MULTIPLE CRITERION OPTIMIZATION WITH STATISTICAL CONSIDERATION

by

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ABSTRACT:

A number of recent papers have described circuit optimization methods in which maximizing yield was the sole design criterion. However, in actual practice there are many competing design criterion such as minimizing power, maximizing speed, area, etc., as well as maximizing yield. In this paper we use the techniques of Multiple Criterion Optimization (MCO) to provide a framework within which to consider all of these objectives simultaneously. Towards this end we develop a new method for estimating yield and the gradient of yield. This method is based upon a combination of the Simplicial Approximation technique of Director and Hachtel and the yield estimation procedure of Bandler and Abdel-Malek. The ideas of MCO and the new yield estimation procedure are applied to the design of a two-input MOSFET NAND gate.

I. INTRODUCTION

During the past ten years concern with the statistical behavior of an electronic circuit design has increased rapidly [1-14]. This increased interest can be traced primarily to the evolution of the integrated circuit industry where the manufacturing (wafer) yield is a prime economic parameter. Furthermore, in integrated circuit design it is generally not possible to specify tighter parameter tolerances in order to increase the yield, thus rendering many of the statistical design techniques developed for discrete circuits invalid for integrated designs. However, yield is not the only concern in integrated circuit design. There are many competing design criteria and in this paper we present a method for considering all the competing design criteria including yield simultaneously.

In addition to yield, LSI design is concerned with such factors as power dissipation, area, and dynamic and steady-state electrical characteristics. We shall call these factors performance criteria to distinguish them from yield. If we did not consider yield we could state the design problem in the form:

$$\min_{\underset{\sim}{x}} \begin{bmatrix} f_1(\underset{\sim}{x}) \\ f_2(\underset{\sim}{x}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_m(\underset{\sim}{x}) \end{bmatrix}$$

$$\text{s.t.} \quad \underset{\sim}{g}(\underset{\sim}{x}) < \underset{\sim}{0}$$

$$\underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{0}$$

where $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ are the various designable parameters, $g(\cdot)$ and $h(\cdot)$ the (nonlinear) circuit and performance constraints and the $f_i(\cdot)$'s the various competing performance criteria. Thus (1) expresses the desire to simultaneously minimize all the performance criterion subject to a certain set of constraints.

Design (optimization) problems of the form (1) are known as Multiple Criterion Optimization (MCO) problems or Multiple Criteria Decision Making (MCDM) problems [15-24]. Techniques for dealing with problems of this form have been developed and successfully applied to the design of electronic circuits [15,16,22]. The main characteristic of these MCO problems is that there is generally no optimum solution, rather we have a set of noninferior [23,24] or optimal trade-off solutions. The final design should be chosen from among this set of optimal trade-off designs. MCO techniques are concerned with the generation of all or certain members of this set of trade-off designs.

In this paper we will consider the MCO problem when yield is one of the competing criterion. The constraints in (1) define the feasible region Ω,

$$\Omega = \{ \underset{\sim}{x} \mid \underset{\sim}{g}(\underset{\sim}{x}) \leq \underset{\sim}{0}, \quad \underset{\sim}{h}(\underset{\sim}{x}) = \underset{\sim}{0} \} \tag{2}$$

The nominal set of parameters is $\underset{\sim}{x}_o$ and these parameters are subject to statistical variation described by a joint probability density function (p.d.f.), $F(\underset{\sim}{x}, \underset{\sim}{x}_o)$, where $\underset{\sim}{x}_o$ is the nominal point and $\underset{\sim}{x}$ the random variable. Thus we have:

<u>Definition</u>: The <u>yield</u>, $Y(\underset{\sim}{x}_o)$, of a circuit whose nominal parameters $\underset{\sim}{x}_o$ have a p.d.f. $F(\underset{\sim}{x}, \underset{\sim}{x}_o)$ and whose feasible region is Ω (2), is given by

$$Y(\underset{\sim}{x}_o) = \int_{\Omega} F(\underset{\sim}{x}, \underset{\sim}{x}_o) d\underset{\sim}{x}. \tag{3}$$

Since our main concern here is the design of integrated circuits, which have a fixed p.d.f. (to first order), our operational definition of yield maximization will be:

Definition Yield maximization is the attempt to increase the yield of a design exclusively through the adjustment of the nominal parameters, $\underset{\sim}{x}_o$.

Therefore the problem we will address in this paper is finding the yield and gradient of yield so as to be able to solve the following MCO problem:

$$\min_{\underset{\sim}{x}_o} \begin{bmatrix} f_1(\underset{\sim}{x}_o) \\ f_2(\underset{\sim}{x}_o) \\ \cdot \\ \cdot \\ \cdot \\ f_m(\underset{\sim}{x}_o) \\ 1-Y(\underset{\sim}{x}_o) \end{bmatrix}$$

subject to

$$\underset{\sim}{h}(\underset{\sim}{x}_o) = \underset{\sim}{0}$$
$$\underset{\sim}{g}(\underset{\sim}{x}_o) \leq \underset{\sim}{0}$$

(Note that minimization of $1-Y(\underset{\sim}{x}_o)$ is equivalent to maximization of $Y(\underset{\sim}{x}_o)$).

In Section II we will discuss techniques for estimating yield, concentrating on techniques due to Director, Hachtel and Brayton [1-4] and Bandler and Abdal-Malek [11-12]. Section III presents our technique for estimating yield which uses techniques from both of the previous methods. In Section IV we apply the techniques of Section III to the MCO design of a MOSFET NAND gate where yield is one of the design criterion. Finally in Section V we present a brief summary and suggestions for future research.

II. 1 SIMPLICIAL APPROXIMATION

Given the feasible region in input space, ft, and a set of m points on the boundary, 8ft, of ft, where

$$3ft » \{x \; | g_{\pm}(x) \leq 0 \text{ for all i and } g_j(x) = 0 \text{ for}$$
$$\text{at least one j; } 1,j\epsilon\{l,2,...,m\}\} \qquad (5)$$

the simplicial approximation, SA, to ft is the convex hull of these points. Specifically the simplicial approximation is defined by

$$SA = \{ x \; | \; n_j^* x \, £ \, b_{\pm} , \; i = 1,2,..., N_F> \qquad (6)$$

where the n_i are outward pointing normals to the bounding hyperplanes defined by the points on the boundary of ft, and the b_i are the distances, in some appropriate norm, between these hyperplanes (or "faces") in the approximation and the origin. Under the assumption that ft is convex and compact

$$SA \; £ \qquad ft \qquad (7)$$

Fig. 1 illustrates the simplicial approximation in two dimension. This approximation derives its name from the fact that each face of the polyhedron is a simplex. Specifics on how this approximation can be generated are given in [1].

Once an adequate simplicial approximation to ft has been generated a variety of statistical design problems can be attacked. Of particular interest here is the use of the simplicial approximation for yield maximization. In order to proceed with this discussion we need the concept of a norm body. In what follows we assume that the probability density function, p.d.f., of the parameters, $F(x,x_o)$ is unimodal and bounded, i.e.

$$0 \leq F(x,x_o) \leq M < \infty$$

where x_o is the nominal point. Let $L_F(a,x_o)$ denote the level set, or level contour, of the p.d.f., i.e.,

