The Motion of a Pushed, Sliding Object
Part 1: Sliding Friction

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Abstract

In many robotic applications, manipulation planning for an object free to slide on a surface is an import oblem. Physical analysis of the object's motion is made difficult by the absence of information about stribution of support of the object, and of the resulting frictional forces. This paper describes a n proach to the analysis of sliding motion.

The instantaneous motion of the object can be described as a pure rotation about a center of rotation (CO mewhere in the plane. In this paper we find the locus of CORs for all possible distributions of supp rces. We assume zero friction at the pusher-object contact, and we assume that the support fo stribution is confined to a disk.

In one application to robotic manipulation, bounds on the distance an object must be pushed to come i gnment with a robot finger or a fence are determined.

. Background

iding operations are encountered in several facets of robotics. It is almost inevitable that when t ›sition of an object that is to be acquired by a robot is not perfectly known, a sliding phase will occ ›fore the robot can acquire the object. This phase need not be considered an undesirable b ıavoidable fact of life. The examples which follow show how sliding operations can be us ›nstructively to manipulate and acquire objects, without sensing, and despite uncertainty in t ientation and position of the object. In each example, properties of sliding are used to reduce t ıcertainty in position and orientation of an object.

a typical grasping operation, the robot opens a two-jaw gripper wide enough to accommodate bo e object to be grasped, and any uncertainty in the object's position. Then the gripper closes. If t ›ject happens to be exactly centered between the jaws initially, both jaws will make contact t ›ject simultaneously, and there will be no sliding phase. But this is unlikely. Usually, the object v › closer initially to one jaw than to the other, and the closer jaw will make contact first.

ıere follows a sliding phase until the second jaw makes contact. During the sliding phase the obje likely to rotate, especially if the face of the jaw is in contact with a corner of the object rather thar ıt facet. Once both jaws come into contact with the object, sliding on the table becomes le ıportant than *slipping* of the object with respect to the faces of the jaws. The behavior of an obje ıring both phases is discussed in (Brost, 1985). Brost finds grasp strategies which bring the obje to a unique configuration in the gripper, despite substantial uncertainty in its initial configuration.

ıother example of the use of sliding is the interaction of an object on a moving belt with a fixe anted, fence across the belt. (Equivalently, the object may be stationary on a surface and the fen oving, under control of a robot.) One of many possible behaviors of the object when it hits the fen to rotate until a flat edge is flush against the fence, and then to slide along the fence (if the fence ıfficiently slanted) until it reaches an obstruction at an edge of the belt. Another behavior is to r ong the fence instead of sliding. Or it may stop rotating and simply stick to the fence.

ıe behavior of objects on encountering a fence has been considered in (Mani, 1985) and (Bro 985). In (Mani, 1985), strategies for manipulation are developed which can orient an object or ble by pushing it in various directions with a fence. Each push aligns a facet of the object with t nce.

P. Paul demonstrated a clever grasping sequence on a hinge plate, illustrated in figure 1-1. T rategy makes use of sliding to simultaneously reduce the uncertainty of a hinge plat onfiguration to zero, and then to grasp it (Paul, 1974) (Mason, 1985). To understand this and simi erations, Mason determined the conditions required for translation, clockwise (CW) rotation, a ounter-clockwise (CCW) rotation of a pushed object (Mason, 1985). Mason's results are us tensively in both (Mani, 1985) and (Brost, 1985), and also in this work.

l of the work discussed above may be called "qualitative" in the best sense of that term: the auth d the exact conditions determining which of various qualitatively different behaviors will occur. T havior in question may be that of rolling along a fence vs. slipping along a fence, or it may be that tating CCW vs. rotating CW as an object is pushed.

e work presented here is complementary to the work discussed above. One result of this work antitative bounds on the *rate* at which the predicted motion occurs. Without rate information t gorithms found in (Mani, 1985) cannot produce guaranteed manipulation strategies. In some cas object may have to be pushed a great distance before it will align with the fence pushing it. oduce a guaranteed strategy we need an upper bound on the distance an object must be pushed gn.

ost's work is less dependent on rate information because the second phase (slipping relative to t ws) is relatively insensitive to the interaction with the table (sliding). However certain of the gra ategies analyzed in (Brost, 1985) do rely on sliding. Since pushing distances are finite the ategies can only be guaranteed if rotation rates are known.

ason and Paul's hinge grasp (figure 1-1) works only for a certain range of initial hinge orientatio r orientations outside of this range, the jaws will be closing too fast for the hinge plate to comple CW rotation into alignment, before the jaws close. To find the range of orientations for which t asp will work, for a given convergence angle of the jaws, we need to know the slowest possi tation rate of the hinge plate as it is pushed.

gure 1-2 illustrates an application of the results of the present work. In the figure, a fence is pushi ectangular object, which may slide on the table. The rectangle will rotate clockwise (CW) (Maso 85), and will cease to rotate when the long side becomes parallel to the fence (Brost, 1985) (Ma 85). So we can orient the rectangle by pushing it in this way. But how far do we have to push it?

