STABLE FINITE ELEMBTS
FOR THE NAVIER-STOKES EQUATIONS

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SUMMARY. The use of arbitrary spaces to represent the velocities and pressures in the Navier-Stokes equations typically leads to unstable finite element approximations. We show in this paper that if spaces of piecewise polynomial functions are used and if the grid for the velocity field is sufficiently fine compared to the grid for the pressure, then the resulting finite element approximations are stable and converge at the optimal rates.

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§1. Introduction

This paper is concerned with the identification of finite element spaces which yield stable and convergent approximations to the Navier-Stokes equations. It has been known for several years that the selection of arbitrary finite element spaces will typically lead to instabilities in the pressure ([1] - [2]). Only special choices will work. This is analogous to the situation that exists with finite difference approximations, where only specially constructured schemes will be stable.

Previous work has identified a number of special spaces which yield stable approximations. For example, in the planar case it has been shown [3] that if the space of velocities consist of continuous piecewise quadratics and the space of pressures are piecewise constants, then the resulting finite element scheme is stable. A similar result holds if the space of velocities consist of continuous piecewise linear functions augmented by suitable trilinear functions. These results do require that the grid be regular and satisfy an angle condition, but there are no other restrictions on the shape of the elements. On the other hand, for specially shaped elements such as the crisscross pattern the smaller space consisting only of continuous piecewise linear functions is stable [4].

The work contained in this paper is an extension of first type of element cited above in the sense that there are no restrictions on the shape of the elements. In particular, we show that the finite element scheme is stable provided that the dimension of the space of velocities is sufficiently large compared to the dimension of the space of pressures. This can readily be translated into a mesh ratio condition which is familiar from previous work on hybrid finite methods [5].

In this last section we show that similar results also apply to the Poisson equation when written as a first order system.

§2. The Brezzi Condition

Let Q be a bounded region in 3R^n (n = 2 or 3). We consider an incompressible flow in D where u denotes the velocity, p the pressure, jE the body forces, and v the viscosity. We shall be interested in the nonlinear case where the equations of motions take the form

(2.1) -v A u + (u*grad)u + grad p = £ in Q

(2.2) div u = 0 in Q

(2.3) u = Q on dQ

It is known that (2.1)-(2.3) has a unique solution provided the generalized-Reynolds number [6] is sufficiently small. We shall assume this without further comment in the sequel. We shall also be interested in the linear case where the terra u*grad is replaced with $\hat{u}$*grad for some known divergent free velocity field $\hat{u}$. In the latter case the restriction on the Reynolds number is not needed.

To define the approximation procedure we let

(2.4) 3J(Q) = {V : grad V € L^2(Q)$_f$ V = £ on ∂Ω}

and

(2.5) L^2_0(Q) = {q : q € L^2(Q) , ∫ q = 0}.

We select two finite dimensional subspaces

$$\vec{V}_h \subseteq \vec{H}^1_0(\Omega), \qquad S_H \subseteq L^2_0(\Omega), \tag{2.6}$$

and seek a pair

$$\underline{u}^h \in \vec{V}_h, \qquad p_H \in S_H \tag{2.7}$$

such that

$$\int_\Omega \{\text{grad}\, \underline{u}^h \cdot \text{grad}\, \underline{v}^h + (\underline{u}^h \text{grad}\, \underline{u}^h) \cdot \underline{v}^h\} - \int_\Omega p_H \,\text{div}\, \underline{v}^h = \int_\Omega \underline{f} \cdot \underline{v}^h \tag{2.8}$$

$$\int_\Omega q^H \,\text{div}\, \underline{u}^h = 0 \tag{2.9}$$

holds for all $\underline{v}^h$ in V_h *and* q^H in S_H. Once a basis has been chosen for $\vec{V}_h$ and S_H, the above reduces to a set of nonlinear algebraic equations [3].

