substitution property, let $a_i \equiv b_i \pmod{[B]_0}$,
 i.e. $a_i - b_i \in [B]_0$, $1 \leq i \leq n$, with $a_i, b_i \in B$, $c_i \wedge d_i \in B$. Then,
 since B is a subalgebra, $f(a_1, \dots, a_n) \in B$ and $f(d_1, \dots, d_n) \in B$

holds for all $f \in F$; so $f(a_1, \dots, a_n) - f(d_1, \dots, d_n) \in [B]_0$.
 This shows that $f(a_1, \dots, a_n) \equiv f(d_1, \dots, d_n) \pmod{[B]_0}$.
 q. e. d.

Appendix to remark 16: It is helpful to realize that the congruence-
 blocks of $\mathcal{A} \pmod{[B]_0}$ are obtained by fixing some θ -block
 and adjoining all intersecting 0-blocks. Thus there is a natural
 1-1 correspondence between the θ -blocks and $\mathcal{A} \pmod{[B]_0}$ -
 blocks, a correspondence of the type that led to the first isomor-
 phism theorem (see diagram).



The solid lines represent θ , the
 dotted lines $\mathcal{A} \pmod{[B]_0}$. If we squiggle
 the $\mathcal{A} \pmod{[B]_0}$ -blocks, then the
 1-1 correspondence is quite clearly
 reflected: Every θ -block determines
 uniquely a squiggled block. So one

is kind of forced to write down the next remark.

Remark 17: If θ is a subalgebra of G , ϕ a congruence relation of H , ψ a congruence relation of G such that $\theta \leq \psi$, then

$$[\theta/\theta]_{[\psi]\theta} \circ \phi \circ \theta_{[\psi]\theta} \cong \theta/\psi.$$

proof: We define $\langle \varphi: \theta \rightarrow [\psi]\theta / \theta_{[\psi]\theta} \cong \theta/\psi \rangle$ by $\varphi(b) =$

$[\psi]\theta / \theta_{[\psi]\theta} \circ \phi \circ \theta_{[\psi]\theta}$; $\langle \varphi \rangle$ is quite clearly an epimorphism with kernel ϕ . The homomorphism theorem then settles the matter, q. e. d.

Remark 17 is really the meat of the 3rd isomorphism-theorem which follows next.

Theorem 9 (3rd isomorphism-theorem or Zassenhaus¹ lemma): Let θ and ψ be subalgebras of G such that $\theta \leq \psi$. Then $\langle \theta/\psi \rangle$ is a subalgebra of G . If $\theta \in C(\theta)$, $\psi \in C(\psi)$ and

$$\psi = \theta_{\theta/\psi} \vee \theta_{\theta/\psi} \in C(\theta/\psi) \quad \text{then} \quad \theta_{\theta/\psi} \wedge \psi \vee \theta_{\theta/\psi} \wedge \psi \quad \text{and} \\ [\theta/\psi]_{\theta/\psi} \circ \psi \circ \theta_{\theta/\psi} \cong \theta/\psi \cong [\theta/\psi]_{\theta/\psi} \circ \psi \circ \theta_{\theta/\psi}$$

proof: By remark 17 θ/ψ is isomorphic to each of the other two algebras. q. e. d.

An immediate corollary, sometimes called "Zassenhaus¹ lemma" itself, is the following:

Corollary 1: Let θ and ψ be subalgebras of G with $\theta \leq \psi$. If $C(\theta/\psi)$ is commutative (i.e. $\theta \langle X \rangle \theta = \theta \langle X \rangle \theta$ for all $\theta, X \in C(\theta/\psi)$) and $\theta \in C(\theta), \psi \in C(\psi)$, then,

$$[\theta/\psi]_{\theta/\psi} \circ \psi \circ \theta_{\theta/\psi} \cong \theta/\psi \cong [\theta/\psi]_{\theta/\psi} \circ \psi \circ \theta_{\theta/\psi}$$

Proof: Since $* = 9^{\wedge} V *_{\text{Diffi}} = G^{\wedge} o *_{\text{DfiE}}$ (due to the commutativity of $C(\text{flDe})$; see §2, following def. 4) we conclude

that $e_{[DnE]0} \ll *_{DnE} \ll 9_{[DnE]9} = e_{[DnE]} e^{\circ e} DnE^{\circ} *_{DDE} \circ \theta_{[DnE]0} =$

$\theta_{[DnE]0} \circ \psi \circ \theta_{[DnE]0}$. Similarly, $\theta_{[DnE]0} \circ \psi \circ \theta_{[DnE]0} =$

$\theta_{[DnE]0} \circ \psi \circ \theta_{[DnE]0}$. So the $i^{30111017?11 \wedge 3111}$ follows from theorem 17. q. e. dL

Corollary 2 (Zassenhaus^r lemma for groups):

Let Q be a group with subgroups H_1 and H_2 such that H_1 is a normal subgroup of $H_1 H_2$ is a normal subgroup of M^{\wedge} . Then

$$\langle N_1(H_1 \cap H_2) / N_1(H_1 \cap H_2); \cdot, {}^{-1}, 1 \rangle \cong \langle H_1 \cap H_2 / (H_1 \cap H_2) \cdot (H_2 \cap N_1); \cdot, {}^{-1}, 1 \rangle \cong$$

$$\langle N_2 \cdot (H_2 \cap H_1) / N_2 \cdot (N_1 \cap H_2); \cdot, {}^{-1}, 1 \rangle.$$

16« The Jordan-Hölder-Schreier-Theory.

The task to decompose algebras of all kinds into simpler components (be it via normal chains, direct sums, subdirect sums...) is one that keeps coming up in all branches of mathematics. So we shall engage in a discussion of at least some of the most important results known on the subject matter in this and the next section.

Def» 17; If \mathfrak{f}_t is a subalgebra of an algebra G , then $(G_0, G_1, \dots, G_n, \theta_0, \theta_1, \dots, \theta_n)$ is a normal chain from \mathfrak{f}_t to G modulo θ_i if

(i) all G_i are subalgebras of G

(ii) θ_i is a congruence relation on G_i such that $[A_{i-1}]_{\theta_i} = A_{i-1} = [a_{i-1}]_{\theta_i}$ for every $i = 1, \dots, n$ and $a_{i-1} \in A_{i-1}$.

The algebras G_i/θ_i constitute the so-called factors of the normal chain.

If $T_1 = (G_0, G_1, \dots, G_n; \theta_0, \dots, \theta_n)$ and $T_2 = (G_0, \dots, \mathfrak{f}_t, \theta_0, \dots, \theta_m)$ are normal chains with $\theta_0 = \theta_0$, then $T_1 + T_2$ denotes the normal chain $(G_0, \dots, G_n, \mathfrak{f}_t, \dots, G_n; \theta_0, \dots, \theta_n, \theta_0, \dots, \theta_m)$.

Def» 18; If $C = (G_0, \dots, G_i, G_{i+1}, \dots, G_n; \theta_0, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n)$ is a normal chain from \mathfrak{f}_t to G and $C' = (G_0, \dots, G_{i-1}, G_i, \dots, G_n; \theta_0, \dots, \theta_{i-1}, \theta_i, \dots, \theta_n)$ is a normal chain from G_{i-1} to G_i such that (i) $[A_{i-1}]_{\theta_i} = A_{i-1}$ (ii) $\theta_{i+1} \leq \theta_i$ for all $j = 0, \dots, m-1$ (C_{i+1} is then called a strongly normal chain with respect to A_{i+1}), then $R = (G_0, \dots, G_i, G_{i+1}, \dots, G_n; \theta_0, \dots, \theta_n)$ is

called a refinement of the chain C (and is, of course, again a normal chain).

Def. 19: The normal chains $(S = G_0, \dots, G_n = G; \theta_0, \dots, \theta_n)$ and $(C = B_0, \dots, B_m = G; \phi_0, \dots, \phi_m)$ are isomorphic if $n = m$ and there is a permutation π of $\{1, \dots, n\}$ such that $G_i / \theta_i \cong B_{\pi(i)} / \phi_{\pi(i)}$.

The three definitions just given contain the basic concepts underlying the so-called Jordan-Hölder-Schreier theorems. It is a useful exercise to rewrite them in the language of, e.g., groups. We should point out that $G_i / \theta_i \cong B_{\pi(i)} / \phi_{\pi(i)}$ in def. 18 unless we have the trivial "refinement" $(G, \theta_0 = \theta_1 = \dots = \theta_n = \theta; \theta_0 / \psi, \dots, \theta_n / \psi)$. The next remark contains some information on refinements.

Remark 18; (1) If $T = (C = G_0, \dots, G_n = G; \theta_0, \dots, \theta_n)$ is a strongly normal chain from θ to G with respect to $0 \in C(G)$ (see def. 18), then $T_\psi = (\mathcal{E} / \theta_\psi = G_0 / \theta_{\psi_0}, \dots, G_n / \theta_{\psi_n} = G / \psi; \theta_0 / \psi_{\theta_0}, \dots, \theta_n / \psi)$ is a normal chain from $\mathcal{E} / \theta_\psi$ to G / ψ .

(2) Vice versa: If θ is a congruence relation on G and $T = (B_0, \dots, B_n = G / \theta; \theta_0, \dots, \theta_n)$ is a normal chain from B_0 to G / θ then $T_2 = (B_0, \dots, B_n = G; \theta_0, \dots, \theta_n)$ is a strongly normal chain from B_0 to G if (1) $\theta \cap \theta_j = \langle \theta_j^{-1}(B_j); \theta \rangle$,

- (2) θ_0 is the kernel of $\varphi: B_0 \rightarrow G / \theta$ mapping b to $[b]_\theta$.

Moreover: $T_{\theta_0} = T$ and $T = T_{\theta_0}$ holds for all chains T .

Corollary: If \mathfrak{A} is a subalgebra of G and θ a congruence relation of G , then there is a one-to-one correspondence between the normal chains from $\mathfrak{A}/\theta_{\mathfrak{A}}$ to G/θ and the strongly normal chains from \mathfrak{A} to G (with respect to θ).

Note: Would we (what we don't) assume the elements of category theory, then we could simply say that the 1-1 correspondence is achieved by the pull-back with respect to θ .

proof of remark 18:

(1) Since $G/\theta_{\mathfrak{A}} = \text{Tr.}(G)$, it is, of course, clear that $\mathfrak{A}/\theta_{\mathfrak{A}} \subseteq G/\theta_{\mathfrak{A}} \subseteq \dots \subseteq G/\theta$. Furthermore, if $a_{i-1} \in \mathfrak{A}_{i-1}$, i.e. $[a_{i-1}]_{\mathfrak{A}_{i-1}} \in \mathfrak{A}_{i-1}/\theta_{\mathfrak{A}_{i-1}}$, then $[a_{i-1}]_{\mathfrak{A}_{i-1}} \vee \theta_{\mathfrak{A}_{i-1}} = [b]_{\mathfrak{A}_{i-1}}/\theta_{\mathfrak{A}_{i-1}}$ for $b \in \mathfrak{A}_{i-1}/\theta_{\mathfrak{A}_{i-1}}$ (i.e. $b = [b]_{\mathfrak{A}_{i-1}}$ for some $b \in \mathfrak{A}_{i-1}$) is equivalent to $a_{i-1} \vee \theta_{\mathfrak{A}_{i-1}} b_i(0_i)$, i.e. $b_i \in \mathfrak{A}_{i-1}/\theta_{\mathfrak{A}_{i-1}}$. Hence, $[[a_{i-1}]_{\mathfrak{A}_{i-1}}]_{\mathfrak{A}_{i-1}} \vee \theta_{\mathfrak{A}_{i-1}} = [A_{i-1} \wedge \wedge \wedge I \ 0 \ \wedge]_{\mathfrak{A}_{i-1}} = \mathfrak{A}_{i-1}/\theta_{\mathfrak{A}_{i-1}}$.

This establishes part (1). Part (2) is established in a similar fashion. q. e. d.

We can now prove what is most commonly known as Schreier's Refinement Theorem¹¹, although in some cases (for obvious reasons) Zassenhaus' name is attached to it.

Theorem 10 (Schreier's Refinement Theorem);

Let G be an algebra such that $C(\mathfrak{A})$ is commutative (with respect to θ) for every subalgebra \mathfrak{A} . Then any two normal chains from \mathfrak{A} to G have isomorphic refinements.

proof: Let (1) = $(e = G_0 \wedge G_1 \wedge \dots \wedge G_n = G; \theta_0, \theta_1, \dots, \theta_n)$ and

$$(2) = (e = fB \circ \theta_{i-1} \dots \theta_i, \theta_i = G; \phi_0, \dots, \phi_m)$$

be the two normal chains in question. Then we consider the chain

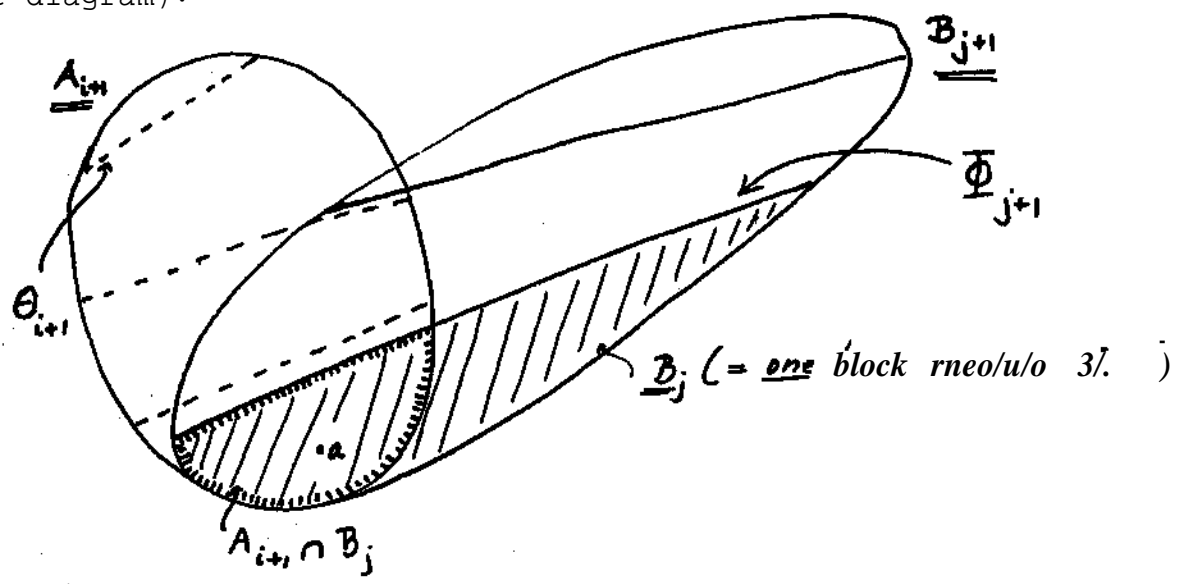
$$(li) = (G_i = [G_{i+1} \cap \theta_i] \theta_{i+1}, [G_{i+1} \cap \theta_1] \theta_{i+1}, \dots, [a_{i+1} \cap B_j] \theta_{i+1}, \dots, [G_{i+1} \cap B_n] \theta_{i+1})$$

$$G_{i+1} \theta_{i+1} \circ \phi_0 \circ \theta_{i+1}, \dots, \theta_{i+1} \circ \phi_j \circ \theta_{i+1}, \dots, \theta_{i+1} \circ \phi_m \circ \theta_{i+1}$$

where $\theta_{i+1} \circ \phi_j \circ \theta_{i+1} = \overset{\vee}{A} \bullet \bullet$ is an abbreviation for the more precise notation $\overset{\vee}{A} \bullet \bullet$ 3-5 D

$$\theta_{i+1} \mid \quad j \mid A_{i+1} \cap B_j \circ \theta_{i+1} \mid [A_{i+1} \cap B_j] \theta_{i+1}$$

Since both the carriers of the algebras $[G_{i+1} \cap \theta_i] \theta_{i+1}$ and the blocks of the corresponding congruences $\overset{\vee}{A}_{i,j}$ are (by definition) unions of θ_{i+1} -blocks, we know that (li) is a strongly normal chain from G_i to G_{i+1} once we know that it is a normal chain. To show the latter point we merely need to show that $[a] / \overset{\vee}{A}_{i,j+1} = [A_{i+1} \cap B_j] \theta_{i+1}$ for all $a \in A_{i+1} \cap B_j$. But this is immediately seen (see diagram):



$$[a]_{i,j+1}^\chi = \{x; x \in [A_{i+1} \cap B_{j+1}]^{\theta_{i+1}} \text{ and } a^{\theta_{i+1}} c^{\theta_{j+1}} d^{\theta_{i+1}} x$$

for some $c, d \in A_{i+1} \cap B_{j+1}\}$,

while

$$[A_{i+1} \cap B_j]^{\theta_{i+1}} = \{x; x \in A_{i+1} \text{ and } f \theta_{i+1} x \text{ for some } f \in A_{i+1} \cap B_j\}.$$

$[A_{i+1} \cap B_j]^{\theta_{i+1}} \in [a]_{i,j}^{\chi}$ is immediately seen since $f \theta_{i+1} x$ and $f \in A_{i+1} \cap B_j$ implies $a \theta_{i+1} a^{*_{j+1}} f \in A_{i+1} \cap B_j$.

Vice versa: $a \theta_{i+1} c^{*_{j+1}} d \theta_{i+1} x$ and $c, d \in A_{i+1} \cap B_{j+1}$ implies $a^{*_{j+1}} c \theta_{i+1} d \in A_{i+1} \cap B_{j+1}$ (since $\theta_{i+1} \circ \theta_{j+1} = \theta_{j+1} \circ \theta_{i+1}$ on $A_{i+1} \cap B_{j+1}$). But then $c^* \in A_{i+1} \cap B_j$ (since B_j is one block modulo θ_{j+1}) and $a^{*_{j+1}} c^* \in A_{i+1} \cap B_j$. Proves that $x \in [A_{i+1} \cap B_j]^{\theta_{i+1}}$ i.e. $[a]_{i,j+1}^\chi \subseteq [A_{i+1} \cap B_j]^{\theta_{i+1}}$.

We recapitulate: (li) is a strongly normal chain for every $i = 0, \dots, n - 1$, which, if inserted into chain (1), yields a refinement. In short (see def. 17):

(A) = (10) + (11) + ... + (1,n-1) is a refinement of (1).

Similarly one can show that

(B) = (20) -f (21) + ... + (2,m-1) is a refinement of (2) if

$$(2j) = (B_{j+1} = [G \cap B_{j+1}]^{\theta_{j+1}}, [G_1 \cap B_{j+1}]^{\theta_{j+1}}, \dots, [G_n \cap B_{j+1}]^{\theta_{j+1}} = B_{j+1}; \theta_{j+1} \circ \theta_0 \circ \theta_{j+1}, \dots, \theta_{j+1} \circ \theta_n \circ \theta_{j+1}).$$

Both chains (A) and (B) have equal length $n \cdot m$. Moreover, a typical algebra in (A) is of the form $[G_i \cap B_j]^{\theta_i}$ and carries the congruence relation $\theta_i \circ \theta_j \circ \theta_i$; also, a typical algebra in (B)

is of the form $[G_i \cap B_j] \theta_j$ and carries the congruence relation $\langle \theta_j \circ \theta_i \circ \theta_j \dots \rangle$. By Zassenhaus* lemma (in the form of corollary 1 to theorem 9), we know that

$$[G_i \cap B_j] \theta_i / \theta_i \cong [G_i \cap B_j] \theta_j / \theta_j$$

i. e. (A) \cong (B) . q. e. d.

If a normal chain has a trivial factor $G_i / 0_i$ (i.e. $G_{i-1} = G_i$ and $0_i = i$) then, of course, every isomorphic normal chain has such a trivial factor. Thus, if we drop all trivial factors (whose presence or non-presence is totally up to the whim of the person using them) isomorphic chains will be transformed into isomorphic chains,

Def. 20; A normal chain without trivial factors which permits no proper refinement without trivial factors is called a composition series.

Corollary 1 (Jordan-Hölder-Theorem):

If G is an algebra all of whose subalgebras have a commutative congruence lattice then any two composition series from t to G are isomorphic.

Corollary 2; If G is as above and there exists a composition series from t to G , then every normal chain from t to G can be refined to a composition series.

In case we deal with a group $Q = \langle G; *, \sim^1, 1 \rangle$ instead of an arbitrary algebra G , we realize that all subalgebras (= subgroups) have a commutative congruence lattice, since congruence relations

are represented by normal subgroups and the composition of congruence relations corresponds to the usual multiplication of normal subgroups which is indeed commutative. Thus, we have the assumptions crucial for the Jordan-Hölder-Schreier-Theory. Moreover, a normal chain from the subgroup \mathfrak{G} to Q is now simply a sub-group chain

$$\mathfrak{G} = \mathfrak{G}_0 \subseteq \mathfrak{G}_1 \subseteq \dots \subseteq \mathfrak{G}_n = Q$$

where Q_i is normal in Q_{i+1} (and determines the congruence-relation which we used to denote by \sim_{i+1} (the general theory)). We have the following simple special case:

Corollary 3:

If $\mathfrak{G} = Q_0 \triangleleft Q_1 \triangleleft \dots \triangleleft Q_n = Q$ and $t = M_0 \triangleleft \dots \triangleleft M_m = Q$ are normal chains from the group t to the group Q , then there are refinements R_1 and R_2 of the two chains, say

$$R_1: \mathfrak{G} = Q_0 \triangleleft \dots \triangleleft Q_s = Q \quad \text{and}$$

$$R_2: M_0 \triangleleft \dots \triangleleft M_t = Q$$

such that $s = t$ and $Q_i/G_i \cong M_i/H_i$ for a suitable permutation ir .

Corollary 4: If there exists a composition series from t to Q , then every normal chain from the group t to the group Q can be refined to a composition series. Moreover: Any two composition-series are isomorphic.

To conclude the section let us point our attention to the outstanding importance of groups with composition-series from the

trivial subgroup to the whole group. If the factors of that chain are Abelian, the groups are known as ^{1f} solvable groups¹¹ and play the known core rôle in the classical Galois-theory.

kl. Birkhoff's Subdirect Decomposition Theorems-

A property of an algebra G is called a "residual property" if it is true for G provided it is true for a family $\{G/\theta^i; i \in I\}$ where θ^i are congruence relations on G and $\bigcap_{i \in I} \theta^i = \omega$. The study of residual properties of an algebra G is closely related to the decomposability of G into subdirect products as was shown by G. Birkhoff. So assume that $G \subseteq \text{Tr}(G_i; i \in I)$ is a subdirect product of the algebras $G^i; i \in I$. Then the i -th projection $\pi_i: A \rightarrow A_i$ is an epimorphism with kernel, say, θ_i . Moreover: if $f, g \in A$ then $f \equiv g \pmod{\bigcap_{i \in I} \theta_i}$ is equivalent to $\pi_i(f) = \pi_i(g)$ for all $i \in I$, i.e. to $f = g$; thus, $\bigcap_{i \in I} \theta_i = \omega$. Since, by the homomorphism theorem, $G_i \cong G/\theta_i$ we can say that an arbitrary subdirect representation of G determines a family θ_i of congruences on G such that $\bigcap_{i \in I} \theta_i = \omega$ and the components are (up to isomorphism) the algebras G/θ_i . The converse is true as well as is summed up in the next theorem.

Theorem 11; If $G \subseteq \text{Tr}(G_i; i \in I)$ is a subdirect product of the algebras $G_i, i \in I$, then $\bigcap_{i \in I} \ker \pi_i = \omega$ and $G \cong \prod_{i \in I} G_i / \ker \pi_i$.

Vice versa: If $\{\theta^i; i \in I\}$ is an arbitrary family of congruence relations on G such that $\bigcap_{i \in I} \theta^i = \omega$, then G is isomorphic to a subdirect product of the algebras G/θ^i via the isomorphism $\langle \pi(a) = ([a]_{\theta^i})_{i \in I}$.

proof: The first half of the theorem was established before the theorem, the second half is (since $\langle \pi$ is given) a simple matter of verification. q. e. d.

Due to this internal characterization of subdirect representations of algebras, we can in the future assume that all subdirect representations are given in the manner just considered.

Def. 21: $G = \prod_{i \in I} (A_i / \theta_i)$ expresses the fact that the algebra G is expressed as subdirect product of the factor algebras A_i / θ_i in the sense of theorem 11. A set of congruence relations $\{\theta_i; i \in I\}$ on G is a separating set if $\bigcap_{i \in I} \theta_i = \omega$.

The task of giving an external, more constructive characterization of subdirect products was taken up by Fuchs[1], Fleischer[2] and Wenzel[1] using an idea of Wedderburn. Since this construction meets harsh limits in case of more than two factors, we will not engage in its discussion.

Thus, to find subdirect representations of universal algebras G we need only to look for separating sets of congruence relations on G . We can, of course, always waste our time by including ω into the set of congruence relations, thus obtaining on the one hand a separating set of congruence relations, on the other hand a worthless decomposition of G ... since one of the components is $G/\omega \cong G$. If this should be the only way to obtain a separating set, then with good right we consider the algebra as M subdirectly irreducible.¹¹

Def. 22: The algebra G is subdirectly irreducible if every separating set of congruence relations on G contains ω .