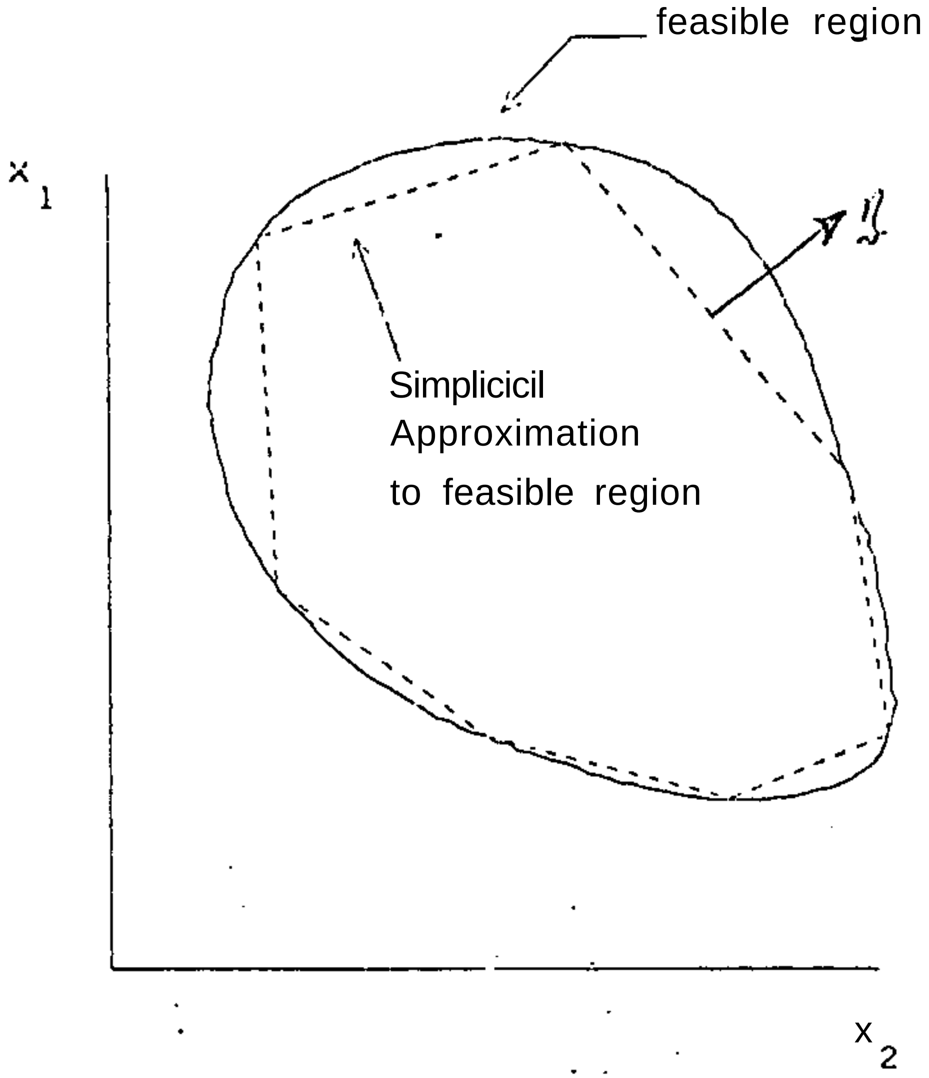


Fig. 1 Simplicial Approximation, SA, to the feasible region.

$$L_F(\alpha, x_0) = \{x \mid F(x, x_0) \geq \alpha,\ 0 \leq \alpha \leq M < \infty\} \qquad (8)$$

Under the above assumptions, $L_F(\alpha)$ is a closed convex body.
For example, if F was an independent Gaussian p.d.f. with equal variance, $L_F(\alpha, x_0)$ would be a circle in two dimensions, a sphere in three dimensions, and in general a hypersphere in n dimensions. Since a norm $n(\cdot)$ can be associated with any closed convex body the level contour defined by (8) can also be written as

$$L(r, x_0) = \{x \mid n(x - x_0) \leq r\} \qquad (9)$$

where the norm has the properties:

i) $n(x) > 0$ for all x

ii) $n(x+y) \leq n(x) + n(y)$

iii) $n(ax) = an(x)$ for $a > 0$

and r is related to α^{-1}.

Note that the norm in the above definition is sometimes called a Minkowski norm [29] because unlike a standard norm, there is no requirement of symmetry about the origin. In this way a much larger class of p.d.f.'s can be described than would be possible with a standard or equilibrated norm [4,29]. In view of (8) and (9) we shall call $L_F(\alpha)$ a norm body or yield body.

In [1] and [3] a procedure for yield maximization was described which was based upon the assumption that the maximum yield point coincides with the center of the maximum yield body which could be inscribed in SA. If this assumption were valid there would be a direct correlation between the size of the inscribed yield body and the actual yield and yield body inscription could form the basis for a greatly simplified procedure for yield estimation. Unfortunately, as we now demonstrate, this assumption is not, in general, valid. Under the assumption the SA is a good approximation to R, we take the yield, at a given nominal, to be

$$Y(\underset{\sim}{x}_o) = \int_{SA} F(\underset{\sim}{x},\underset{\sim}{x}_0)\, d\underset{\sim}{x}, \tag{11}$$

Let $V\{\cdot\}$ denote Euclidean volume, so that $V\{L_F(\alpha,x_o) \cap SA\}$ is the volume of the intersection of the level set $L_F(\alpha,x_o)$ with the approximation SA.

Rewriting (11) as a Lebesgue-Stieltjes integral [14,31,32] yields

$$Y(\underset{\sim}{x}_o) = \int_{SA} F(\underset{\sim}{x},\underset{\sim}{x}_o)d\underset{\sim}{x} = \int_0^M V\{L_F(\alpha,\underset{\sim}{x}_o) \cap SA\}d\underset{\sim}{x} \tag{12}$$

Now define the inner level contour, L_I, as the level contour of the largest yield body with center $\underset{\sim}{x}_o$ which can be inscribed in SA:

$$L_I(x_o) = L_n(r_I, x_o) \tag{13a}$$

where

$$r_I(\underset{\sim}{x}_o) = \max \{r \mid L_n(r,\underset{\sim}{x}_o) \cap SA = L_n(r,x_o)\} \tag{13b}$$

and the outer level contour, L_O, as the level contour of the smallest yield body centered at x which just contains SA:

$$L_O(x_o) = L_n(r_O, x_o) \tag{14a}$$

where

$$r_O(\underset{\sim}{x}_o) = \min \{ r \mid L_n(r,x_o) \cap SA = SA\} \tag{14b}$$

Note that $r_I \leq r_O$. The inner and outer level contours can also be expressed in terms of $L_F(\alpha,\underset{\sim}{x}_o)$:

$$L_I(\underset{\sim}{x}_o) = L_F(\alpha_I , \underset{\sim}{x}_o) \tag{15a}$$

where

$$\alpha_I(x_o) = \min \{\alpha \mid L_F(\alpha,x_o) \cap SA = L_F(\alpha,x_o)\} \tag{15b}$$

and

$$L_0(\underset{\sim}{x}_o) = L_F(\alpha_0, \underset{\sim}{x}_o) \tag{16a}$$

where

$$\alpha_0(\underset{\sim}{x}_o) = \max \{\alpha | L_F(\alpha, \underset{\sim}{x}_o) \cap SA = SA\} \tag{16b}$$

Note that $\alpha_0 \leq \alpha_I$.

With these definitions (12) can be rewritten as

$$Y(\underset{\sim}{x}_o) = \int_0^M V\{L_F(\alpha, \underset{\sim}{x}_o) \cap SA\} d\alpha = \alpha_0(\underset{\sim}{x}_o) V\{SA\}$$

$$+ \int_{\alpha_0(\underset{\sim}{x}_o)}^{\alpha_I(\underset{\sim}{x}_o)} V\{L_F(\alpha, \underset{\sim}{x}_o) \cap SA\} d\alpha$$

$$+ \int_{\alpha_I(\underset{\sim}{x}_o)}^{M} V\{L_F(\alpha, \underset{\sim}{x}_o)\} d\alpha \tag{17}$$

Observe that choosing the nominal point to be the center $\underset{\sim}{x}_I$ of the largest yield body which can be inscribed into SA, is equivalent to minimizing $\alpha_I(x_o)$ which maximizes the third term on the right hand side of Eq. (17). In order to truly maximize yield we should also maximize the first term on the right hand side of (17), i.e. maximize $\alpha_0(x_o)$. This implies choosing the nominal point to be the center, x_0, of the smallest yield body which contains SA, $\underset{\sim}{x}_0$. Only if these two points coincide, i.e. $\underset{\sim}{x}_0 = \underset{\sim}{x}_I$, will either one be the true maximum yield point. Anderson [32] has shown that if the levels sets of $F(\underset{\sim}{x}, \underset{\sim}{x}_o)$ are symmetric about the nominal and if the SA is symmetric about some point $\underset{\sim}{x}^*$, then maximum yield occurs when $\underset{\sim}{x}_o = \underset{\sim}{x}^*$. However, in the general case, the second term on the right hand side of (17) will determine the maximum yield point as a trade-off between $\underset{\sim}{x}_I$ and $\underset{\sim}{x}_0$. In fact, direct differentiation of (17) yields

$$\frac{d}{dx_0}\left(Y(x_0)\right) = \int_{\alpha_0(x_0)}^{\alpha_1(x_0)} \text{Vd}^{\wedge}\text{Cct}_{\tau}\,x^{\wedge}Q\ \text{SA}\}\,da \tag{18}$$

While the yield maximization technique associated with Simplicial Approximation will not satisfy our requirements for an MCO design problem, use of the SA as an approximation to the feasible region is convenient because the general nonlinear constraints associated with the MCO problem could be replaced by a set of linear constraints. This "linearization" of the constraints could greatly reduce the work associated with the constrained optimizations which occur in solving (1). However, this savings must be weighed against the cost of generating a sufficiently accurate approximation. Fortunately for problems with small numbers of statistical parameters (such as IC design where only a few basic physical parameters actually vary) this may be a very useful first step in adding yield as a criterion to the MCO design problem.

