MULTIPLE CRITERION OPTIMIZATION OF ELECTRONIC CIRCUITS

by

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ABSTRACT

The basis of most engineering design is making trade-offs among competing factors. This is especially true in the design of electronic circuits. In this paper we examine Multiple Criterion Optimization (MCO), one aspect of the Multiple Criterion Decision Making (MCDM) problem. After presenting relevant definitions and the concept of a noninferior solution to the MCO problem we develop a family of weighted p-norms for generating noninferior solutions. The prime importance of the weighted p-norm methods is the interpretation of the weights. We develop a canonic weight for the p-norm family. Interpretation of the canonic weight allows the development of various weight selection heuristics. Finally, the techniques of MCO are applied to the design of a MOSFET NAND gate.

I. INTRODUCTION

The essence of a large part of the practice of engineering and design is decision making. Further the typical design situation faced by an engineer is one involving many competing factors. In spite of this fundamental aspect of design it has been, for the most part, economists, mathematicians, operation researchers, and others who have studied and developed the ideas of decision making under competing objectives [1-5]. In this paper we propose to review the ideas of Multiple Criterion Optimization (MCO) and develop new interpretations, techniques, and applications of MCO ideas for the design of electronic circuits.

Multiple Criterion Optimization - one aspect or tool in the decision making (design) process - addresses the simultaneous minimization (maximization) of several competing criterion or objectives. The ideas of competing objectives are not new in the EE literature [e.g. 6-9], however, in most cases the multiple objectives were assigned weights and summed to form a single sealer objective. As will be shown, this weighted sum technique is only one, and the least powerful- of a family of methods for solving MCO problems.

The area of MCO also provides a characterization of a "solution" to problems with competing objectives. Further, using the ideas of MCO and the family of methods developed in this paper it is possible to give a definite meaning to the weights used in various MCO approaches. The interpretation of weights allows the development of meaningful heuristics with which to examine the various "optimal" trade-off solutions. It is our hope that bringing the ideas and techniques of MCO to the attention of the EE community will lead to a better use of optimization as a tool for circuit design and a raising of the consciousness of engineers about an area of potentially great application.

In the next section we introduce the fundamental concepts and terminology of MCO. In Section III we present a family of methods, based upon weighted p-norms, for generating various trade-off solutions to the MCO problem. Section IV presents an interpretation and various selection heuristics for the weights used in the weighted p-norm MCO methods. Section V applies the ideas of MCO to the design of an MOSFET NAND using two different members of the p-norm family of methods. Finally Section VI presents a summary and conclusions.

II. MCO FUNDAMENTALS

The competition between multiple criterion gives rise to the distinguishing difference between MCO and traditional single objective optimization• In the traditional optimization of a single function $f(x)$, $x*$ is a minimum if $f(x^{\#})$ £ $f(x)$ for all x of interest. This inequality is a statement of the fact that the real numbers can be completely ordered by the ^{f}less than or equal to $(.\leq.)^f$ relation. However, if several functions are considered simultaneously it is

possible that one function may decrease while another increases. Thus multiple criterion cannot be completely ordered using a '<₄' relation. This lack of a complete order is the chief characteristic of MCO problems.

In order to make the preceding ideas concrete, we will introduce the following notation. The n designable parameters, $x^{\wedge}$, is1,2,...,n, in the multiple criterion optimization problem will be denoted by the n-vector x:

$$x = (x^{\wedge}, X2? \quad \bullet \bullet \bullet i \; ^{X}j^{\wedge})\; \bullet$$

It is convenient to view x as a point in the n-dimensional input space I. The m design objectives, fj , j=1,2,,,.,m, will be denoted by the

m-vecLor $\underset{\sim}{1}$:

$$\pounds = \pounds <\overset{*}{>} = (^\wedge(x), f_2{}^{(i)^f} \bullet\bullet\bullet\bullet V \underset{\sim}{i}))$$

It is convenient to view $\underset{\sim}{f}$ as a point in the m-dimensional <u>output</u> <u>space</u> 0.

In general the optimization problems under consideration will be subject to certain constraints which will be expressed as

$$gj(\underset{\sim}{x}) \leq 0 \qquad i=1, \ldots, k$$

$$h_j(\underset{\sim}{x}) = 0 \qquad j=1, \ldots, 1$$

or, more succinctly, as

$$g(\underset{\sim}{x}) \; 1 \; \underset{\sim}{0}$$

$$\underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{0},$$

The MCO problem can now be stated as

$$\min \quad f(x)$$

$$\text{subject to} \qquad g(\underset{\sim}{x}) \leq \underset{\sim}{0}$$

$$\underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{0}.$$

(1)

That is, we wish to simultaneously minimize the individual components of $\underset{\sim}{f}$ subject to the given constraints. As previously mentioned, if some (usually all) of the components of $\underset{\sim}{f}$ are competing there will be no point $\underset{\sim}{x}$ that simultaneously minimizes all the components of $\underset{\sim}{f}$.

In other words, when objectives comp< te there is no 'optimal solution*
to the multiple criterion optimizati' n problem.

Instead of optimality, the concept of noninferiority [10,11]
(also known as efficiency [12]. Pareto optimality [13,14], minimality
[14,15] and nondominance [16,17]) is used to characterize a solution
to the MCO problem. In order to concisely define noninferiority we
introduce the following two definitions.

Definition. The feasible region in input space,
ft, is the set of all designable parameters that satisfy
the constraints, i.e.

$$n = U\lfloor h(x) = 0, \ g(x) \leqslant 0\}$$ (2)

Definition. The feasible region in output space,
A, is the image by f of the feasible region 8 in input
space, i.e.

$$A = \{f|f = f(x), \ x \ e \ \ll\}.$$ (3)

We are now in a position to define local and global noninferior
solutions.

Definition. A point x*ea is a local noninferior
point if and only if for some neighborhood of x^* there does not
exist a Ax such that (x^*+Ax) eQ and

$$f_i(x^* + Ax) \leq fjCx^*) \quad i=1, \ ..., \ m$$

$$f_j(x^* + Ax) < f_j(x^*) \quad \text{for some j.}$$ (4)

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solution.

Definition. A point $x^* \in Q$ is a global noninferior
point if and only if there does not exist an $x \in \Omega$ such
that

$$q^{(x)}_i \wedge (x^*) \quad i=1, \ldots, m$$

$$f^*(x) < f_j(x^*) \quad \text{for some } j.$$

(5)

In general there are an infinite number of noninferior points for
a given MCO problem. The collection of noninferior points is the
noninferior set. The image of the noninferior set by f is called the
noninferior solution set or, the noninferior or trade-off surface. As
with scalar optimization procedures, procedures for solving MCO problems
will only generate local noninferior points. However, in our discussion non-
inferior will imply global noninferior unless otherwise stated. Figures 1-2
illustrate the feasible regions in input and output space, an inferior solu-
tion and local and global noninferior solutions.

At this point it is useful to discuss how the typical weighted sum
approach fits into the MCO framework. As with most MCO methods, it is
best to consider the weighted sum technique in output space. Recall that
in the weighted sum approach a weight vector $u \wedge 0$ is chosen and. the follow-
ing problem solved

$$\min \omega^T f(x)$$

$$\text{s.t.}$$

$$g(x) \leq 0$$

$$h_j(x) = 0$$

(6)

In output space this becomes

$$\min_{\underline{f}} \; \underline{\omega}^T \underline{f} \qquad\qquad (7)$$

subject to

$$\underline{f} \in A$$

If we examine (7) we see that the set of $\underline{f}$ in output space that yield a constant value for the objective function, i.e.:

$$\{\underline{f} \in 0 \mid \underline{\omega})^{\sim} \underline{f} = c\}$$

is a hyperplane with normal $\underline{\omega}$. Thus, in order to solve (7), we must find the smallest value of c for which there exists an $\underline{f} \in A$. Clearly this is the point where the hyperplane with normal $\underline{\omega}$ just touches A as it proceeds outward from the origin. This situation is illustrated in Figure 3.

A problem with the weighted sum technique arises when the lower boundary of A is not convex, as illustrated in Figure 4, because not every noninferior solution will have a supporting hyperplane. This situation corresponds to a duality gap in the normal theory of Lagrange multipliers [18,19] and historically indicated the first practical difference between the solution of single objective and multiple objective optimization problems.

In many cases the uses of optimization for electronic circuit design has been in the context of a weighted sum minimization. Clearly if the noninferior surface is nonconvex the weighted sum method may yield poor designs no matter what weight or optimization technique is used. For this reason and for the insight gained into the decision making aspect of circuit design we believe that the study of MCO techniques and ideas is very important for circuit design and CAD in general.

In this section we will d(vclop a family of methods for generating noninferior solutions. Because many existing techniques are contained in this family of methods, its properties are useful in unifying and extending existing methods of generating noninferior solutions [20].

For the development in this section we will assume that A_f the feasible region in output space, is a compact simply connected set. Vie also assume that the minimum of each component of $\underline{f}$ is unique. Furthermore, we assume, without loss of generality, that A is i_n the first quadrant, i.e., $\underline{f}eA$ implies $\underline{f} > \underline{0}$. (Note that since A is compact we can translate A so that it is entirely within the first quadrant .) Finally, whenever the normal to the surface of A is employed we assume that the surface of A is such that the normal is uniquely defined,

We begin by defining a class of utility functions [21] based upon weighted p-norms (also known as ℓ_p norms), i.e.,

$$U_p^W(\underline{f}) = 1\,|W\underline{f}||_p \qquad 1 \le p \le \infty .$$

where $W = \text{diag}(w^\wedge, W2_t.\bullet._f w_m)$ $w^\wedge \wedge 0$ $i=t,2,\ldots,m$

and

$$||W\underline{f}||_p \left\{ I \ M \right\}^{1/P}$$

Noninfcrlor points can now he fti:n"r;itf!(l by minitnJ v.\ n# $U^{*\prime}_{\Gamma}$. 'lhl:? minimization can be interpreted &< ometrically by examining the лсvα sets of $U^{v/}_p(\underline{f})$, i.e.

$$L^W_p(a) = \left\{\underline{f}\middle| \ t^u_{\underline{f}}j(\pounds) \leq^*\right\}. \tag{9}$$

Figure [5] illustrates this level set for W=I and p=1,2 and °°. The level sets for various weights are illustrated in Figure 6.

