ANALYSIS OF A FAMILY OF ALGORITHMS FOR THE EVALUATION OF A POLYNOMIAL AND SOME OF ITS DERIVATIVES

Mary Shaw
J. F. Traub

Department of Computer Science
Carnegie-Mellon University

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ABSTRACT

We have previously presented a new one parameter family of algorithms and a program for evaluating the first m derivatives of a polynomial of degree n. In this paper we analyze that family of algorithms and present practical algorithms for selecting optimal or good values of the parameter q.

A program for selecting the optimal value of q under the constraint that q divides n+1 is given. We also analyze a program that eliminates that constraint and a simple program that selects a good, but not always optimal, value of q. We derive bounds on how close to optimal the "good" value will be.

The above results apply for $n > 12$. We extend the results to all n by tabulating the cost function for $n \leq 12$.

Some open questions on extensions of our results are stated.

1. INTRODUCTION

We have previously presented a one parameter family of algorithms for evaluating the first m derivatives of a polynomial [Shaw and Traub, 1974]. In this paper we analyze that family of algorithms and present a practical algorithm for selecting optimal or good values of the parameter.

The classical algorithm for evaluating a polynomial and its derivatives is Horner's Rule, which requires O(mn) additions and O(mn) multiplications. In 1972 we presented an algorithm which requires the same number of additions but is linear in multiplications and divisions (M/D) [Shaw and Traub, 1972]. This result and concurrent research on applications of fast algorithms related to the discrete Fourier Transform led to an improving sequence of asymptotic results for the special case of all derivatives. The first of these was $O(n \log^3 n)$ in total arithmetic operations [Borodin, 1972]; this was reduced to $O(n \log^2 n)$ by Kung [1973], Strassen [1972], and Borodin [1973]. New results by Aho, Steiglitz, and Ullman [1974] and Vari [1974] are $O(n \log n)$.

The algorithms analyzed in this paper differ from most fast algorithms in that the benefits can be obtained for small as well as large n; 3n-2 M/D suffice for any value of n.

Furthermore, our algorithms are applicable for all m, whereas the asymptotic fast methods apply only if all derivatives are calculated. On the other hand, the most important cases in practice are for small m and for m=n.

There are many open questions on extensions of these results. Questions concerning only the calculation of all derivatives include:

1. Is there an algorithm using only a linear number of additions?
2. It is easy to show at least n+1 multiplications are required. An

upper bound is 3n-2 multiplications and divisions. Can these bounds be tightened?

3. How many multiplications are required if divisions are not permitted?

Let P denote a polynomial of degree n. Define a normalized derivative as $P^{(i)}/i!$. When we refer to derivatives in this paper we always mean normalized derivatives. We consider P itself to be the zeroth derivative $P^{(0)}$.

To compute the first m derivatives of P, let

$$P(x) = \sum_{i=0}^{n} a_{n-i}x^i,$$

let p and q be such that n+1 = pq, and define

$$s(j) = (n-j) \bmod q, \quad j=0,1,\ldots,n.$$

Then Shaw and Traub [1974] presented the

<u>Algorithm</u>

$$(1.1) \quad T_i^{-1} = a_{i+1}x^{s(i+1)}, \quad i=0,1,\ldots,n-1$$

$$T_j^j = a_0x^{s(0)}, \quad j=0,1,\ldots,m$$

$$T_i^j = T_{i-1}^{j-1} + T_{i-1}^j x^{s(i-1)-s(i-j-1)+1}, \quad j=0,1,\ldots,m, \quad i=j+1,\ldots,n$$

and proved that $P^{(j)}(x)/j! = T_n^j/x^{j \bmod q}$.

Stewart [1971] has performed an analysis of the effects of rounding errors in the use of the iterated Horner's rule for root shifting. This corresponds to the special case q=1, m=n. He concludes that zeros near the shift are not unduly perturbed. Despite the divisions used to obtain the derivatives from T_n^j, Wozniakowski [1974] has established a similar result for the family defined by the algorithm.

A program (Program D1) to perform this calculation was presented previously, and the number of operations required to evaluate P(x) and its first m derivatives was established [Shaw and Traub 1974]. The number of additions is $(m+1)(n-\frac{m}{2})$ independent of q. In this paper we consider only the number of M/D required, and we show how to choose optimal or good values for q.

Given n and m, we showed that the best choice of q can be obtained by minimizing the function

$$f_{m,n,r}(q) = n - 1 + m\frac{n+1}{q} - (m+2)r + q\frac{r^2+r+2}{2}$$

where $r = \lfloor m/q \rfloor$ and q is an exact divisor of n+1. For fixed n, the cost function is a surface in three dimensions with coordinates m, q, and $f_{m,n,r}(q)$. The requirement that q divide n+1 will be relaxed in Section 4. It is maintained until then to simplify the analysis.

Known results on the evaluation of polynomials and their derivatives are special cases (often for poor choices of the parameter q) of this family of algorithms. These examples were presented as special cases in [Shaw and Traub, 1974]. They are summarized in Table 1; the m and q coordinates are displayed and the surface $f_{m,n,r}(q)$ is suggested by specific values.

Special properties of the algorithm allow a saving of one M/D each in column n+1 and row n. These savings are not shown in Table 1, nor will they be included in the analysis below except where mentioned explicitly.

In Section 2 we view $f_{m,n,r}(q)$ as a piecewise continuous function of m and q. This continuous model is used in later sections to help explain the behavior of the discrete parameter case.

In Section 3 we restrict m and q to integer values, and analyze the selection of q for Program D1. We present a rule (Program S1) for selecting

m \ q	1	$\sqrt{n+1}$	$\frac{n+1}{2}$	n+1
0	"Horner" [Newton, 1711] (n)			Naive $(2n)$
1		Munro [1971] $(n+2\sqrt{n})$		
k	$(k+1)(n-\frac{k}{2})$			$(2n+k)$
n	Horner [1819] $\left(\frac{n^2+n}{2}\right)$		Shaw-Traub [1974] $(3n-2)$	Shaw-Traub [1974] $(3n)$

Table 1
Number of M/D Required by Some Members of the Family of Algorithms for Computing Polynomials and Derivatives

an optimal value of q for $n > 12$.

In Section 4 we relax the constraint that q divide n+1. We present and analyze a new program (Program D2) that works without the constraint. We show that for Program D2 a simple rule (Program S2) selects a good, but not always optimal, value for the parameter, and we derive bounds on how close to optimal the "good" value will be. Given m and n such that $n > 12$ and $0 \le m \le n$, the following program selects a good value $\hat{Q}$ as the family parameter for Program D2:

PROGRAM S2:

<u>begin</u>

<u>if</u> $m = 0$ <u>then</u> $\hat{Q} \leftarrow 1$

<u>else</u> <u>if</u> $m < \frac{n+1}{6}$ <u>then</u> $\hat{Q} \leftarrow \text{round}\left(\sqrt{m(n+1)}\right)$

<u>else</u> $\hat{Q} \leftarrow \left\lceil \frac{n+1}{2} \right\rceil$

<u>end</u>

The theorems developed in Sections 2, 3, and 4 usually apply for $n > 12$. In Section 5 we extend the results to all n by tabulating $f_{m,n,r}(q)$ for $n \le 12$. For comparison purposes, we also tabulate the function for a few representative values of $n > 12$.

2. THE CONTINUOUS MODEL

If we relax the requirement that m and q be integers, we can use techniques of calculus to understand the behavior of the function $f_{m,n,r}(q)$. Restoring the integer constraint then forces us to consider only particular achievable points. For example, the minima of the continuous function do not always correspond to integer values of m and q, but knowledge of the locations of those minima can guide the search for the minima of the discrete function. Note, however, that r is always a nonnegative integer, $r = \lfloor \frac{m}{q} \rfloor$.

Since the optimal result for $m = 0$ is known to be $q = 1$, we assume $m \geq 1$ unless otherwise noted. We assume throughout that $m \leq n$. Also, since q is treated as continuous we always have $n+1 = p \cdot q$ for some p.

We now study the locations of the minima of

$$(2.1)\quad f_{m,n,r}(q) = n-1 + \frac{m(n+1)}{q} - r(m+2) + q\,\frac{(r^2+r+2)}{2}$$

for given m and n.

Let n,m be fixed. Then r is determined by q. For each value of r, $f_{m,n,r}(q)$ is a rational and continuous function of q. The m-q plane is divided into regions by the values of r. These regions are specified by

$$(2.2)\quad \begin{cases} m < q \quad , r = 0 \\ \frac{m}{r+1} < q \leq \frac{m}{r}, \; r = 1,\ldots,n. \end{cases}$$

The cost function $f_{m,n,r}(q)$ is piecewise continuous.

<u>In the remainder of this paper we often abbreviate $f_{m,n,r}(q)$ by $f_r(q)$.</u>

We investigate the minimum value of $f_r(q)$ in each of the r-regions. Now,

$$f_r'(q) = -\frac{m(n+1)}{q^2} + \frac{r^2+r+2}{2} .$$

Denote by Q_r the points at which $f_r'(q) = 0$. Then

$$(2.3) \quad Q_r = \sqrt{\frac{2m(n+1)}{r^2+r+2}} ,$$

and

$$(2.4) \quad f_r(Q_r) = n-1-r(m+2) + 2\sqrt{\frac{r^2+r+2}{2}}\sqrt{m(n+1)}$$

<u>Observe that if n is fixed, then the graph of Q_r as a function of m is a parabola for each value of r.</u> Since $f''(q) = (2m(n+1))/q^3 > 0$,

there is at most one interior minimum. If it exists, this interior minimum lies at Q_r. The following theorem describes the location of the minima of $f_r(q)$.

THEOREM 2.1. The minimum of $f_r(q)$ is:

1. at the interior point Q_0 if $r = 0$,
2. at the interior point Q_1 if $r = 1$, $m > \frac{n+1}{2}$,
3. at $q = m$ if $r = 1$ and $m \le \frac{n+1}{2}$,
4. at $q = \frac{m}{r}$ for $r \ge 2$.

PROOF

1. Let $r = 0$. Then $m < q \le n+1$.

Now,

$$f_0'(n+1) = 1 - \frac{m}{n+1} > 0$$

$$f_0'(m_+) = 1 - \frac{m(n+1)}{m_+^2} < 0$$

Hence $f_0(q)$ has an interior minimum at $Q_0 = \sqrt{m(n+1)}$.

2. Let $r = 1$, $m > \frac{n+1}{2}$. Then $\frac{m}{2} < q \leq m$.

$$f_1'(m) = 2 - \frac{n+1}{m} > 0$$

$$f_1'(\tfrac{1}{2}m_+) = 2\left(1 - \frac{2(n+1)}{m}\right) < 0.$$

Hence $f_1(q)$ has an interior minimum at $Q_1 = \sqrt{\frac{1}{2}m(n+1)}$.

3. Let $r = 1$, $m \leq \frac{n+1}{2}$

$$f_1'(m) = 2 - \frac{n+1}{m} \leq 0$$

$$f_1'(\tfrac{1}{2}m_+) = 2\left(1 - \frac{2(n+1)}{m}\right).$$

Therefore $f_1(q)$ is monotonically decreasing for $\frac{1}{2}m < q \leq m$ and the minimum lies at $q = m$.

4. Let $r \geq 2$. Note that

$$f_r'\left(\frac{m_+}{r+1}\right) = (r+1)\left[\frac{r}{2} - (r+1)\frac{(n+1)}{m}\right] + 1 < 0 \text{ for } \forall r \geq 0.$$

We show $f'_r\left(\frac{m}{r}\right) < 0$, $\forall r \geq 2$. Now,

$$f_r'\left(\frac{m}{r}\right) = -r^2\,\frac{(n+1)}{m} + \frac{r^2+r}{2} + 1 = w(r)$$

Hence

$$w(r) = r^2(\tfrac{1}{2}-x) + \frac{r}{2} + 1,$$

where $x = \frac{n+1}{m}$. Since

$$w(\infty) < 0,\ w(2) < 0,$$

$$w'(r) = 2r\left(\frac{1}{2}-x\right) + \frac{1}{2} < 0, \quad \text{for } x \geq 1,\ r \geq 2$$

we conclude

$$w(r) = f_r'\left(\frac{m}{r}\right) < 0, \ \forall r \geq 2.$$

Hence for $r \geq 2$, the minimum of $f_r(q)$ lies at $q = m/r$. QED

Theorem 2.1 shows how to find the minimum when given a region. The next three theorems show how to choose a region. We first show that only two regions are interesting, then give a rule for deciding between those two regions.