II.2 QUADRATIC APPROXIMATION

In this method, proposed by Bandler and Abdel-Malek [11,12], approximations are made to both the feasible region and the yield integral over the feasible region. The method assumes that the p.d.f., $F(x,x_0)$, is truncated or adequately represented by an orthotopic (or hyperrectangular) truncated distribution over a fixed region R, thus

$$\int_R F(x,x_0)\,dx \approx 1$$

$$\int_{R\cap\Omega} F(x,x_0)\,dx = Y(x_0)$$

$$\int_{R-\Omega} F(x,x_0)\,dx = 1-Y(x_0) \tag{19}$$

Eq. (19) is an evaluation of the failure rate and will be denoted by F. To be able to accommodate arbitrary p.d.f.'s Bandler and Abdel-Malek regionalize R into a number of nonintersecting orthotopic cells, R_i, i.e.

$$R = \bigcup_i R_i \qquad (20)$$

$$i=1,2,\ldots,N$$

$$C\backslash R_{\pm} = q() \qquad (21)$$

Next a weight w_i, is associated with each cell. This weight is the probability of a parameter falling into the ith cell, i.e.

$$W_i = P\{x \in R_x\} = \int_{R_i} F(x, x_0)\,dx. \qquad (22)$$

This integral is evaluated by Monte Carlo techniques [25-28] (note that no circuit simulations are involved here) and is invariant with respect to the nominal point x_o.

Next a quadratic estimate of the feasible region, ft, is generated. The approximation is generated dynamically, updated as necessary and only generated in those areas thought to have a high probability of failure. If we denote this approximation to ft by $\tilde{ft}$, an estimate to the failure rate (19) is

$$1-Y(x_o) \approx \int_{R-\tilde{ft}} F(x,x_o)\,dx \approx 1-Y(x_o). \qquad (23)$$

Since evaluation of (23) is difficult a further refinement is made as follows. First, the points aiong the edges of the orthotope R which intersect $\tilde{ft}$ are found and $\tilde{ft}$ is linearized about these points• Let $\tilde{ft}$ denote the linearized approximation to $\tilde{ft}$. Then (23) can be approximated by

$$1-Y(x_o) \approx \int_{R-\tilde{\Omega}_\ell} F(x,x_o)\,dx, \qquad (24)$$

Next, the volume of each orthotopic cell, R^ outside $\tilde{ft}_\ell$ VCR^$\tilde{ft}$^}, can be can he estimated and the following calculation carried out

$$1-Y(\underset{\sim}{x}_o) \approx \int_{R-\tilde{\Omega}_\ell} F(\underset{\sim}{x},\underset{\sim}{x}_o)\,d\underset{\sim}{x} = \sum_i \int_{R_i-\tilde{\Omega}_\ell} F(\underset{\sim}{x},\underset{\sim}{x}_o)\,d\underset{\sim}{x} \approx \sum_i w_i\, v(R_i - \tilde{\Omega}) \qquad (25)$$

The last term of (25) embodies the essence of the Bandler Abdel-Malek method. Notice that because the failure rate (25) has an analytical rather than probabilistic form it is possible to differentiate this expression in order to estimate the gradient of the yield with respect to the nominal point.

The Bandler, Abdel-Malek method has a number of interesting features including a deterministic estimate of yield and its gradient, an approximation of the feasible region, and the capability of handling arbitrary statistical distributions. There are several problems with this method that prevent its direct use in MCO problems. First, the approximations associated with the proposed method (esp, (24) and (25)) are only accurate when the maximum yield point is approached. This leads to questionable usefulness when yield is being traded-off against other criterion. Next, once a particular quadratic approximation to ft is generated, it must be constantly updated as various trade-off solutions are found. This means that the approximations cannot readily be used to replace the nonlinear constraints of the MCO and thus the cost of the approximation cannot be amortized over the entire MCO solution process.

III. A YIELD ESTIMATION PROCEDURE FOR USE WITH MCO

In this section we describe a yield estimation procedure which is useful in an MCO setting. This procedure employs the simplicial approximation as a linearization of the feasible region thus allowing us to replace the nonlinear constrained optimizations required for MCO by linearly constrained optimizations. (In general, we replace most of the nonlinear constraints by linear constraints, see Sec. IV. and [15,16]). Thus there is a significant reduction in the cost of the optimization required in generating trade-off solutions. Further, we adopt the Bandler, Abdel-Malek idea of partitioning the p.d.f. over a region. This scheme allows the generation of closed form estimates of the yield which can be differentiated to obtain closed form estimates of the gradient of the yield.

By construction, each face of the approximation, SA, is an (n-1) dimensional simplex. Thus, as shown in Fig. 2, each nominal point $\underset{\sim}{x}_o$, interior to SA, induces a unique interior simplicial decomposition of SA. Let us consider various methods of approximating the yield using this decomposition.

The yield over the ith interior simplex, is

$$Y_i(\underset{\sim}{x}_o) = \int_{SA_i} F(\underset{\sim}{x},\underset{\sim}{x}_o)\,d\underset{\sim}{x} \tag{26}$$

where $SA_i(\underset{\sim}{x}_o)$ is the ith component of the interior simplicial decomposition induced by $\underset{\sim}{x}_o$. One approach to estimating (26) is to form a piecewise linear approximation of the integral. Generically, this approximation would have the form

$$Y_i(\underset{\sim}{x}_o) \approx V\{SA_i(\underset{\sim}{x}_o)\}\left\{\sum_{k=1}^{\ell} d_k F(\underset{\sim}{x}_{ik},\underset{\sim}{x}_o)\right\} \tag{27}$$

where $V\{SA_i(\underset{\sim}{x}_o)\}$ is the Euclidean volume of the ith segment, $F(\underset{\sim}{x}_{ik},\underset{\sim}{x}_o)$ the value of the p.d.f. at the point $\underset{\sim}{x}_{ik}$ and d_k is a weight. The gradient of (27) could serve as an approximation to the gradient of yield:

$$\frac{\partial Y_i(\underset{\sim}{x}_o)}{\partial \underset{\sim}{x}} \approx V\{SA_i(\underset{\sim}{x}_o)\}\left\{\sum_{k=1}^{\ell} \nabla_{\underset{\sim}{x}_o} d_k F(\underset{\sim}{x}_{ik},\underset{\sim}{x}_o)\right\} + \nabla_{\underset{\sim}{x}_o} V\{SA_i(\underset{\sim}{x}_o)\}\left\{\sum_{k=1}^{\ell} d_k F(\underset{\sim}{x}_{ik},\underset{\sim}{x}_o)\right\}, \tag{28}$$

Upon examination of (27) and (28), it is seen that in order to efficiently estimate yield and its gradient over the ith region we must efficiently evaluate