We get immediately the following remark.

Remark 19: The algebra G is subdirectly irreducible if and only if $G = 1_T$ or the congruence-lattice of G is atomic with exactly one atom.

Note: We recall that the element α of a lattice with 0 is an atom if $\alpha > 0$ and the only element I of the lattice satisfying $I < \alpha$ is $I = 0$. The lattice is atomic if for every element $I \neq 0$ there exists an atom $\alpha(I)$ such that $\alpha(I) \leq I$.

proof: If G is subdirectly irreducible and $|A| > 1$ then $C(G)$ has at least one more element than ω , so $C(G) \setminus \{\omega\}$ is non-empty and $0(0; B \in C(G) \setminus \{\omega\}) \neq \emptyset$. Clearly, $\alpha = \text{fl}(\text{fl}; 0 \in C(G) \setminus \{\omega\})$ is an atom and every $0 \in C(G) \setminus \{\omega\}$ satisfies $0 \wedge \alpha = 0$. Vice versa: If $G = 1_T$ then G is, of course, subdirectly irreducible. If $G \neq 1_T$ but $C(G)$ is atomic with a single atom α , then $0(0; 0 \in C(G) \setminus \{\omega\}) = \alpha > \omega$, i.e. every separating family of congruence relations on G must contain ω . In other words: G is subdirectly irreducible. q. e. d.

We are now able to state and prove the following fundamental and useful theorem which is also due to G. Birkhoff.

Theorem 12: If G is a universal algebra then G is isomorphic to a subdirect product of subdirectly irreducible universal algebras.

proof: We can assume that $G \neq 1_T$. In example 8, § 2, did we introduce the congruence relations θ_a for $a, b \in A$, $a \theta_a b$. If $p = \bigwedge_{a \in A} (x \wedge y \wedge z \wedge a)$, then $x \theta_a y(p)$ is equivalent to

either $x = y$ or $x = y(0_{x,y})$. The latter being impossible,

we conclude that $p = \infty$, i.e. $\{0_{a, \mathcal{D}} \mid a \wedge b, a, b \in A\}$ is a separating family of congruence relations on G . By theorem 11, $G \cong \text{if}_S(G/0_{a, \mathcal{D}} \mid a \wedge b, a, b \in A)$. So we are done if we can show that $G/0_{a, \mathcal{D}}$ is always subdirectly irreducible, i.e. that $C(G/0_{a, \mathcal{D}})$ is always atomic with a unique atom δ . By remark 15 (§5) we only have to show that $[0_{a, \mathcal{D}})$ is atomic with a unique atom. But this is clear since $G \in [0_{a, \mathcal{D}})$ and $A \wedge j) \cdot$ implies $0_{a, \mathcal{D}} \wedge b, i \vee \delta_{a, \mathcal{D}}$. Thus, $\delta_{a, \mathcal{D}} = 0_{a, \mathcal{D}} \cdot \bigvee_{a, \mathcal{D}} 0_{a, \mathcal{D}}^{cL, \mathcal{D}}$ is the unique atom of $[0_{a, \mathcal{D}})$ which is ≤ 0 for every $0 > 0_{a, \mathcal{D}}$. q. e. d.

The proof establishes a fact which is worth noticing of its own:

Remark 20: If $G / 1_{\mathcal{T}}$ then G is subdirectly irreducible if and only if $0_{a, \mathcal{D}} = \infty$ for some $a, b \in A, a \wedge b$.

proof: If $0_{a, \mathcal{D}} = \infty$ then $G/0_{a, \mathcal{D}} = G/\infty \wedge \delta$, and the proof of theorem 12 shows that G is subdirectly irreducible. If G is subdirectly irreducible then $0(0_{a, \mathcal{D}} \mid a, b \in A \text{ and } a \wedge b) = \infty$ implies $0_{a, \mathcal{D}} = \infty$ for some $a, b \in A$. q. e. d.

We remark that, although $0_{a, \mathcal{D}}$ is not a uniquely determined congruence relation (but rather one chosen out of a family of candidates), still $0_{a, \mathcal{D}} = \infty$ or $0_{a, \mathcal{D}} \wedge \infty$ are unambiguous statements.

(Ex. 11): If $\mathcal{L} = \langle L; V, A \rangle$ is a lattice then every prime filter D determines a congruence-relation $0_{\mathcal{D}}$ by defining $a \sim b(0_{\mathcal{D}})$ if and only if either both a and b are in D or both are not

in \mathfrak{p} . Thus, if we additionally assume \mathfrak{f} to be distributive then every ultra-filter (being prime) defines such a congruence relation. If a, b are two different elements of L then there are essentially two possibilities: (i) $a > b$ or (ii) a and b are incomparable. In both cases, Stone's theorem (theorem 4) assures the existence of an ultra-filter D containing a and not containing b . Thus, in both cases $a \cdot k(6_D) \neq 0_D$. Since $0_D / 60$ unless $|L| = 2$, we conclude that the distributivity of \mathfrak{f} together with $|L| \geq 3$ implies $0_{a,b} \neq 60$. Hence, the only subdirectly irreducible distributive lattices are the 1- and 2-element-lattices. This, by the way, implies trivially that every distributive lattice is a sublattice of the lattice-retract of some Boolean algebra.

4s. Free Algebras, Polynomials, Equational Classes.

We begin our section with a rather technical construction whose importance, on the other hand, is so outstanding that we recommend extreme patience while absorbing it. The starting point is essentially the class of all ordinals? namely we choose a symbol e_a for every ordinal a , call it a projection symbol and collect all the projection symbols in the class \overline{E} (Since the ordinals do not form a set, therefore, of course, E does not). \overline{E} is the starting point for a wider class $\overline{P(T)}$ of symbols to be constructed and associated with any fixed type r . $\overline{P(r)}$ is constructed inductively as follows:

(1) $\overline{E} \subseteq \overline{P(T)}$, (2) If $f_1, \dots, f_{n_y} \in \overline{P(r)}$ and $f_y \in F$ then the symbol $f_y(g_1, \dots, g_{n_y}) \in \overline{P(T)}$. (3) $\overline{P(T)}$ consists exactly of all elements that can be obtained using (1) and (2) in a finite number of steps. Equality is formal equality.

As is clear from the definition, every element g of $\overline{P(T)}$ (called a polynomial symbol) is composed of a well-determined finite number of operation symbols f_y and projection symbols e_a (each counted as often as it appears). These numbers are called the operational rank ($o.rk(g)$) and projection symbol-rank ($p.rk(g)$), respectively. The rank of g ($rk(g)$) is defined as the sum of the two previous ranks:

$$rk(g) = o.rk(g) + p.rk(g).$$

It is quite clear that, if $g = f_y(g_1, \dots, g_{n_y}) \in \overline{P(r)}$, then $rk(g) = 1 + \sum_{i=1}^{n_y} rk(g_i)$.

It is the fact that these ranks are finite which makes these polynomial symbols so valuable in many situations. In particular, the finiteness of the rank is frequently used to prove statements concerning polynomial symbols by induction on that rank as we will immediately demonstrate in the proof of the next remark:

Remark 21: If $p \in P(T)$ involves only projection symbols e_γ with $\gamma < a$ and α is a permutation of $\{0, 1, \dots, \gamma, \dots\}_{\gamma < a}$ then $p^\alpha \in P(r)$ where p^α is the symbol that results from p by consistently replacing e^γ by $e^{\sigma(\gamma)}$.

proof: (i) If $\text{rk}(p) = 1$, i.e. $p = e_\gamma$ for some $\gamma < a$, then $p^\alpha = e_{\sigma(\gamma)} \in P(r)$ holds by definition. (ii) Assume $f_1, \dots, p_n \in P(T)$ and we know already that $f_1^\sigma, \dots, p_n^\sigma \in P(r)$; then $(f_1 \cdot \dots \cdot p_n)^\alpha = f_1^\sigma \cdot \dots \cdot p_n^\sigma \in P(r)$ again by definition. q. e. d.

Def. 23: If $g \in P(r)$ involves actually $e^{\gamma_1}, \dots, e^{\gamma_m}$ then $\gamma = \max\{\gamma_i\}$ is called the leader of g , say $\text{ld}(g)$.

$P(T)$ serves as huge ressort of elements which we will use to create new algebras of type r ... the so-called polynomial-symbol algebras. To do so we fix some ordinal $\alpha \geq 1$ and define $P^{(\alpha)}(T) \subseteq P(r)$ to consist of all polynomial-symbols involving only projection-symbols e_γ with $\gamma < a$. If $f_\gamma \in F$ and $p_1 \cdot \dots \cdot p_n \in P^{(\alpha)}(r)$ then we define $f(p_1, \dots, p_n) =: -f_\gamma(B_1 \cdot \dots \cdot B_n) \in P^{(\alpha)}(T)$, thus turning $P^{(\alpha)}(T) = \langle P^{(\alpha)}; F \rangle$ into an algebra in $K(T)$.

Def. 24: The algebra $P^{(a)}(r) = \langle Y^{a*}(T); E \rangle$, as just constructed, is called the algebra of g-ary polynomial symbols. The elements of $P^{(a)}(r)$ are the a-ary polynomial symbols.

The next remark shows a mere comprehension of the definitions:

Remark 22:

- (1) We have a subalgebra-chain $P^{(1)}(T) \subset P^{(2)}(r) \subset \dots \subset P^{(a)}(r) \subset \dots$ if a runs through all ordinals.
- (2) $P^{(\omega_0)}(T) = U(P^{(n)}(r); n=1,2,3,\dots)$.

Let us remark that a set of subalgebras of an algebra G is called local if it is directed by inclusion and its union equals A . A property for algebras of type T is called a local property of G if G has the property provided every algebra of some local set of subalgebras of G has the property. Statement (2) of remark 22 then says that $\{P^{(n)}(r); n=1,2,3,\dots\}$ is a local set of subalgebras of $P^{(\omega_0)}(r)$.

These algebras $P^{(a)}(r)$ play the rôle of the suns in the sky $K(T)$ of algebras G of type r . Less poetically, $P^{(a)}(r)$ induces on every algebra $G \in K(r)$ an algebra $P^{(a)}(G) \dots$ the so-called algebra of a-ary polynomials. . . . in the following fashion: Every $f \in P^{(a)}(r)$ induces a function $f^{G,a}: A^a \rightarrow A$ according to the agreement that (i) $f^{G,a}(x_0, \dots, x_\delta, \dots)_{\delta < a} = x_\gamma$ and (ii) if $f_1^{G,\alpha}, \dots, f_n^{G,\alpha}$ are already defined then $f_\gamma^{G,\alpha}(x_0, x_1, \dots, x_\delta, \dots)_{\delta < \alpha}$ induces $f_\gamma^{G,\alpha}(x_0, x_1, \dots, x_\delta, \dots)_{\delta < \alpha}$.

Then $P^{(a)}(G) = \{ (p^{G,a}; p \in P^{(a)}(T)) \}$; moreover, if $f_y \in F$ and $p \in P^{(a)}(G)$ then $f_y \circ p = (f_y(p_1, \dots, p_n))^{G,a}$ makes f_y a well-defined operation.

Def. 25; The algebra $P^{(a)}(G) = \langle P^{(a)}(G); F \rangle$, as just constructed, is called the algebra of g-ary polynomials on G.

Note; By definition, e_y^u induces $e_y^a: A^a - A$ in $P^W(G)$ and (if, e.g., $p > a$) $e^p: A^p - A$ in $P^{(p)}(r)$. If there is no possibility of confusion, we will frequently drop the upper indices. Thus, e.g. $p(a_0, a_1, \dots, a_{m-1})$, $a_i \in A$, makes it clear that p stands for $p^{G,a}$.

If we fix $G \in K(r)$ then we have for every epimorphism $\varphi: P^{(a)}(T) \rightarrow P^{(a)}(G)$ mapping p to $p^{G,a}$ a kernel which identifies polynomial symbols inducing the same polynomial on G .

Def. 26; The kernel of $\varphi: P^{(a)}(r) \rightarrow P^{(a)}(G)$ as introduced above is denoted by $J_{G,a}^{(a)}$.

Corollary; $P^{(a)}(T) / J_{G,a}^{(a)} \cong P^{(a)}(G)$.

It is now a recommended (since simple) exercise to verify the next remark:

Remark 23;

- (1) If $p \in P^{(cx)}(T)$ and p is built up from e_{r_0}, \dots, e_{m-1} then there exists a $p \in P^{(m)}(r)$ such that $p \wedge (a_0, a_1, \dots, a_{m-1}) \in P^{(m)}(G)$ for every $G \in K(T)$.
- (2) if $p \in P^{(a)}(T)$, $a \geq \omega$, then there exists some $p_n \in P^{(a)}(r)$.

and $\gamma_0 < \gamma_1 < \dots < \gamma_n < \dots < a^* n < a)_{Q_T}$ such that

$$p^{(\alpha)}(a_0, a_1, \dots, a_\gamma, \dots)_{\gamma < \alpha} = p_1^{(\alpha)}(a_{\gamma_0}, a_{\gamma_1}, \dots, a_{\gamma_n}, \dots)$$
 for every $G \in K(T)$. If $a < 60_Q$ then, of course, $p^{(\alpha)}_{(r)Sz} \circ (T)$.
 In this sense, every polynomial on $G \in K(r)$ is M essentially M
 an c_0 -ary polynomial.

(3) If $G \in K(r)$ and $0 \leq j \leq n$ then $[S] = (p(s_0, \dots, s_{n-1}) ; p \in P^{(a)}(T)$ for some $n \in N, s_i \in S$).

(4) If $\langle p: G \rightarrow IB$ is a homomorphism and $jp \in P^{(a)}(T)$ then, for all $a_i \in A$, $\langle p(p(a_0, \dots, a_\gamma, \dots))_{\gamma < \alpha} = p(\langle p(a_0), \dots, \langle p(a_\gamma), \dots \rangle_{\gamma < \alpha}$.

(5) If $b_i, a_i \in A, 0 \in C(G)$ and $a_i = b_i(0), 0 \leq i < a$, then $p(a_0, \dots, a_i, \dots)_{i < a} = q(a_0, \dots, a_i, \dots)_{i < a}(e)$ for all $g \in P^{(a)}(T)$.

We now come to a point of extreme importance, namely to the so-called M identities¹¹ satisfied by a given class L of algebras of type r . If we, e.g., inquire how the class Q_p of groups is singled out in the class $K(\langle 2, 1, 0 \rangle)$ of all algebras $G = \langle A; *, \sim^1, 1 \rangle$ of type $r = \langle 2, 1, 0 \rangle$, then we unavoidably recall the defining M axioms¹¹ (i) $(x-y)-z = x*(yz)$, (ii) $x*x\sim^1 = 1$, (iii) $x\sim^1*x = 1$, (iv) $x*1 = x$, (v) $1*x = x$ for all x, y, z in the given group $Q = \langle G; \#, \sim^1, 1 \rangle$. If we take a thorough look at these axioms then they really boil down to the identity of certain polynomials on the group. E.g., (i) reads that $p^{**} = q^{G, a}$ if $p = (e^*e_0)*e_0$ and $q = e \cdot e \cdot (e, *e_0)$. (ii)-(v) can be similarly reformulated. Thus, the identities of polynomials that hold or do not hold in a certain algebra or in a class of algebras are of profound importance and information. We therefore are led to

define (and we are motivated by remark 23, (2)) the set of identities valid in a class of algebras $L \subseteq K(r)$, say $Id(L)$, as $\{(\ell, 3); \ell, 3 \in P^{\circ}(r) \text{ and } p^{\circ} = q^{\circ} \text{ for all } G \in L\}$. If $L = \{G\}$ consists of a single algebra we just get (see Def. 26) that $Id(G) = \overset{\circ}{V}$. Since $Id(L) = n(id(G); G \in L)$ we arrive at the following short definition:

Def. 27 If $L \subseteq K(T)$ then the set $Id(L)$ of identities of L is defined to be $\mathfrak{I}_T = n(\mathfrak{I}^{\circ}; a \in L) \subseteq P^{\circ}(T) \times P^{\circ}(T)$. More general: If α is an ordinal ≥ 1 then $Id^{\alpha}(L) = J^{\alpha} =: 0(\mathfrak{I}^{\circ}; G \in L)$ is called the set of g-ary identities of L.

The distinguished position of $\overset{\circ}{\omega}$ among all ordinals in the last definition is quite sufficiently explained by remark 23.

Clearly, the more we narrow our class L the more identities will we in general pick up and the tighter is the algebraic structure of the algebras subjected to those identities. It is this reasoning that causes us to call an algebra G (we are still intuitive) free in L (better $\hat{=}$ the free as possible in L^n) if $Id(L) = Id(G)$. But we need to be a bit more precise. To this end we pick some $\langle f \rangle \in S \subseteq A$ with $G \in L \subseteq K(T)$ and define what we mean with the local identities of S .

Def. 28: If $\hat{0} \wedge S \subseteq A$, $G \in L \subseteq K(T)$ and α is the initial ordinal of $|S|$ then we define the local identities of S , say $Id_{\alpha}(S)$, as follows:

$Id_{\alpha}(S) =: \{(\ell, \langle f \rangle); (1) \ell \in P^{\circ}(T) \times P^{\circ}(T), (2) td(g), \langle td(\zeta_j) \rangle < \alpha, (3) \text{ if } g, \hat{=} \text{ actually involve } \underline{e}^{\wedge}, \dots, e. \text{ then } p(s_0, s_1, \dots, s_n, \dots)_{n < \omega_0} = q(s_0, s_1, \dots, s_n, \dots)_{n < \omega_0} \text{ for some } s \in S^{\omega_0} \text{ with } |\{s_{\underset{y_0}{\cdot}}, \dots, s_{\underset{7}{t}}\}| = t+1\}$.

If $\text{id}_{\text{loc}}(S) = \text{ld}(L) \cap (P^{(a)}(T) \times P^{(a)}(T))$, the ultimately best we can expect, we call S a free set in L .

Def. 29t If $S \subseteq A$, $G \in L \subseteq K(T)$ and the initial ordinal of $|S|$ is a then S is called free in L if $\text{Id}_{\text{loc}}(S) = \text{ld}(L) \cap (P^{(a)}(r) \times P^{(a)}(T))$. In that case, we call $\langle L, S \rangle$ a free algebra over the basis S if $\bigcup_j J^j(S) \in L$.

Corollary 1; If $\hat{\alpha}(S)$ exists in L , $S = \{s_\alpha \dots, s_\gamma \dots, \}_{\gamma < \alpha}$, then $\text{id}_{\text{loc}}(S) = \text{id}(3_L(S)) \cap (P^{(a)}(r) \times P^{(a)}(r)) = \text{id}(L) \cap (P^{(a)}(r) \times P^{(a)}(r))$. proof: The proof of the corollary follows evidently from the fact that cardinals \aleph_α have initial ordinals ω_α .

Corollary 1; If $\hat{\alpha}(S)$ exists in L and S is an infinite set, then $\text{Id}(3_L(S)) = \text{Id}(L)$.

(2) If $\aleph_\alpha > \aleph_\beta$ then $\text{Id}^{(\aleph_\alpha)}(L) \supseteq \text{Id}^{(\aleph_\beta)}(L) = \text{Id}(L)$.

In the same manner as we associated with every $L \subseteq K(T)$ a set $\text{Id}(L) \subseteq P^{(\omega)}(T) \times P^{(\omega)}(T)$ one can associate with every subset $S \subseteq P^{(\omega)}(T) \times P^{(\omega)}(T)$ a class $M(L) \subseteq K(r)$, called the model-class of E .

Def. 31; If $E \subseteq P^{(\omega)}(T) \times P^{(\omega)}(T)$ then the model-class $M(E)$ is defined as the class of all algebras G of type r such that $\text{Id}(G) \subseteq E$.

We therefore have the two mappings $\text{Id}: 2^{K \times r} \rightarrow 2^{P^{(\omega)}(T) \times P^{(\omega)}(T)}$ and $M: 2^{P^{(\omega)}(T) \times P^{(\omega)}(T)} \rightarrow 2^{K \times r}$; which constitute a Galois-connection as precisely stated in the next remark:

Remark 24; Let T be a fixed type, $L_1 \subseteq K(T)$ and $S_i \in P^{(CO)}(T)XP^{(CO)}(T)$. Then

- (1) $L_x \subseteq L_2$ is equivalent to $\text{Id}(L_1) = \text{Id}(L_2)$,
- (2) $T_i \in E_2$ is equivalent to $M(L) \subseteq M(\Sigma_2)$,
- (3) $M(\text{Id}(L)) \subseteq L$ for all $L \subseteq K(T)$,
- (4) $\text{Id}(MCD) \in E$ for all $E \subseteq P^{(CO)}(T)XP^{(CO)}(T)$.

The natural question is now evident: When do we have equality in (3),(4)? The algebraist is interested in knowing when a class of algebras is characterized by its identities, i.e. when $M(\text{Id}(L)) = L$. The logician wants to know the characteristics of a set L of identities which is characterized by its model-set, i.e. which satisfies $\text{Id}(M(L)) = L$. We will in this section narrow our attention to the first of the two questions and derive the famous result of G. Birkhoff stating that $M(\text{Id}(L)) = L$ is equivalent to requiring that L be closed under taking subalgebras ($S(L) \subseteq L$), homomorphic images ($H(L) \in L$), and direct products ($P(L) \subseteq L$). Since $S(H(L)) \in H(S(L))$, $P(H(L)) \subseteq H(P(L))$ and $P(S(L)) \in S(P(L))$ can be easily verified (see, e.g., Grätzer [1], chapter 3, j23) for classes L which are closed under taking isomorphic copies (so-called "algebraic classes"). we can restate the latter condition as $L = H(S(P(L))) =$ (shortly) $= \text{HSP}(L)$.

To deduce the results we first engage in a discussion concerning existence and uniqueness of free algebras.

Theorem 13: Let $G \in LCK(r)$, $\$ t S \subseteq A$ and $[S] = A$. G is free in L over the basis S if and only if every mapping $\langle p: S \rightarrow B, B \in L$, can be extended to a homomorphism $\langle p^* : A \rightarrow B$.

proof: (1) If $S = \{s_1, \dots, s_\gamma, \dots\}$. then every element $a \in A = [S]$ is of the form $a = p(s_{x_0}, \dots, s_{x_\gamma}, \dots)^a$ for some $p \in P^G$ (T) with $\langle \text{id}(D) \rangle a$. Thus if there exists an extending homomorphism at all then it is necessarily given by $\langle p'(a) = p(s_{x_0}, \dots, s_{x_\gamma}, \dots) \rangle$. Again: $\langle p'$, thus defined, is clearly a

homomorphism extending $\langle p$ provided it is well-defined. To prove this latter point we assume that $a = p(s_{x_0}, \dots, s_{x_\gamma}, \dots)$ where $p \in P^G$ (T). Then $(s_{x_0}, \dots, s_{x_\gamma}, \dots) \in \text{Id}_{loc}^{(a)}(S)$ which, by assumption, implies that $(p, q) \in \text{Id}^{(a)}(L)$. Hence $p(\langle p(s_{x_0}, \dots, s_{x_\gamma}, \dots) \rangle) = q(\langle p(s_{x_0}, \dots, s_{x_\gamma}, \dots) \rangle)$.

(2) We assume now that every $\langle p$ can be extended and have to show that $\text{Id}^{(a)}(L) \cong \text{id}_{loc}(S)$. Assume in the contrary that $(p, q) \in \text{Id}_{loc}(S) \setminus \text{Id}^{(a)}(L)$. Then $p(s_{x_0}, \dots, s_{x_\gamma}, \dots) \neq q(s_{x_0}, \dots, s_{x_\gamma}, \dots)$ while $p(b_{x_0}, \dots, b_{x_\gamma}, \dots) = q(b_{x_0}, \dots, b_{x_\gamma}, \dots)$ for some $b_{x_0}, \dots, b_{x_\gamma} \in B, B \in L$. Hence, $\langle p: S \rightarrow B$ defined by $\langle p(s_{x_0}) = b_{x_0}$ cannot be extended to a homomorphism, ... which yields a contradiction, q. e. d.

