ON k-PATH HAMILTONIAN PIANAR GRAPHS*

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ABSTRACT

We give a simple upper bound on k for k-path-hamiltonianness of a graph. Also given are exact values for maximal planar graphs.

1. Introduction

In this paper we consider only finite undirected graphs without loops or multiple edges, and our terminology and notation is standard (i.e., [4] and [1]) except as indicated.

A path P with k edges is called a length k path. Following Kronk [5], a graph G is k-path-hamiltonian if every length k path of G lies on a hamiltonian cycle. Such graphs are characterized in [5]; however, these conditions are similar to Dirac [2] type sufficient conditions for a graph to be hamiltonian. Let $k^*(G)$ be the largest k for which the graph G is k-path hamiltonian. In this paper we shall present a simple upper bound on k^* and discuss exact values for maximal planar graphs.

2. Upper Bound on k*

A path $P_n = (V_n, E_n)$ in a graph G is called separating if $G - V_n$ is disconnected. Let $n_{min}(G)$ denote the length of the shortest separating path in the graph G. (It is possible that the path is closed.)

Theorem 1: $k^*(G) \leq n_{min}(G)$

Proof: Let $G = (V,E)$ be a 2-connected graph, and let $P_n = (V_n, E_n)$ be any shortest separating path in G (i.e., $n-1 = n_{min}(G)$). Let G_1 and G_2 be two components of $G' = G - V_n$. Now let v_1 and v_2 be any pair of vertices in G_1 and G_2, respectively. Clearly, there exists no v_1-v_2 path in G'. Thus G can not contain any cycle $C = (V_c, E_c)$ such that $(V_n \cup \{v_1, v_2\}) \subseteq V_c$.

The upperbound above is, clearly, the best possible. The graph G of <u>Figure 1</u> can be separated by the path $v_r v_s v_t$; hence $n_m{}^{\wedge}{}_n = 2$. Note that even though G is 1-path-hamiltonian, there exists no hamiltonian cycle containing the $v_r v_s v_t$ path. A simple consequence of the above result is a necessary condition for a graph to be k-path-hamiltonian:

<u>Corollary 1.1</u>: A k-path-hamiltonian graph can not be separated by deleting the vertices of a length k-1 path.

We now state some auxiliary results:

<u>Lemma 1</u> (Harary [4, p. 104]): If G is a planar graph, then G has at least four vertices of degree not exceeding 5.

<u>Lemma 2</u> (Harary [4, p. 104]): If G is a maximal plane graph, then every face of G is a triangle.

It follows from <u>Lemma 2</u> that;

<u>Lemma 3</u>: If G is a maximal planar graph and v_i is any vertex of G, then the set of the vertices adjacent to v_i induce a cycle in G.

Let K(G) denote the connectivity number of the graph G.

<u>Lemma 4</u> (Whitney [7]): If G is maximal planar, then $3 \leq <(G) \leq 5$.

Finally we need the following auxiliary result:

<u>Lemma 5</u>: If $G = (V,E)$ is a maximal planar graph and $S \subset V$ is a minimal separating set, then S induces a cycle in G.

<u>Proof</u>: Let $s = |S|$. Clearly, $3 \leq s \leq 5$ (<u>Lemma 4</u>). Let $S = \{u_1, u_2, \ldots, u_k\}$ where $k = s$. Given any embedding of G in the plane, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the components of $G-S$, as shown in <u>Figure 2(a)</u>. Clearly, exactly two vertices in S must lie in the exterior face of G, say u_1 and u_s. Now; suppose that, in G, neither G_1 nor G_2 lies in the interior of a cycle induced by S. Then, since $V_1, V_2 \neq \emptyset$, the exterior face of G can not be triangular, thereby contradicting <u>Lemma 2</u>. Thus, u_1 must be adjacent to u_s, as shown in <u>Figure 2(b)</u> or <u>2(c)</u>. Suppose (u_1, u_s) is as shown in <u>Figure 2(b)</u>. Since S is a separating set, there can not be an edge $(v', v'') \varepsilon E$ as as shown in <u>Figure 2(c)</u>. Then, via <u>Lemma 2</u>, u_1 must be adjacent to u_2 in G. Using similar arguments, we can show that G contains a $u_1 u_2 \ldots u_s$ path of length $(s-1)$. Thus S induces a cycle in G. //

Another consequence of <u>Theorem 1</u> deals with maximal planar graphs:

<u>Corollary 1.2</u>: If G is maximal planar, then $k^*(G) \leq 3$.