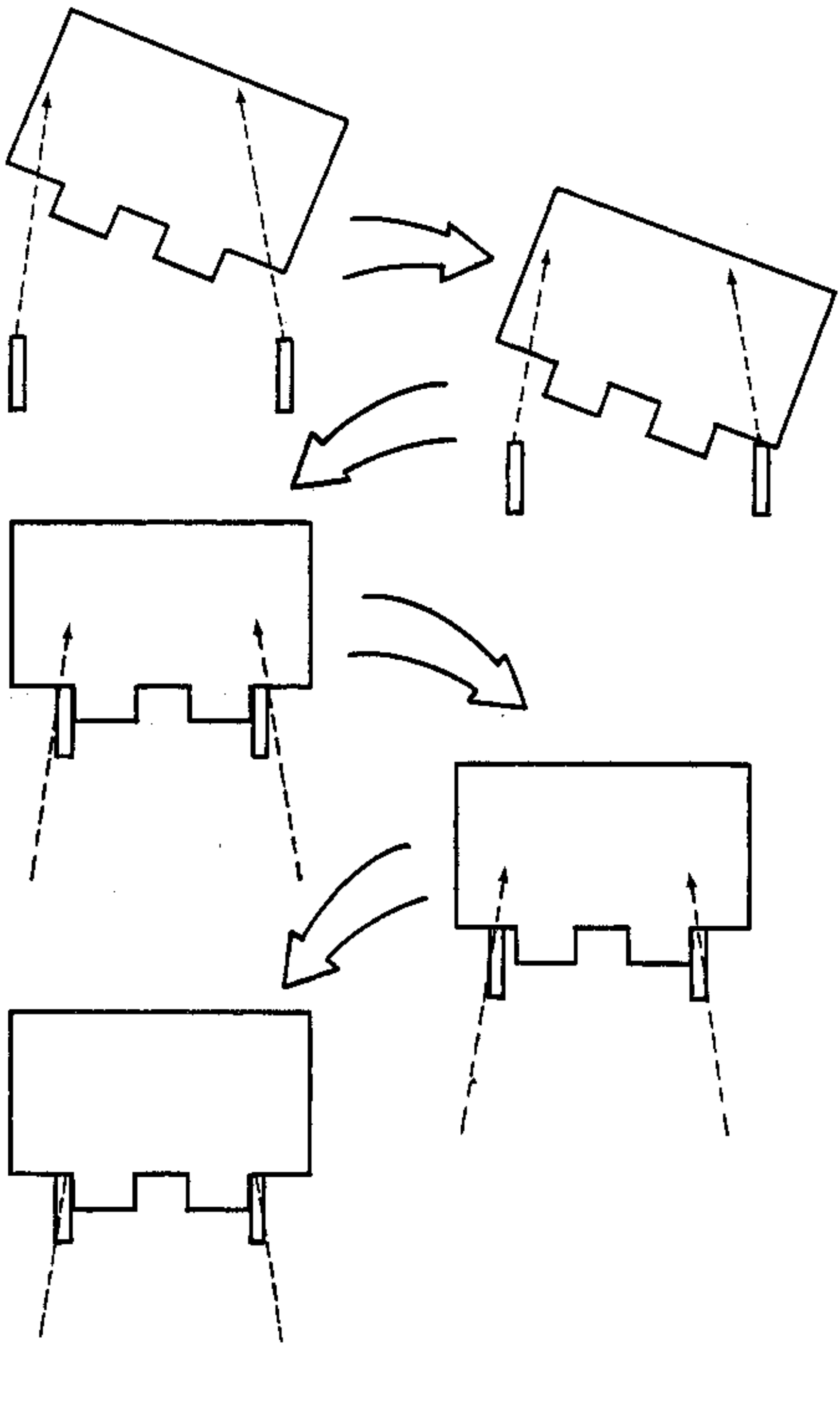


Figure 4.1: Hinge grasp strategy

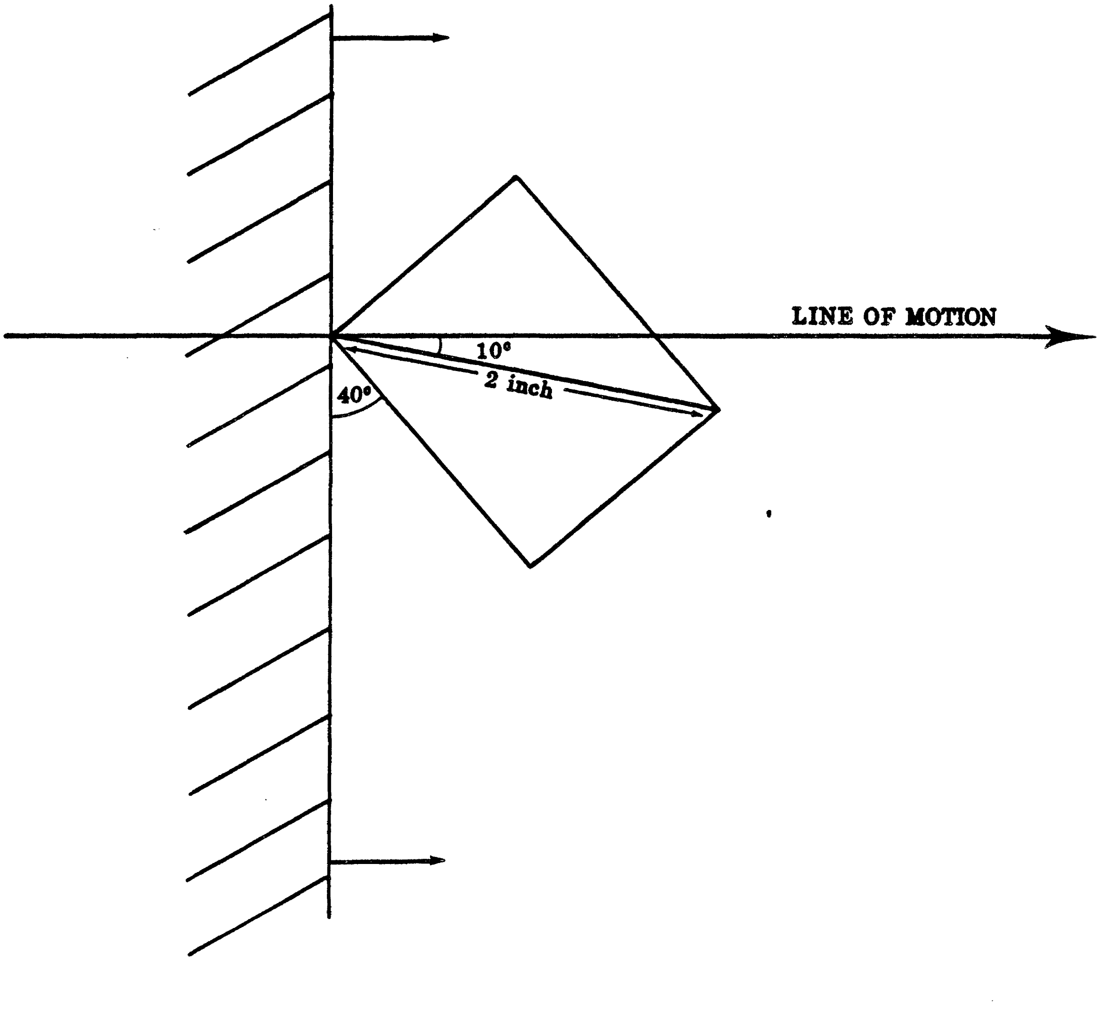


Figure 1-2: The fence must advance 2.02 inches to assure that the rectangle is aligned