Minimizing $U^W_p(\underline{f})$ over the feasible region A will generate at least one point on the surface of A where the level set $L^W_p(\underline{f})$ just touches A as it expands about the origin. Figure 7 illustrates the minimization of various weighted p-norms over A in terms of the level set of the norm•

From the preceding discussion it should be clear that if we set p=», i.e., if we use the max norm, we can generate all noninferior solutions by varying the weights and minimizing the max norm subject to the constraints. This is true because, as shown in Figure 8, the level sets of the max norm act as translations of the nonpositive orthant, and can be used for an alternate statement of the conditions for Pareto optimal!ty or noninferiority. Because of the importance of these utility functions we now formalize the preceding discussions. The assumptions stated at the beginning of the section are still valid. Further, when <j>(f*) < 4>(f) for all feasible $\underline{f}$ we call $\underline{f}$* the unique minimizer of <Ф.

$f^* \epsilon A$ is noninferior if and only if there exists a $W = \text{diag}(w)$, $w \geq 0$ such that f^* is the unique minimizer of $||Wf||$ over A.

Proof. (Sufficiency) If for some $w \geq 0$ f^* uniquely minimizes $||Wf||_\infty$ over A and f^* is not noninferior then

there would exist some $f \epsilon A$ such that

$$f_i \leq f_i^* \quad \text{for all } i$$
$$f_j < f_j^* \quad \text{some } j.$$

This implies

$$w_i f_i \leq w_i f_i^* \quad \text{for all } i \tag{10}$$

Hence

$$||Wf||_\infty \leq ||Wf^*||_\infty$$

which is a contradiction.

(Necessity) Alternately, if f^* is a noninferior point, let $W = \text{diag}(1/f_1^*, 1/f_2^*, \ldots, 1/f_m^*)$. Then $||Wf^*||_\infty < ||Wf||_\infty$ for all $f \epsilon A$ for if this were not true there would exist a $f \epsilon A$, $f \neq f^*$ such that

$$||Wf||_\infty \leq ||Wf^*||_\infty = 1 \tag{11}$$

or $f_i \leq f_i^*$ for all i .

But since i/f* this implies $f_j < f_j^*$ for some j and

hence $\tilde{f}^*$ would not be noninferior, which is a

contradiction, thus establishing the result.
 Using the notation

$$L_\infty^W(\underline{f}^*) = \{\underline{f} | \ U \underline{M}. i 1 1 T 1 1 J \qquad \text{X12)}$$

we can restate the preceding theorem as

Corollary 2.2. A point $\underline{f}^*$ is noninferior if and only

if there exists a V$\underline{M}$) such that

$$L_\infty^W(\underline{f}») H \ A = \{\underline{f}»\}. \qquad (13)$$

We cut j/orHM'.'i 1 j /(? th<: prrci'dinf. 'i::i:u:;:;ion I.o any we i/',hl.i:d p-norm as follows.

Theorem 2.3.

A point $\underline{f}^*$ £ A is noninferior if and only if there exists a W >, 0 and a 1 <. p <. "° such that

$$L_p^{\lvert tj} U_{\underline{~}}^*)^{n \ A} = \{£^*\} \bullet \tag{14}$$

Proof. (Sufficiency) Clearly by the prucrding corollary if $\underline{f}^*$ is noninferior this condition is satisfied.

(Necessity) If

$$L;_p^V(f^*)n \ A = \{\underline{f}^*\} \quad 1 \ i \ p < ao$$

and J* was not noninferior then there would exist a f r f, $\underline{f}$ £ A such that

$$q \leq f_i^* \quad \text{for all i}$$

$$f_{,j} < \underline{f}_{,j}^* \quad \text{some j}$$

but then

$$v/._1 \underline{f}._1 "._{\underline{~}} < w._i \underline{f}._1 ^* \quad \text{for all i}$$

$$\underline{w} \geq 0 \ .$$

Thus $||\text{W}\underline{f}||_p \leq ||\text{W}\underline{f}^*||_p$, and so

$$\underline{f} \ e \ L_p^W \ (f^*)nA \ ,$$

which is a contradiction, thus establishing the result.

$\bullet$

Various aspects of the preceding discussion have been mentioned by others. Bowman [22] discusses the use of • ^ to generate noninferior solutions as does Benayoun F23]. *Xu* [24], Zeleny [26], and others [25,27] in the theory of compromise solutions use families of p norms. The use of general utility functions to find noninferior solutions is discussed by Geoffrion [12,29], Beeson [30] and others [21,28}. General least $\mathfrak{p}^{th}$ optimization has been used in the electrical engineering literature [6,7]. However, the development of a family of weighted I_p norms (with the weight interpretation of the next section) to generate noninferior solutions has not, to our knowledge, been presented previously. We feel this method is an excellent bridge between the traditional approach taken by electrical engineers using single objective optimization and the view of circuit design as a multiple criterion optimization problem.

IV. INTERPRETATION AND SELECTION *OV* WE[CUTS

In any optimization technique, where different solutions are generated by varying a set of weights, the interpretation and selection of these weights is critical. In this section we will give a precise interpretation to the weights used in the weighted p-norm generation of noninferior solutions. Based upon this interpretation we will discuss weight selection heuristics for the 1- and «-norms. Finally we will present an example comparing ℓ_1 and I_∞ methods on a simple two criterion problem.

IV.1 Canonical Weights for p-Norms

We know that if the point f^* is noninferior and is found by minimizing a weighted p-norm, then (without loss of generality) we require the gradient of the p-norm at f^* be equal and opposite to the outward pointing normal to the noninferior surface at f^*. Let n^* be the inward pointing normal to the noninferior surface at f^* and normalize n^* so that

$$n^{*T} f^* = 1.$$

Thus if f« were on a convex portion of the noninferior surface and $\min |\,|w(1)f|\,|^\wedge = 1$, the minimum would occur at $f = f^*$ when $W(1) = n^*$ (assuming f^* is a unique minimizer of $||w(1)f||_2$).

We now develop a formula for a canonical weight $W(p)$ for each p norm so that if f^* was noninferior and could be found by minimizing $1|W(p)f||_p$ then the minimum would be 1 and occur at $f = f$«. We require

$$\left. V_f ||W(p)f||_p \right|_{f=f^*} = n^* \qquad (15)$$

which implies

$$\frac{w_i^{*(p-1)}}{||W(p)f^*||_p^{*(p-1)}} = n_i^*$$

But we assumed that

$$\|W(p)\underline{f}^*\|_p = 1,$$

thus

$$w_{ii}(p) = \frac{n_i^{*1/p}}{f_i^{*\frac{p-1}{p}}}, \quad i = 1, 2, \ldots, m . \tag{16}$$

This weight will be called the <u>canonical weight associated with</u> $\underline{f}^*$ for

the p-norm. Examining (16) shows that if p=1, $W(p)$=diag($\underline{n}^*$) as expected.

Furthermore

$$\lim_{p \to \infty} w_{ii}(p) = \frac{1}{f_i^*} \qquad (f_i^* > 0)$$

and we have the canonical weight associated with the max norm.

Various interpretations of (16) will be discussed below and used to generate

weight selection heuristics. We note in passing that many MCO techniques and

weight selection methods can be shown to be related either directly or

indirectly to the 1- and ∞-norm generation scheme [20]. However, use of

intermediate norms has not been pursued and both weight selection heuristics

and the usefulness of these intermediate methods remains an open area for

research. (For a group decision making interpretation of the unity weighted

norms see [24,25].)

IV.2 <u>Weight Selection Heuristics</u>

Using the canonic weight for the p-norm we now have a tool for choosing

weights to generate various noninferior solutions. However, we have also

seen that, in general, there are an infinite number of noninferior solutions.

The designer must choose one design from among the set of noninferior designs.

Various techniques [e.g. 30-36] have been presented for generating a representative

set of noninferior solutions in order to aid the designer in making his final choice of noninferior design. A discussion of some of these methods can be found in [20].   Ibr the purposes of this paper we will discuss some heuristics for the 1-norm method, our suggestions for ∞-norm heuristics and present a compurational comparison of the 1- and »-norm generation methods,

a. $\underline{f_1}$ Weights

We first discuss weight selection heuristics for the 1-norm or weighted sum method. One of the early workers in interactive weight selection was Dyer [31,32] who assumed that an unknown utility function $U(f)$ existed but could not be specified by the decision maker. Dyer considered the following:

$$\nabla_{\underline{f}} U(\underline{f}) = \left(\frac{\partial U}{\partial f_1}, \frac{\partial U}{\partial f_2}, \cdots, \frac{\partial U}{\partial f_{n,}} \right)^T \qquad (17)$$

If we could approximate this gradient direction we could take a step in output space to minimize $U(\underline{f})$.   Consider the following vector which is collinear with $\nabla_f U(\underline{f})$:

$$\underline{w} = \left(1, \frac{\partial U/\partial f_0}{\partial U/\partial f_1^2}, \cdots \frac{\partial U/\partial f}{\partial U/\partial f_{j1}^m} \right) \qquad (18)$$

or

$$\underline{w} = \left(1, \frac{\partial f_1}{2} \qquad \frac{\partial f_1}{m} \right)$$

i.e., $\underline{w}$ is a vector of marginal substitution rates. At a given point

in A, Dyer aaks the decision maker ln<w much he would give up in i_1 'Ior an increase in f^ and approximates

$$3f_1 \quad , \quad ^A f_1$$

Using this approximation to $V_{\underset{\sim}{f}}U(\underset{\sim}{f})$, Dyer takes a steepest descent step searching for a better solution and queries the user again on his preferences.