As noted in the introduction this system will in general be unstable, and only special choices for V_h and S_H will lead to convergent approximations. The condition for stability was first formulated by Brezzi [7], and it takes the following form (see also [1] for an alternate but in this context equivalent formulation):

$$\sup \{\int_\Omega q_H \,\text{div}\, \underline{v}^h\} \geq \beta \|q_H\| \tag{2.10}$$ (1)

Here the sup is taken over all $\underline{v}^h$ in $\vec{V}_h$ with

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(1) In the sequel we shall use standard Sobolev space notation with $\|\cdot\|_r$ denoting the norm on $H^r(\Omega)$ or $\vec{H}^r(\Omega)$.

and q_R is any element in 8_H. The number 3 should satisfy $0 < p < co$ and should be independent of q_H. In addition, it should be bounded away from zero as the dimension of the spaces $\vec{U}_h \times g_H$ approaches infinity.

§3, A class of finite element spaces.

We are now prepared to state and prove our main result. Here we assume that $\underline{V}_h$ and S_H are finite element spaces with h and H being mesh spacings. It is assumed that these spaces have the standard approximation properties; i.e.,

(3.1) $$\inf \|\underline{u} - \underline{v}^h\|_1 \leq C_A h^{k-1} \|\underline{u}\|_k$$

(3.2) $$\inf \|p - q^H\|_0 \leq C_A H^{\ell} \|p\|_{\ell}$$

for suitable intergers $1 \leq k \leq K$, $1 \leq \ell \leq L$, and for a positive constant C_A independent of h, H, $\underline{u}$ and p.

For spaces S_H of piecewise polynomial functions we have

(3.3) $$S_H \subset H^{\epsilon}(\Omega)$$

for some $\epsilon > 0$. For example, if S_H consists of discontinuous piecewise polynomial functions (such as piecewise constants), then ϵ can be any number in the range $0 < \epsilon < 1/2$. In addition, if the grid for S_H is quasi regular, then an inverse property is valid. More precisely there is a number $0 < C_I < \infty$ independent of H such that

(3-4) $$\|q_H\|_{\epsilon} \leq C_I H^{-\epsilon} \|q_H\|_0$$

holds for all q_H in S_H.

Theorem 1. Let (3.1)-(3.4) hold. Then there is a constant C independent of H, h, $\underline{u}$, and p such that if

(3.5) $$Ch/H < 1,$$

then the Brezzi condition (2.10) is valid.

Remark. In short this result states that the approximation will be stable provided the mesh spacing h for the velocities is sufficiently fine compared to the mesh spacing H for the pressure. The condition (3.5) is familiar from other results on mixed and hybrid finite element methods [5].

The starting point in the proof of Theorem 1 is a result due to Leray which in effort states that the Brezzi condition (2.10) is valid in the infinite dimensional case where $\vec{U}_n$ is replaced with $\vec{H}^1_0(\Omega)$ and $*_\Omega$ is replaced by $L^2_0(\Omega)$. It is normally stated in the context of the ability to stably decompose a vector field into a divergence free part plus a curl free part. Here we give an equivalent form the proof of which can be found in [8].

~~Theorem 2. Leray.~~ Let $f \in L^2_0(\Omega)$. Then there is a $\vec{v} \in \vec{H}^2(\Omega)$ such that

(3.6) $$\operatorname{div} \vec{v} = f \quad \text{in} \quad \Omega$$

(3.7) $$\vec{v} = 0 \quad \text{on} \quad \partial\Omega$$

with

(3.8) kix i CJI fUo'

where $0 < C_{Li} < \infty$ is a constant independent of f and $\underline{v}$.

Strictly speaking, the result is valid only for smooth regions Ω that for example are free of re-entrant corners such as convex regions or regions with C^∞ boundaries. In such cases, the smoothness of $\underline{v}$ increases with the smoothness of f. In particular, we have the following

<u>Corollary</u>. Let $f \in H^{\delta}(\Omega) \cap L_0^2(\Omega)$. Then there is a $\underline{v} \in \vec{H}^{1+\delta}(\Omega)$ satisfying (3.5)-(3.6) and

(3.9) $$\|\underline{v}\|_{1+\delta} \leq C_L \|p\|_{\delta}$$

for $0 \leq \delta \leq 1$.