the example shown, we must push at most [illegible] inches to assure alignment. This result dependent of the mass of the object, the coefficient of friction between the object and the table, a details of the bottom surface of the object (e.g., screw heads, cut-out sections, etc.). Howeve epends on the assumption that the coefficient of friction between the pusher and the object is zerc

. Overview

.1. The Center of Rotation

n object sliding on a plane has three degrees of freedom. Its position may be specified by the (x splacement and angle θ of a coordinate frame fixed in the object relative to a coordinate frame fix the plane. The instantaneous motion of the object may be described as infinitesimal changes in splacements and rotation.

this paper we will treat the object as a two-dimensional rigid body, since we are only concern ith the interaction of the object with the plane on which it is sliding. All pushing forces will stricted to lie in the plane. The results may be applied to three-dimensional objects, so long as ertical component of the pushing force is negligible, and so long as the point of contact is near ble.

the general case, when an object is being pushed there is only one point of contact between oject and the pusher. The contact may be where the front edge of a pushing fence touches a cor the object (figure 1-2), or it may be where a pushing point touches an edge of the object (fig 1). In most of this paper we will assume that the pusher is a point in contact with an edge of oject. The analysis applies equally well if the pusher is a fence in contact with a corner of the obje many figures (e.g., figure 2-3) an 'edge' will be drawn, and it will be left ambiguous whether this e pushed edge of the object or the front edge of a fence.

is assumed the pusher will move along a predetermined path in the plane, i.e., it is under posit ontrol. The object retains two degrees of freedom, with the third degree of freedom of its mot ced by the contact maintained between the pusher and the object.

nese two degrees of freedom are most conveniently expressed as the coordinates of a point in ane called the *center of rotation* (COR). Any infinitesimal motion of the object can be expressed rotation $\delta\theta$ about some COR, chosen so that the infinitesimal motion of each point $\vec{w}$ of the objec

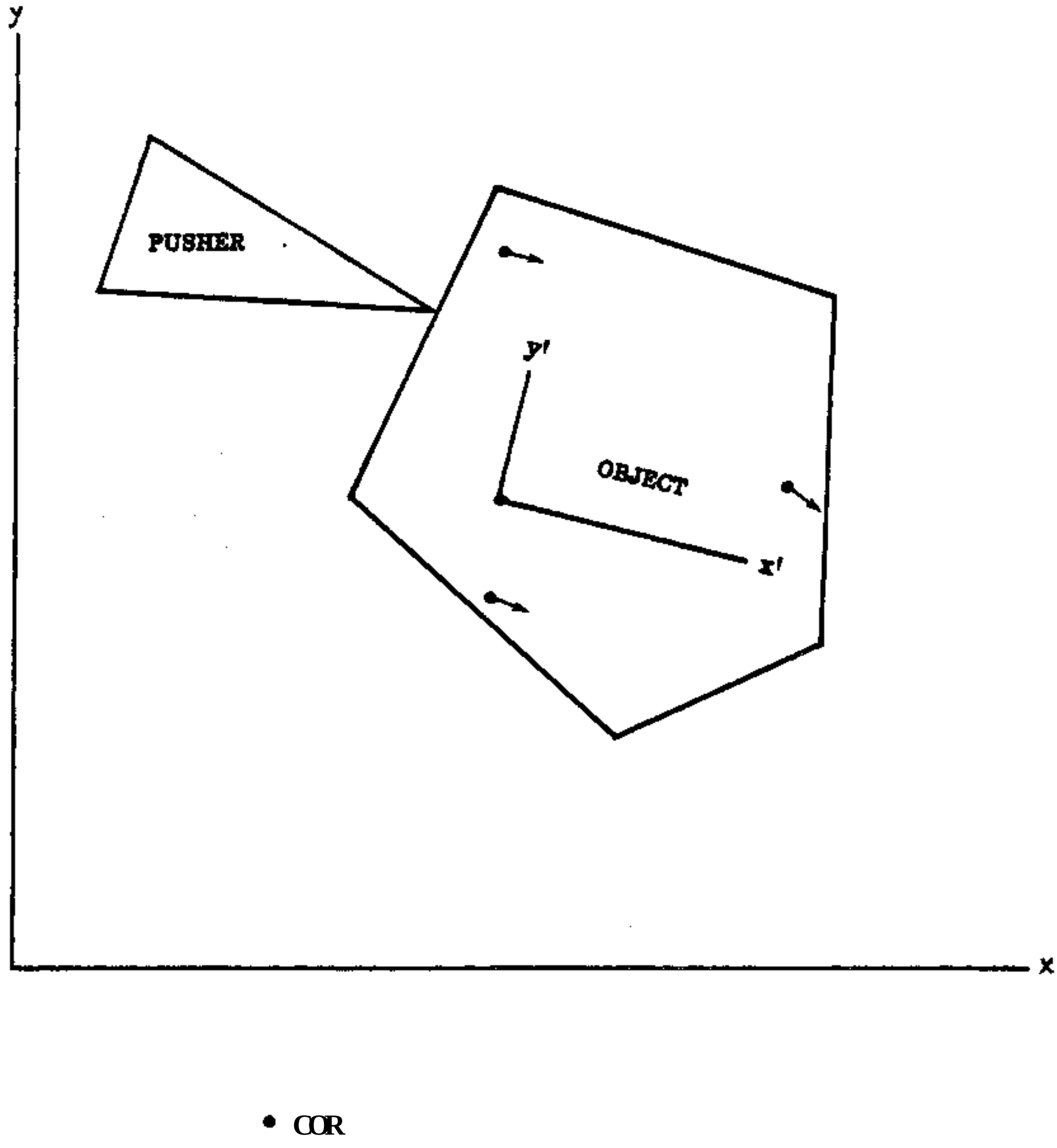


Figure 2-1: An object being pushed has two degrees of freedom

erforms is pure rotation in place, the COR is at the center of the disk. Motions we might describe nostly translation" correspond to CORs far from the point of contact. In the extreme case, p anslation occurs when the COR is at infinity.

his paper is concerned with finding the COR.