We can easily apply Dyer's method to weight selection by noting that if we choose $||\underset{\sim}{Wf}||j$ as a utility function (for generating noninferior solutions) that

$$V_f U(\underset{\sim}{f}) = \underset{\sim}{w} .$$

So we could pick a $W=\text{diag}(\underset{\sim}{w})$, minimize $||\underset{\sim}{Wf}||j$, and then ask the user how much of fj he would give up to decrease f^ and form a new weight

$$\underset{\sim}{\hat{w}} = \left(1, \frac{^A f_1}{\Delta f_2} , \cdots , \frac{^A f_2}{\Delta f_m} \right) \tag{21}$$

and again minimize $||\underset{\sim}{\hat{W}f}||_L$ to find a new noninferior solution. This **allows** the user to **fine tune** the choice of weights as he learns more about the problem. However, in a method such as this where it is possible that only a small portion of the noninferior surface will be explored, there is the danger of the decision being made based upon grossly incomplete information.

Another weight selection heuristic suggested by lirayton and Director [37] and developed and extended by Fraser [36] is based upon the concept of the weight being normal to the supporting hyperplane at the noninferior solution. Suppose that we first minimize each function separately in order to find the extremes or boundaries of the noninferior surface in output space. Let these points be f^{1*}, f^{2*},..., f^{m*}. These points can now be used to define a plane in output space by solving the following set of equations

$$
\begin{bmatrix}
\cdots\cdot f^{1*}\cdots\cdots \\
\cdots\cdots f^{2*}\cdots\cdot \\
\bullet\bullet\bullet\bullet f_x^{m*}\bullet\bullet\bullet\bullet\bullet
\end{bmatrix} . w - 1 . \qquad <22>
$$

The plane implied by this set of equations and the noninferior solution found by using this weight are shown in Fig. 9.

Fraser begins with a boundary search and an extra minimization using the normal to the plane defined by the boundary search as does Brayton. A vector is now formed whose entries are the minimum values found for each criterion.

$$
F_{min} = \left(f_i^{min} , f_2^{min} , \cdots , f_m^{rain} \right) \qquad (23)
$$

Similarly a vector of maximum values is formed

$$
F_{max} = \left(f_1^{max} , f_2^{max} , \cdots , f_m^{max} \right) \qquad (24)
$$

If the designer is not satisfied with the newly found noninferior solution f , he is asked to specify which components of f he would

like to reduce say, i and j. Then the differences

$$\Delta f \Leftarrow f^k - F_{\alpha} \qquad a = i \; J$$

$$\Delta f_m = F_{max_m} - f_{in}^k \qquad \text{,, } \gg \text{1f}_{\,jJ} \tag{25}$$

are formed and the new weight vector becomes

$$w^{k+1} = \begin{bmatrix} \dfrac{i}{1} & \dfrac{\Delta f_i}{\Delta f_{\checkmark}} & \dfrac{\Delta f_i}{\Delta f_3} * \bullet\bullet' \bullet \dfrac{\Delta f_1}{\Delta f_m} \end{bmatrix} \tag{26}$$

The similarity of this procedure with Dyer's (21) should be clear. The weight vector (26) attempts to use existing information and designer preference to assess the marginal substitution rates.

b. I_∞ Weights

Although the use of minimax optimization is relatively common in the EE literature ([e.g. 6,7,38,39]), it has not been viewed as a tool for generating noninferior solutions to an MCO problem. Further, the overt use of minimax methods to solve MCO problems is very limited. Bowman 22 showed that a weighted minimax could be used to generate all noninferior solutions but he gave no indication as to how to choose the weights.

If we write the minimax problem

$$\min_{x \in ft} \max_i \{w_i \cdot f_i(x)\} \qquad (27)$$

as the equivalent

$$\min_{Y, \, x \, \in ft} Y \qquad (28)$$

$$\text{s.t.} \quad a_i f_i(x) \leq Y \qquad i=1,\ldots,m$$

Then minimax methods can be related [20] to the Goal Attainment method [40] a variant of Goal Programming [41]. But again no discussion of weight selection has been presented for this approach.

In our discussion of weight selection for minimax methods we will work under the assumption that any weight used will end up being a multiple of a canonic weight. This means that the solution will occur at the positive corner of the hyperrectangular level set associated with the weighted I norm, or, alternately, at the solution we have

$$w_1 f_1* = w_2 f_2* = \cdots = V m* - \qquad (29)$$

The case where this holds and an example of where it does not hold are shown in Figure 10. We make this assumption without loss of generality because under the conditions previously given, any point which is a unique minimizer that is found using a weighted $^\wedge$ method will be noninferior. Furthermore, if the weight is not canonic we know there exists a canonic weight which will generate that point.

Assuming a canonic weight, the effect of the minimax method is to find a minimum solution on a line specified by the weights. If $w_1 =**2^{s}* \bullet \bullet ^{=w}m^{=1}$ then the line of search is a ray 45° from each axis, otherwise a different angle is specified. This is illustrated in Figure 11. So, one way of choosing weights would be to ask the decision maker to specify a direction of search and then transform this direction into a weight. However, the range of possible angles is not clear. Also, specifying angles in higher dimensions is not intuitively obvious and has little direct connection to the problem being solved. Therefore we now consider an alternate weight selection heuristic.

As was discussed previously, the canonic weight associated with a noninferior solution $\underline{p}\}$ for *« is

$$w_{*ii} " 7 \underset{1}{\overset{1}{\sim}} * \qquad 1 - 1, \text{..-}, m.$$

Clearly any positive multiple of $\underline{w}^*$ is also a valid weight. Thus we can take any set of m previously found noninferior solutions

(initially the points found by the 1-undary search) ;ird ask the uncr
to assign a weight, $\alpha_i > 0$, to each point indicating how much he
favors that particular solution. The weights a can be normalized so
that $1 < *_{\pm} = 1$ and then we form

$$\hat{\underline{f}} = \sum a_{\pm}f \ . \tag{30}$$

$\hat{\underline{f}}$ will be in the plane specified by the m points, i.e., if
$7r = \{f | c^T f = 1\}$, where $\underline{c}$ is the normal to the plane, then

$$c^T \hat{f} = \sum \mathbf{L}\mathbf{W} = 1 \tag{3D}$$

alternately $\hat{\underline{f}}$ is in the cone of directions specified by the m
solutions. At this point $\underline{f}$ can be shown to the user to see if f is the
type of trade-off he desires. If not, then the a_i's can be readjusted. This
consistency check prior to any optimization can be very useful in preventing

excessive optimization runs.

If f passes the consistency check we form the weight

$$\hat{w}_{ii} = |- \qquad i = 1,\ldots,m \tag{32}$$
$$f_i$$

f there is a noninferior solution along the ray extending through $\underline{f}$,
the weighted $1^\wedge$ method using $\underline{w}$ will find it. If $\underline{w}$ is not a scaled
canonic weight for any noninferior solution, a local noninferior solution
will still be found (assuming the minimum is unique) and we have valuable
information concerning the noninferior surface in the direction specified.
Notice that this method of weight selection is much more direct than weighted
sum methods because we are not specifying a normal to the support plane at the
desired solution, but an actual value, $\underline{f}$, for the solution which, if the
weight is a scaled canonic weight, will be a scaled version of the solution.

A number of variations on this method are clearly possible. For instance, if the user examined the existing noninferior solutions he could simply specify a new desired solution $\hat{\underline{f}}$ - this assumes the user has some understanding of noninferior solutions - then a weight $\hat{w}_{ii} = \dfrac{1}{\hat{f}_i}$ is immediately generated. Also the cone in which the user specifies a direction (solution) does not have to be m dimensional. Any two or more noninferior solutions that the user prefers can be weighted to give a new value, $\bar{\underline{f}}$, which automatically specifies a weight.

Thus we see that the ℓ_∞ methods lend themselves to a more direct and flexible specification of a weight and the weight (when a scaled canonical weight) is intimately associated with the value of the noninferior solution and not the normal to the surface of Λ at the noninferior solution. Clearly more computational experience with ℓ_∞ methods is needed in order to further refine the weight selection process and provide more user oriented methods. But we feel that the weighted ℓ_∞ methods have been shown to be exceptionally powerful and intuitively direct methods for solving MCO problems.

V. NAND GATE EXAMPLE

In this section we apply the ideas developed in the previous sections to the optimization of an MOSFET NAND gate circuit. The particular circuit we consider has been used as a time domain optimization example by Director and Brayton [47], an example for Simplicial Approximation by Director and Hachtel [48], and as an example for weighted sum Multiple Criterion Optimization by Fraser [36] .

The two input MOSFET NAND gate used as an example is shown in Figure 14. The first step in designing the NAND gate is to choose a model for the transistors. Based upon the work of Fraser [36] we chose a four terminal model that includes the effect of substrate bias. This model and its defining equations are presented in Figure 15.

There are many possible sets of designable parameters that could be used in designing the NAND gate, for example, the lengths and widths of all the devices as well as the flat band voltages of the devices. We choose (as do Director and Hachtel [48]) the flat band voltage, V_{Ff}, the width of the bottom two transistors, W_2^3, (constrained to be the same) and the width of transistor $T^\wedge V^\wedge$, as the designable parameters. Table 5 presents the range of values of the designable parameters as well as other constants needed to analyze the NAND gate (these are the values used in [48]).

Table 5 Parameter values, constants, and constraints
used in the MOSFET NAND gate example.

PSI=.5771

k=.5

GM=.OO6

$\wedge$ =12.7 microns

L =5.08 microns $-2 < VFB < -1_{Jm}$
 23

VDD=-6.5 volts $5 \leq W_1 \leq 50$

VGG=-14.5 volts

                                   $50 \leq W_{23} \leq 250$

Vm=-6 volts (i_n ON state)

A_f =1.03657

C_L=5 pF

The objectives in our design will be: to minimize the area used by the transistors, to minimize the switching time of the gate, and to require the ON voltage V_o to be as close to zero as possible. The desire to minimize area is obvious since we are considering that this gate will be a part of a more complex integrated circuit ana thus should take up as little chip ai")a as possible. Because we would like to include the NAND gate as part of a larger logic circuit, reducing the switching time of the gate increases the potential speed of the logic circuit. Finally, in order to have an adequate noise margin as well as to be able to drive the stages connected to the NAND gate, we would like the ON voltage to be as close to zero as possible. Reducing the power dissipated by the gate would also seem to be a reasonable objective, however, our own simulations as well as reports in the literature [36]indicate that power dissipation tracks with the area and thus it is only necessary to consider one of these an an objective.