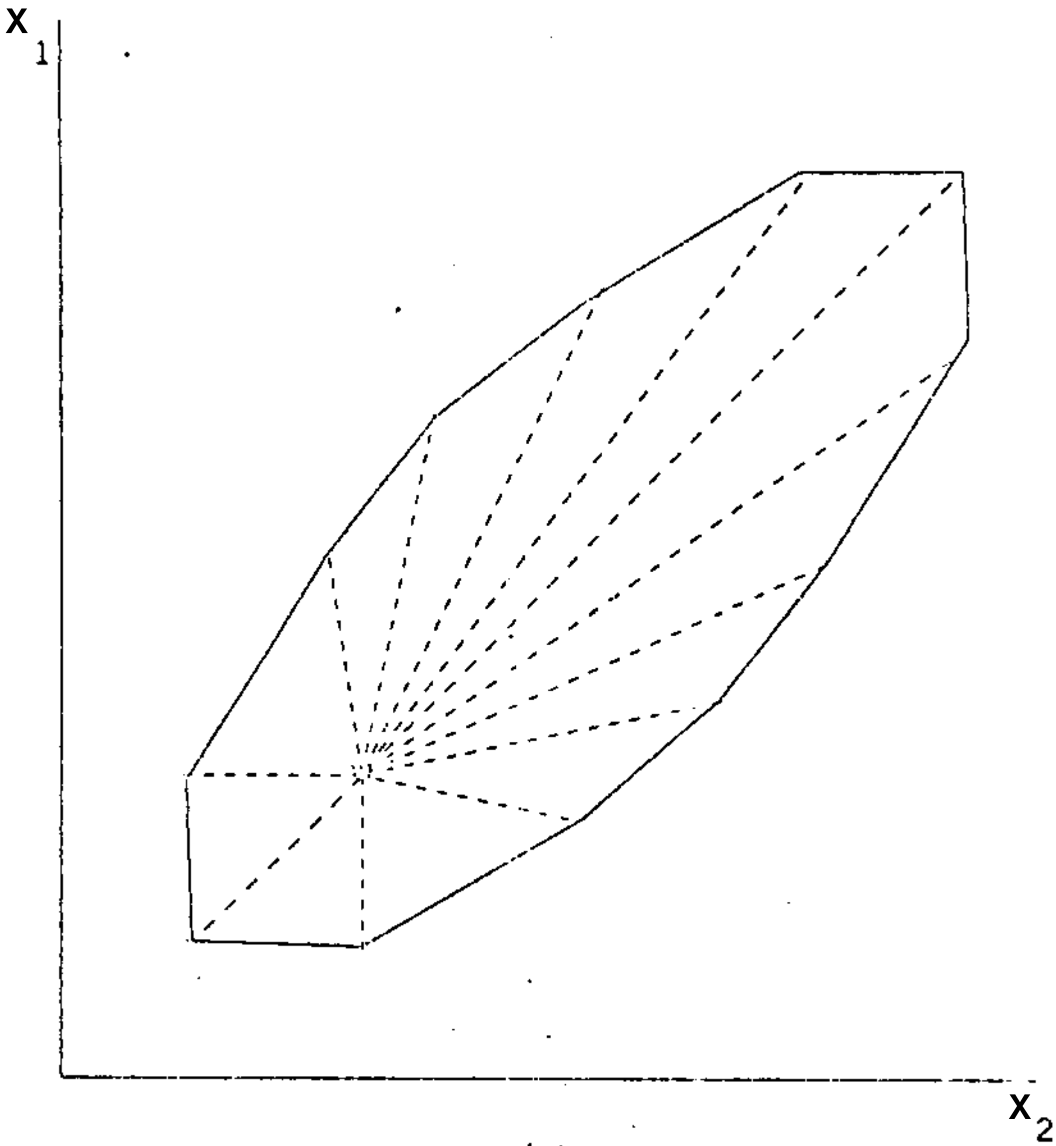


Fig. 2 An interior simplicial decomposition of the Simplicial Approximation induced by a nominal point.

$V\{SA_i(x_o)\}$ and $\nabla V\{SA_i(x_o)\}$. Let X_i be a matrix whose columns are the coordinates of the vertices of $SA_i(x_o)$ exclusive of x_o. This matrix is independent of the nominal. Observe that the volume of a unit simplex which has one vertex at the origin and the remaining vertices at unity on each coordinate is $1/n!$. By using X_i and x_o to form the affine transformation that takes the unit simplex into $SA_i(x_o)$ it can be shown that

$$V\{SA_i(x_o)\} = \frac{1}{n!}\,|\det(X_i)|\,\left|1 - x_o^T X_i^{-T} e\right| \tag{29}$$

where n is the number of designable parameters, e is a vector with all ones and $\det(X_i)$ is the determinant of X_i. The gradient of (29) is

$$\nabla_{x_o} V\{SA_i(x_o)\} = \frac{1}{n!}\,\mathrm{sgn}[\det(X_i)\,\{1-x_o^T X_i^{-T} e\}]\,(-X_i^{-1} e) \tag{30}$$

where sgn is the sign function. Thus to evaluate (29) and (30) we need to find $X_i^{-1} e$ and $\det(X_i)$. This can be easily done for each X_i by solving

$$X_i y = e$$

or by LU factorization

$$L_i U_i y = e \tag{31}$$

$$y = U_i^{-1} L_i^{-1} e = X_i^{-1} e$$

$$\det(X_i) = \prod_{i=1}^{n} u_{ii}$$

where L_i and U_i are determined by triangular factorization of X_i. Note that (31) is a preprocessing step that only needs to be done once for each X_i. Also as we progress from the ith face to its neighbor, only one column of X_i changes and so further economies are possible in these calculations (for similar ideas see [33]).

Let us examine (29) and (30) and relate them to the information contained in the Simplicial Approximation. The term

$$\backslash \quad \backslash$$

corresponds to a normal to the ith face assuming the ith face is one unit from the origin. Therefore

$$1-x^T X^{-1} e$$

is simply the distance of the nominal point to the ith face and, assuming we have a SA, is the same as

$$b_i - x_o^T n_i$$

where n_i is the normal to the ith face of the SA and b_i the corresponding right hand side. Therefore, the only ~~extra~~ work required to find (29) and (30) above that of finding the SA, is the calculation of $\det(X_i)$.

Greater accuracy can be obtained in approximating (26) if each interior simplex is further broken into a number of segments by cutting it with hyperplanes parallel to the included face of the SA. The integral of the yield over each segment can be found by using a piecewise linear estimate of the p.d.f• over the segment. Such a division is shown in Fig. 3. By defining

$$\xi = x + \frac{k}{\ell} (x - x) \qquad \begin{matrix} m=1,\ldots,n \\ k=1,\ldots,\ell \end{matrix} \tag{22}$$

where ℓ is the number of segments desired and x_m, m=l, ..., n, are the vertices of SA_i, the estimate of the yield over the ith simplex becomes

$$Y_i(x_o) = \frac{1}{\ell^n} V\{SA_i(x_o)\} \frac{1}{n+1} \cdot \{F(x_o,x_o) + \sum_{k=1}^{n} F(\xi_{ik\ell},x_o)\} \tag{33}$$

$$+ \sum_{J=2}^{\ell} \frac{j^n-(j-1)^n}{\ell^n} V\{SA_i(x_o)\} \cdot \frac{1}{2n} \left\{ \sum_{k=j-1}^{j} \left[\sum_{m=1}^{n} F(\xi_{imk},x_o) \right] \right\}$$

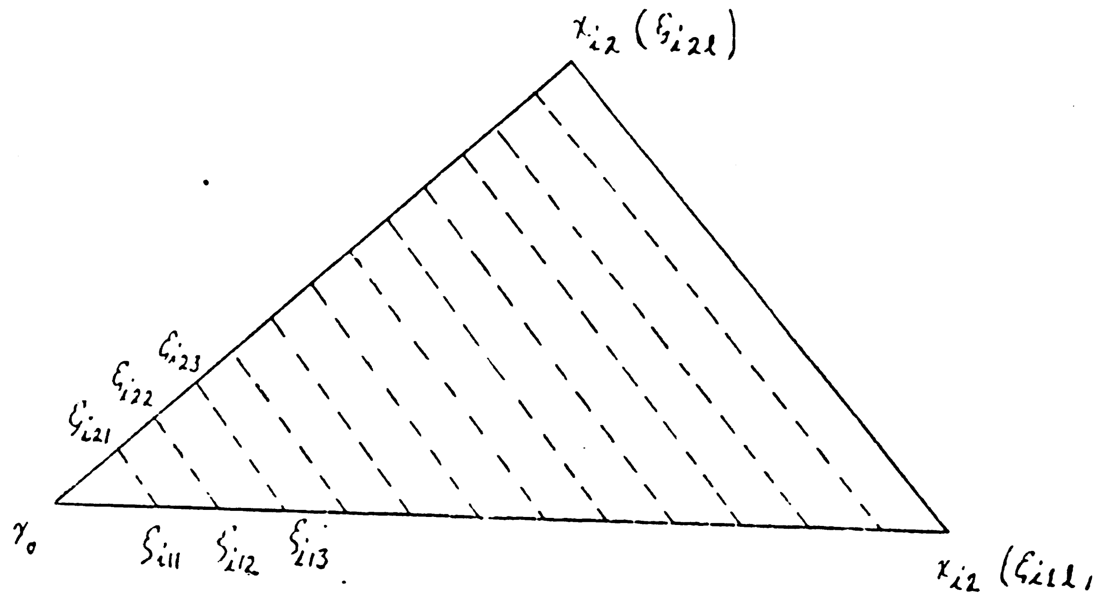


Fig. 3   Division of the ith interior simplicial.