<u>Proof</u>: It follows from <u>Lemmas 1</u> and <u>3</u> that, $n_{min}(G) \leq 4$. Hence, via <u>Lemma 5</u> and <u>Theorem 1</u>, $k^*(G) \leq 3$. //

3. Exact Values of $k^*(G)$, G Maximal Planar

Theorem 2: Let $G = (V,E)$ be a maximal planar graph. Then

$$k^*(G) = \begin{cases} 3 \text{ if and only if } \kappa(G) = 5 \\ 2 \text{ if and only if } \kappa(G) = 4 \\ 1 \text{ or } 0 \text{ if and only if } \kappa(G) = 3 \end{cases}$$

Proof: Suppose $\kappa(G) = 5$. Let $v_r v_s v_t v_u$ be any path in G. It can easily be shown that the elementary contraction G_{st} of G obtained by replacing v_s and v_t with a new vertex v_w is a 4-connected maximal planar graph. (Since $\kappa(H) = 1 + \min_{v \in H} \kappa(H - v)$ if H is connected (see Exercise 5.21 in Harary [4]), delete v_t from G, replace s by w, and join v_w to the vertices that were adjacent to v_t in G). Since 4-connected planar graphs are 2-path hamiltonian (follows from Theorem 2 in Tutte [6]), G_{st} contains a hamiltonian cycle C_n containing the path $v_r v_w v_u$. Clearly, the union of the path $C_n - v_w \subset G_{st}$ and the path $v_r v_s v_t v_u$ is a hamiltonian cycle of G. Thus G is 3-path hamiltonian if $\kappa(G) = 5$. On the other hand, it follows from Lemma 5 that $k^*(G) = 3$ implies $\kappa(G) \geq 5$: the subpath induced by $S \sim \{$any vertex in $S\}$ can not lie on a hamiltonian cycle if $|S| = 3$ or 4. Thus $|S| = 5$ is necessarily true, and, therefore $\kappa(G) = 5$. Now suppose $\kappa(G) = 4$. Then G contains at least one separating 4-cycle (Hakimi and Schemeichel [3]). Hence $k^*(G) \leq 2$ via the first part above. But 4-connected planar graphs are

2-path hamiltonian; thus $K(G) = 4 \Rightarrow k^*(G) = 2$. On the **other hand;** $k^*(G) = 2 \Rightarrow K(G) \geq 4$ (Lemma 5). Hence, together **with** the first part, $k^*(G) = 2 \Leftrightarrow K(G) = 4$. The third part follows from the first two parts. ∎

We have thus established that there exists k-path hamiltonian maximal planar graphs, $k = 2,3$, with p vertices for all $p \geq 10$.

We next investigate maximal planar graphs with $k^*(G) = 0$ or 1. We can construct either types of graphs easily. Start with any 4-connected maximal plane graph G with $p \geq 6$ vertices and insert a vertex in each **one** of exactly p faces of G, and join each new vertex to the vertices of the triangle bounding the respective faces. Let G_1 denote the resulting graph (Figure 3). It can easily be shown that G_1 is hamiltonian. However, G_1 can not contain any hamiltonian cycle containing any edge of at least one of the face-bounding triangles common to both G and G_1; for otherwise, if we insert a vertex in that face as well, the resulting graph would also be hamiltonian. However, G_1 would, then, have $2p + 1$ vertices and the deletion of the vertices common to both G and G_1 would result in a $\bar{K}_{p+1}$, thereby contradicting the 1-toughness (see Theorem 4.2, Bondy and Murty [1]) of hamiltonian graphs. Thus $k^*(G_1) = 0$.

On **the** other hand, start with any 4-connected maximal plane graph G and **insert one** vertex in any face of G and join it to the vertices of the **triangle** bounding that face. Let G_1 be the resulting graph (Figure **4)**. **Since** G **is** 2-path-hamiltonian, it is clear that G_1 is 1-path-hamiltonian. **However,** $k^*(G_1) = 1$ because G_1 can not contain any hamiltonian cycle containing two edges of the triangle bounding that face of G in which a vertex was added, since the degree of the new vertex is 3.

We would also like to add that since $k^*(G)$ £ 1 implies that $K(G) = 3$, characterization of these cases would, in effect, be equivalent to characterizing hamiltonian maximal planar graphs, which remains a rather difficult unsolved problem.