We are now prepared to prove Theorem 1. To do this we must show there is a number β such that for any q_H in $\mathcal{S}_H$ we have

(3.10) $$\int_{\Omega} \{q_H \operatorname{div} \underline{v}_h\} \geq \beta \|q_H\|_0 \|\underline{v}_h\|_1$$

for a suitable $\underline{v}_h$ in $\vec{\mathcal{U}}_h$. Let $\underline{v}$ satisfy

(3.11) $$\operatorname{div} \underline{v} = q_H \quad \text{in} \quad \Omega$$

(3.12) $$\underline{v} = \underline{0} \quad \text{on} \quad \Gamma$$

with

(3.13) $$\|v\|_{1+\epsilon} \leq C_L \|q_H\|_{\epsilon} .$$

where C_L is the constant in Theorem 2. Using the approximation property (3.1) we select $\underline{v}_h$ in $\vec{\mathcal{U}}_h$ satisfying

(3.14) $$\|\underline{v} - \underline{v}_h\|_1 \leq C_A h^{\epsilon} \|\underline{v}\|_{1+\epsilon} \leq C_A C_L h^{\epsilon} \|q_H\|_{\epsilon}$$

Using the inverse property (3.4) this becomes

(3.15) $$\|\underline{v} - \underline{v}_h\|_1 \leq C_* (h/H)^{\epsilon} \|q_H\|_0 .$$

where

(3.16) $$C^* = C_A C_L C_I$$

Since $\operatorname{div} v = q_H$ we also have

(3.17) $$\|q_H\|_0 \leq \|\underline{v}\|_1 .$$

Hence (3.15) implies

(3.18) $$\|\underline{v} - \underline{v}_h\|_1 \leq C_*(h/H)^\epsilon \|\underline{v}\|_1 .$$

We now put these inequalities together to establish (3.10). Indeed, first note that

$$\int_\Omega \{q_H \operatorname{div} \underline{v}\} = \|q_H\|_0^2 \geq C_L^{-1} \|q_H\|_0 \|\underline{v}\|_1 .$$

Thus, using (3.18) we have

$$\int_\Omega \{q_H \operatorname{div} \underline{v}_h\} = \int_\Omega \{q_H \operatorname{div} \underline{v}\} - \int_\Omega \{q_H \operatorname{div} (\underline{v} - \underline{v}_h)\}$$

$$\geq (C_L^{-1} - C_*(h/H)^\epsilon) \|q_H\|_0 \|\underline{v}\|_1 .$$

But

$$\|\underline{v}_h\|_1 - \|\underline{v}\|_1 \leq \|\underline{v} - \underline{v}_h\|_1 \leq C_*(h/H)^\epsilon \|\underline{v}\|_1 .$$

Thus

$$\int_\Omega q_H \operatorname{div} \underline{v}_h \geq (1 + C_*(h/H)^\epsilon)^{-1} (C_L^{-1} - C_*(h/H)^\epsilon) \|q_H\|_0 \|\underline{v}_h\|_1 .$$

Thus, (3.10) holds with

$$\underline{\beta} = (1 + C_*(h/H)^\epsilon)^{-1}(C_L^{-1} - C_*(h/H)^\epsilon) .$$

It follows that $\underline{\beta}$ is bounded above from zero as $h, H \to 0$ provided h/H is sufficiently small. In particular, the constant C in Theorem 1 is

$$C = C_*C_L)^{1/\epsilon} = (C_L{}^2 C_A C_I)^{1/\epsilon}.$$

§4. The Poisson Equation. We now consider the Poisson equation which we write in first order form as follows:

(4.1) $$\underline{u} - \vec{\text{grad}}\ \varphi = 0 \quad \text{in} \quad \Omega$$

(4.2) $$\text{div}\ \underline{u} = f \quad \text{in} \quad \Omega$$

(4.3) $$\varphi = g \quad \text{on} \quad \partial\Omega$$

The weak form of this system is to seek

(4.4) $$\underline{u} \in H(\text{div},\Omega) = \{\underline{v} \in L^2(\Omega) \ : \ \text{div}\ \underline{v} \in L^2(\Omega)\}$$