Theorem 14: if a free algebra over a basis of cardinality \bar{a} exists in $L \subset K(T)$ then it is isomorphic to $P^{(a)}(r)/6_T^\alpha$. If $M(\text{Id}(L)) = IJ$ for a non-trivial class L , then indeed free algebras over bases of arbitrary cardinality exist in L .

proof:

We assume that $B = P^{(\alpha)}(r)/f \in L$ where $\alpha > 1$ is an arbitrary but fixed ordinal number. Evidently $B = [S]$ where $S = \{e_\gamma \mid 0 \leq \gamma < \alpha\}$. Since L is non-trivial we conclude that $P^{(\alpha)}(r)/f \in L$ if $\gamma < \alpha$; thus $[S] = \bar{a}$. If $C \in L$ and we pregive the mapping $\varphi: [e_\gamma] \in L \rightarrow C, \gamma \in C$, from S to C then evidently $\varphi^*: P^{(\alpha)}(r)/f \rightarrow C$ defined by $\varphi^*(\sum_{i \in C} p_i e_i + \sum_{j \in C} c_j y_j + \dots)$ is a homomorphism extending φ . Thus, by theorem 13, B is free over the basis S in L . Since $P^{(\alpha)}(r)/\theta_L^\alpha \in M(\text{Id}(L)) = L$ we have proved the second half of the theorem. If $\mathfrak{F}_T(S)$ is an arbitrary free algebra in L over the basis $S = \{s_0, s_1, \dots, s_\alpha, \dots\}$ then $\langle \varphi: P^{(\alpha)}(r) \rightarrow \mathfrak{F}_T(S) \text{ mapping } e_\gamma \text{ to } s_\gamma \rangle$ has kernel $e^{(g)} = \theta_L^\alpha$, i.e. $P^{(\alpha)}(r)/\theta_L^\alpha \cong \mathfrak{F}_T(S)$.

q. e. d.

Theorem 14 states, a.o., that a free algebra in a class L is uniquely determined (up to isomorphism) by the cardinality of its basis. We will therefore omit in the future to mention the particular basis S in $\mathfrak{F}_T(S)$ and rather use the notation $\mathfrak{F}_L(a)$ where a is the initial ordinal or the cardinality of S .

Def. 32: $\mathfrak{F}_L(a)$ denotes the (up to isomorphism unique) free algebra in the class L over a basis of cardinality \bar{a} .

exists in L for every ordinal α . q. e. d.

We have now arrived at the crucial theorem characterizing
¹¹ equational classes:¹¹

Theorem 16 (G. Birkhoff):

Let L be a non-trivial class of algebras in $K(r)$. Then
 $M(\text{Id}(L)) = L$ is equivalent to $L = \text{HSP}(L)$.

proof; It is easy to verify that $M(\text{Id}(L)) = L$ implies that
 $\text{HSP}(L) = L$. Vice-versa: Assume that $G \in M(\text{Id}(L))$; say, $A =$
 $\{a_0, \dots, a_y, \dots\}_{y < \alpha}$ and $\bar{a} = (a_0, \dots, a_y, \dots)_{y < \alpha}$. Then $\mathfrak{F}_L(\alpha)$

exists (see theorem 15) and $\langle a \rangle \in \mathcal{T}_G(P^{(a)}(T)/0; 0 \in D^a$. As we
 have shown in the proof of theorem 15, $0^a \in D_x$ and $P^{(a)}(T)/6^a \in \mathcal{F}_G$.
 Thus, $G \in \text{HSPI}(L)$, i.e. $G \in \text{HSP}(L) = L$. We have completed the
 proof that $M(\text{Id}(L)) = L$ and are done, q. e. d.

Since therefore the classes of algebras closed under the for-
 mation of subalgebras, products and homomorphic images are exactly
 the ones that are characterized by their identities, they are
 frequently called ¹¹ equational classes¹¹ (better would be: ¹¹ iden-
 tity classes¹¹) or ⁿ varieties¹¹ (the latter term being most
 decisively used in the theory of varieties of groups).

Def. 34: If $L \subseteq K(r)$ satisfies $\text{HSP}(L) = L$ then we call L
 an equational class of algebras.

It follows from our results that every class $L \subseteq K(T)$ is
 contained in a smallest equational class, namely $\text{HSP}(L)$. The
 next remark is clear:

Remark 25: The operator $\text{Equ} = \text{HSP}$ which associates with every class $L \subseteq K(r)$ the class $\text{HSP}(L)$ is a ^{1f} closure-operator¹¹ on $2^{K\langle r \rangle}$, i.e.

- (i) $L \subseteq \text{Equ}(L)$
- (ii) $L_1 \subseteq L_2$ implies $\text{Equ}(L_1) \subseteq \text{Equ}(L_2)$
- (iii) $\text{Equ}(\text{Equ}(L)) = \text{Equ}(L)$.

If we combine our results obtained so far then we get the following equivalent statements:

Remark 26: If $L_1 \wedge L_2 \subseteq K \wedge T \wedge$ are equational classes then the following statements are equivalent:

- (1) $L_1 = L_2$
- (2) $\text{Id}(L_1) = \text{Id}(L_2)$
- (3) $\exists_{L_1} \circ \exists_{L_2}$

proof:

(1) implies (2), (2) implies (3) are clear. If $\exists_{L_1}(\alpha) \wedge \exists_{L_2}(\alpha)$ then $\text{Id}(\exists_{L_1}(\alpha)) = \text{Id}(\exists_{L_2}(\alpha))$ which, by the corollary to def. 30, implies that $\text{Id}(L_1) = \text{Id}(L_2)$, i.e. $L_1 = M(\text{Id}(L_1)) = M(\text{Id}(L_2)) = L_2$. Hence, (3) implies (1). q. e. d.

We will conclude this section with a result due to A. Tarski which reveals a remarkable finiteness-behaviour of the operator " Equ^M ".

Theorem 17 (A. Tarski): Every equational class $L \in K(T)$ is generated by a single algebra $G \in L$; i.e. $L = \text{HSP}(G)$.

proof: If L is trivial then every algebra 1^{\wedge} generates it.

If L is non-trivial then $\hat{L}(\hat{\circ})$ exists and $\text{Id}(L) = \text{Id}(3_L(c0_Q)) = \text{Id}(\text{HSP}(3_L(60_{\circ})))$. This implies by remark 26 that $\text{HSP}(3_L(t0_Q)) = L$.

q. e. d.

J9. Equations. Mycielski's Conjecture.

In an absolutely analogous fashion as we developed the theory of polynomial-symbols, respectively polynomials, can the reader develop the theory of polynomial-symbols (respectively, polynomials) with constants in S defined as follows:

Def. 35: Let $E = (e_{\alpha_0}, \dots, e_{\alpha_1}, \dots)$ be a class of symbols, one for each ordinal α . If $G \in K(T)$ and $S \subseteq A$ then we take the disjoint union $E \dot{\cup} S$. The class $P_{\alpha}(T)$ of polynomial-symbols with constants in S^n is defined as follows:

$$(1) \quad E \dot{\cup} S \subseteq P_{\alpha}(r)$$

(2) If $g_1, \dots, g_n \in P_{\beta}(T)$ then, for every $f_{\gamma} \in F$,

$$f_{\gamma}(E_{r_1}, \dots, E_{r_n}) \in P_{\alpha}(r)$$

(3) $P_{\alpha}(T)$ consists exactly of all elements obtainable by steps (1) and (2) in a finite number of steps.

The reader is doubtless aware of the similarity between the constructions of $P_{\alpha}(r)$ and $P(T)$, resp. As a matter of fact, we have evidently that $P(T) = P/P(r) \cdot A^S$ before, $P_{\alpha}(T)$ plays the role of a huge resort of elements (it is a class rather than a set) by means of which we create new algebras of type r in the by now well-known fashion: We fix some ordinal $\alpha \geq 1$ and define $P_{\alpha}(T) \subseteq P_{\alpha}(r)$ to consist of all elements in $P_{\alpha}(T)$ involving only e_{γ} with $\gamma < \alpha$. If $f \in F$, $p_1, \dots, p_n \in P_{\alpha}(r)$ then $f_{\gamma}(p_1, \dots, p_n) = f_{\gamma}(E_{r_1}, \dots, E_{r_n})$ turns $P_{\alpha}(T) \subseteq P_{\alpha}(r)$ into an algebra of type T :

Def. 36; $P_i^{\alpha}(T)$, as just constructed, is called the algebra of g-ary polynomial symbols with constants in S.

Corollary: $P^{\alpha}(r) = P^{(\alpha)}(r)$ for all ordinals α .

The algebras $P_{\alpha}(r)$ induce an algebra $P^{\alpha}(G)$ for every algebra $G = \langle A; F \rangle$ whose carrier A contains S in the same fashion as $P^{\alpha}(r)$ induced $P^{\alpha}(G)$; the only specification we ought to add concerns the elements of S : If $s \in P^{\alpha}(T) \cap S$ then $s^{*a}: A \rightarrow A$ is the constant function mapping (x_1, \dots, x_s, \dots) to s .

Def. 37: If $K(T; S)$ contains all algebras of type T whose carrier contains S , then $P_i^{\alpha}(G)$, as defined for every $G \in K(T; S)$, is the so-called algebra of g-ary polynomials on G with constants in S.

It is a suggested exercise to verify that polynomials with constants share most of the properties enjoyed by polynomials without constants; in any case, we skip the trivial verifications without depriving ourselves of the right to make use of them.

E.g., if $L \subseteq K(T; S)$ we can consider the identities of L with constants in S, namely $\text{Id}_g(L) = \{ (f, 3); P^{\alpha} \wedge q^{\beta} \wedge \dots \text{ for every } G \in L \} \subseteq P^{\alpha}(T) \times P^{\beta}(T)$. Clearly, $\text{Id}(L) = \text{Id}_{\phi}(L)$. Similarly $\text{Id}_S^{\alpha}(L)$ is defined, etc.

Identities with constants constitute a certain subset of $P_g^{\alpha}(r) \times P_g^{\beta}(r)$ and are defined by inducing the same polynomials on algebras in a certain class L . It might so happen that $(p, q) \in P^{\alpha}(T) \times P^{\beta}(T)$ is not an identity but still

$$p^{G, \alpha}(x_0, \dots, x_\gamma, \dots)_{\gamma < \alpha} = q^{G, \alpha}(x_0, \dots, x_\gamma, \dots)_{\gamma < \alpha}$$
 for some $\bar{x} = (x_0, \dots, x_\gamma, \dots)_{\gamma < \alpha} \in A^a$ in which case we are in agreement with standard terminology if we say that \bar{x} is a solution in G of $(\{j\}Cj)$. This motivates our next definition:

Def. 38: The elements of $P^{(r)} \times P^{(T)}$ are called q -ary equations with constants in S of type T . If $G \in K(T; S)$, $\bar{a} \in A^\alpha$ and $p(\bar{a}) = q(\bar{a})$ we say that \bar{a} is a solution of (g_5c_j) in G . Sol $((p, q); G)$ denotes the solution-set of $(f^* \leq f)$ ⁿ 6.

If $f \in P^{(\alpha)}(T) \times P^{(a)}(r)$ is a non-empty set of equations, $L \subseteq K(T, S)$ and $G \in L$, then $\bar{a} \in A^\alpha$ is a solution of f if it is a solution of every $(\&\%) \in S^*$. If the symbol Sol $(S; G)$ denotes the solutionset of S in G then evidently $\text{Sol}(f; G) = \cap(\text{Sol}((p, q); G); (p, q) \in E)$.

Corollary: If $T \subseteq \text{Id}_0(L)$, $L \subseteq K(T; S)$ and $G \in L$, then $\text{Sol}(r; G) = A^a$.

We are now enabled to give the definition which is crucial for many of our investigations.

Def> 39: (1) Let $G \in K(r; S)$ be a universal algebra of type r with the following property $c(m)$ with respect to the cardinal number m :

¹¹ If $f \in P^{(a)}(T) \times P^{(a)}(T)$ (cardinality of f^m) is finitely solvable (i.e. $\text{Sol}(f; G)$) ¹¹ for each finite $1 \leq m \in T$ then S is solvable in G . "

Then G is called S -equationally m -compact.

(2) If $S = \langle f \rangle$ in (1) we use the phrase: " G is weakly equationally m -compact",^{1t}

(3) In the special case $S = A$ in (1) we call G an equationally m -compact algebra,

(4) If, in (1), $c(m)$ holds for every cardinal m , we call G S -equationally compact and adjust the terminology in (2),(3) accordingly.

(Ex. 12): To illustrate the concepts we investigate, a.o., the very first examples we gave in the introductory part of this chapter.

(a) if L is the trivial class of type r (i.e. L consists of all 1-element algebras) then L is, of course, a class of equationally compact algebras.

(b) If $x = \langle \mathbb{Z}; + \rangle$ is the group of integers, then ex. 1 establishes that z is not equationally ft'_0 -compact, hence not equationally m -compact for any $m \geq \aleph_3$, hence not equationally compact. More yet: z is not $\{1\}$ -equationally- $\hat{\ }_0$ -compact.

(c) Ex. 2 establishes that complete Boolean algebras are equationally compact (the proof is still to be given).

(d) Let \mathfrak{f} be the lattice given in ex. 3 and $K(\mathfrak{f})$ the class of complete lattices containing \mathfrak{f} as sublattice then $K(\mathfrak{f})$ contains no equationally $\hat{\ }$ -compact, not even $\{0,1\}$ -equationally- yV_1 -compact algebras.

On the other hand, \mathcal{A} is an equationally \wedge -compact algebra.

Although the use of the word 'compactness' in the preceding definition is sufficiently justified by the very definition, there is a much stronger reason coming from topology as was observed by J. Mycielski in [1]. Let us recall that when dealing with algebraic structures as semi-groups or groups or rings we are not just interested in arbitrary topologies that we might be able to inflict on the carrier-sets of those algebras.

In general, we want the fundamental operations f to be continuous functions $f : A^n \rightarrow A$ if A is endowed with the Tychonoff-product-topology and we want to be able to have n small enough neighborhoods to separate points, i.e. we insist on Hausdorff-topologies. This is summed up in the next definition:

~~Def. 40~~; If $G \in K(r)$ is an algebra then (G, τ) , shortly G , is a topological algebra of type T if τ is a Hausdorff-topology in A which makes the operations continuous. (G, τ) is a topologically compact algebra if it is topological and every cover of A with open sets has a finite subcover.

We must now, of course, insist on some knowledge of elementary point-set topology to which, e.g., belongs the fact that a topological space (S, τ) consists of subsets of S , the so-called 'open' sets of the topology] is compact if; and only if every family of closed sets which has the finite intersection property has a non-empty intersection (A family of closed sets has the finite intersection property if each intersection of a finite

subfamily is non-empty). Another elementary fact is that $S(f,g)$ is closed if $f: (S^a)^n \rightarrow (S^a)^m$ and $g: (S^a)^m \rightarrow (S^a)^k$ are continuous functions, $S(f,g) = \{x; x \in S_1 \text{ and } f(x) = g(x)\}$ and τ_1, τ_2 are Hausdorff-topologies. This is all we need at this time to derive the following fundamental result:

Theorem 18 (J. Mycielski):

Every topologically compact universal algebra G is equationally compact.

proof; Let $E \subset P^{a,1}(r) \times P^{a,1}(r)$ be a non-empty set of identities with constants in A which is finitely solvable. We well-order the finite subsystems $E_{S_0}, \dots, E_{S_\gamma}, \dots, E_{S_p}$ of T and denote their (non-empty) solution-sets in A^a by $S_0, S_1, \dots, S_\gamma, \dots, S_p$, respectively. Since A is endowed with a compact topology, Tychonoff's product-theorem assures that A^a carries a compact topology. Since $S_\gamma = S^H \setminus S^T$ where $S^V_\gamma = 1, \dots, 1(\gamma)$, are the solution-sets of the different single identities in E_{S_γ} and since (as we have recalled before the theorem) the sets S^V_γ are closed, we conclude that each S_γ is a closed set and the family $\{S_\gamma; \gamma < p\}$ has the finite intersection property. Thus, since A^a is compact, $\bigcap \{S_\gamma; \gamma < p\} \neq \emptyset$. Since $\text{Sol}(I^G) = \bigcap \{S_\gamma; \gamma < p\}$, we are done. $q. e. d.$

Mycielski's observation is really slightly stronger than theorem 18 inasmuch as it states the same conclusion for retracts of topologically compact algebras. It is unfortunate that the concepts of F^0 -retract (see def. 1) and simply retract are

linguistically so close, but it is hoped that no confusion arises.

Def. 41: If G, B are algebras of type r , $\varphi: G \rightarrow B$ is an epimorphism, B is a subalgebra of G and $\varphi|_B$ is the identity then $(p$ is called a ~~retraction~~ and B is called a ~~retract~~ of G . (In
 IB
 2

other words: If $\varphi: G \rightarrow G$ is an endomorphism such that $\varphi^2 = \varphi$ then φ is a retraction $\varphi: G \rightarrow \varphi(G)$ and $\varphi(G)$ is a retract of G).

Corollary 1: Every finite algebra is equationally compact.

Corollary 2 (J. Mycielski):

Every retract of a topologically compact universal algebra is equationally compact.

proof: Let $\varphi: G \rightarrow \varphi(G)$ be the retraction and choose

$E \in \mathcal{P}(A)^{(T) \times \mathcal{P}(A)^{(T)}} \subseteq \mathcal{P}(A)^{(T) \times \mathcal{P}(A)^{(T)}}$ as finitely solvable in $\varphi(G) \subseteq G$. Then, by theorem 18, E is solvable in G . If $\bar{a} \in A^a$ is a solution then evidently $\varphi(\bar{a})$ is a solution and $\varphi(\bar{a}) \in \varphi(A)^a$. q. e. d.

An unsolved question is whether or not the converse of Corollary 2 is true. The converse is true in case of Abelian groups, Boolean algebras and vector-spaces as we shall see. We state the general problem:

Problem 1 (Mycielski's Conjecture): Is every equationally compact universal algebra retract of a topologically compact universal algebra?

It has come to the attention of the author that there seems to exist some counter-example against Mycielsk's conjecture in the general form. Its details are not yet published and unknown to the author at the time. Even so The conjecture remains to be investigated in special classes of algebras, a task which is as interesting as the original one.

Chapter II, Elements of First Order Logic

11. The language of first order logic.

Let $K(r) = [G = \langle A; F, R \rangle]$ be the class of all relational systems of type $r = \langle \tau_1, \tau_2 \rangle^{\text{t} \wedge \text{ien}}$ we can form $P^{(\alpha)}(\tau_1)$ for every ordinal α , as discussed in chapter I. As it happens quite frequently if the same object is approached from different directions, different notations confront each other where the two approaches merge. So it is quite conventional in logic to denote the ft projection symbols e_y^m from which $p^{(\alpha)}(\tau_1)$ is built up by $\text{"} x_y \text{"}$ and to call them n variables, ft a convention we will go along with since no confusion seems possible. If $R^{(\alpha)}(r_2)$ denotes the set of all symbols $R(x_1, \dots, x_m)$, $R \in R$, $0 \leq i_1, \dots, i_m < a$, $a_j < \omega$, then we define the ft algebra of a -ary first order formulas by $\mathcal{F}_r^{(\alpha)} = \langle \{ (P^{(\alpha)}(\tau_1) \times P^{(\alpha)}(\tau_1)) \cup R^{(\alpha)}(\tau_2) \}; \{ (\exists x_y); \neg; \vee \} \rangle$, a universal algebra of type $\gamma \setminus = \langle 1, 1, \dots, 1, 1, 2 \rangle$ (i.e. all operations but \vee are unary) which is generated by $(P^{(\alpha)}(\tau_1) \times P^{(\alpha)}(\tau_1)) \cup R^{(\alpha)}(\tau_2)$ via formal application of the operations. Equality is formal equality. We combine this and further definitions as follows:

Def. 1: If α is a limit ordinal > 0 and r is a fixed type of relational systems then $\mathcal{F}_r^{(\alpha)}$, as just defined, is the algebra of g -ary first order formulas. The elements of

$(P^{(\alpha)}(\tau_1) \times P^{(\alpha)}(\tau_1)) \cup R^{(\alpha)}(\tau_2)$ are called atomic formulas. The operations $(\exists x_y)$ are called existential quantifiers, the operation \neg is referred to as negation, the operation \vee as disjunction.

We read $(\exists x_\gamma)$ (\exists) as: "There exists x_γ such that $\$$ holds"^f and denote $\$$ to be the scope of $(\exists x_\gamma)$. Similarly, \neg (\neg) reads " $\$$ is not true;" \forall (\forall) reads " $\$$ or 0 is true" and is frequently written as $\forall 0$. $L_T \wedge U(L^a; a J > 0)_0$ is the language of type r .

Of course, we feel that our list of quantifiers and connectives (\neg, \forall) is still rather poor. So we pass on to $P^{(0)}$ (70) where A is the type $\langle \{1_\gamma; 1_\gamma = 1, \gamma < a\}, 1, 2 \rangle$ of our language and find a few more of the standard quantifiers and connectives there:

Def. 2: The following elements of $P^{(0)}(A)$ are endowed with particular importance:

$$(1) (\forall x_\gamma) =: \neg(\exists x_\gamma)(\neg(x_{\gamma+1}))$$

$$(2) A =: \neg(\neg(x_0) \vee \neg(x_1))$$

$$(3) - =: \neg T(x_0) \vee x_{11}$$

$$(4) \langle _ \wedge =: (x_0 \rightarrow x_1) A(x_{1L} - x_0)$$

If we pass to $P^{(a)}_0(\{ _ \wedge \})$ we get the following operations:

(1) the unary operation $(\forall x_\gamma)$ (\forall) reading: "For all x_γ \wedge holds true." $(\forall x_\gamma)$ is called the universal quantifier, $\$$ its scope.

(2) the binary operation $A(\$, t/)$, also $\forall A 0$, reading: "Both $\$$ and 0 hold true."^{ff} A is called the conjunction of $\$$ and 0.

(3) the binary operation $\rightarrow(\$, 0)$, also $\$ \rightarrow 0$, reading: " $\$$ implies 0"^{ff} \rightarrow is called implication.

(4) the binary operation $\langle - * \{ * >, \$ \}$, also $\$ * \rightarrow 0$, reading:
¹¹ $\langle \&$ and *if*) are equivalent¹². $* \rightarrow$ is called the equivalence-
 operation.

We have thus special names for the existential quantifiers $(\exists x_\gamma)$ and the universal quantifiers $(\forall x_\gamma)$ and for the so-called connectives \neg, \vee and \wedge . In addition we named the implication-processes. These, together with parentheses, commas and the variables x_γ constitute the building blocks of the so-called first order logic with identity? for instead of the less confusing notation $(\exists x_\gamma) \in P^{f, a_1}(T_1) \times P^{a, a_1}(T_1)$ it is also a (more suggestive) convention to use the notation $g = \langle j \rangle$.

Def. 3: The occurrence of a variable x_α is called bound if

x_γ occurs in $(\exists x_\gamma)$ or in the scope of an existential quantifier $(\exists x_\gamma)$. If it is not bound it is called free. A formula in which every occurrence of every variable is bound (resp., free) is called a sentence (resp., open formula).

(Ex. 13): If $K(r_1)$ is the class of all relational systems $G = \langle A? \{ +, - \}, f \rangle$ of type $T = \langle \langle 2, 2 \rangle; 2 \rangle$ then

(i) $(\exists x_0) ((\forall x_1) (x_0^{*+} x_1 = 3^{\wedge}))$ is a sentence,

(ii) $(x_0 + x_1 = x_0) \vee (x_1 \leq f x_0)$ is an open formula,

(iii) $((x_0 = x_x) \wedge (x_x \wedge x_2)) \vee (ax_3) ((\forall x_0) (x_0 \wedge x_3 - x_1))$

is neither a sentence nor an open formula. Every occurrence of x^{\wedge} is bound, the first occurrence of x_0 is free, the second and third are bound, all occurrences of x^{\wedge} are free, so is the only occurrence of x_2 .

Although equality of formulas is formal equality, we feel that in applications a modified equality-concept should be used. Thus, e.g., the sentences $(\forall x_0) ((\forall x^1 ((\forall x_2) ((x^1 x^1 * x_2 = x_0 - (x_1 - x_2))))))$ and $(\forall x^1 ((\forall x_2) ((\forall x_3) ((x^1 x^1 ' x_3 ^ (x ^ x ^ V(-r (x^1 x_1)))))))$ are clearly equally good to characterize associativity of a binary operation $f^1 \cdot f^t$. In order to make the idea precise, we have to discuss the concept of f^1 satisfiability of a formula. f^1

Def. 4: If $\langle ij \rangle$ is a formula in the language L_f^α , $G \in K(r)$ and $\bar{a} \in A^\alpha$ then we say that \bar{a} satisfies $\langle ij \rangle$ in G if it follows from the following rules constituting an inductive definition:

- (i) $\langle ij \rangle$ is atomic of the form $(\exists D, \langle J \rangle)$, or $p = g$, where $\mathcal{E}, \langle j \rangle \in G, p \in A^{\langle j \rangle}(r_1)$: Then \bar{a} satisfies 0 in G if and only if \bar{a} is a solution of the equation $(\mathcal{E} * 3)$.
- (ii) $\langle j \rangle$ is atomic of the form $R(x_1, \dots, x_m)$: Then \bar{a} satisfies 0 in G if and only if $R_{\mathcal{E}}(a_1, \dots, a_m)$ is true.
- (iii) 0 is of the form $(\exists x^y) (\langle \mathcal{E} \rangle)$: Then \bar{a} satisfies 0 in G if and only if $\bar{a}(y/b)$ satisfies 0 for some $\bar{a}(y/b) \in A^a$ where $\bar{a}(y/b)$ equals \bar{a} in all but the y -th component the latter one being replaced by b .
- (iv) 0 is of the form $\neg \langle \mathcal{E} \rangle$: then \bar{a} satisfies 0 in G if and only if \bar{a} does not satisfy \mathcal{E} .