The gradient of the yield over the ith interior simplex is estimated by taking the gradient of (33), i.e.

$$
\begin{aligned}
\nabla_{x_o} Y_i(x_o) = {} & \frac{1}{\ell^n} V\left\{SA_i(x_o)\right\}\left(\frac{1}{n+1}\right)\left\{ \nabla_{x_o} F(x_o,x_o) + \sum_{k=1}^{n} \frac{\partial F(\xi_{ik\ell},x_o)}{\partial x_o} \right\} \\
& + \frac{1}{\ell^n} \nabla_{x_o} V\left\{SA_i(x_o)\right\} \frac{1}{n+1} \left\{ F(x_o,x_o) + \sum_{k=1}^{n} F(\xi_{ik\ell},x_o) \right\} \\
& + \sum_{j=2}^{\ell} \frac{j^n-(j-1)^n}{\ell^n} V\left\{SA_i(x_o)\right\} \frac{1}{2n}\left\{ \sum_{k=j-1}^{j} \left[\sum_{m=1}^{n} \frac{\partial F(\xi_{imk},x_o)}{\partial x_o} \right] \right\} \\
& + \sum_{j=2}^{\ell} \frac{j^n-(j-1)^n}{\ell^n} \nabla_{x_o} V\left\{SA_i(x_o)\right\} \frac{1}{2n}\left\{ \sum_{k=j-1}^{j} \left[\sum_{m=1}^{n} F(\xi_{imk},x_o) \right] \right\} .
\end{aligned}
\tag{34}
$$

Note that (34). is the exact gradient of (33), Further, since the computational effort of calculating $V\{SA_i(x_o)\}$ has been reduced to an inner product, (see (29)) and $\nabla_{x_o} V\{SA_i(x_o)\}$ to checking a sign (see (30)), the major effort in evaluating (33) and (34) is the computation of $F(x,x_o)$ and $\nabla_x F(x,x_o)$ at various points over $SA_i(x_o)$. Summing (33) and (34) over all interior simplices gives the estimates of yield and gradient of yield for the simplicial approximation.

For a particular simplicial approximation (Fig. 4) we compare a 1000-sample Monte Carlo estimate of yield to the method described above using 10 segments per interior simplex. For a Gaussian p.d.f. with equal variance and no correlation, the results are shown in Table 1. Comparisons are made for two different standard deviations and for several nominal points through the approximation.

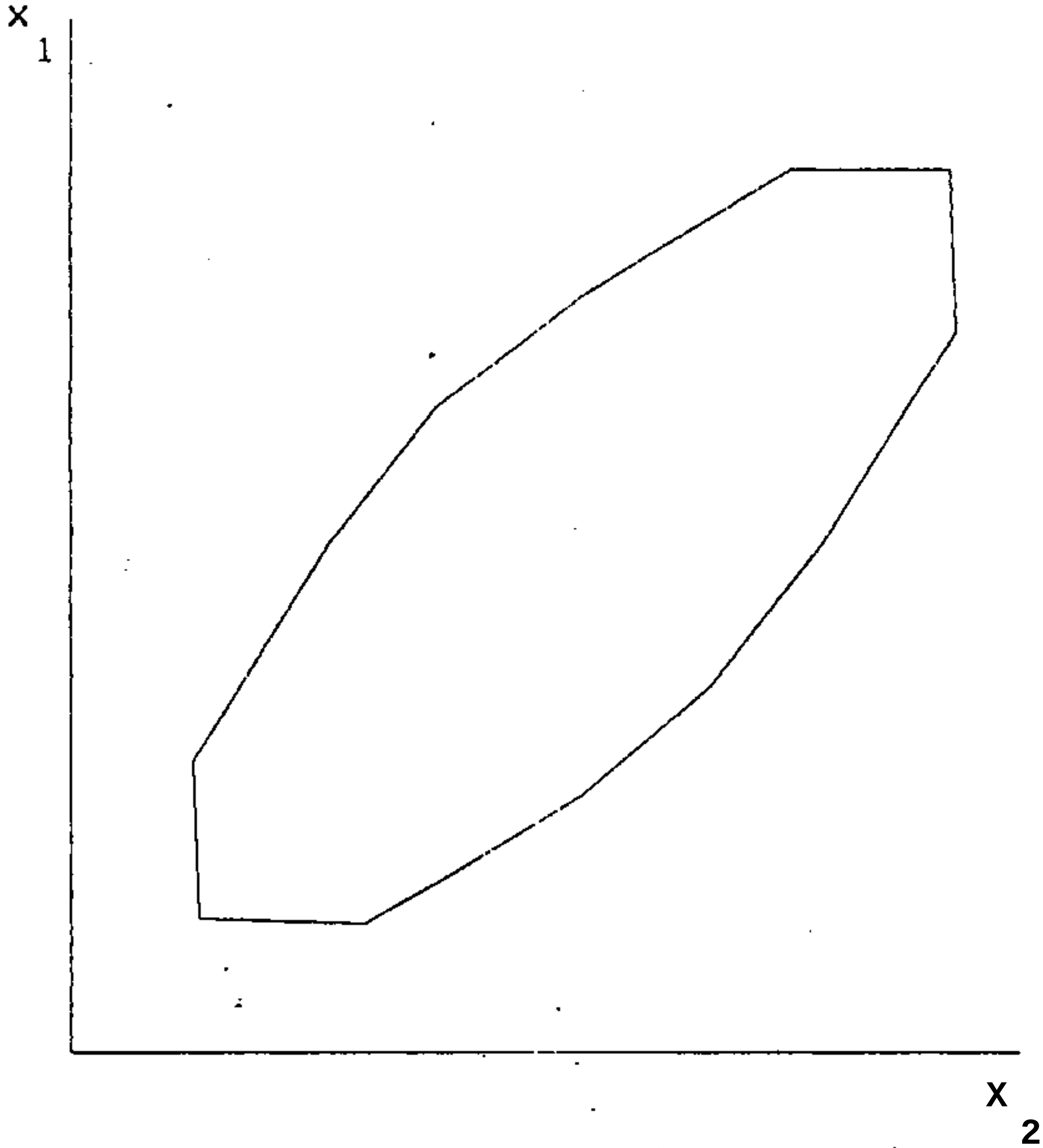


Fig. 4 The Simplicial Approximation used in the comparison of the piecewise linear and Monte Carlo estimates of yield.

Table 1 Comparison of piecewise linear approximation of yield using 10 divisions of each interior simplex with a 1000 sample Monte Carlo over the Simplicial Approximation.

PL Approximation	Monte Carlo (1000 Samples)	Confidence Interval of Monte Carlo
Standard Deviation -- 1		
37.68	42.2	±4
46.78	48.8	± 4.1
96.44	94.9	± 1.8
93.22	93.5	±2
90.24	91.8	± 2.2
Standard Deviation = 2		
26.97	29.1	±3-7
42.9	44.9	±4-1
62.47	63.7	± 4
56.61	56.2	[illegible] *
61).62	60.8	±4

Since much of the effort in the above method for estimating yield and its gradient is preprocessing and hence only need be done once, and because the Simplicial Approximation reduces the work involved in solving constrained optimization problem, we feel that this is a computationally viable approach for use when adding yield as an objective function in a Multiple Criteria Optimization circuit design problem.