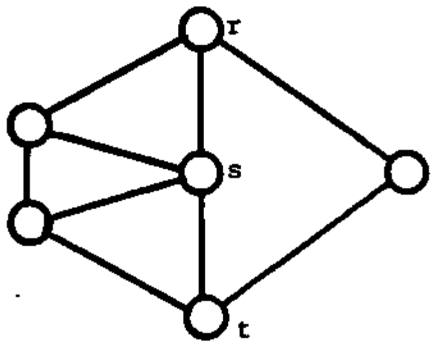


Figure 1; A 1-path hamiltonian graph

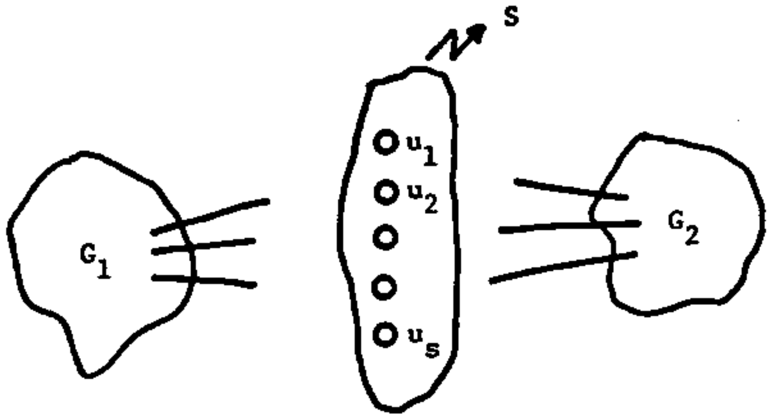


Figure 2(a)

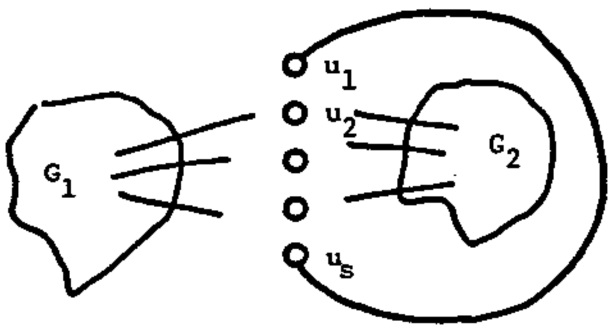


Figure 2(b)

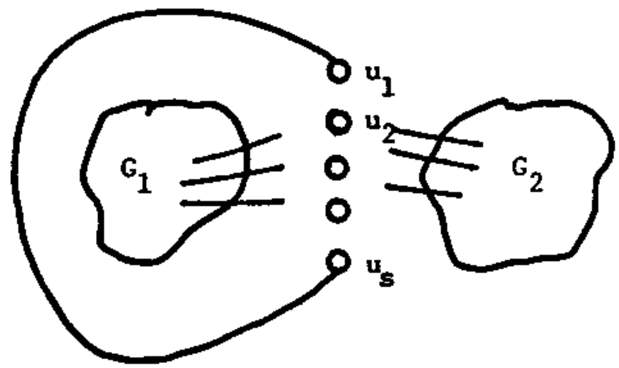


Figure 2(c)