(v) ϕ is of the form $\exists x \psi$. Then \bar{a} satisfies ϕ in G if and only if $\exists L$ satisfies ψ or ϕ_2 .

Corollary: If $\phi \in L_T^{(a)}$ is a sentence, then ϕ is satisfied either by all $\bar{a} \in A^a$ or by none. Accordingly, we say that ϕ is either a true or false sentence in G .

Every formula ϕ in L_T^a induces an a -ary relation $\bar{J}L$ on every relational system $G \in K(r)$: $\phi^u(a)$ holds true if and only if \bar{a} satisfies ϕ in G . If ϕ is a sentence then either $\phi^G = \emptyset$ or $\phi^G = A^a$. We can now loosen our tight requirement on formal equality of formulas and replace it by a more natural specification.

Def. 5: If ϕ_1 and ϕ_2 are formulas in $L_T^{(a)}$, then we say that ϕ_1 and ϕ_2 are equivalent ($\phi_1 \equiv \phi_2$) if $\phi_1^G = \phi_2^G$ for every $G \in K(T)$. If $\phi_1^G \subseteq \phi_2^G$ for every $G \in K(r)$ then we say that ϕ_1 is weaker than ϕ_2 . **write:** $\phi_1 \Rightarrow \phi_2$.

Corollary: $\phi_1 \equiv \phi_2$ holds if and only if $\phi_1 \Rightarrow \phi_2$ and $\phi_2 \Rightarrow \phi_1$. The relation \equiv is an equivalence relation on the set $L_T^{(a)}$.

Of course, instead of taking a single formula ϕ we can take any set E of formulas which again induces an a -ary relation $\bar{J}E$ on every $G \in K(r)$, namely: $\bar{J}E = \{ \phi^G; \phi \in E \}$.

Def. 6: If F_1 and T_2 are sets of formulas in $L_T^{(a)}$ then F_1 and T_2 are equivalent sets if $F_1^G = T_2^G$ for every $G \in K(r)$; we write: $F_1 \equiv T_2$. Similarly, $D_1 \Rightarrow T_2$ is defined. If $\bar{a} \in \Sigma_1^G$ then we say that \bar{a} satisfies T .

The notations \Rightarrow just introduced are applied in one more situation:

Def. 7: If $0 \in L_T^a$ and $0^u = A^a$ then we say that G implies $0'$ and write $G \Rightarrow i|)$. Similarly, $G \Rightarrow S$ is defined for a set of formulas. If $G \Rightarrow^* f$ holds for all $G \in K(r)$ we call f a set of universally true formulas and write simply $r \Rightarrow^* f$. If $0^u = \emptyset$ for every $G \in K(r)$, then we call 0 a universally false formula and write $T \Rightarrow^* (\emptyset)$.

Corollary: (1) If $(p, q) \in \text{Id}(G)$ then $G \Rightarrow^* i v_{\%}$.

(2) If $i j)$ is universally false then $\neg \wedge$ is universally true, and vice versa.

Bypassing the less interesting formal equality of formulas and narrowing the attention to the equivalence \equiv of formulas one can (using simple induction) prove the following so-called prenex normal form-theorem (we skip the proof):

Theorem 1: If $0 \in L_T^{(\alpha)}$ is a formula then there exists always a formula \wedge of the form $Q \wedge Q \wedge Q_3 (\dots (Q_m (*))))))$ [where Q is either some $(\exists x)$ or some $(\forall x)$ and $\$$ contains neither of the two quantifiers] such that $0 \equiv \wedge$.

Def* 8: A formula $0 \wedge$ of the form described in theorem 1 is said to be in prenex normal form. $\$$ is called the matrix of $rb^Q-L(\dots(Q_m \text{ its prefix}$.

Thus, the theorem states that every formula can be assumed to be (up to equivalence) in prenex normal form. This may end our introduction into the language to be used.

§2. Ultra-products and the Compactness-Theorem.

Our aim in this section is to discuss the preservation of formulas under the formation of ultraproducts and to give its most outstanding application: Tarski's proof of the compactness-theorem. To do this we derive an easy, though basic, theorem for whose proper formulation we can conveniently use the next definition:

Def. 9: If $G_i, i \in I$, are relational systems of type r , $G = \prod_{i \in I} G_i$, D is an ultra-filter over I , $\bar{g} = (g_0, g_1, \dots)$ $\in A^a$ and $\ast) \in L^{(a)}$ then the support of \bar{g} with respect to ih is defined to be $S(0, \bar{g}) = \{i; i \in I \text{ and } g(i) = (g_Q(i), \dots, g_{R_i}(i), \dots) \text{ satisfies } \ast \text{ in } G_i\}$.

Theorem 2: If $G = \prod_{i \in I} G_i$ is an ultra-product of the relational systems G_i of type r and $0 \in L^{(a)}$ then $\bar{g} = (g_0, \dots, g_1, \dots) \in A^a$ satisfies 0 in G if and only if $S(0, \bar{g}) \in D$ where $g = (g_0, \dots, g_1, \dots) \in A^a$.

~~Note:~~ See theorem 1.3 for the terminology.

~~proof:~~ We proceed by induction beginning with the atomic formulas:

(i) If $0 = (B_{ra}) \in P^{(a)}(T_1) \times P^{(a)}(T_1)$ then $p^{G, a}\{h = q^{G, a}(J)$ is (as follows immediately from theorem 1.3) equivalent to $\{i; p(g(i)) = q(g(i))\} \in D$ which (by def. 4) proves our point.

(ii) If $0 \in R^{aj}(T_2)$ theorem 1.3 applies immediately.

(iii) Assume that $0 = \neg \bar{1}$ where $\bar{1}$ satisfies the theorem. Then \bar{g} satisfies 0 in G if and only if \bar{g} does not satisfy

\bar{A} in G , i.e. if and only if $SU_{\bar{1},g} \in D$ which (since D is an ultra-filter) is equivalent to $As \wedge \wedge g) = S(ij[,g) \in D$.

(iv) Assume that $if) = 0, \forall \wedge$ where $if)^{\wedge}$ and 0_2 satisfy the \bar{A} .

theorem. Then g satisfies $if)$ in G if and only if g satisfies $-i/j,$ or $)l)_2$ in G , i.e. if and only if $S(0^{\bar{1}},g)$ or $S(itU'S)^{\wedge s}$ contained in D which (since D is an ultra-, hence, prime-filter) is equivalent to $Sd/j^{\wedge}g) \cup S(0_2,g) = S(0,g) \in D$.

(v) Assume that $if) = (3x)(*)^{\wedge}$ where $if)^{\wedge}$ satisfies the theorem.

Then g satisfies 0^{\wedge} in G if and only if $n = g(y^{\wedge}b)$ satisfies 0^{\wedge} in G for some $D \in \bar{A}$ which, in turn, is equivalent to $S(i/),,n) \in D$. (*) Since $S(*//,R) c: S(0,g)$ we conclude that $S(0,g) \in D$.

Vice versa: If $S(0,g) \in D$ then $g(i)$ satisfies $if)$ in G for all $i \in S(0,g)$ i.e. $g(i) (y/b.)$ satisfies 0^{\wedge} in G for all $i \in S(0,g)$ with $b. \in A^a$. Thus, if we define $h \in A^a$ by (a) $h(i) = g(i) (y/b.)$ for $i \in S(A,g)$, (b) $h(i)=g(i)$ for $i \in I \setminus S(>ft,g)y$ then $S(\wedge)^{\wedge},h) \in D$ and we can take up the proof at (*) and go backward to conclude that g satisfies tb in G . q. e. d.

We can, thus, say that a formula is satisfied by an a -tuple of elements in an ultra-product if and only if it is satisfied on ¹¹ nearly all components^M or ^{tf} on a component-set of measure 1, ¹¹ to keep with the terminology of chapter I. We have immediately a few corollaries:

Corollary 1:

If $\bar{1} \forall r (G.; i \in I)$ is an ultra-product of the relational systems $\bar{1}$ G . then it is of finite cardinality n if and only if $\{i; |A., | = n\} \in D$

proof: We consider the sentence θ defined as follows:

$$\begin{aligned}
 & (\exists x_1) (\exists x_2) \dots (\exists x_n) (\forall x_{n+1}) ((x_1 \wedge x_2) \wedge (x_1 \wedge x_3) \wedge \dots \wedge (x_1 \wedge x_n) \wedge (x_2 \wedge x_3) \wedge \dots \\
 & \dots \wedge (x_2 \wedge x_n) \wedge (x_3 \wedge x_4) \wedge \dots \wedge (x_{n-1} \wedge x_n) \wedge \\
 & ((x_{n+1} = x_0) \vee (x_{n+1} = x_1) \vee \dots \vee (x_{n+1} = x_n))
 \end{aligned}$$

Evidently, the sentence θ_n holds in G if and only if $|A| = n$.

Thus, the corollary follows from theorem 2. q. e. d.

Corollary 2 A sentence ϕ holds in $\prod_{i \in I} G_i$ if and only if $\{i \in I \mid \phi \text{ holds in } G_i\} \in D$.

Corollary 3 A sentence ϕ holds in the ultrapower G^I_D if and only if it holds in G .

We ought to mention that our new results yield a new proof for the corollary of theorem I#5? for j_0, D, I is, of course, onto **if** $|AJ| = |A|$.

We now turn our attention to the most important application of all, the so-called compactness-theorem. Instead of giving the theorem and Tarski's proof thereof immediately we derive a useful generalization inspired by a result due to Mycielski, Ryll-Nardzewski and Weglorz (see Weglorz [1]) whose proof is essentially the same Tarski gave for the weaker compactness-theorem. For its formulation we need to give a short discussion of the ideas that lead to the so-called "diagram-language" of Robinson.

If $r = \langle n_0, \dots, n_y, \dots \rangle_{y < a}$, $a = \langle m_0, \dots, m_5, \dots \rangle_{5 < p}$ are 2 types of universal algebras then we can combine the two types to the new type $q = \langle n_{0^*}, \dots, n_y, \dots, n_{0^*}, \dots, n_{a^*}, \dots \rangle_{\substack{0 \leq y < a \\ a \leq b < a+b}}$ with $n_{a^*} = m_e$.

Similarly one can proceed with the types of relational systems.

Def. 10; (1) If T and a are types of algebras as above then $T @ a$ is defined to be p . Similarly, if $r = \langle r_i \rangle_{i \in I}$ and $a = \langle a_i \rangle_{i \in I}$ are types of relational-systems then $r @ a$ is defined to be $\langle T @ a_i \rangle_{i \in I}$.

(2) If $G = \langle A; F, R \rangle$ is of type r , or is any type and $H = \langle B; G, S \rangle$ is of type o , then one can (by suitable defining G and S on A) turn G into a relational system $G^* = \langle A; F \cup G, R \cup S \rangle$ of type $T @ O$. In particular, one can well-order $A = \{a_0, \dots, a_{\epsilon-1}, \dots\}_{\epsilon < p}$, take the type $a = \langle 0, 0, \dots, 0, \dots \rangle_{\epsilon < p}$ for $\epsilon \leq p$ and turn G into $G^\epsilon = \langle A; F \cup \{a_{\nu_0}, \dots, a_{\nu_{\epsilon-1}}\}, R \rangle$ of type $T @ a$ where we interpret a as 0-ary operation in the natural fashion. If we choose $\epsilon = p$ and $a_{\nu_j} = a_j$ then we call $L_T(G)$ the diagram-language of type T over G and write $L_T(G)$. In general if we add the type $cr(B) = \langle 0, 0, \dots \rangle_{\epsilon < p}$ represented by $B = \{a_{\nu_0}, \dots, a_{\nu_{\epsilon-1}}\}_{\epsilon < p}$ as nullary operations with the natural interpretation in G we speak of the language of type T with constants in B and write $L(B; G)$.

Corollary: $L(A; G) = L(G)$ and $L(0; G) = L$.

Theorem 3; If L is a set of formulas in $L(B; G)$ each of

whose finite subsets δ is satisfiable in some relational system G_δ of type $r @ cr(B)$ then there is an ultra-product $\prod_{\delta \in I} G_\delta$ in which L is satisfiable.

proof: Let $I = \{0; P \in \mathcal{L} \text{ and } |G| < \aleph\}$ and $I_\phi = \{0; P \in \mathcal{L} \text{ and } |G| < \aleph\}$ for all $\phi \in \mathcal{L}$. Then $D_\phi = \{I_\phi; P \in \mathcal{L}\} \subseteq 2^I$ has the finite intersection-property and is therefore (corollary 2 to def. 1.1) contained in an ultra-filter D of 2^I . We will show that T_ϕ is satisfiable in $\hat{G}_D(G, \phi \in I)$: If L involves the variables $x_0, \dots, x_\gamma, \dots, y < \alpha$, then for every $0 \in I$ there exists some $\bar{a}(0) = (a_A(0))_{A \in \mathcal{A}}$ satisfying 0 in G_0 . We narrow our attention to the sequence $(\bar{a}_\theta)_{\theta \in I} \in \prod_{\theta \in I} G_\theta = \prod_{\theta \in I} (T_{D_\theta}(A_\theta; G_\theta))$. If $\langle p \in T$ then \bar{a} satisfies $\langle p$ in $\hat{G} \wedge Q; \theta \in I$ if and only if $U = \{0; (a_\theta(A))_{A \in \mathcal{A}} = \bar{a}(0)\}$ satisfies $\langle p$ in $G_Q\} \in D$ (we use theorem 2). Since visibly every 0 containing $\langle p$ is in $\bigcup_{\theta \in I} D_\theta$ we conclude that $U \in D$, which, since $I_\phi \in D \subseteq D$, implies that $u \in D$. Thus \bar{a} satisfies E in $\hat{G}_D(G, \phi \in I)$. q. e. d.

An immediate corollary is the compactness-theorem. We recall that a sentence either holds true in a relational system G or it does not (if it is satisfiable by some string of elements, then it is satisfiable by every string; hence the sentence holds).

Def. 11: If \mathcal{L} is a set of sentences in L then the model-class

$\overline{M(D)}$ of \mathcal{L} is the class of all relational systems of type T in which every sentence of T holds.

Corollary 1: (Compactness-theorem): If T is a set of sentences in L_T each finite subset of which has a model then L has a model.

Corollary 2 (Ryll-Nardzewski): Let \mathcal{L} be a set of formulas in $L_T(G)$. If each finite subset of \mathcal{L} is satisfiable in G then there is an ultrapower of G in which \mathcal{L} is satisfiable.

13. Elementary Extensions*

Closely related to the concept ^{ft} ultra-power,ⁿ the concepts of ¹¹ elementary embedding,^{f1} resp. ^{ff} elementary extension*¹ play an important role in the first order logic. The relevant definitions, as given by A. Tarski, are as follows:

Def. 11: If G and B are relational systems of the same type with $A \subseteq B$ then B is called an elementary extension of G (notation: $G \prec B$) if for each formula $\phi \in L_T^{(\alpha)}$, ϕ is satisfied by $\bar{a} \in A^a$ in B if and only if it is satisfied by \bar{a} in G . G is then an elementary subsystem.

It is useful to realize immediately the following equivalent formulations:

Remark 1: If G and B are relational systems of type T with $A \subseteq B$ then the following statements are equivalent:

- (1) $G \prec B$
- (2) If $\phi \in L_T^{(\alpha)}$ and $\bar{a} \in A^a$ then ϕ is satisfied by \bar{a} in B if and only if ϕ is satisfied by \bar{a} in G .
- (3) If $\phi \in L_T^{(\alpha)}$ and $\bar{a} \in A^a$ ($=$ set of all $\langle a_i \rangle$ -tuples of elements of A which become stagnant from some point on) then ϕ is satisfied by \bar{a} in G if and only if ϕ is satisfied by \bar{a} in B .

proof: Clearly, (1) implies (2) and (2) implies (3). To see that (3) implies (1) we choose $\phi \in L_T^{(\alpha)}$ and $\bar{a} \in A^a$. ϕ depends (by definition) only on finitely many variables, say

$x_1, \dots, x_n \in V$; we indicate this by writing $\phi = \phi(x_1, \dots, x_n)$. Then $\phi' = \exists x_1 \dots \exists x_n \phi$, obtained by consistently replacing x_i by $x_{i'}$, $0 \leq i' \leq n$, is a formula in \mathcal{L}^r . Moreover: $a \in A$ satisfies ϕ in B if and only if $a^{(0)}$ satisfies ϕ' in B which holds (by(3)) if and only if a satisfies ϕ in G . $q, e. d.$

If we apply the definition of $G \equiv B$ to the special formulas $R_y(x_1, \dots, x_m)$ and $(\exists y)(f_y(x_1, \dots, x_n) = y)$ for all $R_y \in R, f_y \in F$, then we deduce immediately the following remark:

Remark 2: If G is an elementary subsystem of B then G is a subsystem of B .

Another obvious corollary of the definition is the fact that, if $G \equiv B$ then a sentence holds in G if and only if it holds in B ; one calls this relationship between G and B an elementary equivalence.

Def 12: The relational systems G and B of type r are called elementarily equivalent, $G \equiv B$, if every sentence that holds in G also holds in B , and vice versa.

Remark 3: $G \equiv B$ implies $G \subseteq B$.

The attempt to really understand the concept of 'elementary extension' leads to the recognition of the fact that it puts really conditions only on formulas involving the existential quantifier. So the following result, as useful as it is, proves to be not too surprising.

Theorem 4 (Tarski): If G is a subsystem of B then B is an elementary extension of G if and only if the following condition is satisfied: For every $\omega \in L_r^0$, $0 \wedge y < 0$ and $\bar{a} \in A^{\langle \omega \rangle}$ (= set of stagnant co_0 -tuples), if \bar{a} satisfies $(\exists x)(\phi)$ in B then there exists some $b \in A$ such that $\bar{a}(y/b)$ satisfies ϕ in B .

proof: To see that the condition in theorem 4 implies $G \models \phi$ we verify condition (3) of remark 1. So let $\omega \in L_r^0$ and $\bar{a} \in A^{\langle \omega \rangle}$.

$a \in A$: If ϕ is an atomic formula then the fact that G is a subsystem of B implies that \bar{a} satisfies ϕ in G if and only if it does so in B . We proceed by induction and assume that ϕ_1, ϕ_2 are two formulas in L_r^0 which are satisfied by C in G if and only if they are satisfied by c in IB for every $c \in A$. Then, clearly, $\neg \phi_1$ and $\forall u \phi_2$ are formulas with that property. Finally, if $(\exists x^y)(\phi)$ is satisfied by \bar{a} in B then (by the condition in theorem 4) $\bar{a}(y/b)$ satisfies ϕ in B , hence in G for some $b \in A$; $\forall i \in \text{Vers}(\omega) \exists x_j d_i b_j$ is satisfied by \bar{a} in G then $\bar{a}(y/b)$ satisfies ϕ in G for some $b \in A$,
 $\langle a \rangle_0$

i.e. $c = \bar{a}(y/b) \in A$ satisfies ϕ in G , hence (by assumption on ϕ) in B . Therefore \bar{a} satisfies $(\exists x^y)(\phi)$ in IB . The converse statement is quite evident. q. e. d.

We should pause at this point to see a few examples:
 (Ex. 14): We suggest a rigorous proof of the following simple fact: If $\theta: P \rightarrow Q$ is an isomorphism of relational systems, $\bar{c} \in C^a$,

Let $\bar{c} \in D^a$ the a -tuple resulting from \bar{c} by componentwise application of $*$, $\$ \in \mathcal{L}^{\alpha}$, then \bar{c} satisfies $\$$ in C , if and only if $0(\bar{c})$ satisfies f in f_s .

Thus, $C \wedge f_1$ implies always $C \wedge A$.

(Ex. 15): We have seen that the free algebra $\mathcal{F}(a)$ exists for arbitrary a if K is an equational class. If $C_0 \leq C_1 \leq C_2$ are ordinals and $\{c_0, c_1, \dots, c_n, \dots\}$ is a basis of $\mathcal{F}_v(a)$ then, as one easily checks, $\langle \{c_0, \dots, c_n, \dots\} \rangle$ is a basis of $\mathcal{F}_v(a)$ and we can therefore assume that $\mathcal{F}_T(a) \subseteq \mathcal{F}_T(a_0)$. As was shown by

R. Vaught, $\mathcal{F}_K(a_2)$ is an elementary extension of $\mathcal{F}_K(a_1)$ in particular: $\mathcal{F}_K(a_2) \models \exists x \exists y (x \neq y \wedge \dots)$. We prove this statement using Tarski's criterion in theorem 4: Let $G = \mathcal{F}_K(a_1)$, $IB = \mathcal{F}_K(a_2)$ and $a \in A$, $b \in B$. If a satisfies $(\exists x \exists y) \dots$ in B , then there exists some $b \in B$ such that $a(y/b) \in B$ satisfies $\$$ in IB .

Remark 1.23 implies the existence of natural numbers n, m such that $\bar{a} \in \langle \{c_0, \dots, c_n\} \rangle$ and $b \in \langle \{c_0, \dots, c_m\} \rangle$ (of course, we assume that we listed only pairwise different c 's.) Thus, if we choose a bijection $\varphi: \{c_0, \dots, c_n\} \rightarrow \{c_0, \dots, c_m\}$ with $\varphi(c_0) = c_0, \dots, \varphi(c_n) = c_n$, and $\varphi(c_{n+1}) = c_{n+1}, \dots, \varphi(c_{n+m}) = c_{n+m}$, which we extend to an automorphism $\varphi: \mathcal{F}_K(a_1) \rightarrow \mathcal{F}_K(a_2)$ then $\varphi(\bar{a}(y/b)) = \bar{a}(y/\varphi(b)) \in A$ (see example 14) satisfies $\$$ in B . q. e. d.

(Ex. 16): In theorem 1.5 we introduced the mapping $j = j_r$: $G \rightarrow G^I_D$ which embedded the relational system G into an arbitrary of its ultrapowers. We can now say even more: j is an elementary embedding, i.e. G^I_D is an elementary extension of G . To see this we use again theorem 4 and assume that $(\exists x \exists y) (\$)$ is satisfied

by $\overline{(a)}_I = \overline{(a_i)_{i \in I}}$ $\delta < \alpha \in (A) \overset{I}{G}_D^X$ in G_D^I , Then $*$ is satisfied
 by $\overline{T^{\wedge}_I (y / (b^{\wedge}_{i \in I}))}$ in G_D^I for some $(b_i)_{i \in I} \in \mathcal{P}^A$. Hence
 (see theorem 2), $S(\$, \overline{(a)}_I (y / (b_i)_{i \in I})) \in D$. Since
 $S(*, \overline{T^{\wedge}_I (y / (b_i)_{i \in I})) \in S(*, \overline{(a)}_I (y / (b_i)_{i \in I}))$ for an arbitrary
 fixed $b \in A$, we conclude that $S(\$, \overline{(a)}_I (y / (b_i)_{i \in I})) \in D$, i.e.
 (again by theorem 2) $\overline{(a)}_I (y / (b_i)_{i \in I})$ satisfies $< \&$ in G_D .
 This concludes the proof.

The conclusion of the last example is so important that we
 decide to give it a special emphasis:

Theorem 5; If G is a relational-system then it can be elementarily
 embedded in each of its ultrapowers $G_{\mathcal{D}}$: Hence, $G \overset{I}{\wedge} G_{\mathcal{D}}$ and
 $G \overset{D}{\wedge} G^*$.

The theorem could be essentially sharpened such as to give
 a characterization of elementary equivalence in terms of ultra-
 powers. We prefer to only mention the relevant result due to
 T. Frayne without giving the proof:

Remark 4 (T. Frayne): Two relational systems $G_5 B$ of the same
 type r are elementarily equivalent if and only if B can be
 elementarily embedded in some ultrapower of G .

Another outstanding application¹¹ of theorem 4 is a simple
 proof of the famed Lowenheim-Skolem-Tarski-theorem:

Theorem 6 (Lowenheim-Skolem-Tarski):

Let G be a relational system with m fundamental relations
 and fundamental operations. If $B \in \mathcal{A}$ has cardinality $J > \max \{m, f(b)\}$
 then there exists $C \in \mathcal{A}$ such that $B \subset C$, $|B| = |C|$ and G

is an elementary extension of C .

proof;

We well-order $A = \{a_\gamma; \gamma < \alpha\}$ and define for every $n \in \mathbb{N}_0$ the set C_n inductively as follows: $C_0 = B$, $C_{n+1} = \{b; b \in A \text{ and } \exists c \in C_n \text{ such that } a_\gamma < b \text{ and } \exists x \in C_n \text{ such that } S(x, a_\gamma, b) \text{ in } G\}$.

there exists some $c \in C_n$ such that b is the least element in the well-ordering of A making $a_\gamma < b$ to satisfy $S(x, a_\gamma, b)$ in G . If we choose some $c \in C_n$, $x = c$, $y = 0$ and $a = (c, c, c, \dots)$ then visibly c satisfies the requirements to belong to C_{n+1} . We conclude that

$$C_{n+1} \supseteq C_n \supseteq \dots \supseteq C_0 = B$$

and set $C = \bigcup \{C_n; n = 0, 1, 2, \dots\}$.