We must now consider the evaluation of these objectives. The evaluation of the area of the gate is, of course, independent of the analysis of the operation of the gate. To find the ON voltage of the gate we only need a d.c. analysis of the NAND gate in the ON state. Thus the only remaining objective to evaluate is the switching time.

The switching time of the gate is a function of the time it takes for the circuit to turn ON and the time it takes the circuit to turn OFF, But the turn OFF time is much larger than the turn ON time and thus we can consider minimization of the turn OFF time - turn OFF propagation delay tp_D - instead of the entire switching cycle. In general, the evaluation of tp_D will require a transient analysis, but

fur tho class ol" MUSKET gates wo ro concvM-nod with an approximation to t_{pn} dons exist.

The approximation to t_{pn} is based upon the assumption that the output » node is dominated by a single load capacitance (independent of voltage). This assumption is used in several analysis programs [49,50], was used by Fraser [36], and is adequate for static MOSFET logic. Fraser develops this approximation assuming that the lower transistors are out of the circuit and the ON state value of the output voltage was zero. The approximation is

$$^{fc}rD = {}^A F \; w7 \; \mathbb{k}^{\frac{-L_i}{}} \qquad , \qquad (33)$$

where

$$\alpha = \frac{1}{(m-1)} \; \ln \; \frac{m-2}{3m-4} \qquad (34)$$

and

$$\tau = \frac{C_L}{Gm \; (V_T - V_{GG})} \cdot \qquad (35a)$$

$$m = \frac{V_{DD}}{(V_{GG} - V_T)} \qquad (35b)$$

with A_F a multiplicative constant, used to match (33) with the delay found using an accurate transient simulation. Using (33)-(35), the turn OFF propagation delay can be approximated (to first order) without performing a circuit simulation (except once to estimate A_F).

Therefore, in order to evaluate the objectives of our design we simply need to perform a d.c. analysis of the NAND gate in the ON state. The final form of our performance objective functions for the MCO design of the NAND gate is

$$\text{ι (eqn. 1)} \tag{36}$$

$$*2 = W_1 " L_1 + 2 L_{23} ' W_{23} \, Urea) \tag{37}$$

$$4). = -V \quad \text{(ON output voltage)} \tag{38}$$
$$3 \quad o$$

where the designable parameters are Wj, W_{23}, and V_{FB} (see Figure 15)
Besides the constraints on designable parameters given in Table 5
an upper limit of 2500 mil$^{\wedge}$was placed upon the area, 110 nsec. was
the maximum acceptable propagation delay, and -.7 volts the smallest
acceptable ON.output voltage (see [36]).

The gradients of Φ_1 and $\Phi^{\wedge}$ were found by considering the circuit
equations as equality constraints and adding them via Lagrange
multipliers to (36) and (37). Direct differentiation of the resulting
equations and proper definition of the multipliers gave the gradients
of Φ_1 and Φ_3 (this is essentially the approach taken by Hachtel,
Brayton and Gustavson [51] and is also equivalent to the adjoint
network [52]. method of calculating gradients).

Tims, 1 iu» MCO problem wo want U .:◇l $_{vr}$; iɔ

$$\min \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \tag{39a}$$

subject to

$$\bullet j \leq 110 \text{ (nsec.)}$$

$$*_2 -\pounds^2 500 \text{ (mils}^2)$$

(39b)

$$4>_3 \leq .7 \text{ (volts)}$$

$$-2 \leq {}^\wedge_B \leq -i \text{ (volts)}$$

$$5 \leq L \, Wj \leq 50 \text{ (microns)}$$

$$50 \pounds W_{23} \pounds 250 \text{ (microns)} .$$

For notational convenience we will let ft be the feasible region in

input space defined by (39b), and x = $(W_n, W_{oo}, V_{\_\_\_})$.

We will generate noninferior solutions to (39) using the

weighted $\underbar{\imath}_1$ and weighted %^ technique. Thus we solve two problems:

Problem 1

$$\min_{\underline{x}} \quad (W_1\phi_1 + W_2\phi_2 + W_3\phi_3)$$

$$\text{s.t.} \quad \underline{x} \, z \, \Omega \tag{40}$$

Problem 2

$$\min_{(\underline{x},Y)} \quad Y$$

$$\text{s.t.} \quad X \, G \, 0,$$
$$1 \leq \Upsilon$$
$$W_2\phi_2 \leq \Upsilon \tag{41}$$
$${}^W3*3 \, i \, {}^\Upsilon$$

A number of different constrained optimization methods could be

used to solve problems 1 and 2. We will use the constrained variable

metric method of Powell [43,44] (using the quadratic program of Canon,

Cullum and Polak [46]). In other experiments [20] we have compared

Powells[1] method with an Augmented Lagrangian technique which used a

first-order update and dynamic stopping criteria [42,43,45]. In all our
experiments we found Powells[1] method to be markedly superior. However, as
in the choice of which p-norm to use, the choice of the best constrained
optimization method is likely to be problem dependent.

V.I[1] <u>Problem 1</u>   ,

The first three optimizations were to find the minimum of ϕ_1, ϕ_2, ϕ_3
separately. The weight for the next optimization was the normal of the
plane passing through the three boundary points in output space.(22)
The results of these runs as well as the initial starting point (from [48])
and initial objective function values are given in Table 6.

Notice that the noninferior solution found using the weight plane
generated from the boundary search is a reasonable trade-off between the
extreme points. It is quite possible that this would not be the case.
For the next run a weight normal to the plane through the first, second,
and fourth noninferior solution was chosen. This optimization resulted
in an excellent trade-off between propagation delay and area but at the price
of output voltage. (At this point the designer might want to consider
alternate designs for later stages that would make the value of output
voltage acceptable).

To test the effect of starting value on the optimization we used the
weight plane from the fourth run but used the final solution of the fifth
optimization as the starting point. Using these values of weight and
starting point we found a solution close to that of run four (the difference
could be due to the optimization, or more likely slight nonconvexities in
the noninferior surface) but with considerably fewer function evaluations.
Further computational experience [20] has indicated that a starting point
in the plane defining the weights for a particular run tends to decrease
the computational expense of generating new noninferior solutions.

Next we choose a weight based upon the results of runs one, three,
and five. The results of this run showed good values for delay and voltage
but a high value for area. Thus we experience the typical complaint with

weighted sum minimization: the inability to choose weights to get new, good trade-offs between all the objectives.

The last two optimizations carried out with the ℓ_1 method used a weight based upon runs two, four, and seven. Run eight terminated abnormally. Run nine using the same weight as run eight but starting from (8.5, 165, -1) also terminated abnormally, although at a different stopping. The results of these two runs (and others 20) have two possible explanations: first, we have detected a nonconvexity on the noninferior surface and the ambiguity of solution is a result of this nonconvexity. Second, we could have found a region of the noninferior surface that is a plane segment parallel to the weight plane thus admitting multiple solutions to the ℓ_1 minimization and causing computational difficulties. Either one of these problems can be circumvented by going to a higher norm - as will be seen below.

V.2 Problem 2

We now use a weighted ℓ_∞ method to generate noninferior solutions to the NAND gate. First we generate boundary points using an ℓ_∞ method. (Usually we start with $\gamma = 0$, however in run two we required $\gamma = 2258 = \phi_2(x_o)$. The second boundary point was more expensive using ℓ_∞ than ℓ_1 methods, but the remaining boundary searches cost the same. The ℓ_∞ results are reported in Table 7. The next optimization was performed using a weight that was the inverse of the centroid of boundary solutions. The result of this run was remarkably close to the result obtained using the ℓ_1 method though the ℓ_∞ method was less expensive. Comparing the fourth ℓ_1 and ℓ_∞ runs gives insight into the convexity and shape of the feasible region in the vicinity of this noninferior solution.

Next in order to compare the computational expensive for finding a particular noninferior solution using ℓ_1 and ℓ_∞ we used the result of the fifth ℓ_1 run (Table 6) to generate the weight for the fifth ℓ_∞ run.

	WGHT			F^*			x^*			ITER	NF	NG
1	1	0	0	56.18	2500	.7	15.46	226.74	-1	5	6	6
2	0	1	0	110	1277.22	.7	7.896	115.84	-1	5	7	6
3	0	0	1	110	2500	.3536	7.825	236.28	-1	15	16	16
4	.0036	.00016	.5651	88.55	1992.26	.5549	9.775	183.873	-1	•24	25	25
5	.004662	.000205	.3217	78.62	1786.99	.7	11.05	162.07	-1	16	17	17
6	.0036	.00016	.5651	88.43	1998.01	.5541	9.787	183.42	-1	5	6	6
7	.00469	$9.066 \times io-^5$	.7283	78.505	2500	.4979	11.008	232.3	-1	19	20	20
8	,00318	.000207	.5518	103.99	1599.57	.5891	8.33	147.02	-1	23	27	22
9	.00318	.000207	.5518	98.787	1787.166	.5545	8.7618	164.95	-1	6	12	5

$$\underset{\sim o}{x} = 10.6, \quad 209 \quad -1.47$$

$$\underset{\sim o}{\phi} = 86.28 \quad 2258.06 \quad .56423$$

Table 6 Results of weighted sum
optimization of NAND gate

The result of the ℓ minimization was identical to the $\ell_{..}$ result at the cost of one extra iteration thus indicating that the power of ℓ_{∞} methods does not require excessive computational effort.

The sixth run is the same as the fourth except the starting point was the solution to the third run, i.e. x_o = 7.825, 236.28, -1. The results of the sixth run again point out the importance of using a reasonable heuristic for starting point selection in order to reduce computational expense. The seventh and last run uses a weight based upon the eighth and ninth ℓ_1 runs which ran into difficulty. No computational problems were encountered using the ℓ_{∞} method, clearly indicating the practical value of ℓ_{∞} techniques, in particular, and ℓ_p methods, p > 1, in general, for the solution of MCO problems.