IV. MCO OF A MOSFET NAND GATE INCLUDING YIELD

We now apply the yield estimation techniques discussed in Section III to the MCO of the two input MOSFET NAND gate shown in Fig. 5. The circuit we considered has been used as a time domain optimization example by Director and Brayton [34], an example for Simplicial Approximation by Director and Hachtel [1], and as an example for MCO design by Fraser [22] and Lightner and Director [15,16]. The MOS model, shown in Fig. 6, includes the effect of substrate bias.

The designable parameters are the width W_{23} of the bottom two transistors (constrained to be the same), the width of T_1, W_1, and the flat band voltage, V_{FB}. Table 2 presents the range of these parameters as well as other constants needed to analyze the NAND gate.

The objectives are: to minimize the area used by the transistors, to minimize the switching time of the gate, to require the ON voltage V_o to be as close to zero as possible, and to maximize the yield.

The switching time of the gate is a function of the time it takes for the circuit to turn ON and the time it takes the circuit to turn OFF. But the turn OFF time is much larger than the turn ON time and thus we can consider minimization of the turn OFF time - turn OFF propagation delay t_{PD} - instead of the entire switching cycle. In general, the evaluation of t_{PD} will require a transient analysis, but for the class of MOSFET gates we are

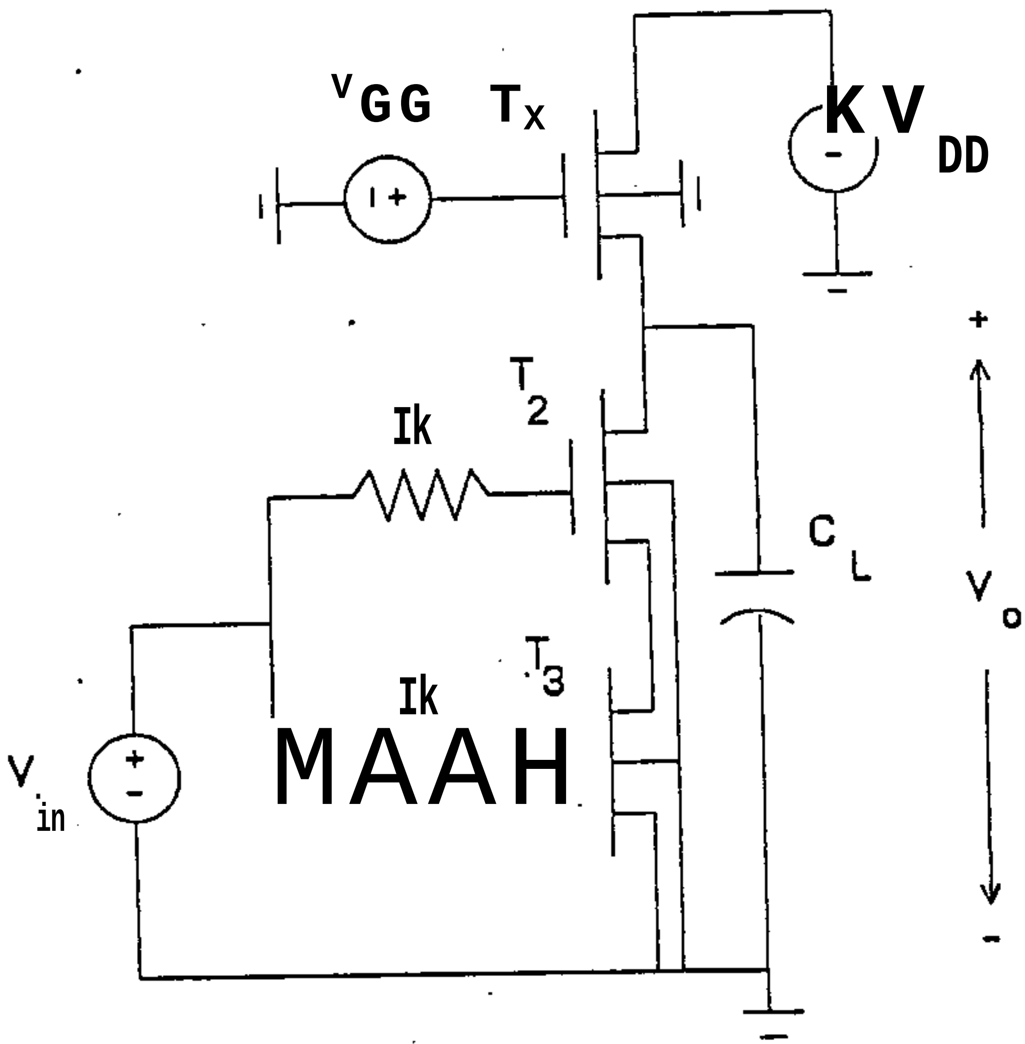
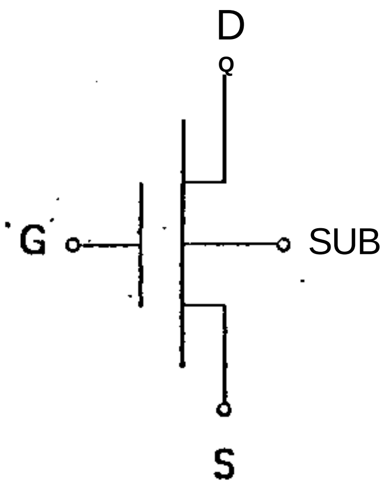


Fig. 5 Two input MOSFET NAND gate used in example.



JDS=GM»WU(VGS-VT)^{2} above pinch-off

JDS=GM.WL*VDS*(VGS-VT-VDS/2) below pinch-off

VT=V_{CQ}+k*(VSSUB+PSI)$^{.5}$
ro

where

GM=Normalized transconductance

WL»Width-to-length ratio of the device

VT=Gate threshold voltage

VGS-Gate-to-substrate voltage

VDS=Drain-to-source voltage

VFB=Flat band voltage

k-constant

PSI=Electrostatic potential on surface at the onset of conduction

V5SUB^Source-to~substrate voltage

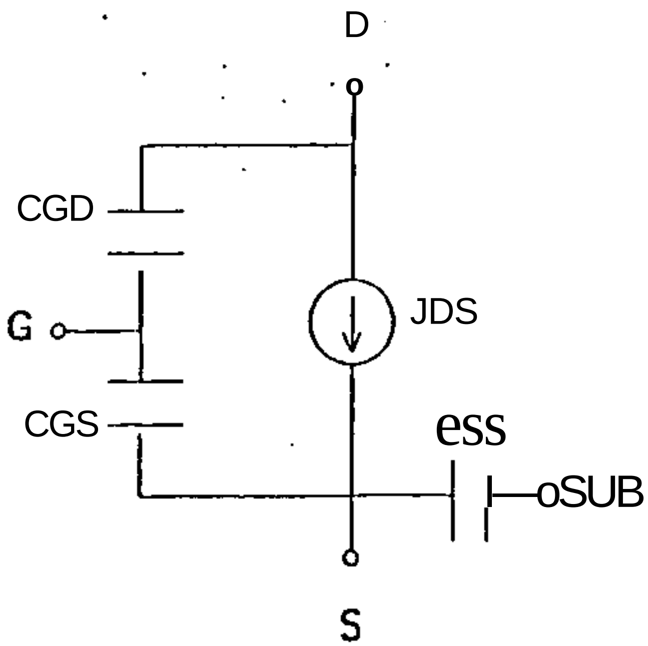


Fig. 6 Model of MOSFET device used in example.

Table 2   Parameter values, constants, and constraints used in the MOSFET NAND gate example.