.2. The Support Distribution

nding the COR is complicated by the fact that changes in the distribution of support forces un e object substantially affect the motion. Intuitively, if the support is concentrated near the cente ass (CM), the object will tend to rotate more and translate less than if the support is unifor stributed over the entire bottom surface of the object. With support concentrated near the C uch less energy is expended in rotating than with more dispersed support.

he distribution of support may be changed dramatically by tiny deviations from flatness of the bott rface of the object (or of the surface it is sliding on). Indeed, if the object and the surface ufficiently rigid and not perfectly flat, they may be expected to make contact at only three poi he three points may be located anywhere on the object's bottom surface, but like the legs o ree-legged stool, the triangle formed by the three points of support must enclose the projection e CM onto the surface. Otherwise the object is unstable. This observation perhaps gives so tuitive justification to a theorem proved by Mason (Mason, 1985) which states that all extreme val the location of the COR may be attained by considering only support distributions consisting ree points of support (a "tripod").

nce any assumption we could make about the form of the support distribution (for instance that i niform under the object as in (Prescott, 1923)) would not be justified in practice, our goal is to f e locus of CORs under *all* possible support distributions. Let the CM be at the origin, and $\vec{w}$ b oint in the plane. All that is known about the support distribution $S(\vec{w})$ is that

- $S(\vec{w})$ is zero outside the object,
- $S(\vec{w}) \geq 0$ everywhere,
- the total support force $\int S(\vec{w})\, d\vec{w} = Mg$, the weight of the object, and
- the first moment of the distribution, $\int S(\vec{w})\, \vec{w}\, d\vec{w} = 0$. (i.e., the centroid of the

he coefficient of friction with the supporting surface $(\{i_s)$ does not affect the motion of the objec e use a simple model of friction. We assume that /x$_s$ is constant over space, that it is independent Drmal force magnitude and tangential force magnitude and direction, and that it is veloc dependent In short, we assume Coulomb friction.

itially we consider only $(i_c = 0$, where /*$_c$ is the friction coefficient between the edge of the obj< id the pusher. (In a subsequent paper we will do the analysis needed to treat /*$_c$ > 0.) We assu at the contact between the pusher and the object is restricted to a single point.

is also assumed that all motions are slow. This *quasi-static approximation* requires that frictio rces on the object due to the coefficient of friction with the surface /x$_s$ quickly damp.out any kine lergy of the object:

$$\frac{Mv^2}{2} \ll XMg\mu_s$$

here v is the velocity of the object and *X* is the precision with which it is desired to calcul stances.

this paper we will assume the object being pushed is a disk with its CM at the center. Given anott >ject of interest (e.g. a square), we can consider a disk centered at the CM of the square, big enou enclose it The radius *a* of the disk is the maximum distance from the CM of the square to any po i the square. Since any support distribution on the square could also be a support distribution e disk, the COR locus of the disk must enclose the COR locus of the square. The locus for the d ovides useful bounds on the locus for the real object.

:her parameters of the problem are the point of contact <T between the pusher and the object, a e angle a between the edge being pushed and the line of pushing, as shown in figure 2-2. T .lues of a and ?sho»vn are useful in considering *Vr* 3 motion of the five-sided object shown inscrib the disk. We do not require the point of contact to be on the perimeter of the disk, as this wo minate applicability of the results to objects inscribed in the disk. Indeed, for generality we do r en require the point of contact to be within the disk. Similarly, we will not require a to be such tt *d* edge being pushed is perpendicular to vector £ as it would be if the object were truly a disk. T sk (with radius *a),* a, *$* and the CM, are shown in figure 2-2. A particularly simple distribution