(4.5) $$\varphi \in L_2(\Omega)$$

such that

(4.6) $$\int_\Omega \underline{u}\cdot\underline{v} + \int_\Omega \varphi\ \text{div}\ \underline{v} = \int_\Gamma g\underline{v}\cdot\underline{\nu}$$

(4.7) $$\int_\Omega \varphi\ \text{div}\ \underline{u} = \int_\Omega \varphi f$$

holds for all $\underline{v} \in H(\Omega;\ \text{div})$ and $\varphi \in L^2(\Omega)$. In (4.6) $\underline{\nu}$ denotes the outer normal to Ω.

As with the Navier-Stokes equations an approximate procedure is obtained by first introducing finite dimensional subspaces

$$\vec{\mathcal{V}}_h \subseteq H(\Omega,\ \text{div}),\ \mathcal{S}_H \subseteq L^2(\Omega).$$

One then seeks

$$\underline{u}_h \in \vec{\mathcal{V}}_h,\ \varphi_H \in \mathcal{S}_H$$

such that (4.6)-(4.7) holds with $\underline{u}$ replaced with $\underline{u}_h$ and φ

replaced with q_H. In addition, $\underline{v}$ and f are restricted to U_h and S_h, respectively.

The stability and convergence of this scheme centers on a condition similar to (2.10) which was formulated in [9]. In particular, there must be an absolute number $0 < \beta < \infty$ for which

(4.8) $$\sup\,(\underline{v}^h, q_H)_{-1} \geq \beta \|q_H\|_{-1} .$$

Here $(\cdot,\cdot)_{-1}$, $\|\cdot\|_{-1}$ denote the inner product and norm on $H^{-1}(\Omega)$, and the sup is taken over all $\underline{v}^h$ in $\mathring{V}_h$ with

(4.9) $$\|\underline{v}^h\|_0 \leq 1 .$$

We now show that the analog of Theorem 1 is valid.

Theorem 3. Let the assumptions in Theorem 1 hold. Then (4.8) is valid.

We first establish the analog of Theorem 2. (Actually, it is a special case of Theorem 2.)

Lemma. Let $f \in \overset{0}{H}(\Omega)$. Then there is a $\underline{v} \in H(\Omega)$ such that

(4.10) $$\operatorname{div} \underline{v} = f \quad \text{in} \quad \Omega .$$

Moreover,

$$\|\underline{v}\|_{1+\delta} \leq C\|f\|_{\delta}$$

for $-1 \leq \delta \leq 1$.

Proof. Solve $\Delta\phi = f$ in Ω, $\phi = 0$ on $\partial\Omega$ and let $\underline{v} = \overrightarrow{\operatorname{grad}}\,\phi$.

To prove (4.8) we let

$$\operatorname{div} \underline{v} = q_H.$$

Then

(4.11) $(\operatorname{div} \underline{v}, q_H)_{-1} = \|q_H\|_{-1}^2 \geq (1/C)\|q_H\|_{-1}\|\underline{v}\|_0.$

We now select a $\underline{v}_h \in \underline{U}_h$ such that

$$\|\underline{v} - \underline{v}_h\|_0 \leq Ch^{\epsilon}\|q_H\|_{-1+\epsilon}.$$

Using the inverse inequality

$$\|q_H\|_{-1+\epsilon} \leq H^{-\epsilon}\|q_H\|_{-1}$$

we obtain

(4.12) $\|\underline{v} - \underline{v}_h\|_0 \leq C(h/H)^{\epsilon}\|q_H\|_{-1}.$

Thus (as in Section 3) we obtain

(4.13) $(\operatorname{div} \underline{v}_h, q_H)_{-1} \geq (C_1 - C_2(h/H)^{\epsilon})\|q_H\|_{-1}\|\underline{v}_h\|_0$

for absolute positive constants C_1 and C_2.

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