We first prove

LEMMA 2.1

If $n > 2$, $r \geq 2$, then

$$f_r\left(\frac{m}{r}\right) < f_{r+1}\left(\frac{m}{r+1}\right).$$

PROOF

$$f_r\left(\frac{m}{r}\right) = (n-1)(r+1) - \frac{m}{2r}(r^2-r-2),$$

$$f_{r+1}\left(\frac{m}{r+1}\right) = (n-1)(r+2) - \frac{m}{2(r+1)}(r^2-r-2) - \frac{r}{r+1}m,$$

$$f_{r+1}\left(\frac{m}{r+1}\right) - f_r\left(\frac{m}{r}\right) = n-1 - \frac{r}{r+1}m + \frac{m}{2}(r^2-r-2)\left[\frac{1}{r}-\frac{1}{r+1}\right] > 0$$

QED

THEOREM 2.2. Let $n \geq 12$. Then the minimum value of $f_r(q)$ lies in one of the regions $r = 0$ and $r = 1$.

PROOF. The proof is divided into three cases:

1. If $n \geq 12$, $m \leq n$, $r \geq 3$, then $f_r\left(\frac{m}{r}\right) > f_0(Q_0)$.

2. If $n \geq 3$, $m < n-2$, $r = 2$, then $f_2\left(\frac{m}{2}\right) > f_0(Q_0)$.

3. If $n \geq 9$, $n-2 \leq m \leq n$, $r = 2$, then $f_2\left(\frac{m}{2}\right) > f_1(Q_1)$.

When we have established these three cases, it then follows that at least one of $\{f_0(Q_0), f_1(Q_1)\}$ is smaller than $f_r\left(\frac{m}{r}\right)$, $\forall m$, $\forall r \geq 2$. The theorem then follows immediately.

By Theorem 2.1, the values of f considered in this proof are the minima of f in their respective regions for the values of m in question.

1. We first establish

$$f_3\left(\frac{m}{3}\right) > f_0(Q_0).$$

Now

$$f_3\left(\frac{m}{3}\right) - f_0(Q_0) = 3(n-1) - \frac{2}{3}m - 2\sqrt{m(n+1)}.$$

The worst case is at m = n; in that case the expression above is positive when $\frac{13}{9}n^2 - 18n + 9 > 0$ which is true if $n \geq 12$. By Lemma 2.1,

$$f_r\left(\frac{m}{r}\right) > f_3\left(\frac{m}{3}\right), r \geq 4$$

which completes the proof of Case 1.

2. $f_2\left(\frac{m}{2}\right) - f_0(Q_0) = 2(n-1) - 2\sqrt{m(n+1)}$.

Now, $(n-1)^2 - m(n+1)$ is linear in m. Thus, when n is fixed there is only one point at which $f_2\left(\frac{m}{2}\right) = f_0(Q_0)$. This point lies between m = n-2 and m = n-3. Hence

$$f_2\left(\frac{m}{2}\right) > f_0(Q_0) \text{ for } m < n-2.$$

3. Let $\Delta(m) = f_2\left(\frac{m}{2}\right) - f_1(Q_1) = 2n + m - 2\sqrt{2m(n+1)}$.

We consider m = n, n-1, n-2.

At m = n, $\Delta(n) = 3n - 2\sqrt{2n(n+1)}$.

Then $\Delta(n) > 0$ when $9n^2 - 8n(n+1) > 0$, which holds for $n > 8$.

At m = n-1, $\Delta(n-1) = 3n - 1 - 2\sqrt{2(n-1)(n+1)}$.

Then $\Delta(n-1) > 0$ when $n^2 - 6n + 9 > 0$, which holds for all $n \neq 3$.

At m = n-2, $\Delta(n-2) = 3n - 2 - 2\sqrt{2(n-2)(n+1)}$.

Then $\Delta(n-2) > 0$ when $n^2 - 4n + 20 > 0$ which holds for all n. QED

THEOREM 2.3 Let $n \geq 23$, $m \leq \frac{1}{2}(n+1)$. Then the minimum value of $f_r(q)$ lies in the $r = 0$ region at $Q_0 = \sqrt{m(n+1)}$.

PROOF. From Theorem 2.2 the minimum value of $f_r(q)$ lies in one of the regions $r = 0$ and $r = 1$. From Theorem 2.1, it lies at Q_0 if $r = 0$ and at $q = m$ if $r = 1$, $m \leq \frac{1}{2}(n+1)$. We need only compare $f_0(Q_0)$ and $f_1(m)$. We show under the hypotheses of this theorem that $f_0(Q_0) < f_1(m)$. Now

$$\Delta(m) = f_0(Q_0) - f_1(m) = 2\sqrt{m(n+1)} - (n+m-1).$$

If m is small, $\Delta(m) < 0$. The worst case is at $m = \frac{1}{2}(n+1)$. Then $\Delta(\frac{1}{2}(n+1)) = n\left(\sqrt{2}-\frac{3}{2}\right) + \sqrt{2}+ \frac{1}{2}$ which is negative for $n \geq 23$. QED

THEOREM 2.4 Let $n \geq 23$, $m > \frac{1}{2}(n+1)$. Then:

1. If m is near $\frac{1}{2}(n+1)$, the minimum value of $f_r(q)$ lies in the $r = 0$ region at $Q_0 = \sqrt{m(n+1)}$. If m is near n, the minimum value of $f_r(q)$ lies in the $r = 1$ region at $Q_1 = \sqrt{\frac{m(n+1)}{2}}$.

2. For n fixed, there is a unique value of m where the crossover from $r = 0$ to $r = 1$ occurs. The crossover point is given by $m + 2 + 2(1 - \sqrt{2})\sqrt{m(n+1)} = 0$. As $n \to \infty$, this crossover moves to $m/n = 4(3 - 2\sqrt{2}) \doteq .686$.

PROOF

1. By Theorems 2.1 and 2.2 we need only compare $f_0(Q_0)$ and $f_1(Q_1)$. Now,

$$\Delta(m) = f_0(Q_0) - f_1(Q_1) = m + 2 + 2(1-\sqrt{2})\sqrt{m(n+1)} .$$

Since $\Delta((\frac{n+1}{2})_+) < 0$ for $n \geq 23$, $\Delta(n) > 0$ for $n \geq 0$, the conclusion of Part 1 follows

2. Since

$$\Delta'(m) = 1 + (1-\sqrt{2})\sqrt{\frac{n+1}{m}}$$

is positive for $\frac{1}{2}(n+1) < m \leq n$ there is, for fixed n, a unique value of m at which the crossover of the minimum from the $r = 0$ to the $r = 1$ region occurs. This crossover is given by

$$\Delta(m) = 0 = m + 2 + 2(1-\sqrt{2})\sqrt{m(n+1)}.$$

As $n \to \infty$, the crossover moves to $m/n = 4(3 - 2\sqrt{2}) \doteq .686$. QED

3. THE DISCRETE PARAMETER CASE

In the previous section we treated $f_{m,n,r}(q)$ as if it were continuous in m and q. However, the function makes sense in the intended interpretation - multiplication counts for an algorithm - only if m and q are integers. In this section we constrain m, n, and q to integer values, and further require q to be an integer divisor of n+1. We will relax this divisibility restriction in the next section. In order to distinguish evaluations of f on the restricted values of q from the general evaluations of the previous section, we denote by q^* the values of q that are restricted to integers.

Given m and n, the optimality problem for the discrete parameter case is to select an <u>integer value</u> Q^* so as to minimize $f_{m,n,r}(q^*)$. It is not sufficient to select an integer q' close to the value of q ($q'=\lfloor q \rfloor$, for example) that minimizes the continuous case; there may be another integer, q'', such that $f_{m,n,r}(q'') < f_{m,n,r}(q')$.

The development in this section parallels Section 2. We first show (Theorem 3.1) that the best q lies in one of the regions $r = 0$ and $r = 1$. We then show in Theorems 3.2, 3.3, and 3.4 how to select the best q^* within a region. Next, we show how to choose between the regions $r = 0$ and $r = 1$ (Theorem 3.5). Finally, we present an algorithm for choosing Q^* given m and n.

Throughout this section we prove results for moderate to large n - at worst, $n > 12$. Precise statements for smaller n can be made by carrying more detail in the proofs or by enumeration. We do not take advantage of the special cases $m = n$ and $Q^* = n+1$; a one-multiplication saving can be made in each of these cases.

THEOREM 3.1. Let $n > 12$ and assume $q^* \mid (n+1)$. Then the minimum value of $f_r(q^*)$ lies in one of the regions $r = 0$ and $r = 1$.

PROOF. We will show that if $m \le n$, $n > 12$, $r \ge 2$ then $f_0(n+1) < f_r(q^*)$. It follows that the best choice of q^* will lie in $r = 0$ or $r = 1$.

From the analysis of the continuous case we have the results

(a) for $r \ge 2$, $\min\limits_{\frac{m}{r+1} < q \le \frac{m}{r}} f_r(q) = f_r\left(\frac{m}{r}\right)$

(b) for $n > 2$, $r \ge 2$, $f_r\left(\frac{m}{r}\right) < f_{r+1}\left(\frac{m}{r+1}\right)$

It follows from (a) that $\min\limits_{\frac{m}{r+1} < q^* \le \frac{m}{r}} f_r(q^*) \ge f_r\left(\frac{m}{r}\right)$ when such q^* exists.

There are two cases: $m \le n - 4$ and $n - 3 \le m \le n$.

<u>Case 1: $m \le n - 4$</u>

Note that $q^* = n+1$ always lies in the region $r = 0$. If $m \le n - 4$ then

$$f_0(n+1) = 2n+m < 3n-3 = f_2\left(\frac{m}{2}\right) \le f_2(q^*) \text{ for any } q^* \text{ in the region } r = 2.$$

Since $f_2\left(\frac{m}{2}\right) < f_r\left(\frac{m}{r}\right)$ for $r > 2$,

$$f_0(n+1) < f_r\left(\frac{m}{r}\right) \le \min_{\frac{m}{r+1} < q^* \le \frac{m}{r}} f_r(q^*) \quad \text{for } r \ge 2$$

<u>Case 2: $n - 3 \le m \le n$</u>

<u>Case 2a: $r \ge 3$, $n > 12$</u>

By the same reasoning as for Case 1,

$$f_0(n+1) = 2n+m < 4(n-1) - \frac{2}{3}m = f_3\left(\frac{m}{3}\right) \le f_3(q^*)$$

for any q^* in the region $r = 3$, provided $n > 12$.

Case 2b: $r = 2, n \ge 12$

When $r = 2$, q^* may assume values in the following ranges:

$m = n$	$\frac{n}{3} < q^* \le \frac{n}{2}$
$m = n - 1$	$\frac{n}{3} - \frac{1}{3} < q^* \le \frac{n}{2} - \frac{1}{2}$
$m = n - 2$	$\frac{n}{3} - \frac{2}{3} < q^* \le \frac{n}{2} - 1$
$m = n - 3$	$\frac{n}{3} - 1 < q^* \le \frac{n}{2} - \frac{3}{2}$

Since the integer divisors of n+1 are $q^* = \frac{n+1}{k}$, for these cases (and provided $n \ge 11$) the only possible value for q^* is $\frac{n+1}{3}$. But $f_0(n+1) = 2n+m < \frac{7}{3}n+m - \frac{11}{3} = f_2\left(\frac{n+1}{3}\right)$ when $n \ge 12$ and so $f_0(n+1) < f_2(q^*)$. QED

Theorem 3.2 shows how to select a Q^* from feasible values in a given region.

THEOREM 3.2. Let $u < v$ be factors of n+1 and fix r. Then

$$f_r\left(\frac{n+1}{v}\right) < f_r\left(\frac{n+1}{u}\right) \text{ if and only if } m < (n+1)\frac{r^2+r+2}{2uv} .$$

PROOF. Note that fixing r restricts the range of m to $r\left(\frac{n+1}{u}\right) \le m < (r+1)\left(\frac{n+1}{v}\right)$. Now,

$$f_r\left(\frac{n+1}{k}\right) = n - 1 + mk - (m+2)r + \left(\frac{n+1}{k}\right)\left(\frac{r^2+r+2}{2}\right)$$

Let $\Delta(u,v) = f_r\left(\frac{n+1}{u}\right) - f_r\left(\frac{n+1}{v}\right)$

$$= (v-u)\left[(n+1)\left(\frac{r^2+r+2}{2uv}\right) - m\right]$$

Since $v-u > 0$,

$$\Delta(u,v) > 0 \text{ exactly when } m < (n+1)\frac{r^2+r+2}{2uv} \qquad \text{QED}$$

For the only two cases of interest, this specializes to

$$f_0\left(\frac{n+1}{v}\right) < f_0\left(\frac{n+1}{u}\right) \quad \text{when } m < \frac{n+1}{uv}$$

$$f_1\left(\frac{n+1}{v}\right) < f_1\left(\frac{n+1}{u}\right) \quad \text{when } m < \frac{2(n+1)}{uv}$$

The next theorem further restricts the values of q^* that must be considered when selecting Q^* within the region $r = 1$.