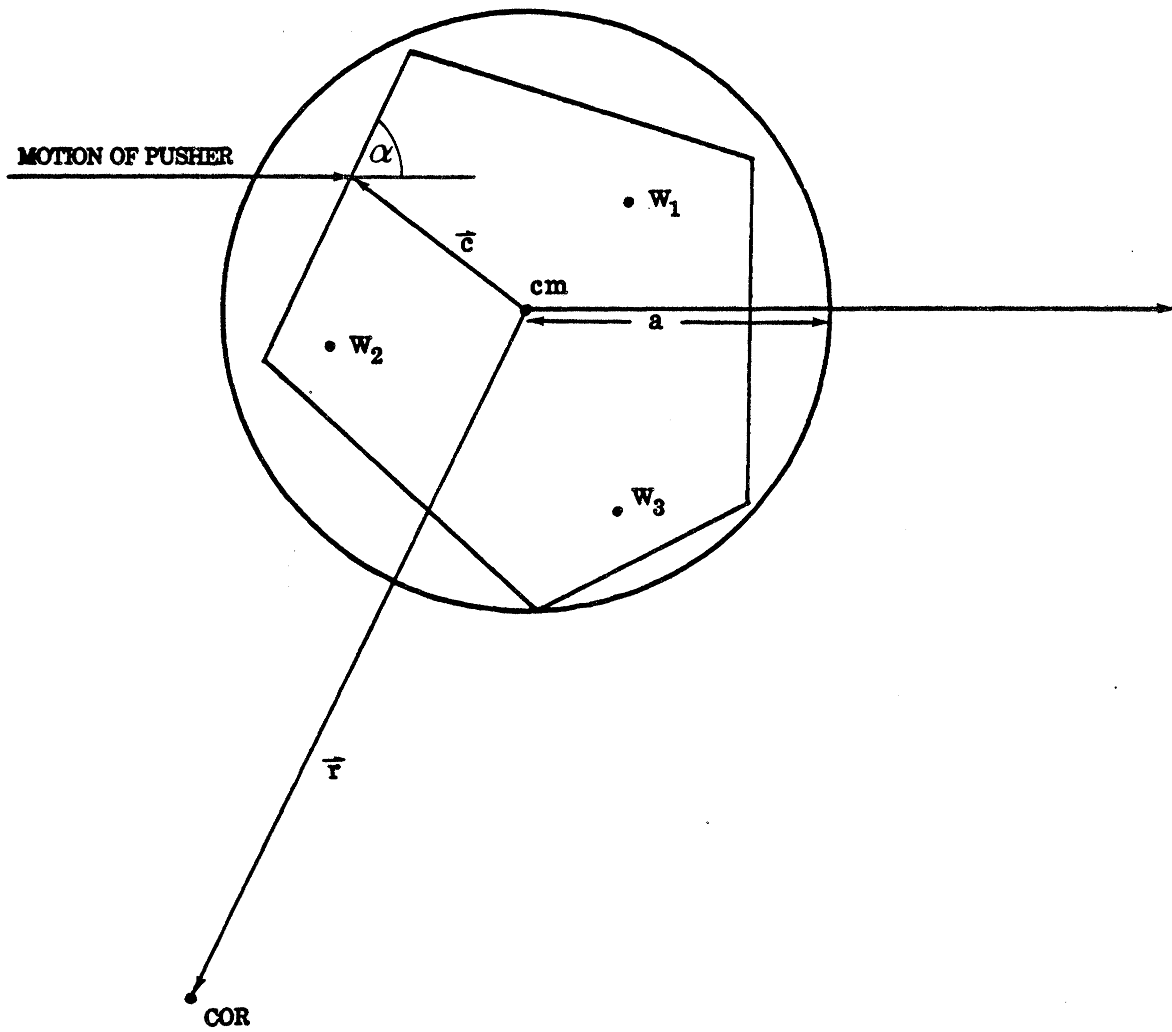


Figure 2-2: Parameters of the pushing problem

dicated, along with the COR for that distribution of support.

section 3, we derive the energy E_f lost to friction when the pusher advances a distance *8x.* This function of the presumed location of the COR *?.* The system will rotate about the COR whi inimizes $£_f$, as proved in the appendix. We will find the COR by setting $\nabla E_f = 0$,

sing hundreds of thousands of randomly selected tripods satisfying the conditions listed in secti 2, we can find the COR by numerically minimizing E_f for each tripod. Plotting the resulting CO ves a rough idea of the COR locus, for a single choice of $\vec{c}$ and a (figure 2-3).

hile Mason's theorem 5 states that *extremal* values (i.e., the boundary) of the COR locus may und by considering only tripods of support, we see that in fact all values interior to the locus < tained in this way as well. The dense central region of the numerically generated locus in the figi is been replaced by hashing to facilitate printing.

section 4, we express the equation $\nabla 2f_r=0$ in terms of a new construction called the *quoth cus.* Each element of the quotient locus is a quotient of two moments of the support distribution *£*

surprising discovery is that all elements of the quotient locus for the disk may be attained insidering only *pairs* of points of support (dipods), instead of tripods. This statement therefc >plies to the COR locus, as well Using this information we calculate the analytic form of t jundaries of the quotient locus.

ice the quotient locus is found for a given object and pushing geometry, we calculate (in section e boundaries of the COR locus from it. The greatest distance from any point in the COR locus to t d, and its direction, are found analytically. These results, all that are needed for many applicatioi e of simple form and have a convenient graphical interpretation.

section 6 we solve some sample problems using the results. One result is a simple formula for t aximum distance an object must be pushed by a fence to assure that it has rotated into alignme th the fence.