If $c_1, \dots, c_n \in C$, say in C_m , and $f \in F$, then $b = f(c_1, \dots, c_n)$

is the unique, hence smallest element which makes a_{n+1}/b

satisfy $f(x_1, \dots, x_n) = x_{n+1}$ in G if $a = (c_1, \dots, c_n, \dots)$.

Thus, $f(c_1, \dots, c_n) \in C_{n+1} \subseteq C$; this proves that $C = \langle C; F, R \rangle$

is a subsystem of $G = \langle A; F, R \rangle$. Since the criterion of theorem 4 is immediately applicable to the subsystem C of G , we conclude that G is an elementary extension of C and, of course, $B \in C$.

We are left with showing that $|B| = |C|$; but this follows from

$$|C_{n+1}| = |C_n| \cdot \max\{|A|, |C_n|\} = |C_n| \cdot |A|$$

$|C| = \aleph_0 \cdot |B| = |B|$. q. e. d.

Corollary: Every infinite relational system $G = \langle A; F, R \rangle$ with $|A| < \aleph_0$ has a countable elementary subsystem. In particular, this holds for all M classicalⁿ structures, as semi-groups, groups, rings, etc.

We want to conclude this section with a short discussion of 'pure subsystems,' a concept closely related to elementary subsystems and retracts, as we shall see.

Def. 13: A relational system G is called a pure subsystem of B (or B a pure extension of G) if G is a subsystem of B and any finite set of atomic formulas in $L_{\tau}(G)$ (see def. 10) which is satisfiable in B is also satisfiable in G . If the same condition holds only for all finite sets of atomic formulas in L_{τ} then we call G a weakly pure subsystem.

Let us assume for a minute that G is an elementary subsystem of B and let S be a finite set of atomic formulas with constants in A , say $S = \{o_1, \dots, o_n\}$. If x_1, \dots, x_m are the free variables occurring in o_1, \dots, o_n then we consider the sentence
$$\exists x_1 \dots \exists x_m (o_1 \wedge \dots \wedge o_n) \in L(G)$$
 which holds in G if and only if it holds in B ; thus $\{o_1, \dots, o_n\}$ is satisfiable in G if and only if it is satisfiable in B . In other words: G is a pure subsystem of B . We sum up:

~~Remark 5:~~ Each elementary subsystem of a relational system B is a pure subsystem of B .

Let us assume that G is not only a pure subsystem of $\langle R \rangle$ but that all sets of atomic formulas in $L_{\tau}(G)$ which are satisfiable in B are satisfiable in G . Then we write down the set of all possible correct polynomial equalities $p(b_1, \dots, b_m) = q(b_1, \dots, b_m)$ and all true relations $R(b_1, \dots, b_m)$

where $B = \{b_1, \dots, b_r, \dots\}$. We associate with each such equality (resp. relation) the atomic formula $P(x_{b_1}, \dots, x_{b_r}, \dots)$ (resp. $R(x_{b_1}, \dots, x_{b_r}, \dots)$) as follows: If $b \in B \setminus A$ then x_b is a free variable; if $b \in A$ then $x_b = b$. Then the set \mathcal{F} of all these formulas is satisfiable in \mathcal{B} by construction. So, by our assumption, \mathcal{F} is satisfiable in G . Let us assume that $(a_{b_1}, \dots, a_{b_r}, \dots)$ is a solution in G (i.e. if $b \in B \setminus A$ then $a_b \in A$ has to be substituted for the free variable x_b ; if $b \in A$ then $a_b = b$): Then we can define the mapping $\langle p: B \rightarrow A$ by $\langle p(b) = a_b$. If $f(b_1, \dots, b_n) = b$ for $f \in F$ then $\langle p(f(b_1, \dots, b_n)) = a_b = f(a_{b_1}, \dots, a_{b_n}) = f(\langle p(b_1), \dots, \langle p(b_n))$. If $R(b_1, \dots, b_n)$ holds true then $R(\langle p(b_1), \dots, \langle p(b_n))$ must hold true. So $\langle p$ is a weak epimorphism.

Visibly, $\langle p$ restricted to A is the identity. Thus, via the next definition (which extends def. 1.41) we showed that every retract of a relational system \mathcal{B} is pure in B .

Def. 14: If $G \subset B$ are relational systems, then a weak epimorphism $\langle p: B \rightarrow G$ (1 is called a retraction if $\langle p = \text{id}_G$. G is then a retract of B .

Remark 6: If G is a retract of the relational system \mathcal{B} then G is pure in B .

(Ex. 17): Apart from our approach one knows in the theory of Abelian groups the concept of a "pure subgroup".^{f1} We recall: $\langle H; +, -, 0 \rangle$ is a pure subgroup of the Abelian group $\langle G; +, -, 0 \rangle$ if every equation $n \cdot x = h$, $n \in \mathbb{N}$, $h \in H$ is solvable in H if-

and only if it is solvable in G (As a matter of fact, one can easily see that it already suffices to require the last property of equations $p * x = h$ where p is a prime power). Since $n \ll x = h$, as just introduced, is an atomic formula in the language of Abelian groups, it is quite clear that every pure subsystem $\langle H; +, -, 0 \rangle$ of the Abelian group $\langle G; +, -, 0 \rangle$ (in the sense of definition 13) is a pure subgroup. But the converse is true too as was shown by Gacscllyi [1]. Thus, in case of Abelian groups the notions of a ¹¹ pure subsystem^M and a ^{f1} pure subgroup¹¹ coincide.

Chapter III. Compactness in Relational Systems and Algebras.

Now that we are familiar with the basic concepts underlying our theory we intend to collect in this chapter the essential results known so far on the impact of the diverse kinds of compactness (introduced in chapter I) on the properties of diverse algebraic structures. Questions as ^M Is Mycielski's conjecture true for Boolean Algebras? ^{tf} or ^{ff} What exactly are the equationally compact Abelian groups? ^{!f} or ^{ft} If a relational system has an equational compactification does it have one in the same equational class? ^{ff} and the like will be answered as far as known, unsolved problems stated, general connections between concepts like ^{ff} equational compactness^M, ^M elementary extension¹¹, ⁿ pure subsystemⁿ, ⁿ ultrapower^{tf} will be established. Many of the core results are due to the Polish mathematicians Weglorz, Pacholski, Ryll-Nardzewski and Mycielski. We begin our chapter with a characterization of equationally compact semi-lattices, a result that does not require any further theory at this point. We agree to write equations $(f \star \langle f \rangle) \text{ in } t^e \text{ form } f = \underset{\sim}{g}$ and to replace e by x .

1. Equationally compact semilattices.

We recall that a semi-lattice is a universal algebra $S = \langle S; V \rangle$ of type $\langle 2 \rangle$ with a commutative, idempotent and associative binary operation V . One defines a partial order on S by specifying $s_1 \leq s_2$ if $s_1 \vee s_2 = s_2$.

Theorem 1 (Grätzer and Lakser [2]): The semi-lattice $S = \langle S; V \rangle$ is equationally compact if and only if the following three conditions are satisfied:

- (1) Every subset $T \subseteq S$ has a least upper bound $V(t; tGT)$.
- (2) Every chain C in S has a greatest lower bound $A(C; CGC)$.
- (3) If $a \in S$ and C is a chain in S then $aV(A(C; CGC)) = A(aV_C; CGC)$.

proof:

To prove the result we have to study systems of equations in $P_i^{a)}(\langle 2 \rangle \times P_{\mathcal{A}}^{*a)}(\langle 2 \rangle)$ (see def. 1.39). Utilizing the identities on semilattices we see immediately that every such equation is of the type $s V x_0 \vee \dots \vee x_{n-1} = t V x_0 \vee \dots \vee x_{m-1}$ where $s, t \in S$ and $n, m \geq 0$ (with the usual convention that $s V x_0 \vee \dots \vee x_{n-1} = s$ in case $n = 0$).

(a) We first show the easier implication, namely the fact that equationally compact semi-lattices satisfy the conditions (1)-(3):

To see (1) we consider the set of equations

$E_1 = \{tVx = x; t \in T\} \cup \{xVu = u; u \in U\}$ if U is the set of upper bounds of T . Since every finite subset $\{t_0 V x = x, \dots, t_m V x = x\} \cup \{x v u_0 = u_0, \dots, x v u_n = u_n\}$ has the solution $x = t_0 V t_1 \vee \dots \vee t_m$ we conclude that E_1 has a solution which is evidently $V(t; tGT)$. To verify (2) we choose the chain C and the set of equations $E_2 = \{cVx = c; c \in C\} \cup \{x \setminus A = x; t \in L\}$ where

L is the set of lower bounds of C . Since every finite subset $(c_0 \vee x = c_0 \dots j \wedge c_m \vee x - c_m \wedge u f^{x \vee} \wedge_0 = x, \dots, x \vee * n = x \wedge$ of T_2 has the solution $x = \min\{c_0, \dots, c_j\}$ we conclude that T_2 has a solution which is evidently $A(c; c \in C)$. Finally, to show (3) we take $s \in S_y$ a chain C_0 and put $c_0 = A(c; c \in C_0)$. Of course, if C_0 is a chain then so is $C_1 = \{c \vee s; c \in C_0\}$ and we put $c_1 = A(c; c \in C_1)$. We have to show that $s \vee c_0 = c_1$ i.e. (since $s \vee c_0 \leq c_1$ is clear) that $s \vee c_0 \geq c_1$. To see this we consider the set $T_3 = \{s \vee x = c_1 \vee x\} \cup \{x \vee d = d; d \in C_0\}$. Since the finite subset $\{s \vee x = c_1 \vee x, x \vee d = d, \dots, x \vee d = d\}$ has the solution $x = \min\{d_0, d_1, \dots, d_m\}$ we conclude that T_3 has a solution e . Hence $s \vee e = c_1 \vee e$, i.e. $s \vee e \geq c_1$ which (since $e \vee d = d$, i.e. $e \leq c_0$ for all $d \in C_0$) implies that $s \vee c_0 \wedge s \vee e \wedge c_1$, i.e. $s \vee c_0 \wedge c_1$.

(b) To prove the converse direction of the theorem we refer to the standard argument which shows that (2) implies that every downward directed set D has a greatest lower bound $A(d; d \in D)$ and (3) implies that $s \vee (A(d; d \in D)) = A(s \vee d; d \in D)$ for every such set (Recall: D is downward directed if $d_1 \wedge d_2 \in D$ implies the existence of some $d \in D$ such that $d_1 \wedge d, d_2 \wedge d$). We sum up:

(2*) if D is downward directed then $A(d; d \in D)$ exists.

(3*) If D is downward directed and $s \in S$ then $s \vee (A(d; d \in D)) = A(s \vee d; d \in D)$.

We are now in a position to state the two lemmas crucial for our proof:

Lemma 1: Let $s \vee x_1 \vee \dots \vee x_{n-1} = r \vee x_0 \vee \dots \vee x_{m-1}$ be an

equation in $P^{(a)} \langle 2 \rangle \times P^{(a)} \langle 2 \rangle$ whose solution-set $K \subseteq S^a$ is downward directed ($S^a = \langle S^a; \vee \rangle$ is the direct product). Then $t = A(k; k \in K)$ is a solution of that equation.

proof: Let $K = \{c_A; A \in A\}$ with $c_A = (c_A(i))_{i \in I_A}$. Then $t(i) = \bigwedge_{A \in A} (c_A(i); A \in A)$; hence, $s \vee t(i_0) \vee \dots \vee t(i_{n-1}) = s \vee (A(c_A(i_n); A \in A)) \vee \dots \vee (A(c_A(i_{n-1}); A \in A)) = (we apply (3')) = MsVc^{\wedge}(i_0) \vee \dots \vee c^{\wedge}(i_{n-1}; A \in A)$. Similarly, $r \vee t(j_0) \vee \dots \vee t(j_{m-1}) = A(rVc_A(j_0) \vee \dots \vee c^{\wedge}(j_{m-1}); A \in A)$. Since, by assumption, $s \vee c^{\wedge}(i_0) \vee \dots \vee c^{\wedge}(i_{n-1}) = r \vee t(j_0) \vee \dots \vee t(j_{m-1})$ for all $A \in A$ we conclude that $s \vee t(i_0) \vee \dots \vee t(i_{n-1}) = r \vee t(j_0) \vee \dots \vee t(j_{m-1})$, i.e. t is a solution of our equation. q. e. d.

Similarly one proves the next lemma:

Lemma 2: Let $p(x) = s \vee x_0 \vee \dots \vee x_{n-1}$, $q(x) = r \vee x_0 \vee \dots \vee x_{m-1}$

and $g(x) = g(x)$ an equation with solution set $K \subseteq S^a$. Then $t = V(k; k \in K)$ is a solution for $p(x) = q(x)$.

proof: As above, we get $p(t) = \bigwedge_{A \in A} (s \vee k(i_0) \vee \dots \vee k(i_{n-1}); A \in A)$ and $q(t) = \bigwedge_{A \in A} (r \vee k(j_0) \vee \dots \vee k(j_{m-1}); A \in A)$. Thus, $p(t) \wedge V(q(k); k \in K) = q(t)$. Symmetrically, $q(t) \wedge p(t) = p(t)$. i.e. $q(t) = p(t)$. q. e. d.

With these two lemmas at our disposal we can now settle the claim that (1), (2), (3) imply the equational compactness of the semi-lattice \mathcal{E} . To this end we choose a set of equations $\{e_i; i \in I\} \subseteq P^{(a)} \langle 2 \rangle \times P^{(a)} \langle 2 \rangle$ which is finitely solvable; i.e. every finite subset

$J?$ of $Y_i \wedge^{as} a$ non-empty solution set $K(T^*)$. By lemma 2, $t(T^*) = V\{k; k \in K(r^*)\} \in K(r^*)$. Moreover, the set $K(JD = \underline{q}) = \{t(P); P \in T, |\wedge^*| < V \text{ and } p = q \in T^*\}$ is downward directed for every $p = q \in T$, since $E^* \subseteq T^{**}$ implies $t(T^*) \Rightarrow t(T^{**})$. Since $K(\underline{p} = \underline{q})$ is the solution-set of $\underline{p} = \underline{q}$ we get (by lemma 1) that $u(g = \underline{c}_j) = A\{k; k \in K(g = 2 \wedge \dots)\}$ is a solution for $\underline{p} = \underline{q}$. Moreover, since $K(\wedge^1 = \underline{j}_j)$ and $K(\wedge_2 = \wedge_2)$ are mutually downward cofinal in S we obtain the fact that $M(\underline{p}_i = \wedge_{-i}) \wedge w(\underline{p}_0 = \underline{q}_0)$ for any two $\wedge^1 = \wedge^1$ $\wedge_2 = \wedge_2 \in T$. Thus, $M^1 = \wedge^1$ is a solution for \wedge , i.e. \wedge is solvable in \mathcal{L} . q. e. d.

Now that we have characterized equationally compact semilattices it is natural to ask for the solution of the following specific open problem:

Problem Zi Is every equationally compact semilattice the algebraic retract of some topologically compact semilattice? (or embeddable in one?).

A trivial application of theorem 1 yields the following corollary concerning equationally compact lattices:

Corollary: If $\langle \wedge = \langle L; V, A \rangle$ is an equationally compact lattice then

(1) \wedge is complete

(2) If $D \subseteq L$ is a downward [upward] directed set then

$$s V(A\{d; d \in D\}) = A(sVd; d \in D) \quad [s A(v\{d; d \in D\}) = v(sAd; d \in D)].$$

(3) If $a > b$ are two elements in L and $\{c_i; i \in I\}$ is a set of elements in L such that i / j implies $c_i \vee c_j = a$, $c_i \wedge c_j = b$.

then $|1| < \hat{\circ}$.

proof: While (1) and (2) follow immediately from theorem 1, (3) follows from the set of equations $T = [x^i = a_i / j, i, j \in I] \cup \{x_i x_j = b_{ij} / j, i, j \in I\}$ [see example 3, chapter I].

Problem 3; Find necessary and sufficient conditions for a lattice to be equationally compact.

Remark: I do not know a lattice satisfying conditions (1),(2),(3) of the above corollary which is not equationally compact.

Although the characterization of equationally compact lattices is open at the time, some results have been obtained on lattice-related systems. But before discussing these results and thus rounding up the picture on the situation in lattices it will prove useful to have some more general results at our disposal which we will now study in section 2.

J2. Equivalence of different kinds of algebraic compactnesses.

In definition 1.39 we got acquainted with the concept of S -equational m -compactness. Equational compactness of an algebra G was the special case of A -equational m -compactness for all cardinals m . We now extend the realm of these definitions as follows:

Def. 1: (a) Let $K \subset L$ be a class of formulas in the language L_T . A relational system $G \in K(r)$ is called weakly K -compact

if every set $T \subset K$ is satisfiable in G provided every finite subset of T is (we then call T finitely satisfiable in G). If $K(G) = K \cup L_T(G)$ and every set $E \subset K(G)$ which is finitely satisfiable in G is satisfiable in G then we call G K -compact.

(b) If K is the class of all equations with constants in A (without constants) then we call [of course, in agreement with def. 1.39] the algebra G (weakly) equationally compact.

(c) If K is the class of all atomic formulas with constants in A (without constants in A) then we call the K -compactness (weak K -compactness) atomic compactness (weak atomic compactness).

(d) If K is the set of all positive formulas [i.e. the set of all formulas whose matrix in the prenex normal form contains not the negation sign-?] with constants in A (without constants in A) then K -compactness (weak K -compactness) is called positive compactness (weak positive compactness).

In this section we derive a few results centered around a characterization of atomic compactness. Analogous results for

the ^{ff} weak¹¹ case are collected in the next section.

In remark 6 of chapter II we established the fact that every retract G of a relational system (B) is pure in IB . It is interesting to notice that if G is atomic compact then G is pure in IB if and only if it is a retract of IB .

Remark 1: If G is an atomic compact relational system then G is a pure subsystem of (B) if and only if G is a retract of IB .

proof: For the one half of the proof see remark II.6. So assume that G is pure in (B) and let \mathcal{E} be a set of atomic formulas with constants in A which is satisfiable in (B) . Then \mathcal{E} is finitely satisfiable in iB , hence (since G is pure in IB) finitely satisfiable in G . Thus (since G is atomic compact), \mathcal{E} is satisfiable in G which (remark II.6) proves that G is a retract of iR . q. e. d.

We have now all the pieces we need to put together a fundamental result due to Weglorz:

Theorem 2 (Weglorz): If G is a relational system of type r then the following conditions are equivalent:

- (1) G is positively compact.
- (2) G is atomic compact.
- (3) G is a retract of every algebraic system in which G is pure.
- (4) G is a retract of every elementary extension of G .
- (5) G is a retract of every ultrapower of G .

proof: (1) implies (2) since every atomic formula is positive.
 (2) implies (3) by remark 1. (3) implies (4) because of remark 11.5.
 (4) implies (5) because of theorem II.5. To prove that (5) implies
 (1) we pick a set of positive formulas $T \subseteq L_{\tau}(G)$ which is
 finitely solvable in G . Then corollary 2 of theorem II.3 assures
 that T is solvable in some ultrapower $G_{\mathcal{D}}^{\mathbf{I}}$ of G . If $\varphi: G \rightarrow G_{\mathcal{D}}^{\mathbf{I}}$
 is the retraction which exists by (5) and (c_y) is a solution
 of T in $G_{\mathcal{D}}^{\mathbf{I}}$ then $(\varphi(c_y))_{y \in \mathcal{A}}$ is a solution of T in G as
 is assured by Marczewski's well-known theorem stating the invariance
 of positive formulas under homomorphisms (see Marczewski [1] or
 Grätzer [1], chapter 7). Thus, T is solvable in G which proves
 our point. q. e. d.

To state one important corollary we need the following concept:

Def. 2: If G is a relational system in the class L of relational systems then G is an absolute retract in L if it is a retract of every extension in L .

(Ex. 18): We recall that a relational system G is injective in a class L if every weak homomorphism $\varphi: B \rightarrow G$, $B \in L$ can be extended to a weak homomorphism $\varphi: C \rightarrow G$ if C is an extension of B in L . Thus, if G is injective in L and $C \in L$ contains G as subsystem then the identity-mapping $1_G: G \rightarrow G$ can be extended to a weak homomorphism, hence retract $\varphi: C \rightarrow G$.

Hence: Every relational system G which is injective in the class L is an absolute retract in L .

Corollary 1 to theorem 2: If L is a class of relational systems of type r which is closed under the formation of ultrapowers (as, e.g., equational classes of algebras) then every absolute retract G in K is atomic compact.

proof; This is clear since G is retract of each of its ultrapowers, q. e. d.

Corollary 2: If L is as above then every injective relational system in L is atomic compact.

proof: This, of course, is clear from example 18 and corollary 1. q. e. d.

Let us conclude with a trivial remark concerning the preservation of atomic compactness under algebraic construction (we skip the proof):

Remark 2: Direct products and retracts of atomic compact algebras are atomic compact.

Before we switch to }2a which contains parallel results on weakly K -compact relational systems we should bring to mind the obvious fact that atomic compactness and equational compactness, although different concepts in relational systems $G = \langle A; F, R \rangle$ with $R / \langle f \rangle$, coincide for universal algebras. Thus, every result we obtained and will obtain on atomic (resp. positive) compactness has direct relevance towards questions concerning equational compactness of algebras.

§2a. Weak atomic compactness.

Using once more Marczewski's theorem stating the invariance of positive formulas under homomorphism one proves easily the next useful remark:

Remark 3: If B, G are relational systems of the same type r , $B \subseteq G$ and $h: G \rightarrow B$ a weak homomorphism then the weak atomic compactness of any one of the three systems $h(G)$ or B or G implies the same property for the remaining two systems.

proof: Assume that G is weakly atomic compact and $T \subseteq L_r$ a set of atomic formulas finitely solvable in $B(h(G))$; then D is solvable in G . If $\bar{a} \in A^a$ is a solution then $h(\bar{a}) \in B^a(h(G)^a)$ is a solution of T in $B(h(G))$; hence, $B(h(G))$ is weakly atomic compact. The remaining two proofs are analogous. q. e. d.

Corresponding to remark 2 (up to the point that we skip its proof) we have remark 4:

Remark 4: The direct product of weakly atomic compact relational systems is again weakly atomic compact.

Before we proceed let us give an example:

(Ex. 19): Every lattice $\mathfrak{L} = \langle L; V, A \rangle$ is weakly atomic (i.e. weakly equationally) compact. This is clear since every element $a \in L$ forms a one-element-sublattice $\mathfrak{L}' = \langle [a]; V, A \rangle$ which is a homomorphic image of \mathfrak{L} . \mathbb{Z}^3 is, of course, weakly equationally compact; thus, remark 3 settles the matter. For the same reason are **all groups** $\mathbb{Q} = \langle G; *, \sim^1, 1 \rangle$ and rings, $\mathfrak{R} = \langle R; +, -, 0, \cdot \rangle$ weakly equationally compact.

A basic theorem (corresponding to theorem 2) concerning characterizations of weakly atomic compact relational systems is the following:

Theorem 3 (Weglorz): If G is a relational system of type T then the following conditions are equivalent:

- (1) G is weakly positively compact.
- (2) G is weakly atomic compact.
- (3) G contains a homomorphic image of every algebraic system ft in which G is weakly pure.
- (4) G contains a homomorphic image of every elementary extension of G .
- (5) G contains a homomorphic image of every ultrapower of G .

proof: (1) implies (2), (3) implies (4), (4) implies (5) and (5) implies (1) for exactly the same reasons as were valid in the corresponding implications of theorem 2. Thus, let us prove that (2) implies (3)! To do so we choose the system \mathcal{E} of atomic formulas in L as follows: We take the set S of all possible expressions $\mathcal{A}(b_1, \dots, b_n)$ where $\mathcal{A} = \mathcal{A}(x_1, \dots, x_n)$ is an atomic formula in the free variables x_1, \dots, x_n and the substitution $b_j \rightarrow x_{i_j}$ satisfies \mathcal{A} in B . For each such expression we introduce free variables x_{p_1}, \dots, x_{p_n} and define $\mathcal{E} = \{ \mathcal{A}(x_{p_1}, \dots, x_{p_n}); \mathcal{A}(b_1, \dots, b_n) \in S \}$. Evidently \mathcal{E} is solvable, hence finitely solvable in ft . Therefore, T is finitely

solvable in G , since G is weakly pure in B . Due to the weak atomic compactness of G we conclude that S is solvable in G by, say, a system $\{a_b - x_b\}_{b \in B}$ (i.e. we have to substitute $a_b - x_b$). Then $h: B \rightarrow A$ mapping b to a_b is a homomorphism. $q. e. d.$

We now go on to continue our discussion of §1 and to collect all results we know so far on compactness in lattice-related structures.