THEOREM 3.3. Assume n+1 is non-prime. Let u be the smallest divisor of n+1 greater than 1 and v be any other divisor. For values of $m \ni \frac{m}{2} < \frac{n+1}{v} < \frac{n+1}{u} \le m$, we have $f_1\left(\frac{n+1}{u}\right) \le f_1\left(\frac{n+1}{v}\right)$. QED

PROOF. If $u > 1$, then $u \ge 2$ and $v \ge 3$ and by Theorem 3.2 $f_1\left(\frac{n+1}{v}\right) < f_1\left(\frac{n+1}{u}\right)$ if and only if $m < \frac{2(n+1)}{uv}$. But if it were the case that $m < \frac{2(n+1)}{uv}$ we would have $m < \frac{2(n+1)}{uv} \le \frac{2}{3}\,\frac{n+1}{u}$, which contradicts the assumption $m \ge \frac{n+1}{u}$. Hence $f_1\left(\frac{n+1}{u}\right) \le f_1\left(\frac{n+1}{v}\right)$. QED

The next theorem further restricts the values of q^* that must be considered when selecting Q^* within the region $r = 0$. Recall $Q_0 = \min_{m < q} f_0(q) = \sqrt{m(n+1)}$.

THEOREM 3.4. If Q_0 is an integer, $\min_{m < q^*} f_0(q^*)$ is at $q^* = Q_0$. Otherwise let $v_1 < v_2$ be the integer factors of n+1 such that $\frac{n+1}{v_1}$ and $\frac{n+1}{v_2}$ are adjacent to and on opposite sides of Q_0. Then Q^* is either $\frac{n+1}{v_1}$ or $\frac{n+1}{v_2}$, that is, $\min_{m < q^*} f_0(q^*)$ is either $f_0\left(\frac{n+1}{v_1}\right)$ or $f_0\left(\frac{n+1}{v_2}\right)$.

PROOF. f_0 is concave up in this region, with minimum at Q_0. If Q_0 is an integer, $q^* = Q_0$ is clearly optimal. Otherwise, of two points on the same side of Q_0, the point of evaluation closer to Q_0 will yield the lower value.

Now we consider the selection of Q^* from values that correspond to different values of r. From Theorem 3.1, we know that the values correspond to r = 0 and r = 1.

THEOREM 3.5. Let $u < v$ be factors of n+1. Let $n > 12$. Consider only values of $m \ni \frac{m}{2} < \frac{n+1}{v} \le m < \frac{n+1}{u}$. (That is, let u correspond to r = 0 and v correspond to r = 1). Then

(a) $f_1\left(\frac{n+1}{v}\right) < f_0\left(\frac{n+1}{u}\right)$ when

(i) $u = 1,\ v = 2$

(ii) $u = 1,\ v = 3,\ m = \frac{n+1}{3},\ \frac{n+1}{3} + 1$

(iii) $u = 1,\ v > 3,\ m = \frac{n+1}{v}$

(b) $f_1\left(\frac{n+1}{v}\right) = f_0\left(\frac{n+1}{u}\right)$ when

(i) $u = 1,\ v = 3,\ m = \frac{n+1}{3} + 2$

(ii) $u = 1,\ v = 4,\ m = \frac{n+1}{4} + 1$

(c) $f_1\left(\frac{n+1}{v}\right) > f_0\left(\frac{n+1}{u}\right)$ otherwise.

PROOF. Let $\Delta(u,v) = f_1\left(\frac{n+1}{v}\right) - f_0\left(\frac{n+1}{u}\right) = (n+1)\left(\frac{2}{v}-\frac{1}{u}\right) + m(v-u-1) - 2$

Case 1 establishes parts a and b and a portion of part c. Case 2 establishes the remainder of part c.

<u>Case 1:</u> u = 1

$$\Delta(1,v) = (n+1)\left(\frac{2}{v}-1\right) + m(v-2) - 2 = \left(m-\frac{n+1}{v}\right)(v-2) - 2$$

By enumeration for $v \le 4$,

$\Delta(1,2) = -2 < 0$

$\Delta(1,3) = m - \frac{n+1}{3} - 2$ so $\Delta(1,3) < 0$ for $m = \frac{n+1}{3}, \frac{n+1}{3} + 1$

$\Delta(1,3) = 0$ for $m = \frac{n+1}{3} + 2$

$\Delta(1,3) > 0$ otherwise

$\Delta(1,4) = 2m - \frac{n+1}{2} - 2$ so $\Delta(1,4) < 0$ for $m = \frac{n+1}{4}$

$\Delta(1,4) = 0$ for $m = \frac{n+1}{4} + 1$

$\Delta(1,4) > 0$ otherwise

For $v > 4$,

$\Delta(1,v) = \left(m - \frac{n+1}{v}\right)(v-2) - 2$ so $\Delta(1,v) < 0$ for $m = \frac{n+1}{v}$

$\Delta(1,v) > 0$ otherwise

<u>Case 2:</u> $u \geq 2$

Since $\Delta(u,v)$ is an increasing function of m, it suffices to show $\Delta(u,v) > 0$ for $m - \frac{n+1}{v}$. Substituting for m, let

$$\delta(\mu,\nu) = (n+1)\left(\frac{2}{\nu} - \frac{1}{\mu}\right) + \frac{(n+1)}{\nu}(\nu-\mu-1) - 2,$$

where μ and ν are potentially continuous. We show $\delta(u,v) > 0$ for all u, v for which $v > u$ and both u and v divide n+1.

Let $j = \frac{n+1}{v}$ and $w = \frac{n+1}{j+1}$. Then $w \geq u$ for integers u, $u < v$. Since

$$\frac{\partial^2}{\partial\mu^2}\delta(\mu,\nu) = -2(n+1)/\mu^3 < 0,$$

the minimum value of $\delta(\mu,\nu)$ for $2 \leq \mu \leq w$ must occur at $\mu = 2$ or $\mu = w$. Now,

$$\delta(2,v) = (n+1)\left(\frac{1}{2} - \frac{1}{v}\right) - 2 > 0, \quad v \geq 3, \quad n \geq 12$$

$$\delta(w,v) = \frac{vn-n-1}{n+1+v} - 2.$$

Since $\frac{\partial}{\partial\nu}\delta(\mu,\nu) = \frac{n+1}{\nu^2}(\mu-1) > 0, \quad \mu > 1,$

$$\delta(w,4) = \frac{n-11}{n+5} > 0, \quad n \geq 12$$

we conclude

$$\mathring{\Delta}(w,v) > 0, \quad v \geq 4, \quad n \geq 12$$

which completes the proof. QED

The results of this section can be summarized in

PROGRAM S1. [selects Q^*, an integral divisor of n+1, to minimize $f_{m,n,r}(q^*)$]

Given n,m such that $n > 12$ and $0 \leq m \leq n$, this algorithm gives Q^*.

Let u be the smallest factor of n+1 ∋ u > 1 (if n+1 is prime, u = n+1). [Theorem 3.3]

Let $Q_0 = \sqrt{m(n+1)}$. If m = 0 set $v_1 = 0$, $v_2 = 1$. If Q_0 is a positive integer, set $v_1 = v_2 = \frac{n+1}{Q_0}$. Otherwise, set v_1 and v_2 ($v_1 < v_2$) to adjacent factors of n+1 ∋ $\frac{n+1}{v_2} < Q_0 < \frac{n+1}{v_1}$. [Theorem 3.4]

To find Q^*:

```
begin                                                    [Horner's Rule]
if m = 0 then Q* ← 1 else
if m ≥ (n+1)/u then
    begin
    if u = 2 then Q* ← (n+1)/2                           [Theorem 3.5a(i)]
    else if u = 3 and (n+1)/3 ≤ m ≤ (n+1)/3 + 1 then Q* ← (n+1)/3   [Theorem 3.5a(ii)]
    else if m = (n+1)/u then Q* ← (n+1)/u                [Theorem 3.5a(iii)]
    else Q* ← n+1                                        [Theorem 3.5b and 3.5c]
    end
else
    begin
    if m < (n+1)/(v1 v2) then Q* ← (n+1)/v2              [Theorem 3.2]
    else Q* ← (n+1)/v1
    end
end
```

4. RELAXING THE DIVISIBILITY CONSTRAINT

The algorithm (1.1) for computing the value of a polynomial and some derivatives does not depend on q being an integer factor of n+1. However, Program D1 [Shaw and Traub, 1974] requires that q divide n+1. In this section we present a program (Program D2) for which q need not divide n+1; we establish the operation counts for this program; and we develop a simple rule (Program S2) for selecting q without regard to the factors of n+1.

When using Program D1, the number of M/D required to evaluate some of the derivatives of P depends strongly on the factors of n+1. If there is no factor near Q_0 or Q_1 then the cost may be much higher than the continuous model of Section 2 might suggest. Even though Program D2 exacts a penalty in extra multiplications for using a q that does not divide n+1, it is often less expensive than using Program D1. This is particularly true when m is small and n+1 is prime. For example, when n+1 = 23, the best selection of q for each program results in the following number of multiplications:

	Program D1		Program D2	
m	q	# mult	q	# mult
0	1	22	1	22
1	1	43	4,5, or 6	31
2	23	46	6 or 8	35
3	23	47	8	38
4	23	48	8 or 12	41
5	23	49	12	43
6-10	23	m+44	12	2m+33
11-12	23	m+44	12	m+43

PROGRAM D2 [for evaluating m derivatives of P of degree n, given q]

<u>begin</u>

[Given: n, the degree of the polynomial

m, the number of derivatives required

q, the splitting parameter

```
            a(i), i=0,...,n, the coefficients

            x, the point of evaluation (x ≠ 0)]

      [Choose p,t such that n+1 = p·q + t, 0 ≤ t < q.
            r = ⌊m/q⌋
            x(i) will be x^i

            T(i,j) will be T_i^j]

[When t = 0, this program specializes to Program D1.]

      x(0) ← 1

      x(1) ← x

      for i = 2,3,...,q

            x(i) ← x * x(i-1)

            [powers of x require q-1 multiplications]

      if t > 0 then

         begin

         for i = 0,1,2,...,t-2

              T(i-1, -1) ← a(i) * x(t-i-1)

         T(t-2, -1) ← a(t-1)

         [this step requires t-1 multiplications; it is not executed if t = 0]

         end

      for i = t, t+q, t+2q,...,t+(p-1)*q

         begin

         for k = 0,1,...,q-2

            T(i+k-1, -1) ← a(i+k) * x(q-k-1)

            [inner loop requires q-1 multiplications each time]

         T(i+q-2, -1) ← a(i+q-1)

         [this step requires total of p * (q-1) multiplications]

         end
```

```
[entire initialization requires (p+1)(q-1) multiplications if t = 0
 or (p+1)(q-1) + (t-1) multiplications if t > 0]
for j = 0,1,...,m
   begin
   [ℓ+1 will be the number of complete groups of q-1 additions and
    one multiplication]
   ℓ ← ⌊(p+q-j-1)/q⌋ - 1
   T(j,j) ← T(j-1, j-1)
   if t > 0 then
       begin
       for i = j+1, j+2,...min(j+t-1, n)
           T(i,j) ← T(i-1, j-1) + T(i-1, j)
       if j+t ≤ n then
           T(j+t, j) ← T(j+t-1, j-1) + T(j+t-1, j) * x(q)
       end
   for i = t+j+1, t+j+q+1, t+j+2q+1,...,t+j+ℓ*q+1
       begin
       for k = 0,1,...,q-2
           T(i+k, j) ← T(i+k-1, j-1) + T(i+k-1, j)
       T(i+q-1, j) ← T(i+q-2, j-1) + T(i+q-2, j) * x(q)
       end
   for i = t+j+(ℓ+1)*q+1, t+j+(ℓ+1)*q+2,...,n
       T(i,j) ← T(i-1, j-1) + T(i-1, j)
   end
[recurrence requires (m+1)(n-½m) additions independent of t and
 (m+1)(p-r-1) +½q·r(r+1) multiplications if t = 0 or
 (m+1)(p-r-1) +½q·r(r+1) + (m+1) multiplications if t > 0]
```

```
[T(n,j) is now x^(j mod q) p^(j)(x)/j!]
for j = 0, q, 2q,...,(r-1) * q
   for k = 1, 2,...,q-1
       T(n, j+k) ← T(n, j+k)/x(k)
for j = r*q+1, r*q+2,...,m
   T(n,j) ← T(n,j)/x(j-r*q)
[m-r divisions used to obtain normalized derivatives in T(j,n)]
end
```

Operation Counts

The arithmetic operation counts for the program are indicated by comments in the program.