PSI=.5771

k=.5

GM=.006

L_1=12.7 microns

L_{23}=5.08 microns

VDD=-6.5 volts

VGG=-14.5 volts

Vin=-6 volts (in ON state)

A_f=1.03657

C_L=5 pF

$-2 \le V_{FB} \le -1$

$5 \le W_1 \le 50$

$50 \le W_2, W_3 \le 250$

concerned with an approximation to t_{PD} does exist. This approximation is based upon the assumption that the output node is dominated by a single load capacitance (independent of voltage), and is adequate for static MOSFET logic. Fraser [22] developed this approximation assuming that the lower transistors are out of the circuit and the ON state value of the output voltage was zero. The approximation is

$$*PD = *F \; \frac{L_1}{\bar{V}T_1} \; Ta \qquad (35)$$

where

$$a = -Ap- \ln \hat{}=1 \quad \begin{matrix}(m-1) & 3m-4\end{matrix} \qquad (36)$$

and

$$\tau = \frac{}{Gm(V}$$

$$m = \frac{V_{DD}}{(V_{GG}-V)} \qquad (37b)$$

with A_F, a multiplicative constant used to match (35) with the delay found using an accurate transient simulation (more informally A_F is known as a Skinner constant [37]). Using (35)-(37), the turn OFF propagation delay can be approximated (to first order) without performing a transient circuit simulation (except once to estimate A_F).

Thus, in order to evaluate the objectives of our design we simply need to perform a d.c. analysis of the NAND gate in the ON state and evaluate the yield by the method of Section III. The objective functions for the NAND gate are

$$\bullet\ l \ ' \wedge \ ' \wedge \frac{1}{W_1} \ Ta \ \text{(eqn. 35)} \qquad (38)$$

$$*2 = V \ L_1 + 2 \ L_{23} \# W_{23} \ \text{(area)} \qquad (39)$$

$$\Phi_3 = -V_o \ \text{(ON output voltage)} \qquad (40)$$

$$\Phi_4 = 1 - Y(W_r W_{23f} V_{FB}) \ \text{(Failure)} \qquad (41)$$

where the designable parameters are V^, W^, and V_{pB} (see Figure 6). Besides the constraints on designable parameters given in Table 2, an upper limit of 2500 $mils^2$ was placed upon the area, 110 nsec. was the maximum acceptable propagation delay, and -.7 volts the smallest acceptable ON output voltage (see [22]).

The gradients of ϕ_1 and ϕ_3 were found by considering the circuit equations as equality constraints and adding them via Lagrange multipliers to (38) and (40). Direct differentiation of the resulting equations and proper definition of the multipliers gave the gradients of ϕ_1 and ϕ_3 (this is essentially the approach taken by Hachtel, Brayton and Gustavson [38] and is also equivalent to the adjoint network [39] method of calculating gradients).

The MCO problem is

$$\min \begin{bmatrix} \phi_1 \\ \phi_2 \\ \psi_3 \\ \phi_4 \end{bmatrix} \tag{42a}$$

subject to

$$\begin{aligned} \phi_1 &\le 110 \text{ (nsec.)} \\ \phi_2 &\le 2500 \text{ (mils}^2\text{)} \\ \phi_3 &\ge .7 \text{ (volts)} \\ -2 &\le V_{ED} \le -1 \text{ (volts)} \\ 5 &\le W_{r} \le 50 \text{ (microns)} \\ 50 &\le W_{23} \le 250 \text{ (microns)}. \end{aligned} \tag{42b}$$

The first step in attempting to find noninferior solutions to (42a) is to generate a simplicial approximation to the feasible region described by the constraints (42b). We used the algorithm described in Director and Hachtel [1] with the starting point 0^-12, W_{23}=230, and V_{Ffi}—1.2), and scales (1/4.6, 1/50.3, 1) to generate an approximation. After generating the initial approximation each objective function was first minimized subject to the constants (42b). Then the constraints were replaced by the simplicial approximation and the objective functions minimized again. The results of these minimizations indicated that the simplicial approximation did not contain the parameter values which were the true minimizers of all the objectives. The situation was caused, in main, by slight nonconvexities in the true feasible region. To remedy this problem we included the minimizers of each objective as vertices of the simplicial approximation. In order to check the extent of the nonconvexity of the feasible region, we performed a circuit simulation at the center of each face of the new approximation. Although some face centers were infeasible, the extent of infeasibility was very small - less than 1.0% in all cases. Based upon the preceding check we assumed that the updated simplicial approximation was an adequate approximation to the true feasible region even though the true feasible region was slightly nonconvex.

Before proceeding, we checked the size of each face. This was done by calculating $\det(X_i)$ which is proportional to the area. In doing this we found four faces quite large compared to the remaining faces. This discrepancy in face size could cause an accuracy problem in our yield algorithm. Therefore we found the centroid of each large face and used it to generate three coplanar (but smaller) faces per large face.

Table 3 lists the normals to each of the 38 faces, the right-hand-side associated with the description of each face and which vertex points define which faces. Table 4 lists the 21 vertices of the approximation and the estimated maximum yield point given by the Simplicial Approximation algorithm.

IV. MCO OF THE NAND INCLUDING YIELD

The problem we want to generate noninferior solutions to is:

Table 3a. Normals to the faces of the Simplicial Approximation.

FACE	n_1	n_2	n_3
1	.307	-.0214	- .581
L.	.0666	.082	- .3789
3	.061	.0837	- .3172
4	.133	-.00282	-1.587
5	-.0334	-.00318	-1.43
6	.127	-.0041	-1.573
7	-.0737	.00051	-1.42
8	.025	.0144	-1.6
9	-.306	-.0002	- .294
10	-.304	-.00025	- .33
11	-.305	-.00019	- .332
12	-.306	-.00017	- .309
13	-.307	.00006	- .314
14	.1204	.096	- .00023
15	.12	.096	- .000196
16	.338	-.0045	-1.06
17	.33	-.0196	- .736
18	.319	-.0216	- .555
19	.323	-.0213	- .61
20	.302	-.0213	- .582
21	-.081	-.0096	- .92
22	-.305	-.00023	- .293
23	-.301	-.00048	- .333
24	.306	-.0208	- .395
25	-.1098	.0468	- .275
26	-.306	-.0001	- .296
27	-.307	.00008	- .313
28	.12	.0963	.0022
29	-.303	-.00019	- .253
30	0	0	3.07
31	0	0	3.07
32	0	0	3.07
33-	.12	.0963	- .00023
34	.12	.0963	- .00023
35	.12	-.0963	- .0002
36	.12	.0963	- .0002
37	.12	.0963	- .0022
38-	.12	.0963	- .0022

Tabie 3b. Right-hand-side of Simpiiciai Approximation and vertices which define faces of SA.

FACE	b_t	VERTICES WHICH FORM FACE		
1	.533	7	8	10
2	20.43	2	3	11
3	20.66	3	5	11
4	3.99	2	4	12
5	1.81	1	8	12
6	3.63	4	8	12
7	2.15	1	11	12
8	6.59	2	11	12
9	- 2.14	6	9	13
10	- 2.09	1	9	14
11	- 2.08	1	11	14
12	- 2.11	9	13	14
13	- 2.07	11	13	14
14	23.7	2	3	19
15	23.7	3	5	20
16	5.25	2	4	15
17	1.40	4	8	15
18	.595	7	10	15
19	.782	8	10	15
20	.503	7	8	16
21	- .832	1	8	16
22	- 2.15	6	9	16
23	- 2.1	1	9	16
24	.395	7	15	16
25	10.48	5	11	17
26	- 2.12	6	13	17
27	- 2.07	11	13	17
2S	23.69	5	21	17
29	- 2.16	6	16	17
30	- 3.07	15	16	18
31	- 3.07	15	17	18
32	- 3.07	16	17	18
33	- 23.7	2	19	15
34	23.7	3	19	15
35	23.7	5	20	15
36	23.7	3	20	15
37	23.69	5	21	15
38	23.69	17	21	15