§3. Lattice-related structures.

If $Z = \langle L; V, A \rangle$ is a lattice, i.e. an algebra in the class L of all lattices, then $Z(\leq)$ denotes the associated poset $\langle L; \leq \rangle$ and $Z_0^1 = \langle L; V, A/0, 1 \rangle$ denotes the smallest lattice with smallest element 0 and largest element 1 containing \mathcal{L} . If $IL(\leq)$ denotes the class of all posets $\mathcal{L}(\mathcal{L})$ then $IL(\leq) \leq (Q$ where 0 is the class of all posets. $IL(\leq)$ consists of the so-called lattice-induced posets. We have the following main-result on these (see Weglorz [3]):

Theorem 4: If $\mathcal{L} = \langle L; V, A \rangle$ is a lattice then the following statements are equivalent:

- (1) $\mathcal{L}(\mathcal{L})$ is atomic compact.
- (2) \mathcal{L} is complete.
- (3) $Z(\mathcal{L})$ is injective in $0_{\#}$
- (4) $Z(\mathcal{L})$ is injective in $|L(\mathcal{L})$.
- (5) $\mathcal{L}(j\mathcal{L})$ is an absolute retract in $IL(\leq)$.
- (6) $Z(\mathcal{L})$ is an absolute retract in \mathbb{D} .

proof:

(1) implies (2) Let $T \subseteq L$ and

$D = \{x \wedge t; t \in T\} \cup \{x \wedge u; u \in U\} \cup \{y \wedge t; t \in T\} \cup \{y \wedge L; L \in L\}$
 $\mathcal{L}(\mathcal{L}(\leq))$ where U is the set of upper bounds of T , L is the set of lower bounds of T . Then $T <$ is finitely satisfiable in $\mathcal{L}(\leq)$ hence satisfiable in $\mathcal{L}(\leq)$. The solution $x = a$, $y = b$ provides clearly a least upper bound a and a greatest lower bound b of T .

(2) implies (3):

Let $G = \langle A; \leq \rangle$, $B = \langle B; \leq \rangle \in \mathcal{O}$, $G \subseteq B$ and $h: A \rightarrow L$ a weak homomorphism of G in $\mathcal{F}(\leq)$. We have to show that h can be extended to B . To that end we apply the standard pattern: Namely we take the set S of all pairs (C, f) where $G \subseteq C \subseteq B$, $C \in \mathcal{O}$, f is a weak homomorphism $f: C \rightarrow L$ and $f|_A = h$. Since $(G, h) \in S$, S is not empty and Zorn's lemma assures the existence of a maximal element $(B_0, h_0) \in S$. Assuming $B_0 \subsetneq B$, say $c \in B \setminus B_0$, we can form $u = \bigcup \{h_0(b) \mid b \in B_0 \text{ and } b \leq c\}$, $v = \bigcap \{h_0(b) \mid b \in B_0 \text{ and } b \geq c\}$, choose an arbitrary element a with $u \leq a \leq v$ and define $h_1: B_0 \cup \{c\} \rightarrow L$ by $h_1|_{B_0} = h_0$ and $h_1(c) = a$; then (as easily verified) $(\langle B_0 \cup \{c\}; \leq \rangle, h_1) \in S$ contradicting the maximality of (B_0, h_0) . We conclude that $B_0 = B$, and $h: B \rightarrow L$ is the desired weak homomorphism extending h .

(3) implies (4) is, of course, obvious. So is the fact that (4) implies (5) (see example 18, this chapter).

(5) implies (6); We recall the McNeille-embedding of arbitrary posets in posets induced by complete lattices (see chapter I, preceding remark 7). Thus, if $P \in \mathcal{O}$ contains $Z(\mathcal{F})$ then we McNeille-embed it in P' . By assumption, there exists a retraction $\varphi: P' \rightarrow \mathcal{F}(\leq)$ hence a retraction $\varphi|_P: P \rightarrow \mathcal{F}(\leq)$ which was to be shown.

(6) implies (1) by corollary 1 to theorem 2 since \mathcal{U} is closed under the formation of ultrapowers. q. e. d.

Looking at the theorem one feels inclined to conjecture that possibly the atomic compact posets are exactly the absolute

retracts in $\hat{0}$ (since this after all holds for posets of the form $\mathcal{L}(\mathcal{L})$). This is not only not true (see next example) but we have even the nice result that $P \in \hat{0}$ is an absolute retract in \hat{D} if and only if $P = \mathcal{L}(\mathcal{L})$ for some complete lattice \mathcal{L} (see next remark).

(Ex. 20): The poset $G = \langle \{a, b, c\}; \mathcal{L} \rangle$ with $a < c$, $b < c$ and unrelated a, b is evidently atomic compact [a short rigorous reason would be the fact that $G^* = \langle \{a, b, c\}; \vee \rangle$ with $a \vee b = c$ is an equationally compact semi-lattice]. Equally evidently G is not an absolute retract in \hat{U} [e.g. G is not a retract of $IB = \langle \{a, b, c, d\}; \mathcal{L} \rangle$ where $d < a, b, c$ and $\langle \{a, b, c\}; \mathcal{L} \rangle = G$].

Remark 5: If the poset $P = \langle P; \mathcal{L} \rangle$ is an absolute retract in \hat{D} then there is a lattice Z such that $\mathcal{L}(Z) = P$.

proof: Again we can assume that P is McNeille-embedded in a (complete) lattice $\mathcal{L} = \langle K; \vee, \wedge \rangle$ such that $P \subseteq \mathcal{L}(\mathcal{L})$. By assumption there exists a retraction $\mathcal{L}(\mathcal{L}) \rightarrow P$. All we have to show is that any two elements $a, b \in P$ have a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$ in P . We know that we have a greatest lower bound c of a and b in $\mathcal{L}(\mathcal{L})$, i.e. (1) $c \leq a, b$, hence, $\mathcal{L}(c) \leq \mathcal{L}(a) = a$, $\mathcal{L}(c) \leq \mathcal{L}(b) = b$ and (2) if $x \in P$ and $x \leq a, b$ then $x \leq c$, i.e. $x = \mathcal{L}(x) \geq \mathcal{L}(c)$. (1) and (2) establish that $\mathcal{L}(c) = a \wedge b$ in P . Dually we can prove that $\mathcal{L}(d) = a \vee b$ in P if d is the least upper bound of a and b in $\mathcal{L}(\mathcal{L})$. q. e. d.

To go back to theorem 4 we recollect that the atomic compactness of the lattice-induced poset $\mathfrak{L}(\langle i \rangle)$ is equivalent to the completeness of \mathfrak{L} . As remarked in fl (see protie.m 3) a characterization of equationally compact lattices is not known at the time, although we know that the mere requirement of completeness is surely not enough (see the corollary to theorem 1). A counter-example is the frequently quoted example 3 which violates condition 3 of the corollary to theorem 1. Another counter-example violating condition (2) of the same corollary is the next one:

(Ex. 21): Let $\mathfrak{L} = \langle L; V, A \rangle$ be the complete lattice defined as follows: $L = \{0\} \cup \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{x\} \cup \{(x, \frac{1}{n}); n \in \mathbb{N}\} \cup \{x, 0\}$

$\cup \{y\} \cup \{A\} \cup \{B\}$ where x, y, A, B are mutually different symbols, B is the largest, A the smallest element in the lattice and

the operations are determined by (1) $OV \frac{1}{n} = \frac{1}{n}$, $OVx = (x, 0)$, $OV(x, \frac{1}{n}) = B$, $OV(x, 0) = (x, 0)$, $OVy = y$, $\mathfrak{L} \vee \frac{1}{m} =$

$$\max\{\frac{1}{n}, \frac{1}{m}\} \quad \mathfrak{L} \vee x = (x, \frac{1}{n}) \quad \mathfrak{L} \vee (x, 0) = \frac{1}{n} \quad \mathfrak{L} \vee (x, \frac{1}{m}) =$$

$$B, \quad \mathfrak{L} \vee y = B, \quad x \vee (x, \frac{1}{n}) = (x, \wedge), \quad x \vee (x, 0) = (x, 0),$$

$$x \vee y = y, \quad (x, \frac{1}{n}) \vee (x, \frac{1}{m}) = (x, \max\{\frac{1}{n}, \frac{1}{m}\}), \quad (x, \frac{1}{n}) \vee y =$$

$$(x, \wedge) \quad \text{and} \quad (2) \quad 0 \wedge \wedge = 0, \quad 0 \wedge x = A, \quad 0 \wedge (x, \wedge) = 0 \wedge (x, 0) =$$

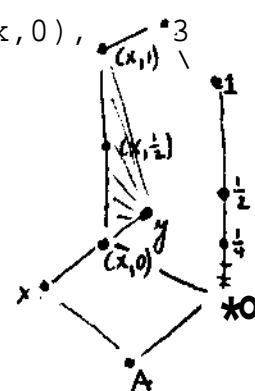
$$0, \quad 0 \wedge y = 0, \quad \frac{1}{n} \wedge \frac{1}{m} = \min\{\frac{1}{n}, \frac{1}{m}\}, \quad \frac{1}{n} \wedge x = A, \quad \frac{1}{n} \wedge (x, \frac{1}{m}) = 0 =$$

$$\frac{1}{n} \wedge (x, 0), \quad \frac{1}{n} \wedge y = 0, \quad x \wedge (x, \frac{1}{n}) = x, \quad x \wedge (x, 0) = x, \quad x \wedge y = x, \quad (x, \frac{1}{n}) \wedge (x, \frac{1}{m}) =$$

$$(x, \min\{\frac{1}{n}, \frac{1}{m}\}), \quad (x, \frac{1}{n}) \wedge (x, 0) = (x, 0), \quad (x, \frac{1}{n}) \wedge y = y, \quad (x, 0) \wedge y = (x, 0).$$

Thus, $XV(A(\frac{1}{n}; n \in \mathbb{N})) = x \vee 0 = (x, 0) \neq y = A(x, \frac{1}{n}; n \in \mathbb{N}) =$

$\wedge(x \vee \frac{1}{n}; n \in \mathbb{N})$ which (in view of the corollary to theorem 1) shows that Z is not equationally compact.



We know much about atomic compactness in $\mathcal{L}(\langle \underline{\quad} \rangle)$, little about equationally compact lattices \mathcal{L} , but we know everything in another lattice-related structure, namely in Boolean algebras $IS = \langle B; V, A, ', 0, 1 \rangle$. The following two theorems contain the crucial information (see Weglorz [1]).

Theorem 5: If $B = \langle B; V, A, ', 0, 1 \rangle$ is a Boolean algebra then the following conditions are equivalent:

- (1) B is complete.
- (2) B is injective in the class \mathcal{J}_1 of all Boolean algebras.
- (3) B is an absolute retract in the class \mathcal{J}_D of all Boolean algebras.
- (4) B is equationally compact.

proof; The fact that (1) implies (2) follows from a standard argument as it was displayed in the proof of theorem 4 when we showed that (2) implies (3) [besides, the result is well-known]. (2) implies (3) and (3) implies (4) was established before [see example 18 and corollary 1 to theorem 2] since \mathcal{J}_D as an equational class is closed under the formation of ultrapowers. If IB is equationally compact then so is the lattice $B' = \langle B; V, A \rangle$ which because of the corollary to theorem 1 in section 1 implies that B is a complete Boolean algebra. q. e. d.

The next theorem establishes that Mycielski's conjecture is true in the class .

Theorem 6: The Boolean algebra B is equationally compact if and only if it is a retract of some topologically compact

Boolean algebra (%)

proof: Of course, we only need to show one direction and assume that B is equationally compact. Then we know that B is complete and can be embedded in a complete atomic Boolean algebra C (more concretely: C can be chosen to be the complete Boolean algebra of all subsets of the Stone space of B). Since, as well-known, C is the direct product of 2-element Boolean algebras B_2 it carries the compact Tychonoff-topology resulting from the discrete topology on B_2^* . Thus, C is topologically compact. Since $H \subseteq C$ and B is an absolute retract in J_5 we proved the result. q. e. d.

The question concerning weak equational compactness of Boolean algebras B is, of course, easily decided. Since $B \geq_{ft_2} \langle \{0,1\}; V, A, ', 0, 1 \rangle$ and B_2 is a retract of B we know that every Boolean algebra is weakly equationally compact because of remark 3 in section 2a. We conclude our section with this obvious remark:

Remark 6: Every Boolean algebra is weakly equationally compact. So is every lattice and, of course, semilattice. So is, in general, every universal algebra with a finite homomorphic image as subalgebra.

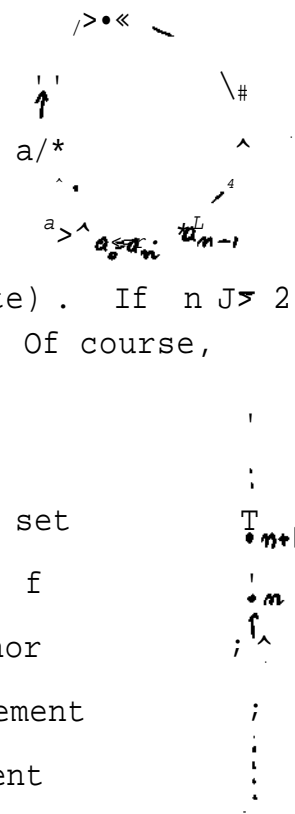
4. Algebras with a single unary operation.

In this section we apply some of the results obtained in section 2 in order to investigate the questions centered around the concept of equational compactness in algebras $G = \langle A; f \rangle$ of type $T = \langle 1 \rangle$ (i.e. f is a single unary operation on A). The results are essentially taken from Wenzel [2] and Pacholski and Weglorz [1]. First we characterize those algebras $G = \langle A; f \rangle$ that are equationally compact. We recall: G is connected if for any two $a, b \in A$ there exist $n, m \in \mathbb{N}_0$ such that $f^n(a) = f^m(b)$. It proves useful to introduce a few suggestive concepts in addition.

Def. 3: If $G = \langle A; f \rangle$ is a unary algebra then $a \in A$ is called a stagnant element if $f(a) = a$. $st(G)$ is the set of all stagnant elements in A . If $n \in \mathbb{N}_0$ and $a \in A$ then the n -periphery $n_G(a)$ is the set of all elements $b \in A$ such that $f^n(b) = a$ and $f^{n-1}(b) \neq a$ (provided $n-1 \geq 0$). In case $a \in A$ satisfies $a = f(b)$ for all $b \in A$ we call a a minimal element. If $a \in A$ is an element such that (1) $f^m(a) \neq a$ for all $m \in \mathbb{N}$ and (2) $n_G(a)$ contains a minimal element then a is said to have order ri in G , i.e. $ri(a) = 2^{n^*}$. Finally, if $a \in A$ satisfies (1) $f^m(a) \neq a$ for all $m \in \mathbb{N}$ and (2) there is an infinite chain $a_0, a_1, \dots, a_m, \dots$ such that $a_i = f(a_{i+1})$, for $i \geq 0$ and $a_m = f(a_{m+1})$, then a is an element of infinite order.

(Ex. 22): The algebra $r_n = \langle C_n; f \rangle$ with n elements $C_n = \{a_0, a_1, \dots, a_{n-1}\}$ whose indices are

determined modulo n and whose operation f is defined by $f(a_k) = a_{k+1}$ is called the cyclic algebra with n elements. It has no stagnant elements unless $n = 1$. None of its elements has an order (be it finite, be it infinite). If $n \neq 2$ then $m_{n,n}(a_i) \neq p\{$ for all $m \in \mathbb{N}$ and $a_i \in C_{in}$. Of course, $m(a_i) / of$ is equivalent to $m = 0$.



(Ex. 23) ; The algebra $\mathcal{A} = \langle \mathbb{Z}; f \rangle$ whose carrier set is, say, the set of integers and whose operation f is defined by $f(n) = n+1$ has neither stagnant nor minimal elements, $m_{n,n}(n)$ consists of the one element $n-m$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}_0$. The arbitrary element $n \in \mathbb{Z}$ has no finite order but is, on the other hand, of infinite order.

We will now state the characterization-theorem of equationally compact unary algebras $G = \langle A; f \rangle$; it will be followed by a couple of lemmas that serve its proof.

Theorem 7: The unary algebra $G = \langle A; f \rangle$ is equationally compact if and only if

- (1) Every element whose finite orders approach infinity is of infinite order .
- (2) G contains either some subalgebra $n_{n,n}(n \geq 1)$ or the subalgebra j .

As we have indicated before we will prepare the proof with some lemmas:

Lemma 1: If IB is an elementary extension of $G = \langle A; f \rangle$ then we have the following relationships:

- (1) $st(G) = \wedge$ is equivalent to $st(ft) = \emptyset$
- (2) For every $a \in A$, ${}^n_G(a) = \$$ is equivalent to ${}^n_{ft}(a) = \emptyset$.
- (3) ft contains a subalgebra isomorphic to r^*_n if and only if G contains such an algebra.

proof:

(1) follows from the fact that $(\exists x)(f(x) = x)$ is a sentence in $L_{\langle 1 \rangle}$ which is true in G if and only if it is true in ft . To see (2) we take the formula $\$ (\exists x_0) (f^n(x_0) = x_0 \wedge f^{n-1}(x_0) \neq x_0)$. If we assume that $\wedge(a) / jtf$ then $\bar{a} = (a_5 a, a, \dots) \in A^\omega$ satisfies $\$$ in $(B; \omega)$; hence \bar{a} satisfies $\$$ in G^ω i.e. there exists $a_1 \in A$ such that $f^n(a_1) = a$, $f^{n-1}(a_1) \neq a$. In short: $\wedge(a) \wedge pf$. (3) follows simply from the sentence $(\exists x)(f^n(x) = x \wedge f^{n-1}(x) \neq x)$. $q, e. d.$

Using an algorithm due to Novotny one can prove the next lemma:

Lemma 2: C_n is retract of every extension $\langle R = \langle B; f \rangle$ that contains no subalgebra (isomorphic to) d_m unless n divides m .

proof: As in example 22, we assume that $C_n = \{a_0, a_1, \dots, a_{n-1}\}$ and $f(a^k) = a_{k+1}$ (all indices are determined modulo n). If B is connected then for every $b \in B$ there exists a unique smallest $m(b) \in \mathbb{N}_0$ such that $f^{m(b)}(b) \in C_n$, say $f^{m(b)}(b) = a_{i(b)}$. We define $\varphi(b) = a_{i(b) - m(b)} \in C_n$ and easily verify that $\varphi: B \rightarrow C_n$ is a retraction. If B is not connected and $B_{\bar{1}}$ is a maximal

connected subalgebra of B not containing any subalgebra C_n with $n \mid m$ then we can, of course, assume that B does not contain the pregiven C_n (otherwise, see the first part of the proof). As we have just seen, there exists a retraction $\rho: B \rightarrow C_n$; since $n \mid m$ there exists a homomorphism $\psi: C_m \rightarrow C_n$. We define in this case $\rho: B \rightarrow C_n$ by $\rho(b) = \psi(\rho(b))$.

Since B is the disjoint union of its maximal connected subalgebras B_i , $i \in I$, we can patch up the homomorphisms $\rho_i: B_i \rightarrow C_n$ (more precisely: If $b \in B$ then there exists a unique maximal connected subalgebra B_i such that $b \in B_i$, and we define $\rho(b) = \rho_i(b)$). q. e. d

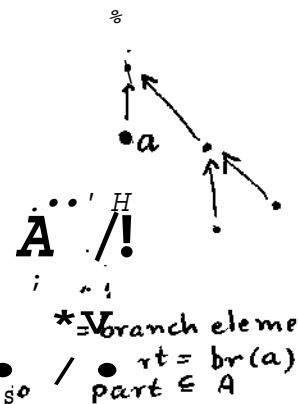
Lemma 3: If $G = \langle A; f \rangle$ contains some cyclic subalgebra C_n with $n \geq 1$, then G is retract of every extension $B = \langle B; f \rangle$ which enjoys the following properties:

- (1) $\langle A; f \rangle$ is equivalent to $\langle C_n; f \rangle$ for $a \in A$ and $n \in \mathbb{N}$.
- (2) $C_m \subseteq B$ implies the existence of $C_n \subseteq B$ such that n divides m .

(3) Every element $a \in A$ whose finite orders approach infinity is of infinite order.

proof: Lemma 2 implies that every maximal connected subalgebra (shortly, connected component) \mathfrak{s}_i of $*A$ which is disjoint from G can be mapped into p_a via a homomorphism $\langle p_i \rangle$. So if $[[G]] = \langle [[A]] \rangle$ is the unique maximal subalgebra of B each of whose components intersects A , then we only need to show that we have a retraction $\langle p: [[G]] \rightarrow G$. Since $\langle \pi(a) = n_{rrr.11}(a) \rangle$ holds true for every $a \in A$ and $n \in N_0$ we conclude that $[[A]] = \bigcup_{n \in N} U(n_0(a); n \in N, a \in A)$. Even more can be said:

If we call an element $a \in A$ a branch element if $l_B(a) \setminus l_Q(a) \neq \emptyset$ and call $U(n_B(a) \setminus n_Q(a); n \in N)$ the branch of a , say $br(a)$ (see diagram), then



$[[A]] = \bigcup_{a \in A} br(a)$; a runs through all branch elements / i \dots H A $!$
 Our aim is to define a homomorphism $\langle p: [[A]] \rightarrow A$ for every branch element a such that $\langle p(a) = a \rangle$ for every branch element a such that $\langle [a]; f \rangle = c_m$ for some $m \in N$, say $[a] = C_m = \{a = a_0, a_1, \dots, a_{m-1}\}$.

If then $b \in br(a) \cap n_k p(a)$ then we define $\langle p^k(b) = a \rangle$ (indices, recall, count modulo m). (2) $f^m(a) \neq a$ for all $m \neq 1$ but there exists some $n \in N$ such that $n_0(a) \neq a$ while $k^{-1}(a) = \bar{a}$ for all $k \neq n+1$. Then, by assumption, $n_Q(a) \neq \emptyset$; say $z \in n^a$. In this case we define $\langle p$ on $br(a)$ as follows: If $Q \leq k \neq n$ and $b \in br(a) \cap n_k p(a)$ then $\langle p^k(b) = f^{n-k}(z) \rangle$ (3) For every n there exists $n' > n$ such that a is of order n' in G . In this

case our assumption implies that a is of infinite order? i.e. there exists a sequence $c = c_0, c_1, \dots, c_n, \dots$ of elements

in A such that $c_n = f(c_{n+1})$ (equivalently $c_n \in n^a$). We then map every $b \in \text{br}(a) \cap n_{\mathbb{R}}(a)$ via $\langle p_a$ to $\langle p_a(t) = c_n$. Thus, in each of the possible three cases we defined $\langle p_a$ on $A \cup \text{br}(a)$ as homomorphism into A which is the identity on A . If we do this for every branch element $a \in A$ in the manner just described then it is a matter of simple verification that the locally defined homomorphisms $\langle p_a : A \cup \text{br}(a) \rightarrow A$ patch up to a retraction

$\langle p : [G] \rightarrow G$ q. e. d

In a quite similar fashion one proves the next result:

Lemma 4; If $G = \langle A; f \rangle$ contains some subalgebra isomorphic to c_9 then G is retract of every extension $(B$ without cyclic subalgebras enjoying properties (1) and (3) of the preceding lemma.

proof of theorem 7:

We first assume that $G = \langle A; f \rangle$ has properties (1) and (2) of theorem 7. Then lemmas 1,3,4 show that G is retract of every elementary extension? this result shows that G is equationally compact (see theorem 2 in J*2). Vice versa, we assume G to be an equationally compact algebra. If then $a \in A$ is an element whose finite orders approach infinity then the infinite set of equations $\mathcal{E} = \{a = f(x_1), x_1 = f(x_2), \dots, x_n = f(x_{n+1}), \dots\}$ is finitely solvable, hence solvable. Thus, a is of infinite order verifying (1). To verify (2) we assume that G contains no cyclic subalgebra c_n . Then $|\{f^n(a), f^{n-1}(a), \dots, f(a), a\}| = n+1$ for every $n \in \mathbb{N}$ and $a \in A$. Thus, the infinite system of equations $T = \{x_0 = f(x_1), x_1 = f(x_2), \dots, x_n = f(x_{n+1}), \dots\}$ is finitely

solvable, hence solvable. If $(a_0, a_1, \dots, a_n, \dots) \in A^{\mathbb{N}_0}$ is a solution of F then evidently $\langle (a_n)_{n \in \mathbb{N}_0} \cup (f^{(1)}(a_0))_{n \in \mathbb{N}_0} \rangle$ is isomorphic to \mathcal{A} . q. e. d.

In order to attack the naturally next question, namely the test of Mycielski's conjecture in the class $K(\langle 1 \rangle)$ we recall that the Stone-Cech compactification βA of a locally compact topological Hausdorff-space is the (up to homeomorphism unique) compact Hausdorff space that contains A as a dense subspace and to which every continuous mapping from A into a compact Hausdorff space B can be continuously extended. So if $G = \langle A; f \rangle$ is a unary algebra then we can endow A with the discrete topology under which f becomes a continuous mapping. If we now extend $f: A \rightarrow A$ to $f: \beta A \rightarrow \beta A$ then $\hat{G} = \langle \beta A; f \rangle$ becomes a unary algebra.