The overall conclusion of many I_{∞} optimizations is that the weight selection heuristic, ease of use, and lack of computational difficulties and expense make the weighted l_m method a very attractive tool for the MCO design of electronic circuits. However verification and modification of all ℓ_p methods await the use of MCO techniques on a variety of design problems.

RUN	WGHT			F*			x*			ITER	NF	NG
1	1	0	0	56.97	8500	.7	15.46	226.74	-1	5	6	6
2	0	1	0	110	1277.22	.7	7.899	115.837	-1	12	13	13
3	0	0	1	110	2500	.3536	7.825	236.28	-1	15	16	16
4	.01086	.00048	1.7107	87.87	1997.08	.5579	9.85	184.25	-1	17	18	18 .
5	.01272	.000559	1.4286	78.62	1786.995	.7	11.05	162.07	-1	17	18	18
6	.01086	.0048	1.7107	87.87	1997.08	.5579	9.85	184.25	-1	11	12	12
7	.01017	.000558	1.8182	98.656	1797.5	.552	8.77	165.95	-1	18	19	19

Table 7 Results of minimax optimization of NAND gate.

VI. SUMMARY AND CONCLUSIONS

In this paper we have discussed the concepts of Multiple Criterion Optimization. In particular, noninferior solutions were defined and their significance in engineering design situations discussed. Next the family of weighted p-norms was presented as a way of generating various noninferior solutions to an MCO problem. Based upon the weighted I_p norms a canonic weight for each noninferior solution was defined. The interpretation of this canonic weight allowed reasonable weight selection heuristics to be developed. Finally, the ideas of MCO were applied to .the design of an MOSFET NAND gate circuit.

Based upon the theoretical and computational work reported in this paper, we feel that the techniques of MCO are a natural and long overdue extension to the traditional optimization techniques applied in the design of electronic circuits. With the ideas and concepts of MCO we feel optimization can be made a computationally viable, user oriented, and powerful partner in the design process.

An aspect of circuit design which has not been mentioned is the yield of the design. Many techniques [e.g. 48] have been developed for considering the yield of a design. It would seem natural based upon the ideas of MCO to include yield (i.e. failure rate) as one of the competing design objectives - which it certainly is. Adding yield as an MCO objective has been discussed in [20]. The combined MCO yield problem can be computationally expensive, however, because it is a natural statement of the circuit design problem and because of potential gain both in results and understanding we feel the combination deserves further consideration.

Finally, and most importantly the development of user oriented heuristics for weight selection and the meta-problem of choosing one particular design from among many noninferior designs deserves continuing research. The work done in economics, operations research, etc. should prove a fruitful source of ideas and methods for the development of MCO as a tool for circuit and system design.