Let

$$n+1 = p \cdot q + t \qquad 0 \le t < q$$
$$r = \left\lfloor \frac{m}{q} \right\rfloor$$

The number of additions required is $(m+1)(n-\frac{1}{2}m)$, independent of q. This is the same number as for Program D1. The number of multiplications and divisions required is given by:

$t = 0$	$t > 0$	step
$(p+1)(q-1)$	$(p+1)(q-1) + (t-1)$	initialization
$(m+1)(p-r-1) + \frac{1}{2}qr(r+1)$	$(m+1)(p-r) + \frac{1}{2}qr(r+1)$	recurrence
$m-r$	$m-r$	divisions to normalize T(i,j)

Now, let $\hat{f}_{m,n,r}(q)$ denote the total number of M/D required to calculate the first m derivatives of a polynomial of degree n if Program D2 is used with splitting parameter q. Then if $t = 0$,

$$\hat{f}_{m,n,r}(q) = (p+1)(q-1) + (m+1)(p-r-1) + \frac{1}{2}qr(r+1) + m-r$$

$$= n-1 + m\frac{n+1}{q} - (m+2)r + q\frac{r^2+r+2}{2}$$

$$= f_{m,n,r}(q)$$

if $t > 0$,

$$\hat{f}_{m,n,r}(q) = (p+1)(q-1) + (t-1) + (m+1)(p-r) + \frac{1}{2}qr(r+1) + m-r$$

$$= n-1 + m\left(\frac{n+1}{q} + \frac{q-t}{q}\right) - (m+2)r + q\frac{r^2+r+2}{2}$$

$$= f_{m,n,r}(q) + m\frac{q-t}{q}$$

so for all t,

$$\hat{f}_{m,n,r}(q) = f_{m,n,r}(q) + m\frac{(q-t)\bmod q}{q}$$

We abbreviate $\hat{f}_{m,n,r}(q)$ by $\hat{f}_r(q)$.

Program D2 can be slightly improved in three special cases. Two are identical to the special cases for Program D1: a savings of one M/D when $m = n$ and when $q = n+1$. In addition, when $t = 1$, the first operation for each derivative is $T^j_{j+1} = T^{j-1}_j + T^j_j x^q$. Since all T^j_j are equal, $T^j_j x^q$ need not be recomputed. This is a saving of m multiplications. As before, the special cases are not considered in the analysis.

We establish a bound on $\hat{f}_r(q)$. Since the optimal result for $m = 0$ is known to be $q = 1$, we again assume $1 \le m \le n$. The proof of the following result is straightforward.

THEOREM 4.1

$$f_r(q) \le \hat{f}_r(q) = f_r(q) + m\frac{(q-t)\bmod q}{q} < f_r(q) + m, \quad 1 \le m \le n, \quad q=1,\ldots,n+1$$

and

$$\text{if } t > 0 \text{ then } f_r(q) < \hat{f}_r(q)$$

The effect of the algorithm is as if q-t leading zero coefficients had been added (enough to make q factor n + (q-t) + 1) and the multiplications by coefficients known to be zero eliminated. The term $m\frac{q-t}{q}$ is a penalty for selecting a q that is not a divisor of n+1. As illustrated above, this penalty may be substantially smaller than the cost of choosing an inappropriate divisor q^*.

As before, we must distinguish evaluations of f and $\hat{f}$ on a restricted set of values for q from general evaluations of f and $\hat{f}$. Let $\hat{q}$ be any integer, $1 \le \hat{q} \le n+1$.

Let $\hat{Q}$ denote the value of $\hat{q}$ which minimizes $\hat{f}_r(\hat{q})$. The next three theorems establish the best selection of $\hat{q}$ for most values of m. Since we frequently use certain values of $\hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right)$, we tabulate them here.

$$r = 0,\ n+1 \text{ odd:}\quad \hat{f}_0\left(\left\lceil\frac{n+1}{2}\right\rceil\right) = n-1 + 2m + \frac{n+2}{2} = \frac{3}{2}n + 2m,\quad m < \frac{n+1}{2}$$

$$r = 0,\ n+1 \text{ even:}\quad \hat{f}_0\left(\frac{n+1}{2}\right) = n-1 + 2m + \frac{n}{2} + \frac{1}{2} = \frac{3}{2}n + 2m - \frac{1}{2},\quad m < \frac{n+1}{2}$$

$$r = 1,\ n+1 \text{ odd:}\quad \hat{f}_1\left(\left\lceil\frac{n+1}{2}\right\rceil\right) = n-1 + 2m - m - 2 + n + 2 = 2n + m - 1,\quad m \ge \frac{n+1}{2}$$

$$r = 1,\ n+1 \text{ even:}\quad \hat{f}_1\left(\frac{n+1}{2}\right) = n-1 + 2m - m - 2 + n + 1 = 2n + m - 2,\quad m \ge \frac{n+1}{2}$$

Note that the value for even values of n+1 is always less than the value for odd values of n+1 with the same r. We remind the reader of the relations

$$n+1 = p\cdot\hat{q} + t \text{ and } m = r\cdot\hat{q} + s$$

THEOREM 4.2 Consider only $\hat{q}$ such that $\left\lceil\frac{n+1}{2}\right\rceil < \hat{q} < n+1$. Then $\hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right) < \hat{f}_r(\hat{q})$.

PROOF. For these values of $\hat{q}$, $p = 1$ and $t = n+1 - \hat{q} > 0$. Let

$$\Delta_r(\hat{q}) = n-1 + 2m - (m+2)r + \hat{q}\frac{r^2+r+2}{2} - f_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right).$$

Since $\hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right)$ is smaller for even n+1 than for odd, it suffices to show $\Delta_r(\hat{q}) > 0$ for n+1 odd.

<u>Case 1</u>: $m < \frac{n+1}{2}$, n+1 odd

For these value of $\hat{q}$ and m, $r = 0$. Thus

$$\Delta_0(\hat{q}) = n-1 + 2m + \hat{q} - (\tfrac{3}{2}n + 2m) = \hat{q} - (\tfrac{n}{2} + 1).$$

Since $\hat{q} > \left\lceil\frac{n+1}{2}\right\rceil = \frac{n}{2} + 1$, $\Delta_0(\hat{q}) > 0$.

<u>Case 2</u>: $m \geq \left\lceil\frac{n+1}{2}\right\rceil$, n+1 odd

For these values of $\hat{q}$ and m, $r = 0$ or $r = 1$.

<u>Case 2a</u>: $r = 1$

$$\Delta_1(\hat{q}) = n-1 + 2m - (m+2) + 2\hat{q} - (2n + m - 1) = 2(\hat{q}-1) - n.$$

Since $\hat{q} > \frac{n}{2} + 1$, $\Delta_1(\hat{q}) > 0$

<u>Case 2b</u>: $r = 0$

$$\Delta_0(\hat{q}) = n-1 + 2m + \hat{q} - (\tfrac{3}{2}n + 2m) = \hat{q} - 1 - \tfrac{n}{2}.$$

Since $\hat{q} > \frac{n}{2} + 1$, $\Delta_0(\hat{q}) > 0$. QED

THEOREM 4.3 $\hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right) \leq \hat{f}(n+1)$, $1 \leq m \leq n$

PROOF. Let $\Delta(\hat{q}) = \hat{f}_r(n+1) - \hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right)$ and show $\Delta(\hat{q}) \geq 0$; the details are similar to the proof of Theorem 4.2. QED

THEOREM 4.4 Restrict values of n, m and $\hat{q}$ to $n > 12$, $1 \le \hat{q} < \frac{n+1}{2}$, and $\frac{n+1}{6} < m \le n$. Then

$$f_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right) \le \hat{f}_r(\hat{q})$$

PROOF. Let $\Delta_r(\hat{q}) = \hat{f}_r(\hat{q}) - \hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right)$. As in Theorem 4.2, since $\hat{f}_r\left(\left\lceil\frac{n+1}{2}\right\rceil\right)$ is smaller for even n+1 than for odd, it suffices to show $\Delta_r(\hat{q}) \ge 0$ for n+1 odd. Note that $\hat{q} < \frac{n+1}{2} \Rightarrow p \ge 2$ and that, by definition of $\hat{q}$, $\frac{n+1}{p+1} < \hat{q}$.

Case 1: $\frac{n+1}{6} < m < \frac{n+1}{2}$

$$\Delta_r(\hat{q}) = p\cdot\hat{q} + t - 2 + m(p+1) - (m+2)r + \hat{q}\frac{r^2+r+2}{2} - (\tfrac{3}{2}n + 2m)$$

$$= m(p-1-r) + \hat{q}\frac{r^2+r+2}{2} - \frac{n+1}{2} - \frac{4r+1}{2}$$

Case 1a: $p \ge r+1$

Substituting $\frac{n+1}{6} < m$ and $\frac{n+1}{p+1} < \hat{q}$,

$$\Delta_r(\hat{q}) > \left(\frac{p-1-r}{6} + \frac{r^2+r+2}{2(p+1)} - \frac{1}{2}\right)(n+1) - \frac{4r+1}{2}$$

or

$$\Delta_r(\hat{q}) > \frac{n+1}{6(p+1)}(p^2 - (r+3)p + (3r^2+2r+2)) - \frac{4r+1}{2}$$

For sufficiently large n, $\Delta_r(\hat{q}) \ge 0$ provided

$$R(p) = p^2 - (r+3)p + (3r^2+2r+2) > 0, \; p \ge 2$$

The roots of R(p) are

$$p = \frac{1}{2}\left((r+3) \pm \sqrt{(r+3)^2-4(3r^2+2r+2)}\right)$$

For $r \ge 1$, the roots are imaginary, so $R(p) > 0 \;\; \forall p$.

For $r = 0$, the roots are $p = \frac{1}{2}(3\pm\sqrt{9-8}) = \{1,2\}$,

so $R(p) > 0$ for $p > 2$. At $p = 2$, $\hat{q} > \frac{n+1}{3}$, so

$$\Delta_0(\hat{q}) > \frac{n+1}{6} + \frac{n+1}{3} - \frac{n+1}{2} - \frac{1}{2} = -\frac{1}{2}\,.$$

Since $\Delta_0(\hat{q})$ assumes only integer values, $\Delta_0(\hat{q}) \geq 0$. To establish the bound on n, note that the worst case occurs when the coefficient of (n+1) in the approximation to $\Delta_r(\hat{q})$ is smallest. Tedious but straightforward operations in algebra and calculus yield $n > 10$.

Case 1b: $p < r+1$

Since $p \geq 2$, $r \geq 2$. Substituting

$$m < \frac{n+1}{2} \text{ and } \frac{n+1}{p+1} < \hat{q},$$

$$\Delta_r(\hat{q}) > \left(\frac{p-r}{2} + \frac{r^2+r+2}{2(p+1)} - 1\right)(n+1) - \frac{4r+1}{2}$$

or

$$\Delta_r(\hat{q}) > \frac{(n+1)}{2(p+1)}(p^2 - (r+1)p + r^2) - \frac{4r+1}{2}$$

For sufficiently large n, $\Delta_r(q) > 0$ provided

$$R(p) = p^2 - (r+1)p + r^2 > 0,\ p \geq 2.$$

But, by inspection, $R(p) > 0$ for $p \geq 2$, $\forall r$.

The bound $n > 12$ is established as for Case 1a.

Case 2: $\frac{n+1}{2} \leq m \leq n$ (necessarily, $r \geq 1$)

$$\Delta_r(\hat{q}) = p \cdot \hat{q} + t - 2 + m(p+1) - (m+2)r + \hat{q}\frac{r^2+r+2}{2} - (2n+m-1)$$

$$= m(p-r) + \hat{q}\frac{r^2+r+2}{2} - (n+1) - 2r$$

Case 2a: $p \geq r$

Substituting $\frac{n+1}{2} \leq m$ and $\frac{n+1}{p+1} < \hat{q}$,

$$\Delta_r(\hat{q}) > \left(\frac{p-r}{2} + \frac{r^2+r+2}{2(p+1)} - 1\right)(n+1) - 2r,$$

or $$\Delta_r(\hat{q}) > \frac{1}{2(p+1)}(p^2 - (r+1)p + r^2)(n+1) - 2r,$$

For sufficiently large n, $\Delta_r(\hat{q}) \geq 0$ provided

$$R(p) = p^2 - (r+1)p + r^2 > 0,\ p \geq 2$$

But (by inspection), $R(p) > 0$ for $p \geq 2$, $\forall r$.