4. Notation

- Vectors are indicated by an arrow, e.g., *$*

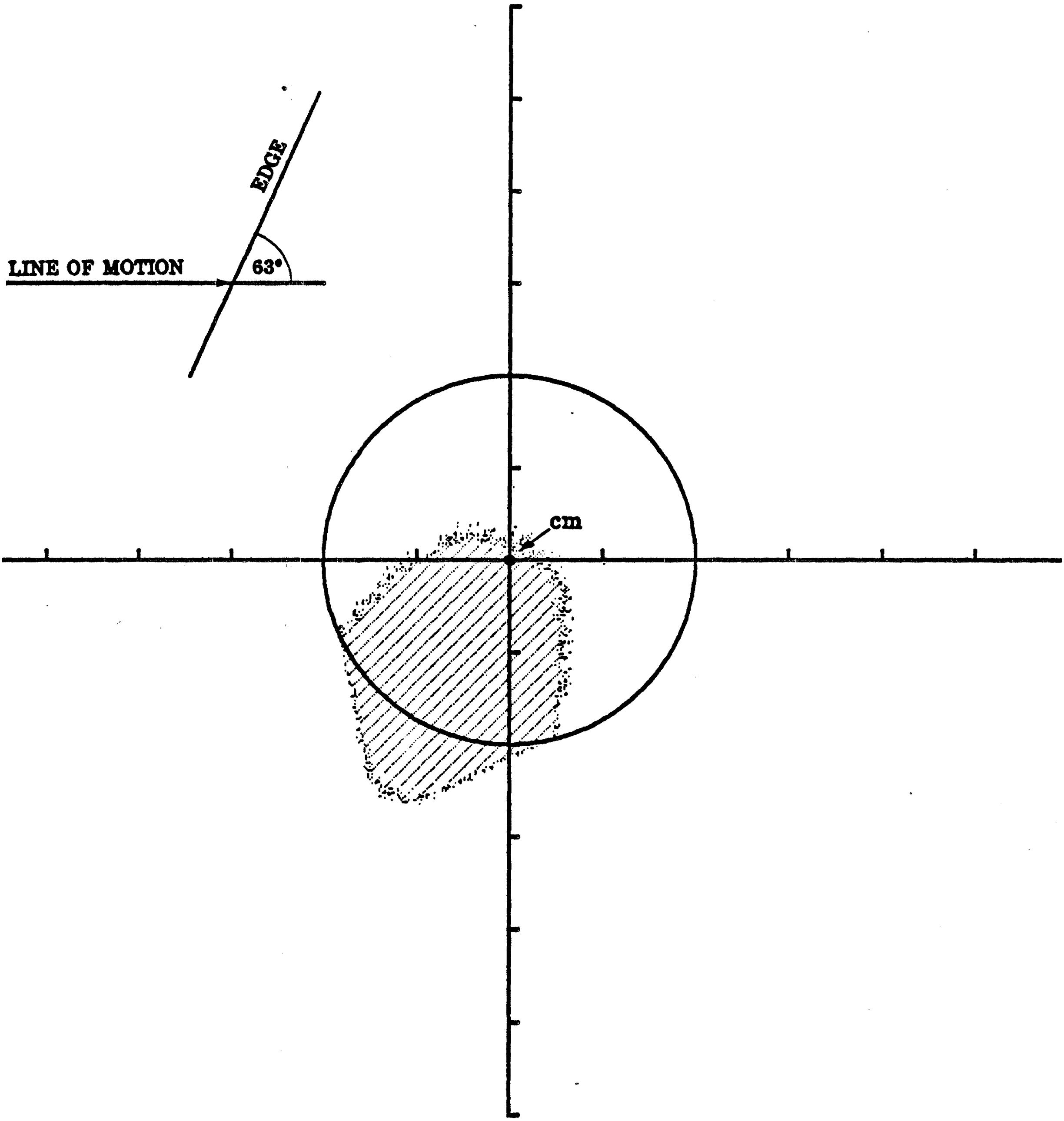


Figure 2-3: COR locus found by iterative minimization (dots)

- A Greek letter is used to represent both an *angle* and a *unit vector* which makes that angle with respect to the x-axis (measured CCW). An arrow is used to indicate the unit vector: $\vec{\alpha} = (\cos\alpha, \sin\alpha)$.

- We indicate functional dependence with subscripts. E_r is a function of $\vec{r}$ (the COR).

- All integrals are over the area of the disk.

- Curly brackets indicate a locus of values of a quantity.

. Minimizing the Energy

this section we compute the energy that is dissipated due to friction when the pusher advance stance δx, for a given center of rotation $\vec{r}$, and for a given distribution of support $S(\vec{w})$. (figure 2-

may help to imagine the disk "pinned" at the COR. This is not difficult to imagine if the C ppens to fall inside the perimeter of the disk, and one's intuition can be extended to include se where the COR is outside the perimeter. Either way, the disk is free to rotate *only about OR*, and the COR itself stays stationary.

ven the COR, the motion of the disk is fully determined when we apply one more constraint: ge being pushed (at $\vec{c}$) must move out of the way of the advancing pusher, but stay in contact.

1. Relation between Motion of Pusher and Rotation of Object

order to accommodate the advance δx of the pusher, the disk will rotate an amount $\delta\theta$ about nter of rotation $\vec{r}$. A rotation of $\delta\theta$ allows an advance of the pusher δx consisting of two parts, own in figure 3-1.

$$\delta x_1 = \delta\theta\,|\vec{c} - \vec{r}|\cos\theta = \delta\theta\,(c_y - r_y)$$

$$\delta x_2 = \delta x_1 \frac{\tan\theta}{\tan\alpha} = \delta\theta \frac{c_x - r_x}{\tan\alpha}$$

$$\delta x = \delta x_1 + \delta x_2 = \delta\theta\,(c_y - r_y + \frac{c_x - r_x}{\tan\alpha})$$

te that δx_2 corresponds to slipping of the point of contact along the object edge.