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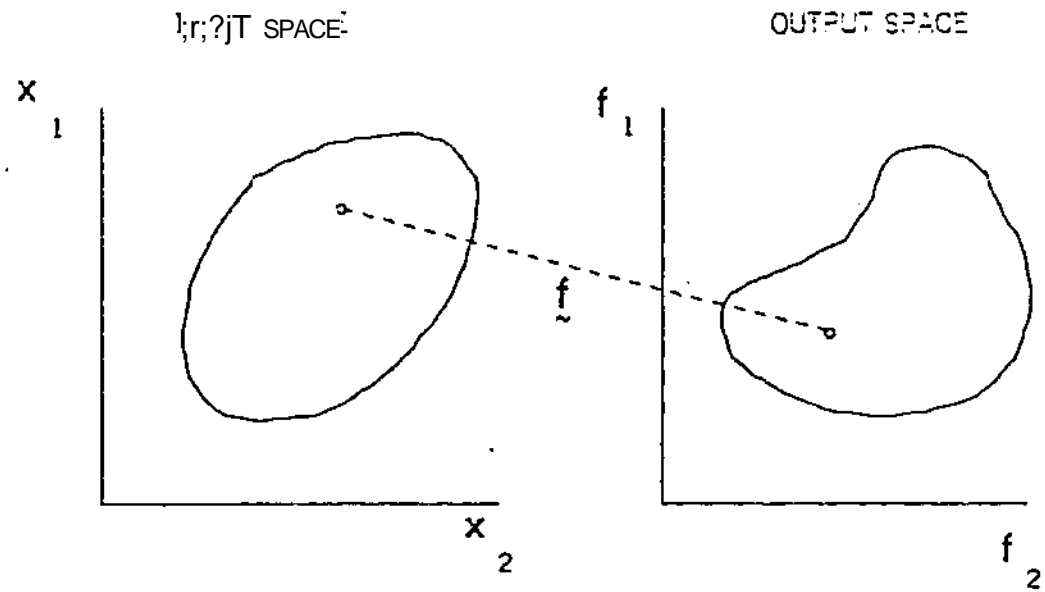
Fig.   1 $\underset{\sim}{f}$ maps input space into output space.
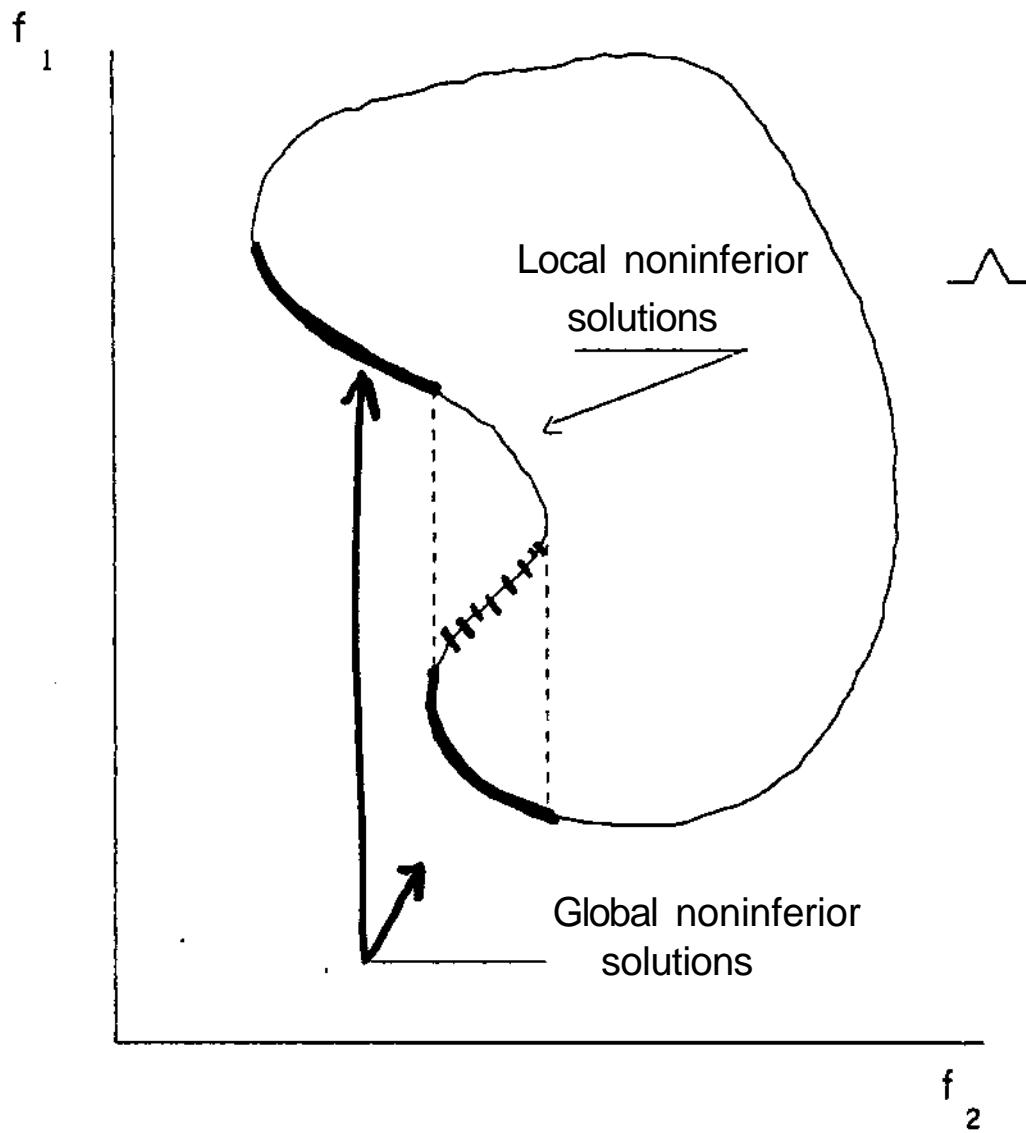
Fig. 2   Global and local noninferior solutions in output space.
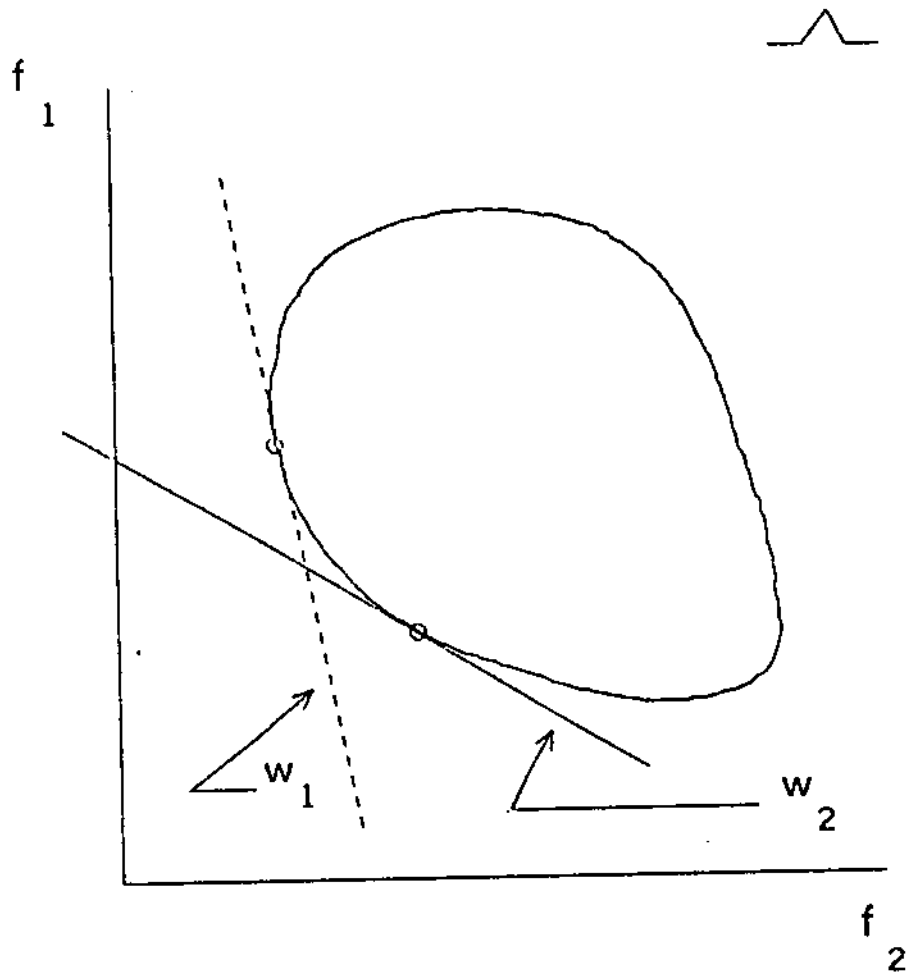
Fig. 3 TWO NONINFERIOR SOLUTIONS FOUND BY CHANGING
THE WEIGHTS IN THE WEIGHTED SUM METHOD.
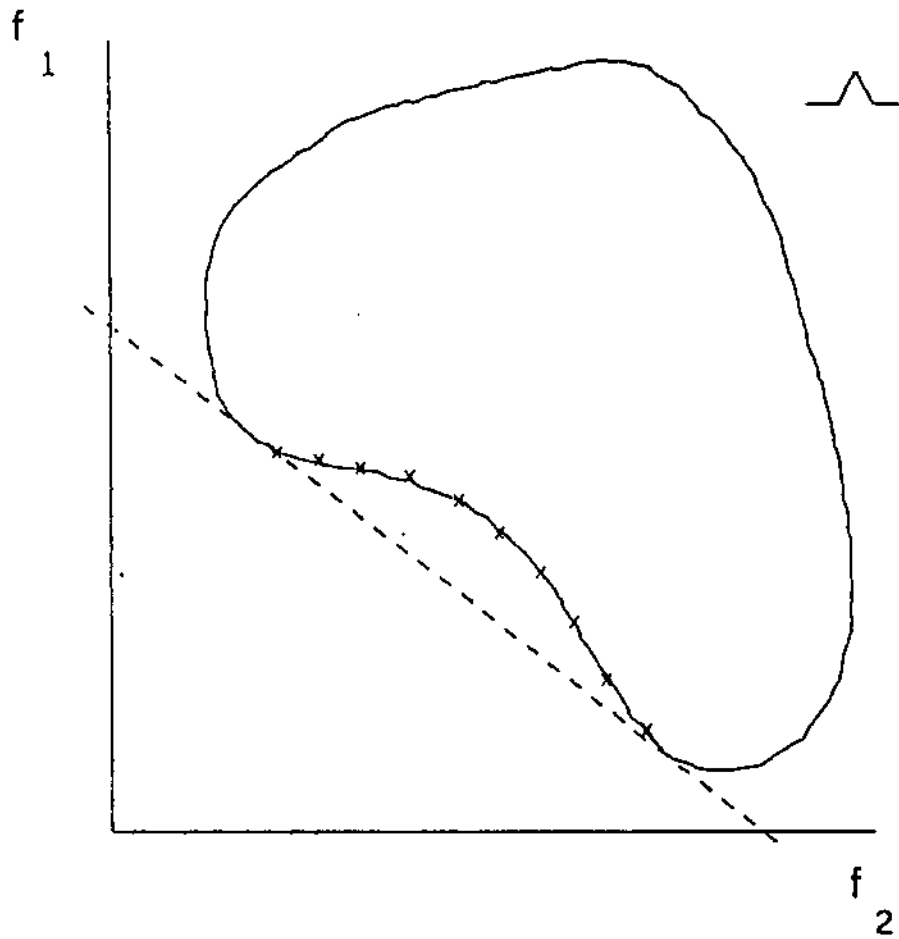
FIG. 4   THE CROSS HATCHED PORTION OF THE NONINFERIOR
          SURFACE CANNOT BE FOUND USING A WEIGHTED SUM APPROACH.
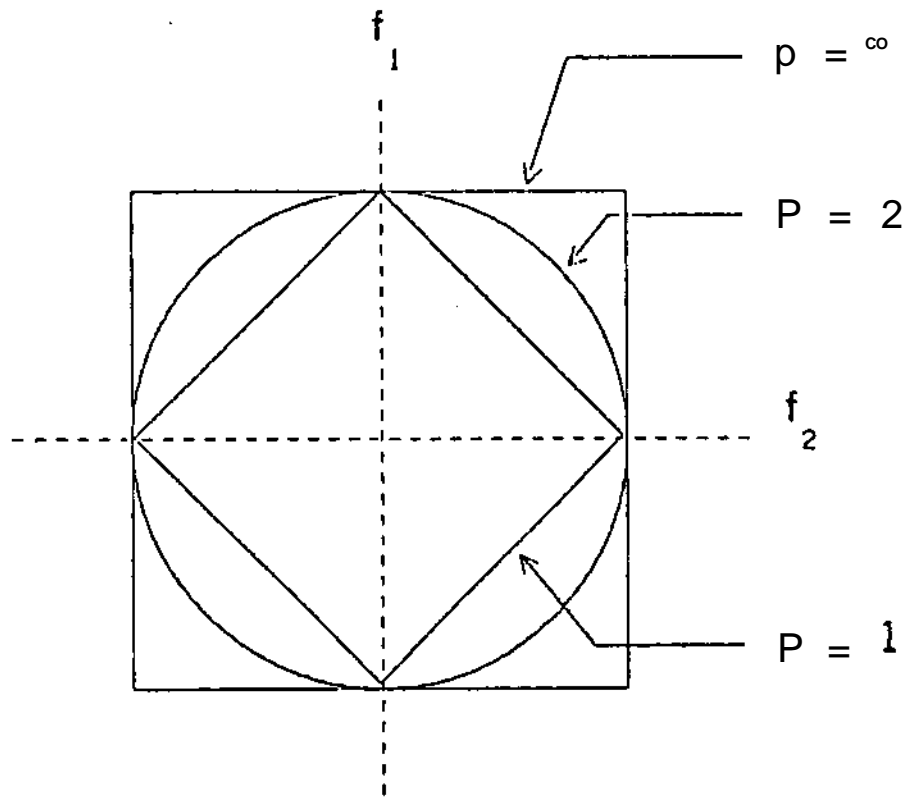
Fig. 5   Level sets of the l-,2-,and a,-norms
with unity weights.
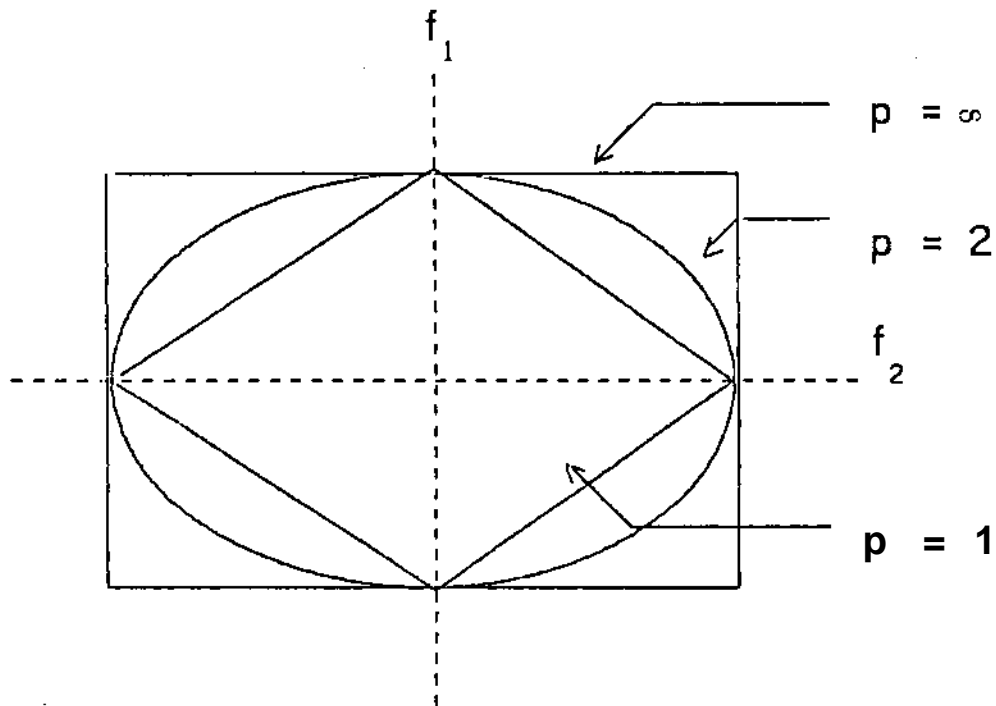
Fig. 6   Level sets of the 1- ,2- and   w -norms
with nonunity and nonequal weights.
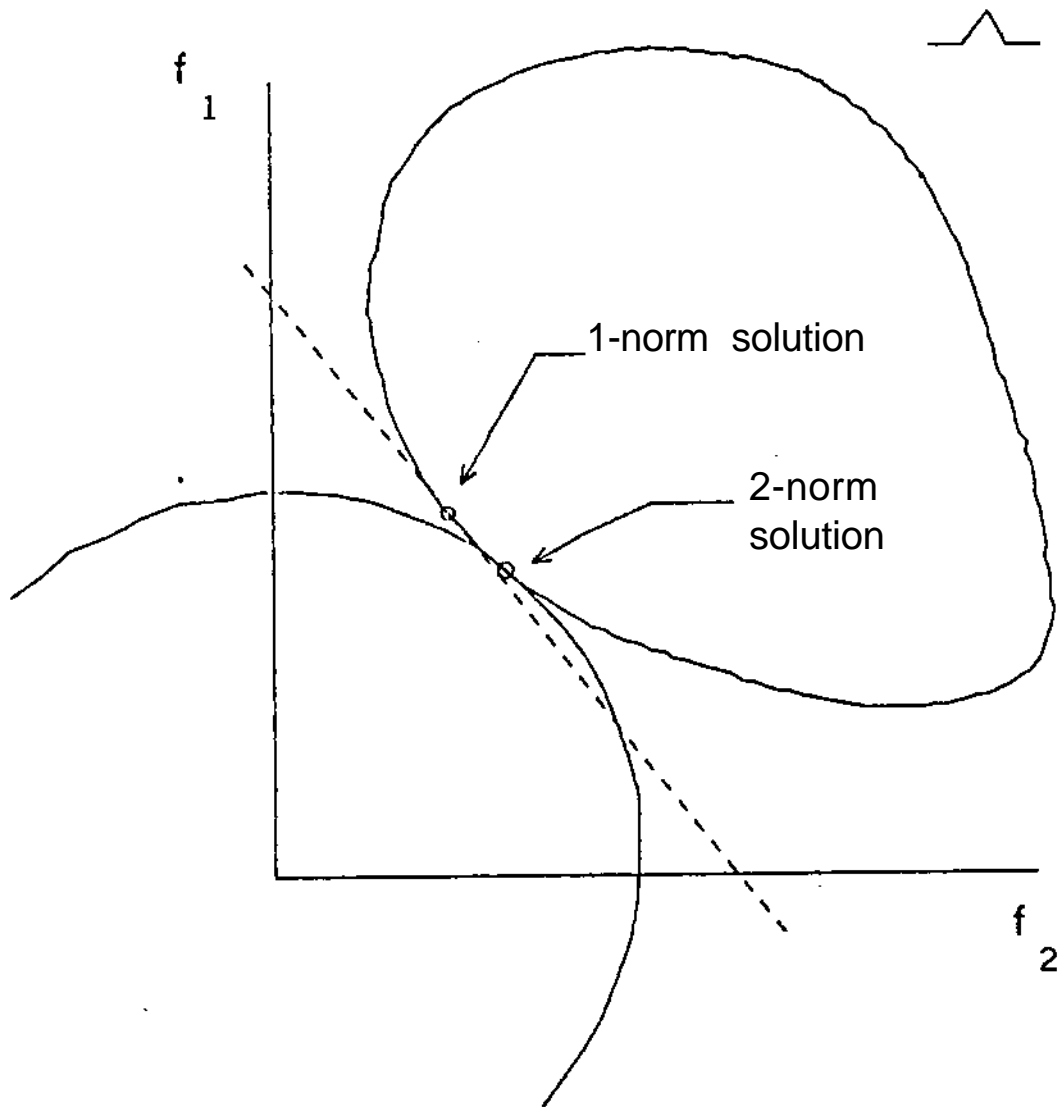
Fig. 7   Noninferior solutions found by minimizing