Def. 4: βG , as just defined, is the Stone-Cech compactification of the unary algebra $G = \langle A; f \rangle$,

Thus, the weaker part of problem 2's question is obvious in this case: Every unary algebra $G = \langle A; f \rangle$ (even $G = \langle A; j \rangle$ with an arbitrary set F of unary operations, for that matter) can be embedded in a topologically compact algebra, namely $G \subseteq \beta G$. Of course, not every one of these embedded unary algebras G is also a retract of βG since not every unary algebra $G = \langle A; f \rangle$ is equationally compact. It is true, however, that every equationally compact unary algebra $G = \langle A; f \rangle$ is retract of βG . We will prove in the sequel this result which answers problem 1 in the affirmative for the type $r = \langle 1 \rangle$.

By theorem 7 every equationally compact algebra $G = \langle A; f \rangle$ contains either some $C_d, c \in J \geq 1$, or j as subalgebra. Thus, in view of lemmas 3 and 4 to theorem 7, we need only to show that (1) $n_G(a) = \langle f \rangle$ is equivalent to $n_G(a) = \langle f \rangle$ for all $a \in A$ and $n \in \mathbb{N}$, (2) $r \gg c \in PC$ implies the existence of $C \subseteq G$ such that n divides $m - c$. The next two remarks serve that purpose:

Remark 7: If $Q = \langle A; f \rangle$ is a unary algebra then $n_Q(a) = \langle \wedge \rangle$ is equivalent to $n_Q(a) = \langle \exists \rangle$ for every $a \in A$, $n \in \mathbb{N}$.

proof: Let $x \in (pA \setminus A) \cap n_G(a)$ for some $a \in A$. Then there is a convergent net (a_d, D, \wedge) in A such that $x = \lim_{d \in D} a_d$ [i.e. $\langle D; \wedge \rangle$ is a directed poset, $a^d \in A$ for all $d \in D$ and for every open set 0_x containing x there is some $d_0 \in D$ such that $a_d \in 0_x$ for all $d \wedge d_0$]. Hence, $a = f^n(x) = \lim_{d \in D} f^n(a_d)$. Since $a \in A$ is an isolated point in pA there exists $d_0 \in D$ such that $f^n(a_d) = a$ for all $d \wedge d_0$. This settles the matter if at least one of the a_d is in $n_Q(a)$ which, of course, is always true if a is not a stagnant element (Since then $a^d \in n_G(a)$ when $f^n(a^d) = a$). If a were stagnant and none of the a_d was in $n_Q(a)$ we would conclude that $f^{n-1}(a_d) = a$ for all $d \wedge d_0$, i.e. $a = \lim_{d \in D} f^{n-1}(a_d) = f^{n-1}(\lim_{d \in D} a_d) = f^{n-1}(x) \wedge a$. This contradiction completes the proof.

Remark 8: If $G = \langle A; f \rangle$ is a unary algebra and r_m^* is a subalgebra of pG then there exists a subalgebra c of G such that n divides m .

proof: We begin the proof with the recollection of a well-known topological fact: If $A_1, A_2 \subset A$ and $A_1 \cap A_2 = \emptyset$ then $\overline{A_1} \cap \overline{A_2} = \emptyset$ (where $\overline{A_1}$ is the closure of A_1 in pA), for $A^{\wedge} \in A$ is an open and closed set in A , thus \overline{A} and $A - \overline{A}$ are complementary open and closed sets in pA (see, e.g., Gillman and Jerison [1], chapter 6.9). Thus, if $y = \lim a$ for some net (a_d, D, \geq) in A , $f = \{a_d; d \in D\}$ and $f \in f^m(A) = f^m$ for some $m \geq 1$, then $f \in f^m(A) = y$ implies that $y \in f^m(A)$, i.e. $y \in f^m(y) = \lim_{d \in D} f^m(a_d)$ [of course, $f^m(A) = \{f(a_d); a_d \in A\}$].

So assume that $0 \gg \wedge G$ for every divisor d of some $m \in N$ and let $x \in pA \setminus A$. We then have to show that $f(x) \wedge x$ to end our proof. Let $G_i, i \in I$, be the connected components of G . The carrier set of every G_i , since, by assumption, it has no stagnant element, can be represented as $A_i^1 = A_i^1 \cup A_i^2 \cup A_i^3$ such that $A_i^j \cap A_i^k = \emptyset$ for $j \neq k$ and $f(A_i^j) \cap A_i^k = \emptyset$, $i, j, k = 1, 2, 3$. If G_i is in the class IK of unary algebras without cyclic subalgebra then this is a lemma by Ryll-Nardzewski (see Pacholski and Weglorz [1]). If, on the other hand, G_i has a cyclic subalgebra $C_{n(i)}^1$, say $C_{n(i)}^1 = \{a_0, a_1, \dots, a_{n(i)-1}\}^*$ then we first subdivide $C_{n(i)}^1$ as follows: If $n(i)$ is even, we take $C_{n(i)}^1 = \{a_0, a_1, \dots, a_{n(i)-2}\} \cup \{a_{n(i)-1}\}$, if $n(i)$ is odd we take $C_{n(i)}^1 = \{a_0, a_1, \dots, a_{n(i)-3}\} \cup \{a_{n(i)-2}, a_{n(i)-1}\}$. In either of the two cases we define $A_i^1 = C_{n(i)}^1 \cup \{a; a \in A_i \setminus C_{n(i)}^1\}$ and $A_i^2 = \{a; a \in (2k) \wedge_{n(i)}(c) \text{ for some } c \in C_{n(i)}^1\}$, $j = 1, 2, 3$. It is an easy matter to check that $A_i^j, j = 1, 2, 3$, thus defined, satisfies the conditions stated at the beginning.

Thus, the carrier set A of $\langle f \rangle$ satisfies $A = A^1 \cup A^2 \cup A^3$,
 $A^1 \cap A^2 = A^1 \cap A^3 = A^2 \cap A^3 = \emptyset$, $A^1 \cap fV^1 = A^2 \cap f^m(A^2) =$
 $A^3 \cap f^m(A^3) = \emptyset$ if $A^j = \bigcup \{A^i \mid i \in I\}$, $j = 1, 2, 3$. Hence, $pA =$
 $\overline{A^1 \cup A^2 \cup A^3}$, and we can now assume that $x \in A^j$ for some
 $1 \leq j \leq 3$. In other words: $x = \lim_{d \in D} a_d$ where (a_d, D, \geq) is a
net in A^j . Since $A^j \cap f^m(A^j) = \emptyset$ we conclude that $f^m(x) =$
 $\lim_{d \in D} f^m(a_d) \in pA \setminus A^j$, i.e. $f^m(x) \notin x$. q. e. d.

~~Appendix:~~ The triple-division of A which we used in the last proof
was stated as Ryll-Nardzewski's lemma in Pacholski and Weglorz [1]
for the case $G \in \{f\}$. It should be noted that in case $G \in K$ there
are actually already two subsets A^1, A^2 such that $A = A_1 \cup A_2$ and
 $f(A_1) \cap A_2 = \emptyset$ and $f(A_2) \cap A_1 = \emptyset$. The proof, of course,
remains elementary.

The next theorem is now easily established. We should note
that part of it follows from a result of Pacholski and Weglorz [1]
stating that pG is an elementary extension of $G = \langle A; f \rangle$ in
case $G \in K$.

Theorem 8: If $G = \langle A; f \rangle$ is an equationally compact unary algebra
then it is a retract of pG .

proof: Remarks 7 and 8 imply the result because of lemmas 3 and
4 to theorem 7 and, of course, theorem 7 itself. q. e. d.

Although we will not give the proof of Pacholski and Weglorz's
result that pG is an elementary extension of every $G = \langle A; f \rangle$
with the property $f^n(x) \notin x$ for all $n \geq 1$, $x \in A$, we feel
motivated to ask for a characterization of the elementary extensions

of arbitrary unary algebras \mathcal{A} purely in terms of the algebraic structure of \mathcal{A} . More specifically: We would like to expand the list of necessary conditions for \mathcal{A} to be an elementary extension of \mathcal{G} begun in lemma 1 to theorem 7 such as to end up with necessary and sufficient conditions.

Problem 4: Given a unary algebra $\mathcal{G} = \langle A; f \rangle$ give a set of structural criteria that are necessary and sufficient for an extension \mathcal{B} of \mathcal{G} to be an elementary extension.

Problem 5: Characterize the equationally compact algebras $\mathcal{G} = \langle A; f^{\wedge} \rangle$ where F is an arbitrary set of unary operations. Test Mycielski's conjecture for that class of algebras (of course, \mathcal{G} can again be embedded in the topologically compact algebra $\beta\mathcal{G}$).

§ 5. f_t -Modules.

In this short section we collect what we know about our questions concerning equational compactness in case of unital (left) modules over rings with identity. More precisely: We make only a few remarks on the general case of such R -modules since not too much is known at the time. Then we turn our attention to vector-spaces (i.e. f_t -modules where f_t is a field), realize that they are always equationally compact and verify the truth of Mycielski's conjecture in their case. We ignore in this section the special case of modules over the ring of integers i.e. the case of Abelian groups, which we will deal with in the next section in a bit more detail. For the results of this section see Weglorz [1].

So let $f_t = \langle R; +, -, 0, \cdot, 1 \rangle$ be a ring with identity and $f_t v = \langle M; \{+\} \cup \{-\} \cup \{0\} \cup \{f_r; r \in R\} \rangle$ a unital R -module; i.e. $f_t^1 = \langle M; +, -, 0 \rangle$ is an Abelian group each f_r an endomorphism on the Abelian group $f_t u$, $f_r =$ identity on M and $f_r \circ f_s = f_{r * s}$. Then, of course, the class \sim_M of all f_t -modules is an equational class and as such closed under the formation of ultrapowers. Thus, by corollary 2 to theorem 2 in section 2, we know that every injective R -module is equationally compact (We call f_t^M injective^{t!} if it is injective in \sim_M). Since, as well-known, every R -module can be embedded in an injective R -module, we can embed every f_t -module in an equationally compact one, i.e. it can be ⁿ equationally compactifiedⁿ (a concept which will still be studied in a later section). We sum up:

Remark 9: Every ft -module tn can be embedded in an equationally compact ft -module (as, e.g., in the injective hull of tn).

From the well-known fact that a ft -module tn is injective if and only if every ft -module homomorphism $\varphi: J \rightarrow tn$ from a left ideal j of ft to tn is of the form $\varphi(j) = j \cdot m_0$ for some $m_0 \in M$ one concludes that vector spaces are always injective, hence equationally compact:

Remark 10: Every vector space is equationally compact.

So to show that IVfycielski's conjecture is true in case of vector-spaces we need to show that every vector-space tn is retract of a topologically compact vector-space. This and a little more follows from the Bohr-compactification for Abelian groups. To this end we recall that every locally compact Abelian group Q can be densely embedded in a topologically compact Abelian group Q' (its Bohr-compactification) such that in particular continuous endomorphisms on Q can be extended to continuous endomorphisms on Q' . That is all we need to know in order to establish the next result:

Theorem 9: (1) Every ft -module tn can be embedded in a topologically compact ft -module tn' .

(2) Every vector space tn over the field ft is a retract of some topologically compact vector space tn' over ft .

proof: If $tn = \langle M; \{+, -, \cdot, 0\} \cup \{f_r; r \in R\} \rangle$ is a ft -module and $\hat{M} = \langle M; \{+, -, \cdot, 0\} \rangle$ the associated Abelian group then we can endow M with the discrete topology thus transforming tn in a

locally compact Abelian group. Since the unary operations $f_{\mathbf{r}}$ are continuous endomorphisms on $\mathbb{T}_{\mathbf{I}}$ we can extend them to continuous endomorphisms $f_{\mathbf{r}}^{\circ}$ on the Bohr-compactification $ft'_{\mathbf{I}}$ of $ft_{\mathbf{I}}$. It follows from the unique extendability of continuous mappings that $f_{\mathbf{r}}^* \circ f_{\mathbf{s}}^{\circ} = f_{\mathbf{r}\mathbf{s}}^{\circ}$ and $f_{\mathbf{1}} = \text{identity}$ are still true. Therefore $ft^{\circ} = \langle M; (+) \cup (-) \cup \{0\} \cup \{f_{\mathbf{r}}; \mathbf{r} \in R \} \rangle$ is an R -module which is topologically compact and contains ft . This settles (1). (2) follows then immediately from the fact that the vector-space ft is injective, hence an absolute retract in \mathfrak{M} . Thus, the vector-space ft is a retract of ft' . q. e. d.

As mentioned before, the special case of Abelian groups will be treated in the next section. Before doing so we mention the following open problem:

~~Problem 6~~: One investigates the structure of equationally compact R -modules aiming at their characterization and a test of Mycielski's conjecture in \mathfrak{M} .

§6. Abelian Groups,

Although our mastery of ft -modules with respect to the study of equational compactness is (as we have realized ini5) rather unsatisfactory at the time, the situation is quite different in case of unital modules over the ring \mathbb{Z} of integers, alias Abelian groups. Both the algebraic structure of equationally compact Abelian groups $Q = \langle G; +, -, 0 \rangle$ has been extensively studied and Mycielski's conjecture has been verified in the class \underline{A} of Abelian groups. For references see, e.g., Kaplansky [1], Loś [1], Balcerzyk [1], Gacsályi [1].

Let us set out with a folklore result (of course we assume familiarity with elementary concepts and results in \underline{A} and confine our attention to our very questions).

Remark 11: The subgroup H of the Abelian group Q is a direct summand of Q if and only if M is a retract of Q .

proof: The converse being obvious we assume that $\langle p: Q \rightarrow H$ is a retraction. If then $g \in G$ then $g = \langle p(g) + g' \rangle$ with unique

$g' \in G$? since $\langle p(g) = \langle p(g) + \nu(g^*) \rangle$ we conclude that $\langle p(g') = 0$, i.e. $g' \in \ker \langle p$ (in accordance with example 8 we denote $\{x; x \in G, \langle p(x) = 0\}$ as $\ker \langle p$). Thus, every element $g \in G$ is the sum of an element in H and an element in $\ker \langle p$. If $g = h_1 + g_2 = h_2 + g_1$ where $h_1, h_2 \in H$ and $g_1, g_2 \in \ker \langle p$ then $(p(h_1) + \langle p(g_2)) = (p(h_2) + \langle p(g_1))$, i.e. $h_1 = \langle p(h_2) = (p(h_2) = h_2$ and, consequently, $g_1 = g_2$. Thus, every element $g \in G$ is in a unique way the sum of an element in H and an element in

ker in, a fact which is commonly expressed by writing $Q = M \oplus \langle \ker \varphi, \dots, 0 \rangle$; in other words: H is a direct summand of Q .
q. e. d.

If we take in account example 18, chapter II, which elaborates on the fact that pure subgroups and pure subsystems coincide in case of Abelian groups and combine this with the above remark and theorem 2 then we conclude immediately the following characterization of equationally compact Abelian groups.

Theorem 10: An Abelian group is equationally compact if and only if it is a direct summand of every extending Abelian group in which it is pure.

Balcerzyk [1] showed that the class of all Abelian groups which are direct summands of every extending Abelian group in which they are pure coincides with the class of all algebraically compact Abelian groups in the sense of Kaplansky [1].

Def. 5 (Kaplansky [1]): An Abelian group Q is algebraically compact if $Q = S \oplus P$, i.e. Q is the direct sum of a divisible group S and an Abelian group P which is the direct product of Abelian groups P_p ($p \in$ set of prime numbers $\subseteq \mathbb{Z}$) where P_p is a module over the ring of p -adic integers \mathbb{Z}_p both without elements of infinite height and complete in its p -adic topology.

We mentioned Balcerzyk's result at this point since it illuminates the fact that theorem 10 is indeed a valuable characterization of equationally compact Abelian groups? for the class of algebraically compact Abelian groups has been intensively studied

and their structure is satisfactorily discerned. However we will delay a proof of Balcerzyk's theorem until the end of this section for two reasons: Firstly, its proof presupposes familiarity with p-adic integers and p-adic topologies, results that we can not derive although we have to use it within the present framework; secondly, while verifying the truth of Mycielski's conjecture for the class \mathcal{A} we have to establish a lemma which we also need to establish Balcerzyk's result. So we decide to aim next at a verification of the fact that every equationally compact Abelian group is direct summand (= retract) of a topologically compact Abelian group. A crucial step in that direction is the following remark of Los [1]:

Remark 12; If C_{p^n} denotes the cyclic group with p^n elements ($p =$ prime number) and C_{∞} , the Prufer-group over the prime number p then each Abelian group Q can be embedded in a direct product M of groups of the type C_{p^a} ($a = 1, 2, 3, \dots, \infty$) such that (1) Q is a pure subgroup of M and (2) Q is a subdirect product of the groups C_{p^a} occurring in M .

proof; The proof is a sharpening of Birkhoff's subdirect representation theorem (theorem 12, chapter I) if applied to our situation. A group Q is subdirectly irreducible (of course, we assume $|Q| > 1$) if and only if the set of all non-trivial subgroups of Q intersects in a non-trivial subgroup E which then, of course, is finite of prime order p (see remark 1.19). If $g \in Q \setminus E$ then $[g] \cap E = E$ shows that $n \cdot g \in E$ for some $n \in \mathbb{N}$ hence

$(pn) \gg g = 0$ and g is of finite order. Thus, Q' is a torsion group and as such a direct sum of primary groups. Since Q' can have no proper decomposition as direct sum we conclude that Q^5 is p -primary. If then Q^5 is of bounded order then it is a direct sum of cyclic groups, hence a cyclic group itself: $Q' = C_{p^n}$ for some $n \in \mathbb{N}$. If Q^5 is not of bounded order then it is either a divisible group or a reduced group (since every Abelian group is the direct sum of its divisible and its reduced part). If Q' is divisible then it is, as well-known, isomorphic to a direct sum of copies of the additive group of rational numbers and of C_{ℓ} where ℓ are prime numbers. Since Q' is p -primary we conclude that Q^* is isomorphic to a direct sum of copies of C_{p^a} , only. If, on the other hand, Q^* is reduced, then Q^* is a reduced torsion group and has as such a finite cyclic non-trivial direct summand; in other words: $Q^* = C_{p^n}$ for some $n \in \mathbb{N}$.

We conclude: The Abelian groups $C_{p^a} = \{1, 2, 3, \dots, \infty\}$ are exactly all subdirectly irreducible Abelian groups and, according to Birkhoff's theorem, every Abelian group Q is isomorphic to a subdirect product of some of these. Thus, we have an embedding of Q in a direct product $\prod_{i=1}^{\infty} C_{p^a} = 1^2, \dots, \infty$; this much we are assured of by merely using Birkhoff's theorem.

We will now show that by adding some more suitable groups Q as direct factors to U then we get an Abelian group M containing Q both as subdirect product and pure subgroup, a claim that will be proved in the next three paragraphs. The very next paragraph serves merely as a refresher of our memory for some elementary

connections:

(1) If $A = \prod_{P^a} \mathbb{F}^{P^a}$ is a factor in the direct product H' then each $\phi \in A$ is isomorphic to some quotient group $Q/Q\phi$. For the purpose of this proof we identify $\phi = Q/\theta_\phi$. Thus, $Q \cong \prod_{\mathfrak{s}} (Q/\theta_{\mathfrak{s}}; \mathfrak{s} \in JS)$ and an isomorphism is given by $\langle P(g) = (tg)^{\mathfrak{s}}$ (see § 7, chapter I). Of course, if we add any new congruence relations $\theta_{\mathfrak{y}}, \mathfrak{y} \in T$, from $C(Q)$ then we have even more that $Q \cong \prod_{\mathfrak{s}} (Q/\theta_{\mathfrak{s}}; \mathfrak{s} \in A \cup T)$, and isomorphism being given by $\theta(g) = ([g]_{\theta})^{\mathfrak{s}}$. If $g \in G$ and p is a prime number then the p -height $h_p(g)$ of g is defined

to be either $n \in \mathbb{N}$ if $g \in p^n G \setminus p^{n+1} G$ or ∞ if such n does not exist. If $g \in G$ then $\overline{P(g)}$ denotes the set of all prime numbers p with $h_p(g) < \infty$.

Let us now assume we can add to $\{0^{\mathfrak{s}} \in A\}$ some more congruence relations from $C(Q)$, say $(\theta_{\mathfrak{y}}; \mathfrak{y} \in T)$ with the following two properties

(P1) $Q/\theta_{\mathfrak{s}} \cong C_{p^{\mathfrak{s}}}$ for all $\mathfrak{s} \in JS$ (p prime number)

(P2) For every $g \in G$ and $p \in \overline{P(g)}$ there exists $\theta_{\mathfrak{s}(g)}, \mathfrak{s}(g) \in \Delta \cup \Gamma$, such that $lgie_{\theta_{\mathfrak{s}(g)}} / \theta$ and $Q/\theta_{\mathfrak{s}(g)} \cong C_{p^{(g)+1}}$

Then (see above paragraph) we have an isomorphism $\theta: Q \cong \prod_{\mathfrak{s}} (Q/\theta_{\mathfrak{s}}; \mathfrak{s} \in A \cup T)$ via $\theta(g) = (tg)_{\theta_{\mathfrak{s}}}$. We claim that then $\theta^{-1}(Q)$ [which we identify with Q] is a pure subgroup of $\mathfrak{G} = \prod_{\mathfrak{s}} (Q/\theta_{\mathfrak{s}}; \mathfrak{s} \in A \cup T)$. Because of example 17 we only need to show that $p^{-x} = g$ ($p =$ prime number, $k \in \mathbb{N}$, $g = ([g]_{\theta})^{\mathfrak{s}}$) is solvable in \mathfrak{H} if and only if it is solvable in $\theta^{-1}(G)$. Thus, let $p^{-x} =$

$g, h' \in H$. If $h_p(g) = \infty$ then it is clear that $p^{-g^*} = g$ for some $g^f \in \#(G)$. If $h_p(g) = n \in \mathbb{N}$ then $p \in P(g)$ and, by (P2), there exists $0_{\mathbb{Q}/\mathbb{Z}}$ such that $[g]_{\mathbb{Q}/\mathbb{Z}} \wedge 0$ and $\mathbb{Q}/\mathbb{Z} \cong C_{n+1}$. Since $p^k [g']_{\mathbb{Q}/\mathbb{Z}} - [g]_{\mathbb{Q}/\mathbb{Z}} \neq 0$ is true in $\mathbb{Q}/\mathbb{Z} \cong C_{n+1}$, we conclude that $k \leq n$ [otherwise $k = (n+1) + v$ and $v \geq 0$ implies $p^k \cdot [g']_{\mathbb{Q}/\mathbb{Z}} = p^v (p^{n+1} [g']_{\mathbb{Q}/\mathbb{Z}}) = p^v \cdot 0 = 0$]. Thus, $k \leq h_p(g)$; i.e. there exists $g^5 \in 0(G)$ such that $p^{-g^j} = g$. Hence, $\mathcal{S}(Q)$ is a pure subgroup of M which (because of (Pi)) settles the matter. We are left with proving that $\{^5?^ \in A\}$ can be extended to $\{Q_y; y \in ZDF\}$ such that properties (Pi) and (P2) are satisfied? this task will be taken care of in the next and last paragraph of this proof:

Let $g \in G$, $p \in P(g)$ and $h(g) = n \in \mathbb{N}$. Then $g \in p^n G \setminus p^{n+1} G$; i.e., if Q is the group $\mathbb{Q}/p^n \mathbb{Z}$, then Q is a group of bounded order in which every element has an order dividing p^{2n+1} and $\bar{g} = [g]_{\mathbb{Q}/p^n \mathbb{Z}}$ has order p^{n+1} . Thus, \bar{Q} is a direct sum of cyclic groups C_p with at least one direct summand $C_{p^{n+1}}$ with $y_0 \wedge p^{n+1}$. Clearly $c_{n+1} = \langle [g]_{\mathbb{Q}/p^n \mathbb{Z}} \rangle$ is a homomorphic image of $C_{p^{n+1}}$ which in turn, is a homomorphic image of \bar{Q} . In short:

(1) $\mathbb{Q}/p^n \mathbb{Z} \cong \mathbb{Q}/p^{n+1} \mathbb{Z}$: some congruence relation $\mathbb{Q}/p^{n+1} \mathbb{Z}$ and (2) $[g]_{\mathbb{Q}/p^n \mathbb{Z}} \wedge 0$. If we add all these congruence relations $\mathbb{Q}/p^{n+1} \mathbb{Z}$ for arbitrary $g \in G$ and $p \in P(g)$ we have a set of congruence relations, namely $\{B^{\wedge}; \delta \in A\} \cup \{p, g; g \in G \text{ and } p \in P(g)\}$, satisfying properties (PI) and (P2). q. e. d.