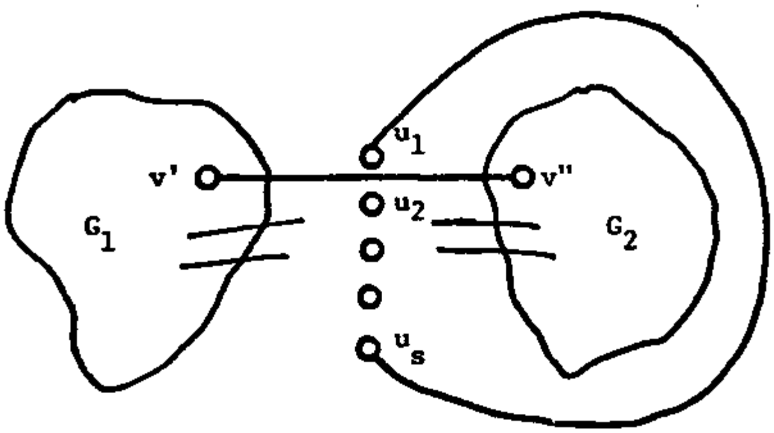
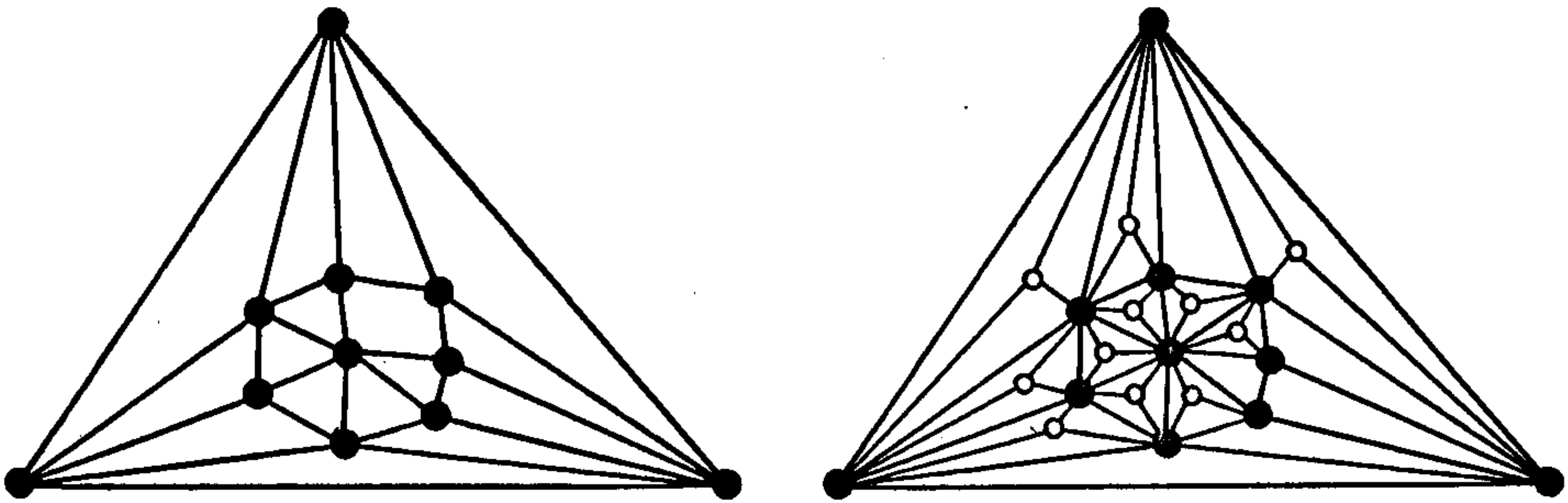


Figure 2(d)



G   G'

<u>Figure 3</u>; Constructing a hamiltonian maximal planar planar graph not every edge of which lies in a hamiltonian cycle

V0

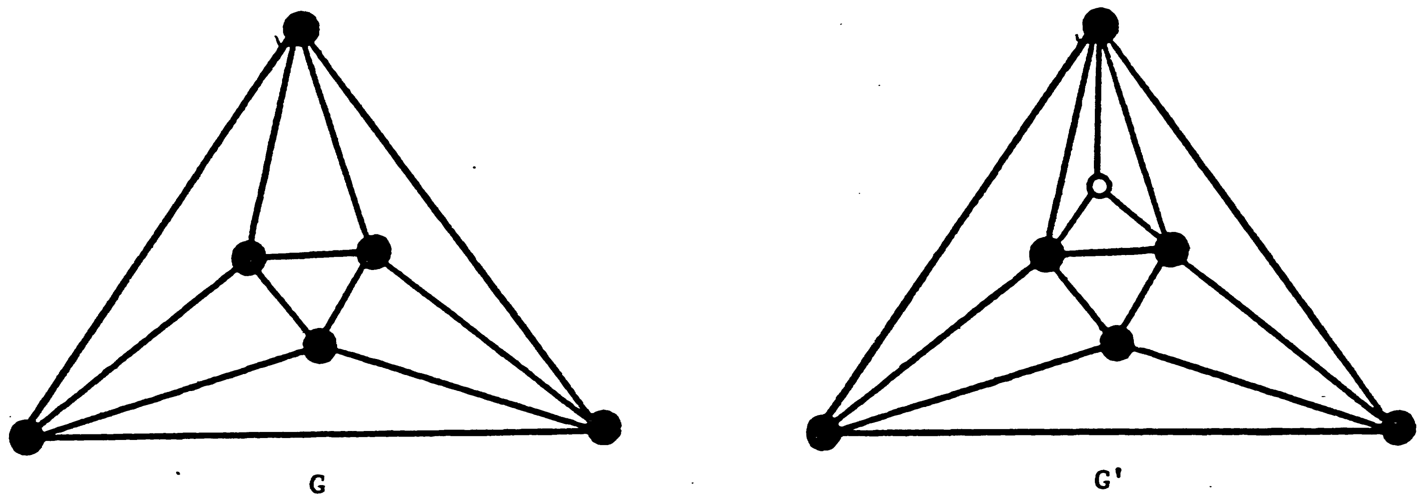


Figure 4: Constructing a connectivity 3 1-path-hamiltonian maximal planar graph

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