1- and 2- norms over the feasible region.
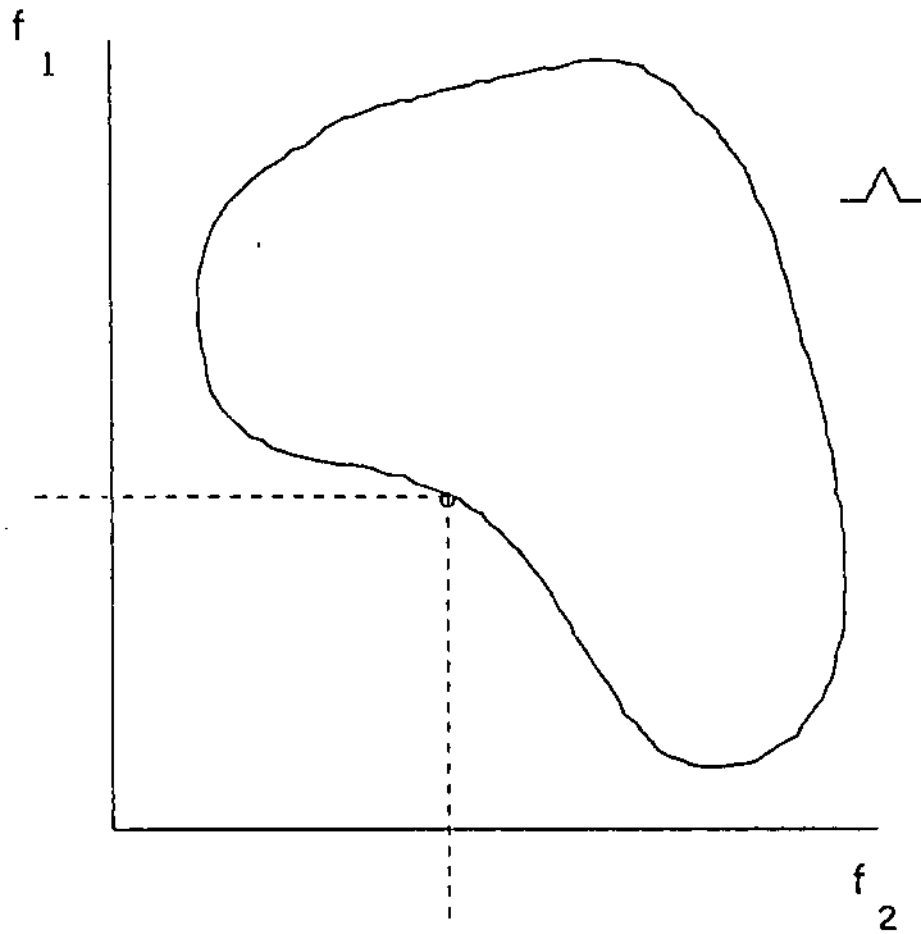
Fig. 8   Noninferior point on nonconvex portion of