The bound $n > 12$ is established as for Case 1b.

Case 2b: $p < r$

By definition of r, $\hat{q} \leq \frac{m}{r}$. By definition of $\hat{q}$, $\frac{n+1}{p+1} < \hat{q}$. It follows that $r < p+1$. Hence the case $p < r$ can never occur. QED

The previous three theorems can be summarized as

THEOREM 4.5 For $n > 12$ and $\frac{n+1}{6} < m \leq n$, $\hat{Q} = \left\lceil \frac{n+1}{2} \right\rceil$ is optimal.

For $m \leq \frac{n+1}{6}$, the selection of $\hat{Q}$ depends strongly on the factors of $n+1$. We do not attempt to analyze the selection of the optimal $\hat{Q}$, but Theorem 4.1 assures us that we can choose a value of $\hat{Q}$ for which $f(\hat{Q})$ is not much larger than the minimum predicted by the continuous model of Section 2. By choosing $\hat{Q}$ = round $(\sqrt{m(n+1)})$, we have

$$f_0(\hat{Q}) \leq f_0(\hat{Q}) < f_0(\hat{Q}) + m, \; 1 \leq m \leq \frac{n+1}{6}$$

Since $\hat{Q}$ is close to $Q_0 = \sqrt{m(n+1)}$ and $f(q)$ is flat near its minimum, $\hat{f}_0(\hat{Q})$ is close to the minimum number of M/D. Small further improvements can often be made by selecting $\hat{Q} = \left\lceil \frac{n+1}{k} \right\rceil$ for a small value of $k \geq 2$. The analysis of such selections is not pursued here.

The results of this section can be summarized in the

PROGRAM S2 [for selecting $\hat{Q}$ to give good values of $f_{m,n,r}(\hat{q})$]

Given n,m such that $n > 12$ and $0 \leq m \leq n$, this algorithm gives $\hat{Q}$.

To find $\hat{Q}$:

begin

if $m = 0$ **then** $\hat{Q} \leftarrow 1$ **else**

if $m < \frac{n+1}{6}$ **then** $\hat{Q} \leftarrow$ round $(\sqrt{m(n+1)})$ **else**

$\hat{Q} \leftarrow \left\lceil \frac{n+1}{2} \right\rceil$

end

5. NUMBER OF M/D FOR SMALL n

Most of the theorems in the previous sections have assumed $n > 12$. In this section we deal with $n \leq 12$ by tabulating the functions f and $\hat{f}$. We also give examples for a few larger values: n = 13, 14, 15, 23.

For each value of n we present all the values of f and $\hat{f}$. In the tabulations of f, labelled "Continuous and Divisible Cases," the columns corresponding to attainable q^* (that is, $q \ni q|n+1$) are flagged with asterisks, the value of q corresponding to the minimum of f for each m is circled, and the values of q^* corresponding to attainable minima are shaded. Note that non-integer values appear in the unflagged columns. These represent values of the continuous function f that are not meaningful in terms of the actual algorithm. In the tabulations of $\hat{f}$, labelled "Arbitrary Divisor Case," the minimum value of $\hat{f}$ for each m is shaded. Since there is no divisibility constraint, each corresponds to an attainable $\hat{q}$.

N = 1 Continuous and Divisible Cases

M\Q	1	2
	*****	*****
0 :	1.0	2.0
1 :	1.0	3.0

N=: 1 Arbitrary Divisor Case

M\Q	1	2
0 :	1.0	2.0
1 :	1.0	3.0

N = 2 Continuous and Divisible Cases

M\Q	1	2	3
	*****		*****
0 :	2.0	3.0	4.0
1 :	3.0	4.5	5.0
2 :	3.0	4.0	6.0

N=: 2 Arbitrary Divisor Case

M\Q	1	2	3
0 :	2.0	3.0	4.0
1 :	3.0	5.0	5.0
2 :	3.0	5.0	6.0

N = 3 Continuous and Divisible Cases

M\Q	1	2	3	4
	*****	*****		*****
0 :	3.0	4.0	5.0	6.0
1 :	5.0	6.0	6.3	7.0
2 :	6.0	6.0	7.7	8.0
3 :	6.0	7.0	7.0	9.0

N=: 3 Arbitrary Divisor Case

M\Q	1	2	3	4
0 :	3.0	4.0	5.0	6.0
1 :	5.0	6.0	7.0	7.0
2 :	6.0	6.0	9.0	8.0
3 :	6.0	7.0	9.0	9.0

N = 4 Continuous and Divisible Cases

M\Q	1	2	3	4	5
	*****				*****
0 :	4.0	5.0	6.0	7.0	8.0
1 :	7.0	7.5	7.7	8.3	9.0
2 :	9.0	8.0	9.3	9.5	10.0
3 :	10.0	9.5	9.0	10.7	11.0
4 :	10.0	9.0	9.7	10.0	12.0

N=: 4 Arbitrary Divisor Case

M\Q	1	2	3	4	5
0 :	4.0	5.0	6.0	7.0	8.0
1 :	7.0	8.0	8.0	9.0	9.0
2 :	9.0	9.0	10.0	11.0	10.0
3 :	10.0	11.0	10.0	13.0	11.0
4 :	10.0	11.0	11.0	13.0	12.0

N = 5 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6
	*****	*****	*****			*****
0 :	5.0	6.0	7.0	8.0	9.0	10.0
1 :	9.0	9.0	9.0	9.5	10.2	11.0
2 :	12.0	10.0	11.0	11.0	11.4	12.0
3 :	14.0	12.0	11.0	12.5	12.6	13.0
4 :	15.0	12.0	12.0	12.0	13.8	14.0
5 :	15.0	13.0	13.0	12.5	13.0	15.0

N=: 5 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6
0 :	5.0	6.0	7.0	8.0	9.0	10.0
1 :	9.0	9.0	9.0	10.0	11.0	11.0
2 :	12.0	10.0	11.0	12.0	13.0	12.0
3 :	14.0	12.0	11.0	14.0	15.0	13.0
4 :	15.0	12.0	12.0	14.0	17.0	14.0
5 :	15.0	13.0	13.0	15.0	17.0	15.0

N = 6 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7
	*****						*****
0 :	6.0	7.0	8.0	9.0	10.0	11.0	12.0
1 :	11.0	10.5	10.3	10.7	11.4	12.2	13.0
2 :	15.0	12.0	12.7	12.5	12.8	13.3	14.0
3 :	18.0	14.5	13.0	14.3	14.2	14.5	15.0
4 :	20.0	15.0	14.3	14.0	15.6	15.7	16.0
5 :	21.0	16.5	15.7	14.8	15.0	16.8	17.0
6 :	21.0	16.0	15.0	15.5	15.4	16.0	18.0

N=: 6 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7
0 :	6.0	7.0	8.0	9.0	10.0	11.0	12.0
1 :	11.0	11.0	11.0	11.0	12.0	13.0	13.0
2 :	15.0	13.0	14.0	13.0	14.0	15.0	14.0
3 :	18.0	16.0	15.0	15.0	16.0	17.0	15.0
4 :	20.0	17.0	17.0	15.0	18.0	19.0	16.0
5 :	21.0	19.0	19.0	16.0	18.0	21.0	17.0
6 :	21.0	19.0	19.0	17.0	19.0	21.0	18.0

N = 7 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7	8
	*****	*****		*****				*****
0 :	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0
1 :	13.0	12.0	11.7	12.0	12.6	13.3	14.1	15.0
2 :	18.0	14.0	14.3	14.0	14.2	14.7	15.3	16.0
3 :	22.0	17.0	15.0	16.0	15.8	16.0	16.4	17.0
4 :	25.0	18.0	16.7	16.0	17.4	17.3	17.6	18.0
5 :	27.0	20.0	18.3	17.0	17.0	18.7	18.7	19.0
6 :	28.0	20.0	18.0	18.0	17.6	18.0	19.9	20.0
7 :	28.0	21.0	18.7	19.0	18.2	18.3	19.0	21.0

N=: 7 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8
0 :	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0
1 :	13.0	12.0	12.0	12.0	13.0	14.0	15.0	15.0
2 :	18.0	14.0	15.0	14.0	15.0	16.0	17.0	16.0
3 :	22.0	17.0	16.0	16.0	17.0	18.0	19.0	17.0
4 :	25.0	18.0	18.0	16.0	19.0	20.0	21.0	18.0
5 :	27.0	20.0	20.0	17.0	19.0	22.0	23.0	19.0
6 :	28.0	20.0	20.0	18.0	20.0	22.0	25.0	20.0
7 :	28.0	21.0	21.0	19.0	21.0	23.0	25.0	21.0

N = 8 Continuous and Divisible Cases

	*****		*****						*****
M\Q	1	2	3	4	5	6	7	8	9
0 :	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0
1 :	15.0	13.5	13.0	13.3	13.8	14.5	15.3	16.1	17.0
2 :	21.0	16.0	16.0	15.5	15.6	16.0	16.6	17.2	18.0
3 :	26.0	19.5	17.0	17.8	17.4	17.5	17.9	18.4	19.0
4 :	30.0	21.0	19.0	18.0	19.2	19.0	19.1	19.5	20.0
5 :	33.0	23.5	21.0	19.3	19.0	20.5	20.4	20.6	21.0
6 :	35.0	24.0	21.0	20.5	19.8	20.0	21.7	21.7	22.0
7 :	36.0	25.5	22.0	21.7	20.6	20.5	21.0	22.9	23.0
8 :	36.0	25.0	23.0	21.0	21.4	21.0	21.3	22.0	24.0

N=: 8 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9
0 :	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0
1 :	15.0	14.0	13.0	14.0	14.0	15.0	16.0	17.0	17.0
2 :	21.0	17.0	16.0	17.0	16.0	17.0	18.0	19.0	18.0
3 :	26.0	21.0	17.0	20.0	18.0	19.0	20.0	21.0	19.0
4 :	30.0	23.0	19.0	21.0	20.0	21.0	22.0	23.0	20.0
5 :	33.0	26.0	21.0	23.0	20.0	23.0	24.0	25.0	21.0
6 :	35.0	27.0	21.0	25.0	21.0	23.0	26.0	27.0	22.0
7 :	36.0	29.0	22.0	27.0	22.0	24.0	26.0	29.0	23.0
8 :	36.0	29.0	23.0	27.0	23.0	25.0	27.0	29.0	24.0

N = 9 Continuous and Divisible Cases

	*****	*****			*****					*****
M\Q	1	2	3	4	5	6	7	8	9	10
0 :	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0
1 :	17.0	15.0	14.3	14.5	15.0	15.7	16.4	17.2	18.1	19.0
2 :	24.0	18.0	17.7	17.0	17.0	17.3	17.9	18.5	19.2	20.0
3 :	30.0	22.0	19.0	19.5	19.0	19.0	19.3	19.7	20.3	21.0
4 :	35.0	24.0	21.3	20.0	21.0	20.7	20.7	21.0	21.4	22.0
5 :	39.0	27.0	23.7	21.5	21.0	22.3	22.1	22.3	22.6	23.0
6 :	42.0	28.0	24.0	23.0	22.0	22.0	23.6	23.5	23.7	24.0
7 :	44.0	30.0	25.3	24.5	23.0	22.7	23.0	24.8	24.8	25.0
8 :	45.0	30.0	26.7	24.0	24.0	23.3	23.4	24.0	25.9	26.0
9 :	45.0	31.0	26.0	24.5	25.0	24.0	23.9	24.2	25.0	27.0

N=: 9 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10
0 :	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0
1 :	17.0	15.0	15.0	15.0	15.0	16.0	17.0	18.0	19.0	19.0
2 :	24.0	18.0	19.0	18.0	17.0	18.0	19.0	20.0	21.0	20.0
3 :	30.0	22.0	21.0	21.0	18.0	20.0	21.0	22.0	23.0	21.0
4 :	35.0	24.0	24.0	22.0	21.0	22.0	23.0	24.0	25.0	22.0
5 :	39.0	27.0	27.0	24.0	21.0	24.0	25.0	26.0	27.0	23.0
6 :	42.0	28.0	28.0	26.0	22.0	24.0	27.0	28.0	29.0	24.0
7 :	44.0	30.0	30.0	28.0	23.0	25.0	27.0	30.0	31.0	25.0
8 :	45.0	30.0	32.0	28.0	24.0	26.0	28.0	30.0	33.0	26.0
9 :	45.0	31.0	32.0	29.0	25.0	27.0	29.0	31.0	33.0	27.0