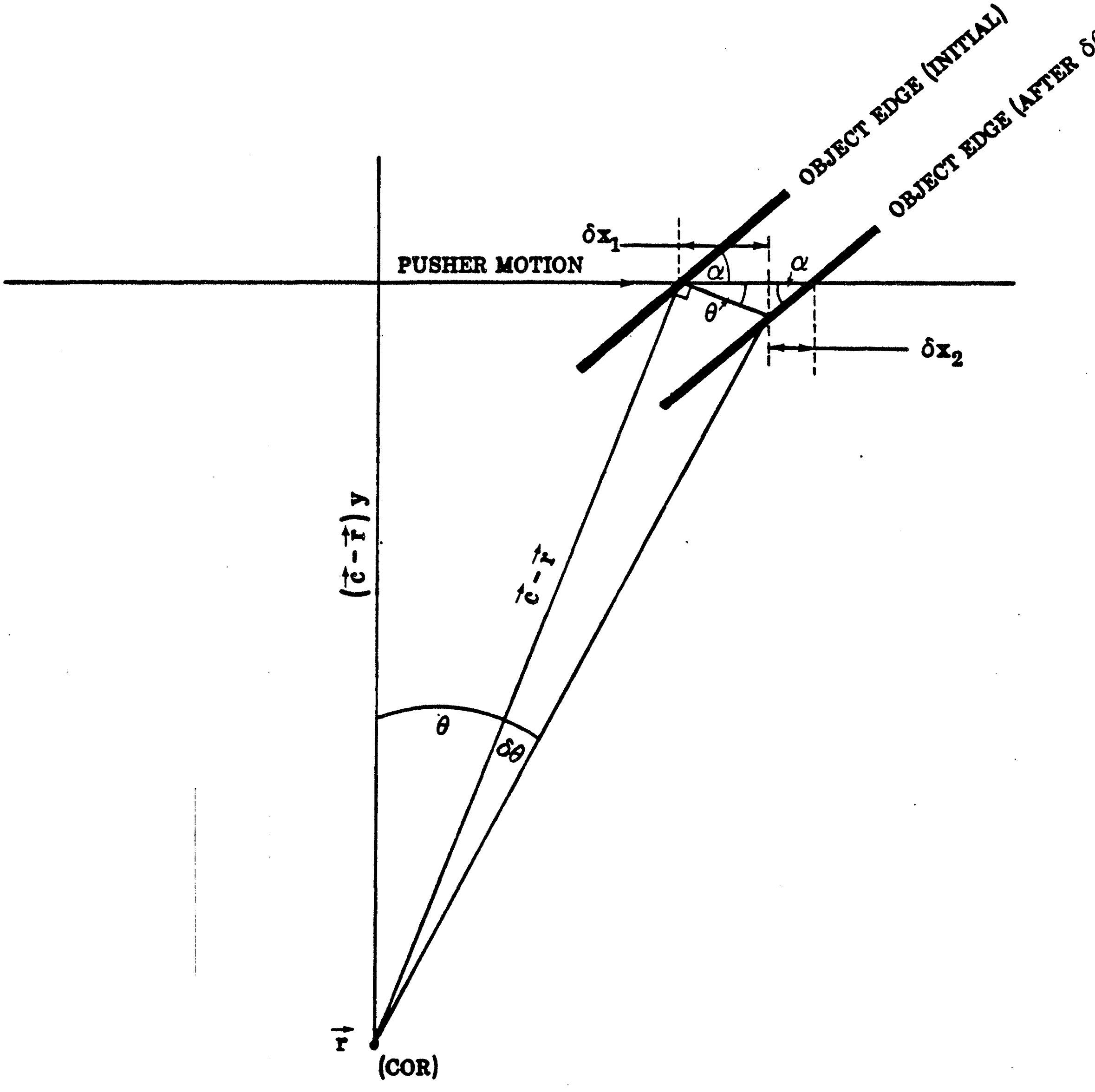


Figure 3-1: Relation between advance of pusher (δx) and rotation about the COR ($\delta\theta$)

$$\delta x = \frac{\delta\theta}{\sin\alpha}\vec{\alpha}\cdot(\vec{c}-\vec{r}).$$

o avoid proliferation cf absolute value signs, henceforth $\vec{\alpha}\cdot(\vec{c}-\vec{r})$ will be taken to be positi onsiderations of symmetry will allow application of the results to cases where $\vec{\alpha}\cdot(\vec{c}-\vec{r})$ is negati nysically, $\vec{\alpha}\cdot(\vec{c}-\vec{r})>0$ corresponds to clockwise rotation of the object as it is pushed.

.2. Energy Lost to Friction with the Table

n element of the disk at $\vec{w}$ supports a force $S(\vec{w})\,d\vec{w}$ normal to the table. The element will slide stance

$$\delta\theta\,|\vec{w}-\vec{r}|$$

ue to the rotation $\delta\theta$ about the center of rotation $\vec{r}$, and in the process will dissipate an amount nergy

$$dE_r = \mu_s\,S(\vec{w})\,d\vec{w}\,\delta\theta\,|\vec{w}-\vec{r}|.$$

tegrating over the area of the disk, the total energy dissipated due to rotation $\delta\theta$ is

$$E_r = \delta\theta\,\mu_s\int S(\vec{w})\,|\vec{w}-\vec{r}|\,d\vec{w}$$

here we write E_r to remind ourselves that the energy is a function of the presumed location of t nter of rotation $\vec{r}$. Substituting for $\delta\theta$, we have

$$E_r = \frac{\delta x\,\mu_s\sin\alpha}{\vec{\alpha}\cdot(\vec{c}-\vec{r})}\int S(\vec{w})\,|\vec{w}-\vec{r}|\,d\vec{w}.$$

e system will find a location for $\vec{r}$ which minimizes E_r. At this minimum the derivatives of E_r w spect to both $\vec{r}_x$ and $\vec{r}_y$ must be zero. Evaluating the derivative of E_r with respect to $\vec{r}$ and settin ual to zero we find

$$\nabla E_r = \delta x\mu_s\sin\alpha\,\frac{[d_r\vec{\alpha}-\vec{v}_r\,\vec{\alpha}\cdot(\vec{c}-\vec{r})]}{[\vec{\alpha}\cdot(\vec{c}-\vec{r})]^2} = 0$$

here

$$d_r = \int S(\vec{w})\,|\vec{w}-\vec{r}|\,d\vec{w}$$

a scalar, can be physically interpreted as the weighted *distance* from the COR to the s
distribution, and

$$\vec{v}_r = \int S(\vec{w})\,\frac{\vec{w}-\vec{r}}{|\vec{w}-\vec{r}|}\,d\vec{w}$$

a vector, can be interpreted as the weighted *direction* from the COR to the support distribution.

Equation 8 expresses the physical principle that the system will execute that instantaneous
which minimizes energy. Energy is not always minimized in dissipative systems. Howev
possible to show that Newton's laws require the numerator of equation 8 to be zero at the
COR (see Appendix.) The energetic derivation above is simpler.

3.3. Iterative Numerical Solution

Minimization of E_r can be carried out in an iterative manner to find the COR for a given s
distribution $S(\vec{w})$. Figure 2-3 shows the locus of CORs obtained in this manner. Each point
COR for a randomly chosen three-point support distribution. Only support distributions consis
three points (a tripod) need be considered since according to Mason's theorem 5 three poi
sufficient. Weights were computed for the three points in such a way as to satisfy the constrai
the CM be at the center of the disk. (If this required any of the weights to be negative, the trip
discarded.) An initial guess was made for the location of the center of rotation $\vec{r}$, and ∇E_r eva
at that point.