noninferior surface found by minimizing max norm.
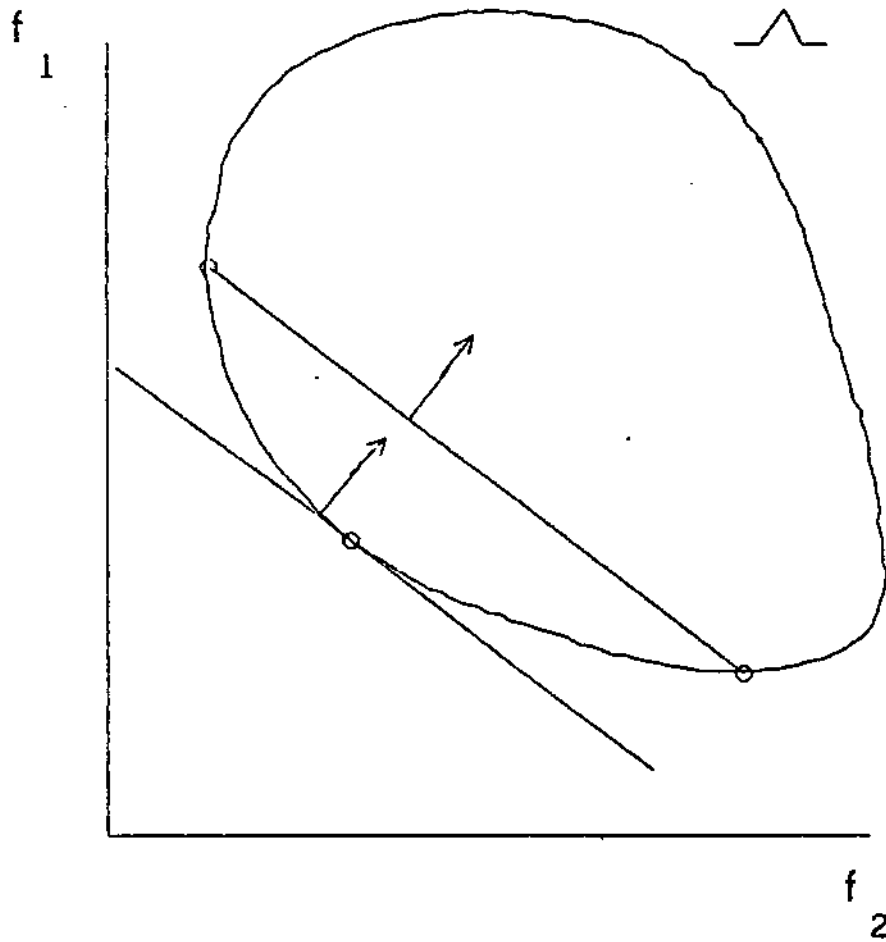
Fig. 9   Weight plane based upon two boundary points and the new noninferior solution found using this weight .
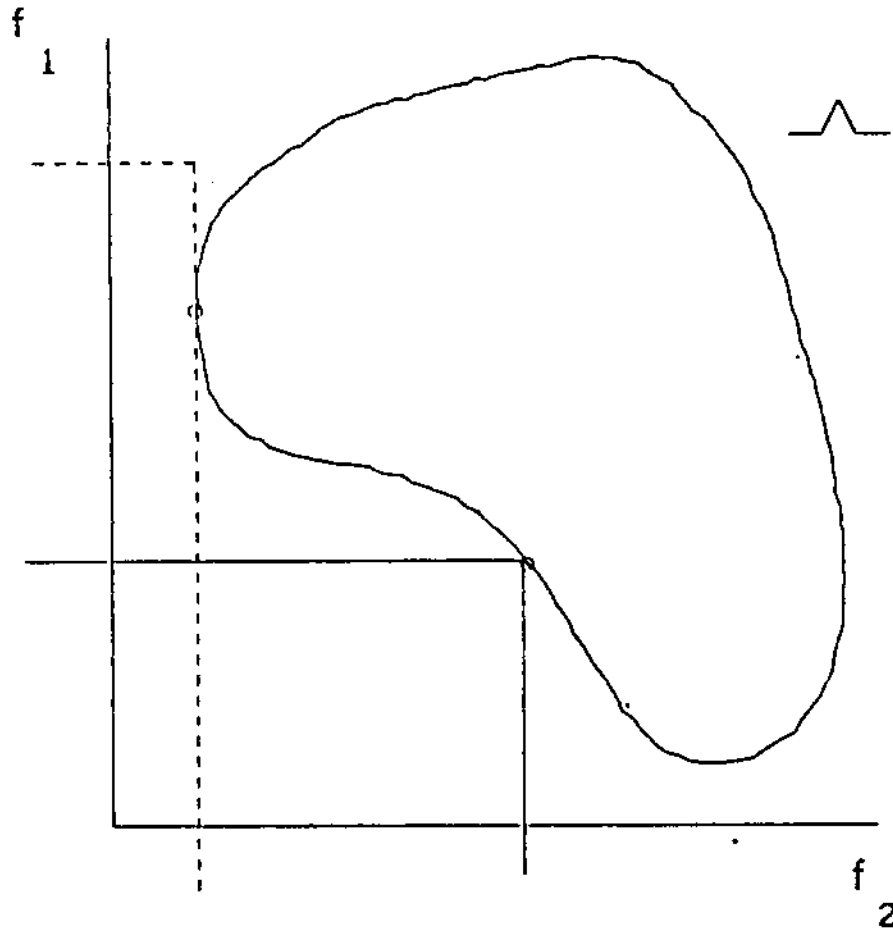
Fig. 10   Noninferior solutions at the comer and at the face of the max norm level set.
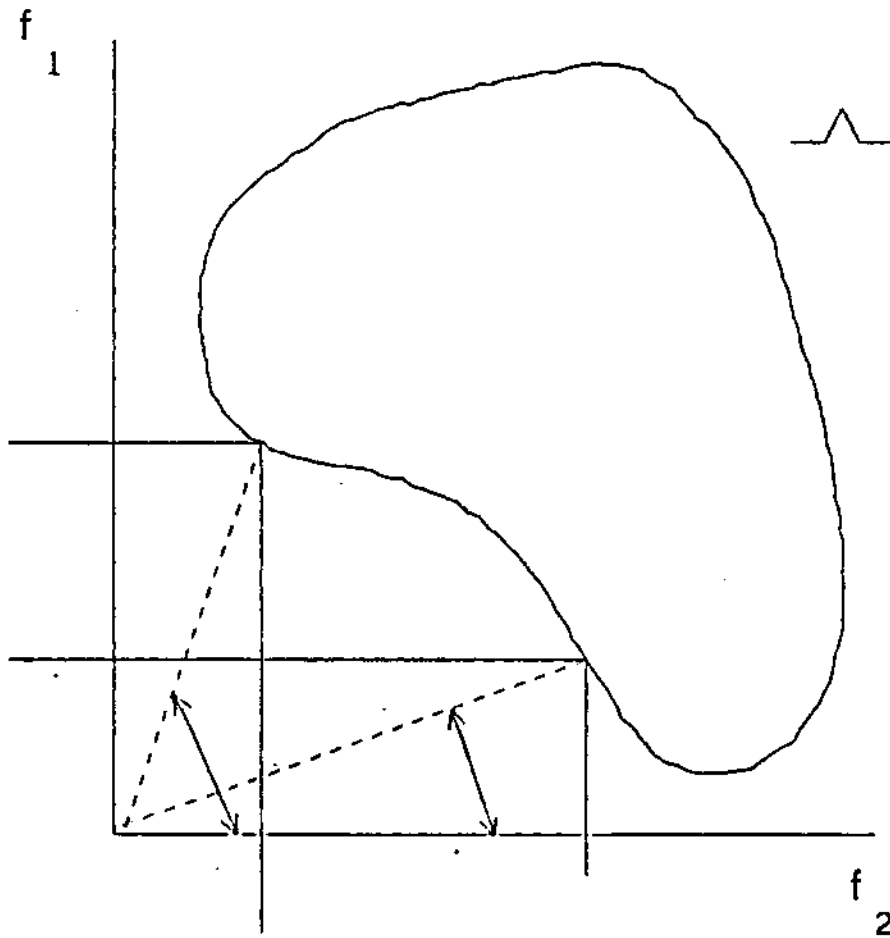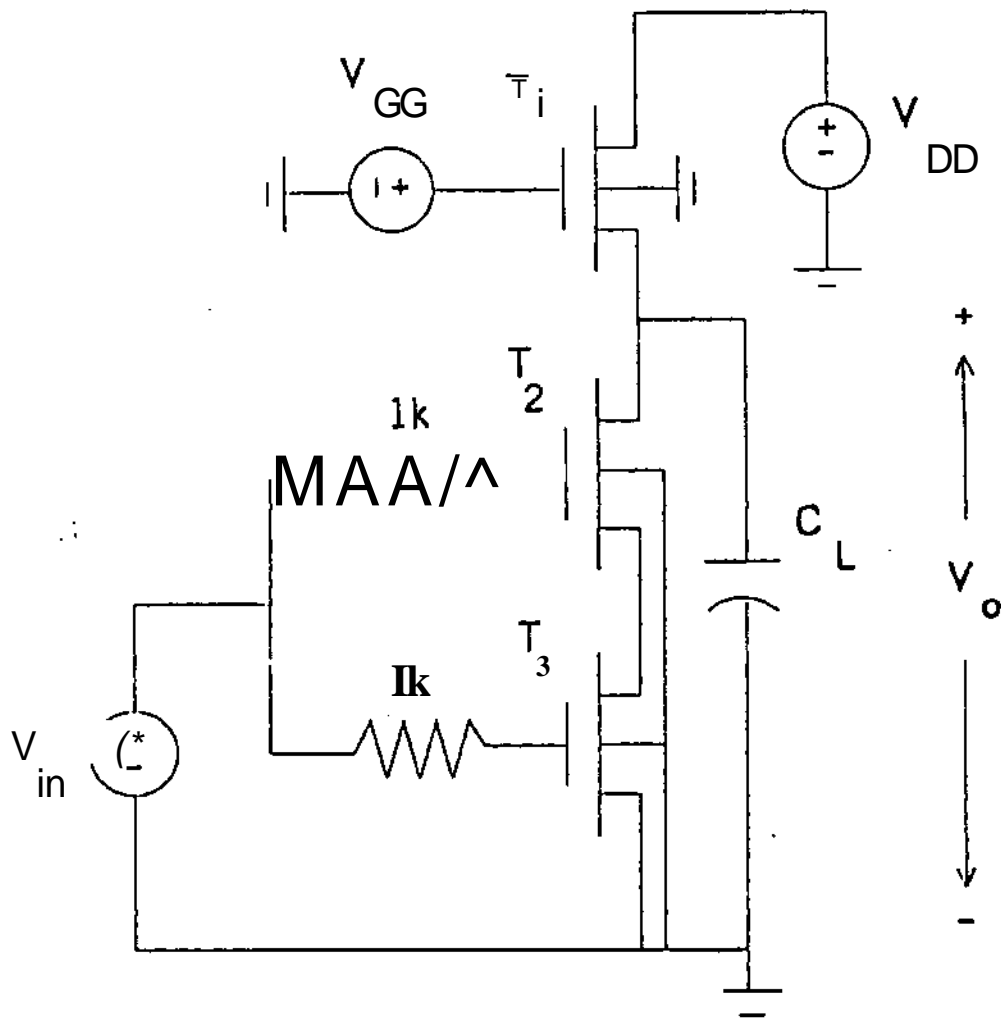
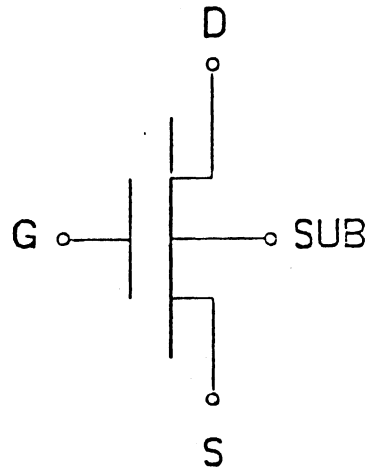Fig. 11   The angles associated with two different

max norm level sets.

Fig. 14   Two input MOSFET NAND gate used in example.

$$JDS=GM*WL*(VGS-VT)^{2} \quad \text{above pinch-off}$$

$$JDS=GM*WL*VDS*(VGS-VT-VDS/2) \quad \text{below pinch-off}$$

$$VT=V_{FB}+k*(VSSUB+PSI)^{.5}$$

where

GM=Normalized transconductance

WL=Width-to-length ratio of the device

VT=Gate threshold voltage

VGS=Gate-to-substrate voltage

VDS=Drain-to-source voltage

VFB=Flat band voltage

k=constant

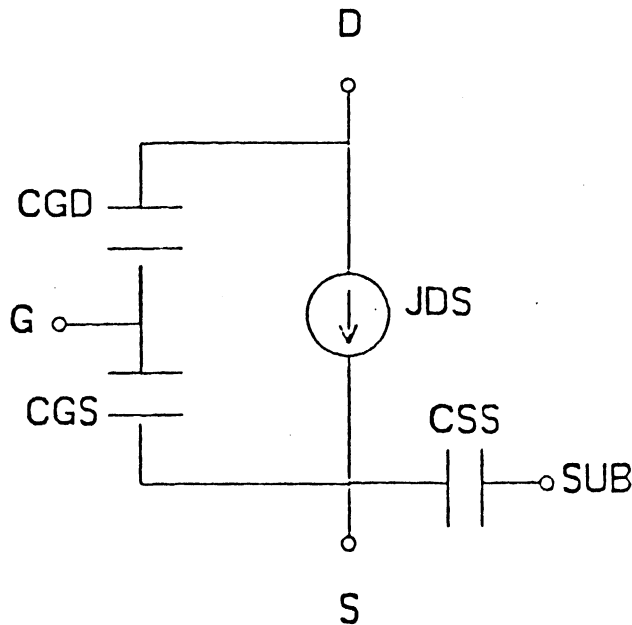PSI=Electrostatic potential on surface at the onset of conduction

VSSUB=Source-to-substrate voltage

Fig. 15   Model of MOSFET device used in example.