Table 4. Vertices of Simplicial Approximation

VERTEX.	w_1	w_2	v_T
1	8.766	194.64	-1.9
[illegible]	12.83	230.03	-1.852
3	10.91	232.43	-1.67
4	12.62	210.01	-1.334
5	8.34	235.6	-1.316
6	7.92	203	-1.089
7	9.71	144.37	-1.102
8	10.65	179.93	-1.91
9	3.23	183.6	-1.404
10	11.27	179.3	-1.57
11	8.72	233.2	-1.887
12	10.787	217.12	-2
13	8.18	201.27	-1.36
14	8.28	199.79	-1.46
15	15.46	226.74	-1
16	7.896	115.8	-1
17	7.82	236.28	-1
13	10.39	192.94	-1
19	13.06	229.7	-1.507
20	11.57	231.6	-1.323
21	10.54	232.90	-1.105

estimated design center

$W_1 = 10.6$; $W_2 = 209$; $V_T = -1.47$

$$\min_{x_o = (W_1, W_{23}, V_{FB})} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ 1-Y(x_o) \end{bmatrix} \tag{43}$$

subject to

$$\eta_i^T x_o \le b_i \quad i=1,\ldots,38$$

where the yield $Y(x_o)$ is estimated using the algorithm presented in Section 3. Because of the success of the minimax method of generating noninferior solutions, [15,16] (43) is restated as

$$\min_{(x_o, \gamma)} \gamma$$

subject to

$$\begin{aligned} w_1\phi_1 &\le \gamma \\ w_2\phi_2 &\le \gamma \\ w_3\phi_3 &\le \gamma \\ w_4(1-Y(x_o)) &\le \gamma \\ \eta_i^T x_o &\le b_i \quad i = 1,\ldots,38 \end{aligned} \tag{44}$$

To proceed with the optimization we must choose the standard deviations of the parameters and the number of subdivisions of each interior simplex.

The choice of standard deviations was, in large part, ad hoc since no industrial data was available. However, it was decided that the two geometric parameters W_1 and W_{23} should have the same standard deviation. The flat band voltage is also subject to variation, but basically independent of the geometric variables. The standard deviations chosen are:

1 " (45)

FB

An independent Gaussian distribution was assumed for all parameters. Finally, after calculating the yield for the standard deviations (45) at various points in the feasible region, and for different number of divisions of the interior simplices, we chose L=20 as being a reasonable trade-off between an accurate estimate of yield and computational expense. The starting values were chosen to be

$$W_x = 10 \quad W_{23} = 200 \quad V_{pB} - -1.3 \tag{46}$$

A discussion of the ideas and concepts of MCO can be found in [15,16]. For our purposes it suffices to state that the weighted minimax approach is one of the most powerful MCO methods available and that specific interpretations of, and selection heuristics for, the weights exist [15,16]. The general approach begins with a minimization of each objective function separately, known as a boundary search. Based upon the results of the boundary search weights are chosen and further optimization performed.

The constrained optimization algorithm we use is the constrained variable metric algorithm due to Powell [40-42]. (The quadratic programming algorithm used in the Powell algorithm was due to Canon, Cullum, and Polak [44]). This algorithm has proven very successful in the optimization of various electronic circuits and on various standard test cases [40] and was far better than a first order Augmented Lagrangian technique used as a comparison [15,44].

Table 5 contains the boundary search data for problem (44) using (46) as nominal starting point.

The first set of weights chosen [see 15,16] was based upon combining the boundary solutions with the maximum yield solution being preferred seven times

Table 5. Results of minimax MCO Design of NAND gate including yield.

RUN #	WEIGHTS				FINAL PARAMETERS W_1	W_{23}	V_{FB}	FINAL PERFORMANCES t_{PD}	AREA	$\|V_o\|$	YIELD %	NITER	NFUNC	NGRAD
1	1	0	0	0	15.46	226.74	-1	56.2	2500	.7	9.22	5	6	6
2	0	1	0	0	7.86	115.8	-1	110	1277	.7	.09	3	4	4
3	0	0	1	0	7.82	236.28	-1	110	2500	.35	10.3	13	14	14
4	0	0	0	1	10.5	200.58	-1.344	85.8	2171	.5712	78.8	9	25	9
5	.0114	.00046	1.74	2.33	10.2	191.53	-1.012	84.8	2076	.56	19.5	11	20	11
6	.01103	.0005	1.675	2.707	9.57	189.7	-1.275	93.3	2049	.546	48.2	19	25	19
7	.01047	.0004	2.06	2.06	9.11	200.1	-1.15	96.4	2150	.49	46.01	5	12	5
8	.0125	.00055	2.5	3.333	9.43	201.3	-1.25	94.3	2165	.508	61.9	21	35	22

more than the other solutions. The factor of seven was chosen because the range of possible values for yield was large and we desired to maintain yield at a moderately high level. However, as Table 5 shows, the factor of seven was not large enough and a small yield resulted.

This run also points out the importance of having a reasonably accurate estimate of the boundary of the feasible region. The trade-off solution could be found anywhere in the feasible region, even on the boundary or exterior to the region, depending on the statistics of the parameters and the shape of the feasible region. Thus during the course of one optimization the nominal point could conceivably go from the center to the edge of the feasible region. It is for this reason we feel the Simplicial Approximation approach to estimating ft, is more economical than the constant updating method of Bandler and Abdel-Malek.

The next run was performed to trade-off between yield and propagation delay. The weight was found by combining the solution of run four and one in a nine-to-one weighting. Notice that a trade-off did occur but certainly not one that would be chosen as a final design since run four produced more preferable results. A trade-off between output voltage and yield was used as the basis for the seventh run in which the noninferior solutions of runs four and three were combined with nine-to-one weighting to generate the weight vector. This run produced a more desirable trade-off in that a significant reduction in output voltage, over the value in run four, was achieved while maintaining the yield moderately high.

Our last run was based on specifying a solution that would be interesting, $t_{pD} = 80$, AREA = 1800, $-V_Q = .4$, and YIELD = 70%, and converting that point into a weight for the minimax optimization [16]. Although requiring a relatively large number of function evaluations, the result of this optimization (run 8, Table 5) was the most interesting point so far obtained. While not achieving the desired performance, the solution is an appealing traderoff point especially when V_o is important.

Clearly many more interesting noninferior designs could be generated by judicious choice of weights. It is interesting to note that the various noninferior solutions generated give the designer an understanding of the potential trade-offs possible. This would be particularly useful in view of the fact that the optimization without considering yield could lead to unacceptably low values for yield (see runs li-3).

V. SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

In this paper we have investigated the addition of manufacturing yield as one of the competing criterion in a Multiple Criterion Optimization of electronic circuits. The desirable characteristics of a yield estimation algorithm in an MCO environment were discussed. Two particular yield estimation algorithms, one due to Director, Hachtel and Brayton and one to Bandler and Abdel-Malek were discussed. A new yield estimation algorithm was presented having some properties of both the previous algorithms which we feel is useful for evaluating yield in an MCO design. This new algorithm was applied to the MCO design of a MOSFET NAND gate. The results of the example were mixed but we feel they indicate that the consideration of yield and various performance criteria simultaneously is an interesting synthesis worthy of future research.

The area of Statistical Design both singly and in connection with MCO is still in its infancy. The method presented in this paper has two main weak points: a "curse of dimensionality," and the requirement that the feasible region in input space be convex. To a greater or lesser extent, all Statistical Design methods suffer from a curse of dimensionality and there does not seem to be any easy fix for this problem. The assumption of convexity of the feasible region in input space is inherent in the Simplicial Approximation method. Some methods have relaxed this requirement slightly but the ability to do statistical design on an arbitrary feasible region needs much further research in order to be realized. Notice further that as new statistical design methods become available which overcome to some extent either of the aforementioned problems, it may take further research so that these methods can successfully be used in an MCO design setting.

Concerning the method of yield estimation presented here at least two improvements are possible. First, instead of dividing each interior simplex by hyperplanes parallel to the face of the SA, subdivisions could be based upon the actual level contours of the p.d.f., thus allowing an adaptive subdivision that might be more accurate and/or efficient. Second, the method of integrating the p.d.f. over each subdivision of the simplex could be modified to improve the accuracy by using higher order approximations. Alternately, use of different methods for the calculation of yield and gradient of yield might improve the accuracy of the gradient information and improve the performance of the optimization.

Finally, more computational experience on circuit design problems is necessary both in yield estimation and MCO problems. However, because of the naturalness, simplicity, and usefulness of considering circuit design from an MCO viewpoint, including yield, we feel that the evolution of various new methods should proceed from this point of view.

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