N = 10 Continuous and Divisible Cases

M\Q	1 *****	2	3	4	5	6	7	8	9	10	11 *****
0 :	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0
1 :	19.0	16.5	15.7	15.8	16.2	16.8	17.6	18.4	19.2	20.1	21.0
2 :	27.0	20.0	19.3	18.5	18.4	18.7	19.1	19.7	20.4	21.2	22.0
3 :	34.0	24.5	21.0	21.3	20.6	20.5	20.7	21.1	21.7	22.3	23.0
4 :	40.0	27.0	23.7	22.0	22.8	22.3	22.3	22.5	22.9	23.4	24.0
5 :	45.0	30.5	26.3	23.8	23.0	24.2	23.9	23.9	24.1	24.5	25.0
6 :	49.0	32.0	27.0	25.5	24.2	24.0	25.4	25.3	25.3	25.6	26.0
7 :	52.0	34.5	28.7	27.3	25.4	24.8	25.0	26.6	26.6	26.7	27.0
8 :	54.0	35.0	30.3	27.0	26.6	25.7	25.6	26.0	27.8	27.8	28.0
9 :	55.0	36.5	30.0	27.7	27.8	26.5	26.1	26.4	27.0	28.9	29.0
10 :	55.0	36.0	30.7	28.5	27.0	27.3	26.7	26.7	27.2	28.0	30.0

N=: 10 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10	11
0 :	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0
1 :	19.0	17.0	16.0	16.0	17.0	17.0	18.0	19.0	20.0	21.0	21.0
2 :	27.0	21.0	20.0	19.0	20.0	19.0	20.0	21.0	22.0	23.0	22.0
3 :	34.0	26.0	22.0	22.0	23.0	21.0	22.0	23.0	24.0	25.0	23.0
4 :	40.0	29.0	25.0	23.0	26.0	23.0	24.0	25.0	26.0	27.0	24.0
5 :	45.0	33.0	28.0	25.0	27.0	25.0	26.0	27.0	28.0	29.0	25.0
6 :	49.0	35.0	29.0	27.0	29.0	25.0	28.0	29.0	30.0	31.0	26.0
7 :	52.0	38.0	31.0	29.0	31.0	26.0	28.0	31.0	32.0	33.0	27.0
8 :	54.0	39.0	33.0	29.0	33.0	27.0	29.0	31.0	34.0	35.0	28.0
9 :	55.0	41.0	33.0	30.0	35.0	28.0	30.0	32.0	34.0	37.0	29.0
10 :	55.0	41.0	34.0	31.0	35.0	29.0	31.0	33.0	35.0	37.0	30.0

N = 11 Continuous and Divisible Cases

M\Q	1 *****	2 *****	3 *****	4 *****	5	6 *****	7	8	9	10	11	12 *****
0 :	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0
1 :	21.0	18.0	17.0	17.0	17.4	18.0	18.7	19.5	20.3	21.2	22.1	23.0
2 :	30.0	22.0	21.0	20.0	19.8	20.0	20.4	21.0	21.7	22.4	23.2	24.0
3 :	38.0	27.0	23.0	23.0	22.2	22.0	22.1	22.5	23.0	23.6	24.3	25.0
4 :	45.0	30.0	26.0	24.0	24.6	24.0	23.9	24.0	24.3	24.8	25.4	26.0
5 :	51.0	34.0	29.0	26.0	25.0	26.0	25.6	25.5	25.7	26.0	26.5	27.0
6 :	56.0	36.0	30.0	28.0	26.4	26.0	27.3	27.0	27.0	27.2	27.5	28.0
7 :	60.0	39.0	32.0	30.0	27.8	27.0	27.0	28.5	28.3	28.4	28.6	29.0
8 :	63.0	40.0	34.0	30.0	29.2	28.0	27.7	28.0	29.7	29.6	29.7	30.0
9 :	65.0	42.0	34.0	31.0	30.6	29.0	28.4	28.5	29.0	30.8	30.8	31.0
10 :	66.0	42.0	35.0	32.0	30.0	30.0	29.1	29.0	29.3	30.0	31.9	32.0
11 :	66.0	43.0	36.0	33.0	30.4	31.0	29.9	29.5	29.7	30.2	31.0	33.0

N=: 11 Arbitrary Divisor Case

M\Q	1	2	• 3	4	5	6	7	8	9	10	11	12
0 :	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0
1 :	21.0	18.0	17.0	17.0	18.0	18.0	19.0	20.0	21.0	22.0	23.0	23.0
2 :	30.0	22.0	21.0	20.0	21.0	20.0	21.0	22.0	23.0	24.0	25.0	24.0
3 :	38.0	27.0	23.0	23.0	24.0	22.0	23.0	24.0	25.0	26.0	27.0	25.0
4 :	45.0	30.0	26.0	24.0	27.0	24.0	25.0	26.0	27.0	28.0	29.0	26.0
5 :	51.0	34.0	29.0	26.0	28.0	26.0	27.0	28.0	29.0	30.0	31.0	27.0
6 :	56.0	36.0	30.0	28.0	30.0	26.0	29.0	30.0	31.0	32.0	33.0	28.0
7 :	60.0	39.0	32.0	30.0	32.0	27.0	29.0	32.0	33.0	34.0	35.0	29.0
8 :	63.0	40.0	34.0	30.0	34.0	28.0	30.0	32.0	35.0	36.0	37.0	30.0
9 :	65.0	42.0	34.0	31.0	36.0	29.0	31.0	33.0	35.0	38.0	39.0	31.0
10 :	66.0	42.0	35.0	32.0	36.0	30.0	32.0	34.0	36.0	38.0	41.0	32.0
11 :	66.0	43.0	36.0	33.0	37.0	31.0	33.0	35.0	37.0	39.0	41.0	33.0

N = 12 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13
	*****												*****
0 :	12.0	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0
1 :	23.0	19.5	18.3	18.2	18.6	19.2	19.9	20.6	21.4	22.3	23.2	24.1	25.0
2 :	33.0	24.0	22.7	21.5	21.2	21.3	21.7	22.3	22.9	23.6	24.4	25.2	26.0
3 :	42.0	29.5	25.0	24.8	23.8	23.5	23.6	23.9	24.3	24.9	25.5	26.2	27.0
4 :	50.0	33.0	28.3	26.0	26.4	25.7	25.4	25.5	25.8	26.2	26.7	27.3	28.0
5 :	57.0	37.5	31.7	28.2	27.0	27.8	27.3	27.1	27.2	27.5	27.9	28.4	29.0
6 :	63.0	40.0	33.0	30.5	28.6	28.0	29.1	28.8	28.7	28.8	29.1	29.5	30.0
7 :	68.0	43.5	35.3	32.8	30.2	29.2	29.0	30.4	30.1	30.1	30.3	30.6	31.0
8 :	72.0	45.0	37.7	33.0	31.8	30.3	29.9	30.0	31.6	31.4	31.5	31.7	32.0
9 :	75.0	47.5	38.0	34.3	33.4	31.5	30.7	30.6	31.0	32.7	32.6	32.8	33.0
10 :	77.0	48.0	39.3	35.5	33.0	32.7	31.6	31.3	31.4	32.0	33.8	33.8	34.0
11 :	78.0	49.5	40.7	36.7	33.6	33.8	32.4	31.9	31.9	32.3	33.0	34.9	35.0
12 :	78.0	49.0	40.0	36.0	34.2	33.0	33.3	32.5	32.3	32.6	33.2	34.0	36.0

N=: 12 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13
0 :	12.0	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0
1 :	23.0	20.0	19.0	19.0	19.0	20.0	20.0	21.0	22.0	23.0	24.0	25.0	25.0
2 :	33.0	25.0	24.0	23.0	22.0	23.0	22.0	23.0	24.0	25.0	26.0	27.0	26.0
3 :	42.0	31.0	27.0	27.0	25.0	26.0	24.0	25.0	26.0	27.0	28.0	29.0	27.0
4 :	50.0	35.0	31.0	29.0	28.0	29.0	26.0	27.0	28.0	29.0	30.0	31.0	28.0
5 :	57.0	40.0	35.0	32.0	29.0	32.0	28.0	29.0	30.0	31.0	32.0	33.0	29.0
6 :	63.0	43.0	37.0	35.0	31.0	33.0	30.0	31.0	32.0	33.0	34.0	35.0	30.0
7 :	68.0	47.0	40.0	38.0	33.0	35.0	30.0	33.0	34.0	35.0	36.0	37.0	31.0
8 :	72.0	49.0	43.0	39.0	35.0	37.0	31.0	33.0	36.0	37.0	38.0	39.0	32.0
9 :	75.0	52.0	44.0	41.0	37.0	39.0	32.0	34.0	36.0	39.0	40.0	41.0	33.0
10 :	77.0	53.0	46.0	43.0	37.0	41.0	33.0	35.0	37.0	39.0	42.0	43.0	34.0
11 :	78.0	55.0	48.0	45.0	38.0	43.0	34.0	36.0	38.0	40.0	42.0	45.0	35.0
12 :	78.0	55.0	48.0	45.0	39.0	43.0	35.0	37.0	39.0	41.0	43.0	45.0	36.0

N = 13 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	*****	*****					*****							*****
0 :	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0
1 :	25.0	21.0	19.7	19.5	19.8	20.3	21.0	21.7	22.6	23.4	24.3	25.2	26.1	27.0
2 :	36.0	26.0	24.3	23.0	22.6	22.7	23.0	23.5	24.1	24.8	25.5	26.3	27.2	28.0
3 :	46.0	32.0	27.0	26.5	25.4	25.0	25.0	25.3	25.7	26.2	26.8	27.5	28.2	29.0
4 :	55.0	36.0	30.7	28.0	28.2	27.3	27.0	27.0	27.2	27.6	28.1	28.7	29.3	30.0
5 :	63.0	41.0	34.3	30.5	29.0	29.7	29.0	28.8	28.8	29.0	29.4	29.8	30.4	31.0
6 :	70.0	44.0	36.0	33.0	30.8	30.0	31.0	30.5	30.3	30.4	30.6	31.0	31.5	32.0
7 :	76.0	48.0	38.7	35.5	32.6	31.3	31.0	32.3	31.9	31.8	31.9	32.2	32.5	33.0
8 :	81.0	50.0	41.3	36.0	34.4	32.7	32.0	32.0	33.4	33.2	33.2	33.3	33.6	34.0
9 :	85.0	53.0	42.0	37.5	36.2	34.0	33.0	32.8	33.0	34.6	34.5	34.5	34.7	35.0
10 :	88.0	54.0	43.7	39.0	36.0	35.3	34.0	33.5	33.6	34.0	35.7	35.7	35.8	36.0
11 :	90.0	56.0	45.3	40.5	36.8	36.7	35.0	34.3	34.1	34.4	35.0	36.8	36.8	37.0
12 :	91.0	56.0	45.0	40.0	37.6	36.0	36.0	35.0	34.7	34.8	35.3	36.0	37.9	38.0
13 :	91.0	57.0	45.7	40.5	38.4	36.3	37.0	35.8	35.2	35.2	35.5	36.2	37.0	39.0

N=: 13 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0 :	13.0	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0
1 :	25.0	21.0	20.0	20.0	20.0	21.0	21.0	22.0	23.0	24.0	25.0	26.0	27.0	27.0
2 :	36.0	26.0	25.0	24.0	23.0	24.0	23.0	24.0	25.0	26.0	27.0	28.0	29.0	28.0
3 :	46.0	32.0	28.0	28.0	26.0	27.0	25.0	26.0	27.0	28.0	29.0	30.0	31.0	29.0
4 :	55.0	36.0	32.0	30.0	29.0	30.0	27.0	28.0	29.0	30.0	31.0	32.0	33.0	30.0
5 :	63.0	41.0	36.0	33.0	30.0	33.0	29.0	30.0	31.0	32.0	33.0	34.0	35.0	31.0
6 :	70.0	44.0	38.0	36.0	32.0	34.0	31.0	32.0	33.0	34.0	35.0	36.0	37.0	32.0
7 :	76.0	48.0	41.0	39.0	34.0	36.0	31.0	34.0	35.0	36.0	37.0	38.0	39.0	33.0
8 :	81.0	50.0	44.0	40.0	36.0	38.0	32.0	34.0	37.0	38.0	39.0	40.0	41.0	34.0
9 :	85.0	53.0	45.0	42.0	38.0	40.0	33.0	35.0	37.0	40.0	41.0	42.0	43.0	35.0
10 :	88.0	54.0	47.0	44.0	38.0	42.0	34.0	36.0	38.0	40.0	43.0	44.0	45.0	36.0
11 :	90.0	56.0	49.0	46.0	39.0	44.0	35.0	37.0	39.0	41.0	43.0	46.0	47.0	37.0
12 :	91.0	56.0	49.0	46.0	40.0	44.0	36.0	38.0	40.0	42.0	44.0	46.0	49.0	38.0
13 :	91.0	57.0	50.0	47.0	41.0	45.0	37.0	39.0	41.0	43.0	45.0	47.0	49.0	39.0