We now can easily prove the next theorem:

Theorem 11: An Abelian group is equationally compact if and only if it is a direct summand (retract) of a topologically compact Abelian group.

proof: Of course, we need only to prove one direction. So we assume that the Abelian group Q is equationally compact and embed it in a direct product M of groups c_p in such a manner

that Q is pure in M (see remark 12). If a is a natural number then c_p^a is finite, hence a topologically compact group

in the discrete topology. If $a = \infty$ we have the Prufer group \mathbb{Z}_p^\wedge which is divisible and isomorphic to the multiplicative group of p^n -th roots of unity among the complex numbers, $n = 0, 1, 2, 3, \dots$. Thus, C_a is a direct summand of \mathbb{C} , the mul-

tiplicative group of complex numbers z with absolute value $|z| = 1$ which, as closed and bounded subset of the complex plane, is a topologically compact group. Thus, if $M = \prod_{p \in I} (C_a / \{a(p) \in \mathbb{Z}\})$; $p \in I$, then J is a retract of $\prod_{p \in I} C_a / \{a(p) \in \mathbb{Z}\}$; $p \in I$ where $C_a^{a(p)} = C_a$ in case $a(p) < \infty$ and $C_a^{a(p)} = C$ in case $a(p) = \infty$. since W^* is a topologically compact group via the Tychonoff-product-topology, Q is a retract of M (Q is equationally compact and pure in M) and J is a retract of M^*

we conclude that Q is a retract of the topologically compact group W^* . q. e. d.

Before we proceed we take time out for a little illustration:

(Ex. 24):

(1) An Abelian group which is the direct product of Abelian groups Q_i $i \in I$, is equationally compact if and only if every Q_i is.

(2) If Q is equationally compact then so is n - Q for every $n \in \mathbb{N}$: To see this we use theorem 11: There is a topologically compact group M such that $tt = Q \otimes Q'$ for some Q' . Hence, $n^*tt = n^*Q \otimes n^*Q'$. Since n^*tt is closed in M it is topologically compact itself, n^*Q is direct summand of the topologically compact group n^*tt and hence, n^*Q is equationally compact.

(3) If Q is equationally compact and n^1, n^2, \dots , are natural numbers then $0((n^1, n^2, \dots) \otimes_k G; k = 1, 2, \dots)$ is equationally compact: As in (2) we see this by realizing that $M = Q \otimes Q'$

implies $(\prod_{i=1}^k n^i) \cdot Q = (\prod_{i=1}^k n^i) \cdot Q \otimes (\prod_{i=1}^k n^i) \cdot Q'$ and therefore $n((\prod_{i=1}^k n^i) \cdot M; k = 1, 2, \dots) = n((\prod_{i=1}^k n^i) \cdot Q; k = 1, 2, \dots) \otimes n((\prod_{i=1}^k n^i) \cdot Q'; k = 1, 2, \dots)$.

Since the left side is topologically compact provided M is, the claim follows again from theorem 11.

(4) If Q is equationally compact, p is a prime number and Q^p is the subgroup of elements of infinite p -height then Q^p is equationally compact: This follows from (3) and $Q^p = n(p^k; k = 1, 2, \dots)$.

(5) It follows, of course, from our considerations that every group c is equationally compact. So is the additive group $+ p^a$

$D = \langle Q; +, -, 0 \rangle$ of rational numbers. So is, in short, every divisible group.

We conclude our investigations with a proof of the equivalence of the concepts "equational compactness"¹¹ and "algebraic compactness" (see definition 5). As indicated before we presuppose familiarity with p -adic integers and p -adic topologies in modules over these rings.

Theorem 12: An Abelian group is algebraically compact if and only if it is equationally compact.

proof: If Q is equationally compact then (by remark 12) we can embed Q in a direct product M of groups C_n such that Q is pure in M , i.e. Q is a direct summand (= retract) of M . If we write $M = \prod_{i \in I} C_{n_i}$ where \prod is the direct product of all direct factors C_{n_i} of M and P is the direct product of all direct factors $C_{n_i}^p$ of M with $n_i \in \mathbb{N}$, then \prod is a divisible group and P is a direct product of finite cyclic groups $C_{n_i}^p$ which, as well-known, are complete in their p -adic topology and, of course, contain no elements of infinite p -height. Thus, \prod is algebraically compact, and so is every retract (as was shown by Kaplansky); in particular, Q is algebraically compact.

Vice versa, let Q be an algebraically compact Abelian group? i.e. $Q = \prod_{i \in I} C_{n_i}^p$ (see definition 5) where $P = \prod_{p \in \mathbb{P}} (C_{n_i}^p; P = 2, 3, 5, 7, \dots)$ and the $C_{n_i}^p$ are \mathbb{Z}_p -modules without elements of infinite height that are complete in their p -adic topology. Since \prod is divisible it is direct summand of every extending Abelian group and is therefore equationally compact. So to show that Q is equationally compact we are left with showing that $P = \prod_{p \in \mathbb{P}} (C_{n_i}^p; p = 2, 3, 5, 7, 11, 13, \dots)$ is equationally compact, i.e. that each $C_{n_i}^p$ is equationally compact. To this end we utilize the fact that every complete \mathbb{Z}_p -module (completeness refers always to the p -adic topology) is the completion of a direct sum of cyclic \mathbb{Z}_p -modules; thus $C_{n_i}^p = (\oplus_{i \in I} H_i)^*$ where each H_i is a cyclic,

hence compact, Z_p -module and $*$ indicates completion in the p -adic topology. $\text{cl}_p(\sum_{i \in I} f_i)$ is then (up to isomorphism) contained in the topologically compact Z_p -module $\sum_{i \in I} \text{cl}_p(f_i)$ and (as Abelian group) pure in $\sum_{i \in I} \text{cl}_p(f_i)$ [we say that $(\sum_{i \in I} f_i)$ is a pure submodule of $\sum_{i \in I} \text{cl}_p(f_i)$]. Thus, $(\sum_{i \in I} f_i)^*$ can be considered to be the topological closure of 0 in $\sum_{i \in I} \text{cl}_p(f_i)$ in $\sum_{i \in I} \text{cl}_p(f_i)$. It is well-known that the pureness of $(\sum_{i \in I} f_i)$ in $\sum_{i \in I} \text{cl}_p(f_i)$ implies the pureness of $(\sum_{i \in I} f_i)^*$ in $\sum_{i \in I} \text{cl}_p(f_i)$. Since therefore $(\sum_{i \in I} f_i)^* = \text{cl}_p(\sum_{i \in I} f_i)$ is a pure submodule of $\sum_{i \in I} \text{cl}_p(f_i)$ which is complete and has no elements of infinite p -height it follows from another known result that $(\sum_{i \in I} f_i)^*$ is a direct summand of $\sum_{i \in I} \text{cl}_p(f_i)$; i.e. $\text{cl}_p(\sum_{i \in I} f_i)$ is direct summand of the topologically compact Abelian group $\sum_{i \in I} \text{cl}_p(f_i)$. In other words: $\text{cl}_p(\sum_{i \in I} f_i)$ is equationally compact. q. e. d.

With the results at our disposal it is an easy exercise to verify the claims made in the next example; the claims can be found as exercises 62-65 in Kaplansky [1].

(Ex. 25):

- (1) A torsion group is equationally compact if and only if it is a direct sum of a divisible group and a group of bounded order.
- (2) No free Abelian group is equationally compact. In particular, neither the group of integers nor any direct product thereof is equationally compact.
- (3) A countable torsion-free group is equationally compact if and only if it is divisible.

Before leaving this section we go one step beyond the necessary and sum up the different characterizations of equationally compact Abelian groups established so far.

Theorem 13: The following statements are equivalent for an Abelian group Q :

- (1) Q is equationally compact,
- (2) Q is direct summand of every extending Abelian group which contains Q as pure subgroup.
- (3) Q is direct summand of some topologically compact Abelian group.
- (4) Q is algebraically compact in the sense of Kaplansky.

In the introductory remarks to theorem 13 we announced to state all characterizations established ^u so far.^{tf} Of course, other interesting characterizations are conceivable and, as a matter of fact, available. So Balcerzyk [2] proved the following remark whose proof we skip:

Remark 13: The Abelian group Q is equationally compact if and only if every finitely solvable set of equations $\mathcal{E} = \{x_{\mathbf{o}} - g_{\mathbf{n}} = nix_{\mathbf{n}}; n = 1, 2, 3, \dots\}$ with $g_{\mathbf{n}} \in G$ is solvable in Q .

Apart from merely providing a different (although quite interesting) characterization of equationally compact Abelian groups, remark 13 has the following noteworthy corollary.

Corollary: An Abelian group is equationally compact if and only if it is equationally $\hat{\mathbf{o}}$ -compact.

Finally we mention three open problems, two of them on the rather obscure situation in non-Abelian groups, which were suggested in Mycielski [1].

Problem 7: In example 4 preceding 1, chapter I, we constructed a set T with ω_1 equations over the ring of integers which was unsolvable although every countable subset was solvable. Does such a set T still exist if we replace the ring of integers by the additive group of integers?

Problem 8: As H. Freudenthal showed, the group of linear substitutions $ax + b$ with rational numbers a, b , $a \neq 0$, is not embeddable in a compact topological group. Is that group embeddable in an equationally compact group? (If the answer was "yes" then this would, of course negate the Mycielski conjecture in its general form).

Problem 9: Study equational compactness, respectively equational m -compactness in non-Abelian connected locally compact topological groups.

Si. Equational Compactifications.

After our investigations concerning the influence of equational compactness on specific algebraic structures as performed in the preceding sections we now turn our attention to a final general problem which comes up quite naturally: If G is a universal algebra which is not equationally compact itself can we always embed it in an equationally compact universal algebra or, at least, in an equationally \aleph -compact one where \aleph is some fixed cardinal number? If the answers were yes could we always remain within the equational class $HSP(G)$ determined by G ? It is questions of this sort that this section is devoted to. We refer to Weglorz [2] and Mycielski [1] as far as the questions and results of this last section are concerned.

Def. 6; An algebra $(B = \langle B; F \rangle)$ is called a (weak) equational \aleph -compactification of G if G is a subalgebra of B and B is (weakly) equationally \aleph -compact ($\aleph \geq$ infinite cardinal). The class of all (weak) equational \aleph -compactifications of G is denoted by $C_{\aleph}^f(G)$ (resp., $C_{\aleph}^{wf}(G)$). We then define $C(G) = \bigcup_{\aleph} C_{\aleph}^f(G)$ (\aleph runs over the class of infinite cardinals) and know, of course, that the elements of $C(G)$ are exactly the equationally compact algebras containing G , called equational compactifications of G . Similarly, $C^{Weak}(G) = \bigcup_{\aleph} C_{\aleph}^{wf}(G)$ (\aleph runs through the class of infinite cardinals) contains the weak equational compactifications of G .

It is quite clear that not every algebra has an equational (or even only a weakly equational) compactification. As example

of such an algebra we could take the lattice in example 3 (preceding section I.I) which is not equationally compact and has visibly no equational compactification. If we add the two elements 0,1 as nullary operations in that lattice then we have already an example of an algebra without a weak equational compactification. We shall see, however, that every algebra G has a (weak) equational λ -compactification for every infinite cardinal number λ . Before pursuing these matters further we introduce classes of algebras related to the ones introduced in definition 5 but weaker in their requirements.

Def. 7: If λ is an infinite cardinal and $G = \langle A; F \rangle$ is a universal algebra then $(B = \langle B; F \rangle$ is a quasi-equational λ -compactification of G if B contains G as subalgebra and every set of f inequations with constants in A is solvable in B provided it is finitely solvable in G . The class $\underline{c}^\lambda(G)$ contains all quasi-equational λ -compactifications of G . Again $\underline{c}^\lambda(G) = \bigcap \{ C(G) \mid \lambda \text{ runs through all infinite cardinal numbers} \}$ consists of the so-called quasi-equational compactifications of G .

Corollary: If G is a universal algebra and λ an infinite cardinal then $\underline{C}^\lambda(G) \subseteq \underline{c}^\lambda(G)$ and $C(G) \subseteq \underline{c}^\lambda(G)$.

Here we face immediately a quite interesting open problem (see Weglorz [2]):

Problem 10: Does $C(G) = J_2^f$ always imply $\underline{c}^\lambda(G) = p^\lambda$

Before we go on we list a few more elementary properties of the operators C , C , c and c .

Remark 14

(a) If $G \subseteq G_1$, then, for each m , $c_m(G) \subseteq c_m(G_1)$ and $C_m(G) \subseteq C_m(G_1)$. This, of course, implies $c(G) \subseteq c(G_1)$ and $C(G) \subseteq C(G_1)$.

(b) Sharpening part of (a) we get that $C_m(G) = C_m(G_0) \cap E(G_1)$ if G_0, G_1 are arbitrary algebras and $E(G_1)$ is the class of all extensions of G_1 .

(c) Both $c(G)$ and $C(G)$ are classes of algebras closed under extensions. This, of course, is not true for $C(G)$ and $C(G)$.

(d) If $W \subseteq W$ then both $c_m(G) \subseteq c_m(W)$ and $C_m(G) \subseteq C_m(W)$.

The statements of remark 14 are quite evident and require no proof. So is, as a matter of fact, the next remark:

Remark 15: The following statements are equivalent for an algebra G .

- (1) G is equationally λ -compact
- (2) $G \in \mathcal{O}(CMB) \rightarrow B \subseteq G = c(G)$
- (3) $E(G) \subseteq c_m(G)$ where $E(G)$ is the class of extensions of G
- (4) $G \in H(C(fc); B \subseteq G) = C_m(G)$.

Appendix: Remark 15 remains valid if we consistently drop the cardinal number λ .

We now will derive the following theorem due to Mycielski. We will, however, give a different proof based on a lemma of Ryll-Nardzewski (see corollary 2 to theorem II.3).

Theorem 14: For an arbitrary algebra G and a fixed infinite cardinal number M it is always true that $C(G) \neq d$, although there are algebras G with $C(G) = \text{fff}$.

In preparation of the proof we first prove the following remark:

Remark 16: For each algebra G and each infinite cardinal number \aleph_α there is a set $X_{\aleph_\alpha, p}$ of equations with constants in A which is finitely solvable in G and has the property that $B \in C(G)$ holds if and only if $G \subset B$ and S_{\aleph_α} is solvable in B .

proof: We define an equivalence relation on the class of all finitely solvable sets of equations E with constants in A satisfying $|X| \leq \aleph_\alpha$ as follows: $E_1 \sim E_2$ holds if and only if there is a bijection φ from the variables actually occurring in E_1 to those occurring in E_2 such that $\varphi(S_{\aleph_\alpha}(E_1)) = E_2$ where $\varphi(S_{\aleph_\alpha}(E_1))$ is the set of equations resulting from E_1 after having replaced every variable x_γ by $\varphi(x_\gamma)$. We then take a complete and irredundant system of representatives of the equivalence classes, say E_i , $i \in I$, and can without loss of generality assume that i / j implies $X_i \cap X_j = \emptyset$ where X_i is the set of variables occurring in E_i . Then the set $T_{\aleph_\alpha} = \bigcup_{i \in I} E_i$ is visibly finitely satisfiable in G ; moreover, E_i is satisfiable in an algebra $B \wedge G$ if and only if, for each $i \in I$, the set S_i

is satisfiable in \mathcal{Q} . The remainder of the proof is now clear and remark 16 proved. q. e. d.

proof of theorem 14;

The fact that $C(G) = 0$ (even $C^{\text{weak}}(\mathcal{Q}) = 0$) can happen was pointed out before. So let G be an arbitrary universal algebra and \aleph_M a fixed infinite cardinal number. We denote the initial ordinal number of \aleph_M by ω_M , choose a limit ordinal $\mu > \omega_M$ such that μ is not confinal with ω_M (i.e. $\exists y_i < \mu$ if each $y_i < \mu$) and define $G_\gamma, \gamma < \mu$, recursively as follows:

(1) $G_0 = G$ (2) if $\gamma = \delta + 1$ then we choose \mathcal{E}_δ according to remark 16. By Ryll-Nardzewski's lemma (corollary 2 to theorem II.3) there exists an algebra, say G_δ , containing G_δ in which \mathcal{E}_δ is solvable. So $G_\delta \in \mathcal{C}(G_\delta)$ (3) if $\gamma = \text{li}(\delta_i; \delta_i < \gamma)$ is a limit ordinal then we define $G_\gamma = U(G_{\delta_i}; \delta_i < \gamma)$. Finally, we define $IB = U(G_\gamma; \gamma < \mu)$ and claim that $IB \in \mathcal{C}(\mathcal{Q})$. To see this we

choose a set S of equations with constants in B which is finitely solvable in B such that $|T| < \aleph_M$. Then the number of constants involved is, of course, $< \aleph_M$ likewise both the number of variables involved is $< \aleph_M$ and the number of finite subsets of T is $< \aleph_M$. So if we take one solution in IB for each finite subset of S and put them together to a set S then the set of constants occurring in the solutions of S can be assumed to be of cardinality $< \aleph_M$. If we finally take in account the equations with constants in $B' (B' \subseteq B)$ which is finitely solvable

in $\langle [B']; F \rangle$ and $|B'| < \mu + \omega = \text{cf}(\mu)$. Assume $B' = (b_0, b_1, \dots, b_\gamma, \dots)_{\gamma < \mu}$ and $b_\gamma \in A_{\nu(\gamma)}$, $\nu(\gamma) < \beta$. Then $B' \subseteq A_{\nu(0) + \nu(1) + \dots + \nu(\gamma) + \dots}$ $\gamma < \mu$. Since f_i is not cofinal with p we know that $S_i / (v) = P^j < P$, hence $\langle [B']; F \rangle \subseteq C_i$. Since $G_{\mu, \mu} \in C(G_{\mu, \mu})$ by construction we conclude that T is solvable in $G_{\mu, \mu}$ i.e. f is solvable in \mathfrak{B} . Hence, $(R \in C(G))$ and $C(G) / X$ -q. e. d.

We ought to re-read the open problem 10 in light of this last theorem, because it shows it in new light. We recollect: $C_{Aw}(G) \wedge /$ (hence, even more $C_{4w}^{\text{weak}}(G) / \text{tf}$ and $C_f(G) \wedge p$) holds for every G and 4^* . On the other hand, $C(G) = /6$ is a definite possibility. It is an interesting result however to note that $C(G) \wedge 0$ is equivalent to $C(G) \text{riHSP}(G) / \&$.

Theorem 15: If $\mathfrak{B} \in C(G)$ then every maximal subalgebra of \mathfrak{B} containing G and belonging to $\text{HSP}(G)$ [and such subalgebras exist!] belongs to $C(G) \text{OHSP}(G)$.

proof: First of all it is a simple application of Zorn's lemma to conclude the existence of maximal subalgebras of \mathfrak{B} containing G and belonging to $\text{HSP}(G)$. We assume G^* to be such an algebra and associate with every infinite cardinal number \aleph_z the set of equations $T_{4^*n, d}^{<}$, of remark 16. We have to show that $G' \in C_{stK}(CM)$ for every 4^*v or, equivalently, that each T_n is solvable in G' . To this end we fix some infinite cardinal number ω and assume that α is an ordinal number such that $\gamma < \alpha$ for every x_γ occurring in E_{-r} . If, finally, $A \geq \text{Id}^\wedge, (\text{HSP}(G'))$

[see def. 1.37 for the notation] then $AU_{XL,r}$ is evidently solvable in B by, say, (b, L, x) . Assume we could show that $G'' = \langle [A'U\{b; y < a\}]; F \rangle \in HSP(G)$ then the maximality of G would imply that $G'' = G'$, hence $E_{m',0}$ was solvable in G^5 , i.e. $G^* G \cong G^{\wedge}$ which would finish our proof. So we are left with showing that $G'' = \langle [A'U\{b_y; y < a\}]; F \rangle \in HSP(G') = HSP(G)$.

To see the last point we select an arbitrary ω_0 -tuple

$(TT_n)_{n < \omega_0} \in A'^* \circ$ where each TT_n is a certain polynomial expression

involving a finite number of c/s and of elements $a^* \in A^5$; say $\wedge^n V \wedge - ' I l n l ' V) \dots \wedge (n)$ with $f_n \in Gpt^0 \circ \wedge \wedge$

If $f = g \in Id(HSP(G)) = Id(HSP(GM))$ then we have evidently that

$$p(p_0(a_1^0, \dots, a_m^0), x_{i_1(0)}, \dots, x_{i_t(0)}(0)), \dots, p_n(a_0^n, \dots, a_m^n), x_{i_1(n)}, \dots, x_{i_t(n)}(n) \wedge \# * V \dots \wedge n < \omega_0 = q(p_0(a_1^0, \dots, a_m^0), \dots, x_{i_t(0)}(0)), \dots, p_n(a_0^n, \dots, a_m^n), \dots, x_{i_t(n)}(n)) \wedge n < \omega_0$$

$\in Id_{A'}^{(\alpha)}(HSP(G^{\wedge}))$. Hence $(b_y; y < a)$ is a solution of that equation

$$\text{and therefore } p(ir_{\omega_0}^{TT_1}, \dots, ir_n, \dots) = q(TT_{\omega_0}^{TT_1}, \dots, TT_n, \dots)$$

In short: $g = g \in Id(G^{\wedge})$. q. e. d.

In a similar fashion one can prove the following analogous result:

Theorem 16: If $B \in c(G)$ then there exists some $f, c \in B$ such that $C \in HSP(G) \cap c(G)$.

To complete the list of results in that direction we mention that continuity considerations assure immediately that every algebra G which can be embedded in a topologically compact

algebra IB (say $B \in C_{top}(G)$) can also be embedded in one of the same equational class. We sum up:

Theorem 17: If $B \in C_{top}(G)$ then $\overline{G} \in HSP(G) \cap C_{top}(G)$ if $\overline{G} = \langle \overline{A}; F \rangle$ is the closure of G in B .

It is interesting to note that the obvious question concerning weak equational compactifications is open at the time:

Problem 11: Does $C^{weak}(G) \wedge \mathcal{H}$ always imply that $C^{weak}(G) \cap HSP(G) \neq \emptyset$?

We conclude both chapter and seminar with a nice application of our results leading to an extension and elegant proof of a lemma of Numakura [1] stating that topologically compact semi-groups with cancellation are groups. We can indeed replace the requirement of topological compactness in that lemma by equational compactness. The technique used in the proof due to Weglorz and Hulanicki is noteworthy since it indeed suggests itself to further application.

Theorem 18 (Numakura, Weglorz, Hulanicki):

Every equationally compact semigroup with cancellation is a group.

proof: We split the proof in two parts, first settling the case of Abelian semigroups.

(1) Let $S = \langle S; * \rangle$ be an equationally compact Abelian semigroup and consider the set of equations $E = \{x = s * x ; s \in S\}$. The finite subset $\{x = s_1^{-1} x, x = s_2^{-1} x, \dots, x = s_n^{-1} x\}$ has evidently the solution $x = s_1 s_2 \dots s_n$, $x = s_1^{-1} s_2^{-1} \dots s_n^{-1} x$, thus, L is finitely solvable, hence solvable in S . if $x = d$,

$x_s = c_s, s \in S$, is a solution then in particular $d = d^{\#}c_d$. Thus, if $c_s \cdot s = t$ for an arbitrary element $s \in S$ we get $d^{\#}c_s \cdot s = dt$, i.e. $d \gg s = d^{\#}t$, i.e. (because of cancellation) $t = s$, hence $cd \cdot s = s$. We therefore have an identity $1 = ca$ in S . To assure the existence of inverses we choose $s \in S$ and know that $(s^{\#}d)^{\#}c_s = d^{\#}$ hence, cancellation implies $s^{\#}c_s = 1$, i.e. $c_{s^{\#}d} = \frac{s^{\#}c_s}{s^{\#}d}$.

(2) if $G = \langle S; \gg \rangle$ is a non-Abelian equationally compact semi-group with cancellation then $S = \cup \{[s]; s \in S\}$ and each $\langle [s]; \# \rangle$ is an Abelian semigroup with cancellation. Since $S \in C(\langle [s]; * \rangle)$ for each $s \in S$ we use theorem 15 to assure for every s the existence of some $g \in HSP(\langle [s]; \cdot \rangle) \cap C(\langle [s]; - \rangle)$ with $\gg 3^s_{c?}$. Each such g of course, is an equationally compact Abelian semi-group with cancellation, hence (by part (1)) a group. Thus, $\mathcal{G} = \cup \{G_s; s \in S\}$ represents S as union of Abelian groups. If we can show that the identity elements 1 of all the groups G_s

G_s coincide then we are done. To see the last point we choose

$s, t \in S$; (i) $1^{s \cdot t} \cdot s^{\#}t = s^{\#}t$ implies $1^{s \cdot t} \cdot s = s$, i.e. $1_{s-t} = 1_s$ (ii) $s^{\#}t, 1_{s-t} = s^{\#}t$ implies $t^{\#}1_{s \cdot t} = t^{\#}1_s$

$1_{s \cdot t} = 1_t$. Thus, $1_{s \cdot t} = 1_s = 1_t$. q. e. d.

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