N = 14 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	*****		*****		*****										*****
0 :	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0	27.0	28.0
1 :	27.0	22.5	21.0	20.7	21.0	21.5	22.1	22.9	23.7	24.5	25.4	26.2	27.2	28.1	29.0
2 :	39.0	28.0	26.0	24.5	24.0	24.0	24.3	24.8	25.3	26.0	26.7	27.5	28.3	29.1	30.0
3 :	50.0	34.5	29.0	28.2	27.0	26.5	26.4	26.6	27.0	27.5	28.1	28.8	29.5	30.2	31.0
4 :	60.0	39.0	33.0	30.0	30.0	29.0	28.6	28.5	28.7	29.0	29.5	30.0	30.6	31.3	32.0
5 :	69.0	44.5	37.0	32.8	31.0	31.5	30.7	30.4	30.3	30.5	30.8	31.3	31.8	32.4	33.0
6 :	77.0	48.0	39.0	35.5	33.0	32.0	32.9	32.3	32.0	32.0	32.2	32.5	32.9	33.4	34.0
7 :	84.0	52.5	42.0	38.3	35.0	33.5	33.0	34.1	33.7	33.5	33.5	33.7	34.1	34.5	35.0
8 :	90.0	55.0	45.0	39.0	37.0	35.0	34.1	34.0	35.3	35.0	34.9	35.0	35.2	35.6	36.0
9 :	95.0	58.5	46.0	40.7	39.0	36.5	35.3	34.9	35.0	36.5	36.3	36.3	36.4	36.6	37.0
10 :	99.0	60.0	48.0	42.5	39.0	38.0	36.4	35.8	35.7	36.0	37.6	37.5	37.5	37.7	38.0
11 :	102.0	62.5	50.0	44.2	40.0	39.5	37.6	36.6	36.3	36.5	37.0	38.7	38.7	38.8	39.0
12 :	104.0	63.0	50.0	44.0	41.0	39.0	38.7	37.5	37.0	37.0	37.4	38.0	39.8	39.9	40.0
13 :	105.0	64.5	51.0	44.8	42.0	39.5	39.9	38.4	37.7	37.5	37.7	38.3	39.0	40.9	41.0
14 :	105.0	64.0	52.0	45.5	43.0	40.0	39.0	39.2	38.3	38.0	38.1	38.5	39.2	40.0	42.0

N=: 14 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0 :	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0	27.0	28.0
1 :	27.0	23.0	21.0	21.0	21.0	22.0	23.0	23.0	24.0	25.0	26.0	27.0	28.0	29.0	29.0
2 :	39.0	29.0	26.0	25.0	24.0	25.0	26.0	25.0	26.0	27.0	28.0	29.0	30.0	31.0	30.0
3 :	50.0	36.0	29.0	29.0	27.0	28.0	29.0	27.0	28.0	29.0	30.0	31.0	32.0	33.0	31.0
4 :	60.0	41.0	33.0	31.0	30.0	31.0	32.0	29.0	30.0	31.0	32.0	33.0	34.0	35.0	32.0
5 :	69.0	47.0	37.0	34.0	31.0	34.0	35.0	31.0	32.0	33.0	34.0	35.0	36.0	37.0	33.0
6 :	77.0	51.0	39.0	37.0	33.0	35.0	38.0	33.0	34.0	35.0	36.0	37.0	38.0	39.0	34.0
7 :	84.0	56.0	42.0	40.0	35.0	37.0	39.0	35.0	36.0	37.0	38.0	39.0	40.0	41.0	35.0
8 :	90.0	59.0	45.0	41.0	37.0	39.0	41.0	35.0	38.0	39.0	40.0	41.0	42.0	43.0	36.0
9 :	95.0	63.0	46.0	43.0	39.0	41.0	43.0	36.0	38.0	41.0	42.0	43.0	44.0	45.0	37.0
10 :	99.0	65.0	48.0	45.0	39.0	43.0	45.0	37.0	39.0	41.0	44.0	45.0	46.0	47.0	38.0
11 :	102.0	68.0	50.0	47.0	40.0	45.0	47.0	38.0	40.0	42.0	44.0	47.0	48.0	49.0	39.0
12 :	104.0	69.0	50.0	47.0	41.0	45.0	49.0	39.0	41.0	43.0	45.0	47.0	50.0	51.0	40.0
13 :	105.0	71.0	51.0	48.0	42.0	46.0	51.0	40.0	42.0	44.0	46.0	48.0	50.0	53.0	41.0
14 :	105.0	71.0	52.0	49.0	43.0	47.0	51.0	41.0	43.0	45.0	47.0	49.0	51.0	53.0	42.0

N = 15 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	*****	*****		*****				*****								*****
0 :	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0
1 :	29.0	24.0	22.3	22.0	22.2	22.7	23.3	24.0	24.8	25.6	26.5	27.3	28.2	29.1	30.1	31.0
2 :	42.0	30.0	27.7	26.0	25.4	25.3	25.6	26.0	26.6	27.2	27.9	28.7	29.5	30.3	31.1	32.0
3 :	54.0	37.0	31.0	30.0	28.6	28.0	27.9	28.0	28.3	28.8	29.4	30.0	30.7	31.4	32.2	33.0
4 :	65.0	42.0	35.3	32.0	31.8	30.7	30.1	30.0	30.1	30.4	30.8	31.3	31.9	32.6	33.3	34.0
5 :	75.0	48.0	39.7	35.0	33.0	33.3	32.4	32.0	31.9	32.0	32.3	32.7	33.2	33.7	34.3	35.0
6 :	84.0	52.0	42.0	38.0	35.2	34.0	34.7	34.0	33.7	33.6	33.7	34.0	34.4	34.9	35.4	36.0
7 :	92.0	57.0	45.3	41.0	37.4	35.7	35.0	36.0	35.4	35.2	35.2	35.3	35.6	36.0	36.5	37.0
8 :	99.0	60.0	48.7	42.0	39.6	37.3	36.3	36.0	37.2	36.8	36.6	36.7	36.8	37.1	37.5	38.0
9 :	105.0	64.0	50.0	44.0	41.8	39.0	37.6	37.0	37.0	38.4	38.1	38.0	38.1	38.3	38.6	39.0
10 :	110.0	66.0	52.3	46.0	42.0	40.7	38.9	38.0	37.8	38.0	39.5	39.3	39.3	39.4	39.7	40.0
11 :	114.0	69.0	54.7	48.0	43.2	42.3	40.1	39.0	38.6	38.6	39.0	40.7	40.5	40.6	40.7	41.0
12 :	117.0	70.0	55.0	48.0	44.4	42.0	41.4	40.0	39.3	39.2	39.5	40.0	41.0	41.7	41.8	42.0
13 :	119.0	72.0	56.3	49.0	45.6	42.7	42.7	41.0	40.1	39.8	39.9	40.3	41.0	42.9	42.9	43.0
14 :	120.0	72.0	57.7	50.0	46.8	43.3	42.0	42.0	40.9	40.4	40.4	40.7	41.2	42.0	43.9	44.0
15 :	120.0	73.0	57.0	51.0	46.0	44.0	42.3	43.0	41.7	41.0	40.8	41.0	41.5	42.1	43.0	45.0

N=: 15 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0 :	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0
1 :	29.0	24.0	23.0	22.0	23.0	23.0	24.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0	31.0	31.0
2 :	42.0	30.0	29.0	26.0	27.0	26.0	27.0	26.0	27.0	28.0	29.0	30.0	31.0	32.0	33.0	32.0
3 :	54.0	37.0	33.0	30.0	31.0	29.0	30.0	28.0	29.0	30.0	31.0	32.0	33.0	34.0	35.0	33.0
4 :	65.0	42.0	38.0	32.0	35.0	32.0	33.0	30.0	31.0	32.0	33.0	34.0	35.0	36.0	37.0	34.0
5 :	75.0	48.0	43.0	35.0	37.0	35.0	36.0	32.0	33.0	34.0	35.0	36.0	37.0	38.0	39.0	35.0
6 :	84.0	52.0	46.0	38.0	40.0	36.0	39.0	34.0	35.0	36.0	37.0	38.0	39.0	40.0	41.0	36.0
7 :	92.0	57.0	50.0	41.0	43.0	38.0	40.0	36.0	37.0	38.0	39.0	40.0	41.0	42.0	43.0	37.0
8 :	99.0	60.0	54.0	42.0	46.0	40.0	42.0	36.0	39.0	40.0	41.0	42.0	43.0	44.0	45.0	38.0
9 :	105.0	64.0	56.0	44.0	49.0	42.0	44.0	37.0	39.0	42.0	43.0	44.0	45.0	46.0	47.0	39.0
10 :	110.0	66.0	59.0	46.0	50.0	44.0	46.0	38.0	40.0	42.0	45.0	46.0	47.0	48.0	49.0	40.0
11 :	114.0	69.0	62.0	48.0	52.0	46.0	48.0	39.0	41.0	43.0	45.0	48.0	49.0	50.0	51.0	41.0
12 :	117.0	70.0	63.0	48.0	54.0	46.0	50.0	40.0	42.0	44.0	46.0	48.0	51.0	52.0	53.0	42.0
13 :	119.0	72.0	65.0	49.0	56.0	47.0	52.0	41.0	43.0	45.0	47.0	49.0	51.0	54.0	55.0	43.0
14 :	120.0	72.0	67.0	50.0	58.0	48.0	52.0	42.0	44.0	46.0	48.0	50.0	52.0	54.0	57.0	44.0
15 :	120.0	73.0	67.0	51.0	58.0	49.0	53.0	43.0	45.0	47.0	49.0	51.0	53.0	55.0	57.0	45.0

N = 23 Continuous and Divisible Cases

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	*****	*****	*****	*****		*****		*****				*****												*****
0 :	23.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0	31.0	32.0	33.0	34.0	35.0	36.0	37.0	38.0	39.0	40.0	41.0	42.0	43.0	44.0	45.0	46.0
1 :	45.0	36.0	33.0	32.0	31.8	32.0	32.4	33.0	33.7	34.4	35.2	36.0	36.8	37.7	38.6	39.5	40.4	41.3	42.3	43.2	44.1	45.1	46.0	47.0
2 :	66.0	46.0	41.0	38.0	36.6	36.0	35.9	36.0	36.3	36.8	37.4	38.0	38.7	39.4	40.2	41.0	41.8	42.7	43.5	44.4	45.3	46.2	47.1	48.0
3 :	86.0	57.0	47.0	44.0	41.4	40.0	39.3	39.0	39.0	39.2	39.5	40.0	40.5	41.1	41.8	42.5	43.2	44.0	44.8	45.6	46.4	47.3	48.1	49.0
4 :	105.0	66.0	54.0	48.0	46.2	44.0	42.7	42.0	41.7	41.6	41.7	42.0	42.4	42.9	43.4	44.0	44.6	45.3	46.1	46.8	47.6	48.4	49.2	50.0
5 :	123.0	76.0	61.0	53.0	49.0	48.0	46.1	45.0	44.3	44.0	43.9	44.0	44.2	44.6	45.0	45.5	46.1	46.7	47.3	48.0	48.7	49.5	50.2	51.0
6 :	140.0	84.0	66.0	58.0	52.8	50.0	49.6	48.0	47.0	46.4	46.1	46.0	46.1	46.3	46.6	47.0	47.5	48.0	48.6	49.2	49.9	50.5	51.3	52.0
7 :	156.0	93.0	72.0	63.0	56.6	53.0	51.0	51.0	49.7	48.8	48.3	48.0	47.9	48.0	48.2	48.5	48.9	49.3	49.8	50.4	51.0	51.6	52.3	53.0
8 :	171.0	100.0	78.0	66.0	60.4	56.0	53.4	52.0	52.3	51.2	50.5	50.0	49.8	49.7	49.8	50.0	50.3	50.7	51.1	51.6	52.1	52.7	53.3	54.0
9 :	185.0	108.0	82.0	70.0	64.2	59.0	55.9	54.0	53.0	53.6	52.6	52.0	51.6	51.4	51.4	51.5	51.7	52.0	52.4	52.8	53.3	53.8	54.4	55.0
10 :	198.0	114.0	87.0	74.0	66.0	62.0	58.3	56.0	54.7	54.0	54.8	54.0	53.5	53.1	53.0	53.0	53.1	53.3	53.6	54.0	54.4	54.9	55.4	56.0
11 :	210.0	121.0	92.0	78.0	68.8	65.0	60.7	58.0	56.3	55.4	55.0	55.0	55.3	54.9	54.6	54.5	54.5	54.7	54.9	55.2	55.6	56.0	56.5	57.0
12 :	221.0	126.0	95.0	80.0	71.6	66.0	63.1	60.0	58.0	56.8	56.2	56.0	57.2	56.6	56.2	56.0	55.9	56.0	56.2	56.4	56.7	57.1	57.5	58.0
13 :	231.0	132.0	99.0	83.0	74.4	68.0	65.6	62.0	59.7	58.2	57.4	57.0	57.0	58.3	57.8	57.5	57.4	57.3	57.4	57.6	57.9	58.2	58.6	59.0
14 :	240.0	136.0	103.0	86.0	77.2	70.0	66.0	64.0	61.3	59.6	58.5	58.0	57.8	58.0	59.4	59.0	58.8	58.7	58.7	58.8	59.0	59.3	59.6	60.0
15 :	248.0	141.0	105.0	89.0	78.0	72.0	67.4	66.0	63.0	61.0	59.7	59.0	58.7	58.7	59.0	60.5	60.2	60.0	59.9	60.0	60.1	60.4	60.7	61.0
16 :	255.0	144.0	108.0	90.0	79.8	74.0	68.9	66.0	64.7	62.4	60.9	60.0	59.5	59.4	59.6	60.0	61.6	61.3	61.2	61.2	61.3	61.5	61.7	62.0
17 :	261.0	148.0	111.0	92.0	81.6	76.0	70.3	67.0	66.3	63.8	62.1	61.0	60.4	60.1	60.2	60.5	61.0	62.7	62.5	62.4	62.4	62.5	62.7	63.0
18 :	266.0	150.0	112.0	94.0	83.4	76.0	71.7	68.0	66.0	65.2	63.3	62.0	61.2	60.9	60.8	61.0	61.4	62.0	63.7	63.6	63.6	63.6	63.8	64.0
19 :	270.0	153.0	114.0	96.0	85.2	77.0	73.1	69.0	66.7	66.6	64.5	63.0	62.1	61.6	61.4	61.5	61.8	62.3	63.0	64.8	64.7	64.7	64.8	65.0
20 :	273.0	154.0	116.0	96.0	85.0	78.0	74.6	70.0	67.3	66.0	65.6	64.0	62.9	62.3	62.0	62.0	62.2	62.7	63.3	64.0	65.9	65.8	65.9	66.0
21 :	275.0	156.0	116.0	97.0	85.8	79.0	74.0	71.0	68.0	66.4	66.8	65.0	63.8	63.0	62.6	62.5	62.6	63.0	63.5	64.2	65.0	66.9	66.9	67.0
22 :	276.0	156.0	117.0	98.0	86.6	80.0	74.4	72.0	68.7	66.8	66.0	66.0	64.6	63.7	63.2	63.0	63.1	63.3	63.8	64.4	65.1	66.0	68.0	68.0
23 :	276.0	157.0	118.0	99.0	87.4	81.0	74.9	73.0	69.3	67.2	66.2	67.0	65.5	64.4	63.8	63.5	63.5	63.7	64.1	64.6	65.3	66.1	67.0	69.0

N=: 23 Arbitrary Divisor Case

M\Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0 :	23.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0	31.0	32.0	33.0	34.0	35.0	36.0	37.0	38.0	39.0	40.0	41.0	42.0	43.0	44.0	45.0	46.0
1 :	45.0	36.0	33.0	32.0	32.0	32.0	33.0	33.0	34.0	35.0	36.0	36.0	37.0	38.0	39.0	40.0	41.0	42.0	43.0	44.0	45.0	46.0	47.0	47.0
2 :	66.0	46.0	41.0	38.0	37.0	36.0	37.0	36.0	37.0	38.0	39.0	38.0	39.0	40.0	41.0	42.0	43.0	44.0	45.0	46.0	47.0	48.0	49.0	48.0
3 :	86.0	57.0	47.0	44.0	42.0	40.0	41.0	39.0	40.0	41.0	42.0	40.0	41.0	42.0	43.0	44.0	45.0	46.0	47.0	48.0	49.0	50.0	51.0	49.0
4 :	105.0	66.0	54.0	48.0	47.0	44.0	45.0	42.0	43.0	44.0	45.0	42.0	43.0	44.0	45.0	46.0	47.0	48.0	49.0	50.0	51.0	52.0	53.0	50.0
5 :	123.0	76.0	61.0	53.0	50.0	48.0	49.0	45.0	46.0	47.0	48.0	44.0	45.0	46.0	47.0	48.0	49.0	50.0	51.0	52.0	53.0	54.0	55.0	51.0
6 :	140.0	84.0	66.0	58.0	54.0	50.0	53.0	48.0	49.0	50.0	51.0	46.0	47.0	48.0	49.0	50.0	51.0	52.0	53.0	54.0	55.0	56.0	57.0	52.0
7 :	156.0	93.0	72.0	63.0	58.0	53.0	55.0	51.0	52.0	53.0	54.0	48.0	49.0	50.0	51.0	52.0	53.0	54.0	55.0	56.0	57.0	58.0	59.0	53.0
8 :	171.0	100.0	78.0	66.0	62.0	56.0	58.0	52.0	55.0	56.0	57.0	50.0	51.0	52.0	53.0	54.0	55.0	56.0	57.0	58.0	59.0	60.0	61.0	54.0
9 :	185.0	108.0	82.0	70.0	66.0	59.0	61.0	54.0	56.0	59.0	60.0	52.0	53.0	54.0	55.0	56.0	57.0	58.0	59.0	60.0	61.0	62.0	63.0	55.0
10 :	198.0	114.0	87.0	74.0	68.0	62.0	64.0	56.0	58.0	60.0	63.0	54.0	55.0	56.0	57.0	58.0	59.0	60.0	61.0	62.0	63.0	64.0	65.0	56.0
11 :	210.0	121.0	92.0	78.0	71.0	65.0	67.0	58.0	60.0	62.0	64.0	55.0	57.0	58.0	59.0	60.0	61.0	62.0	63.0	64.0	65.0	66.0	67.0	57.0
12 :	221.0	126.0	95.0	80.0	74.0	66.0	70.0	60.0	62.0	64.0	66.0	56.0	59.0	60.0	61.0	62.0	63.0	64.0	65.0	66.0	67.0	68.0	69.0	58.0
13 :	231.0	132.0	99.0	83.0	77.0	68.0	73.0	62.0	64.0	66.0	68.0	57.0	59.0	62.0	63.0	64.0	65.0	66.0	67.0	68.0	69.0	70.0	71.0	59.0
14 :	240.0	136.0	103.0	86.0	80.0	70.0	74.0	64.0	66.0	68.0	70.0	58.0	60.0	62.0	65.0	66.0	67.0	68.0	69.0	70.0	71.0	72.0	73.0	60.0
15 :	248.0	141.0	105.0	89.0	81.0	72.0	76.0	66.0	68.0	70.0	72.0	59.0	61.0	63.0	65.0	68.0	69.0	70.0	71.0	72.0	73.0	74.0	75.0	61.0
16 :	255.0	144.0	108.0	90.0	83.0	74.0	78.0	66.0	70.0	72.0	74.0	60.0	62.0	64.0	66.0	68.0	71.0	72.0	73.0	74.0	75.0	76.0	77.0	62.0
17 :	261.0	148.0	111.0	92.0	85.0	76.0	80.0	67.0	72.0	74.0	76.0	61.0	63.0	65.0	67.0	69.0	71.0	74.0	75.0	76.0	77.0	78.0	79.0	63.0
18 :	266.0	150.0	112.0	94.0	87.0	76.0	82.0	68.0	72.0	76.0	78.0	62.0	64.0	66.0	68.0	70.0	72.0	74.0	77.0	78.0	79.0	80.0	81.0	64.0
19 :	270.0	153.0	114.0	96.0	89.0	77.0	84.0	69.0	73.0	78.0	80.0	63.0	65.0	67.0	69.0	71.0	73.0	75.0	77.0	80.0	81.0	82.0	83.0	65.0
20 :	273.0	154.0	116.0	96.0	89.0	78.0	86.0	70.0	74.0	78.0	82.0	64.0	66.0	68.0	70.0	72.0	74.0	76.0	78.0	80.0	83.0	84.0	85.0	66.0
21 :	275.0	156.0	116.0	97.0	90.0	79.0	86.0	71.0	75.0	79.0	84.0	65.0	67.0	69.0	71.0	73.0	75.0	77.0	79.0	81.0	83.0	86.0	87.0	67.0
22 :	276.0	156.0	117.0	98.0	91.0	80.0	87.0	72.0	76.0	80.0	84.0	66.0	68.0	70.0	72.0	74.0	76.0	78.0	80.0	82.0	84.0	86.0	89.0	68.0
23 :	276.0	157.0	118.0	99.0	92.0	81.0	88.0	73.0	77.0	81.0	85.0	67.0	69.0	71.0	73.0	75.0	77.0	79.0	81.0	83.0	85.0	87.0	89.0	69.0

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REFERENCES

Aho, Steiglitz, and Ullman [1974]
A. V. Aho, K. Steiglitz, and J. D. Ullman, "Evaluating Polynomials at Fixed Sets of Points," Bell Laboratories Technical Report.

Borodin [1972]
A. Borodin, private communication.

Borodin [1973]
A. Borodin, "On the Number of Arithmetics Required to Compute Certain Functions - Circa May 1973." Appears in Complexity of Sequential and Parallel Numerical Algorithms, ed. J. F. Traub (Academic Press, New York, 1973).

Horner [1819]
W. G. Horner, "A New Method of Solving Numerical Equations of all Orders by Continuous Approximation," Philosophical Transactions, Royal Society of 109 (1819), 308-335.

Kung [1973]
H. T. Kung, "Fast Evaluation and Interpolation." Computer Science Department Report, Carnegie-Mellon University (1973).

H. T. Kung, "A New Upper Bound on the Complexity of Derivative Evaluation." Information Processing Letters 2 (1974), 146-147.

Munro [1971]
J. I. Munro, "Some Results in the Study of Algorithms." Technical Report 32, Department of Computer Science, University of Toronto (1971).

Newton [1711]
I. Newton, Analysis per Quantitatem Series (London 1711), 10.

Shaw and Traub [1972]
M. Shaw and J. F. Traub, "On the Number of Multiplications for the Evaluation of a Polynomial and Some of its Derivatives," Computer Science Department Report, Carnegie-Mellon University (1972).

M. Shaw and J. F. Traub, "On the Number of Multiplications for the Evaluation of a Polynomial and All its Derivatives," 13th Annual Symposium on Switching and Automata Theory (1972), 105-107.

Shaw and Traub [1974]
M. Shaw and J. F. Traub, "On the Number of Multiplications for the Evaluation of a Polynomial and Some of its Derivatives," Journal of the ACM 21,1 (1974), 161-167. (This is the published version of the 1972 Report.)

Stewart [1971]
G. W. Stewart, "Error Analysis of the Algorithm for Shifting the Zeros of a Polynomial by Synthetic Division," Mathematics of Computation (1971), 135-139.

Strassen [1972]

V. Strassen, "Die Berechnungskomplexität von Elementarsymmetrischen Funktionen und von Interpolationskoeffizienten," Numerische Mathematik 20 (1973), 238-251.

Vari [1974]

T. M. Vari, "Some Complexity Results for a Class of Toeplitz Matrices," York University Technical Report (1974).

Wozniakowski [1974]

H. Wozniakowski, "Rounding Error Analysis for the Evluation of a Polynomial and Some of its Derivatives. SIAM Journal of Numerical Analysis, 11, 4 (1974), 780-787.

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We have previously presented a new one parameter family of algorithms and a program for evaluating the first m derivatives of a polynomial of degree n. In this paper we analyze that family of algorithms and present practical algorithms for selecting optimal or good values of the parameter q. A program for selecting the optimal value of q under the constraint that q divides n+1 is given. We also analyze a program that eliminates that constraint and a simple program that selects a good, but not always optimal, value of q. We derive bounds on how close to optimal the "good" balue will be. The above results apply for n>12. We extend the results

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