The minimization technique used requires computation of $\nabla(\nabla E_r)$, the second derivative o
two-by-two matrix, which can be obtained analytically. A new guess for $\vec{r}$ is then made by ad
the old guess

$$\Delta\vec{r} = \frac{-\nabla E_r}{\nabla(\nabla E_r)}.$$

This method usually converged quickly if the initial guess was sufficiently close to the correct a
By moving only one leg of the tripod at a time, and by only a small amount, the value of $\vec{r}$ fo

it? iripuu ^uuiu ue u^eu eus CLII milieu yuessi iur my next, riyurt? £-o [eprt25eMu> oyuuuu inpuus, taki
CPU hours on a VAX-780. Similar figures, done with four points of support instead of tripods, <
entical.

umerical methods may converge to a local rather than a global minimum, if more than one minimi dsts. Here $\nabla E_f = 0$ if and only if Newton's laws are satisfied for a given value of the COR, i.e. e forces balance. Distinct locations of the COR correspond one-to-one with distinct motions of 1 jshed object. There can be only one motion of the object satisfying Newton's laws, so there can **ily** one minimum of E_r.

. The Quotient Locus

ssuming our analytical discussion from section 3.2, we set $\nabla E_f = 0$ in equation 8. The const* rms drop out leaving

$$r \quad a= \quad q_r[«(c- \quad r))$$ (

here we define the *quotient moment,* a vector, as

$$\vec{q}_r = r^2 \frac{\vec{v}_r}{d_r}.$$ (

is a function of $\vec{r}$ and the distribution $S(\vec{w})$, and has units of distance. In this section we *hold t)nter of rotation $\vec{r}$ fixed,* and analyze the quotient moment for all support distributions $S(\vec{w})$.

le *quotient locus* $\{\vec{q}_r\}$ is the set of $\vec{q}_r$ for all possible choices of the distribution S consistent w e requirements listed in section 2.2. It is still a function of $\vec{r}$, but the dependence on S has be moved.

e will always plot the quotient locus displaced by $\vec{r}$, i.e., based at the COR. $\{£\}$ may be plotted region of space, if we remember tha* ^ yiven ifc $\{\vec{q}_f\}$ is a *vector* with its tail at the COR and its he tywhere in that rcqicn.

e will find the boundary of the quotient locus. The results will allow us to find the boundary of t 3R locus in section 5.

$$Mg = \int S(\vec{w})\, d\vec{w} = 1 \,. \qquad (1$$

ıce multiplying the support distribution S by a constant factor changes both numerator ar nominator of $\vec{q}_r$ by that same factor, the assumption is harmless. Physically, the mass of the di: s no effect on the motion, so we can choose it arbitrarily.

1. Extrema of the Quotient Locus

ıce $\vec{v}_r$ (equation 10) can be interpreted as a weighted average of *unit vectors* from the COR to tl pport distribution, the greatest magnitude $\vec{v}_r$ can have will be 1, and will be attained when tl pport distribution is concentrated at the CM. In all other cases the direction to elements of tl pport distribution varies, and so some cancellation is inevitable. When the magnitude of $\vec{v}_r$ aximal, it must be directed from the COR to the CM.

ıe smallest magnitude $\vec{v}_r$ can achieve depends on whether the COR is inside or outside the di: ., on whether $r>a$ or $r<a$, where a is the radius of the disk. In either case we wish to achieve t aximum amount of cancellation of direction possible. If $r>a$ this occurs when the supp stribution consists of two points at opposite edges of the disk, providing the minimum possil reement on direction between the two vectors, as shown in figure 4-1.

$r<a$, we can arrange for $\vec{v}_r$ to be zero. Indeed we can arrange for $\vec{v}_r$ to point from the C(aximally *away* from the CM by making a two-point support distribution as shown in figure 4-2. e figure the distance from w_2 to the COR is infinitesimal.) The two vectors $\vec{w}_1$ and $\vec{w}_2$ point ›posite directions. To maintain the centroid of the support distribution at the CM, we find eights of $\vec{w}_1$ and $\vec{w}_2$ are

$$S_1 = \frac{r}{r+a}, \text{ and} \qquad ($$

$$S_2 = \frac{a}{r+a}.$$

ıerefore $\vec{w}_2$ is more heavily weighted than $\vec{w}_1$, and

$$\vec{v}_r = S_1\vec{v}_1 + S_2\vec{v}_2 = (S_2 - S_1)\vec{v}_2 = \frac{a-r}{a+r}\vec{v}_2 \qquad ($$

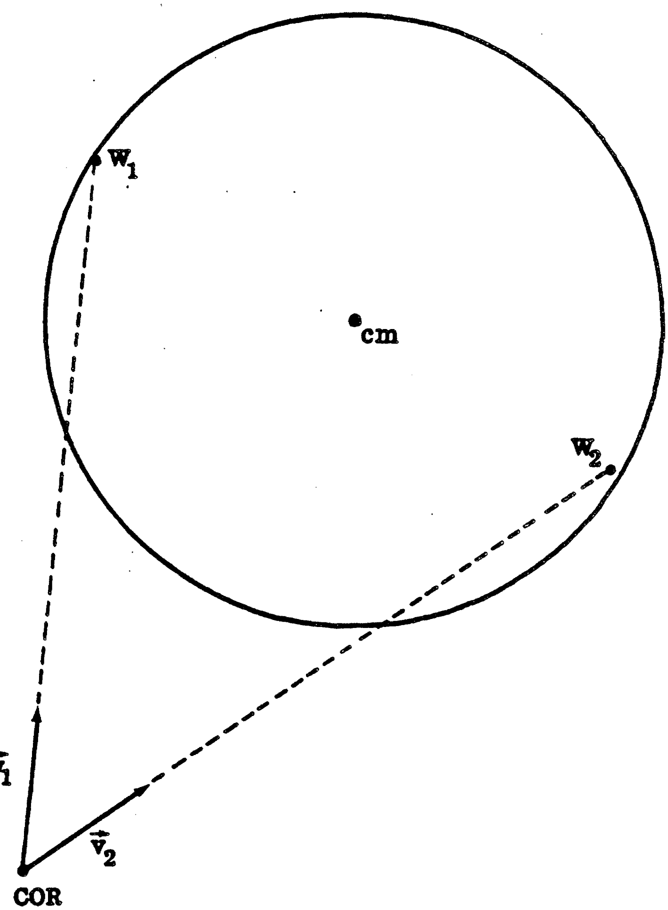


Figure 4-1: Dipod responsible for the smallest value of $\vec{v}_r$, for $r > a$

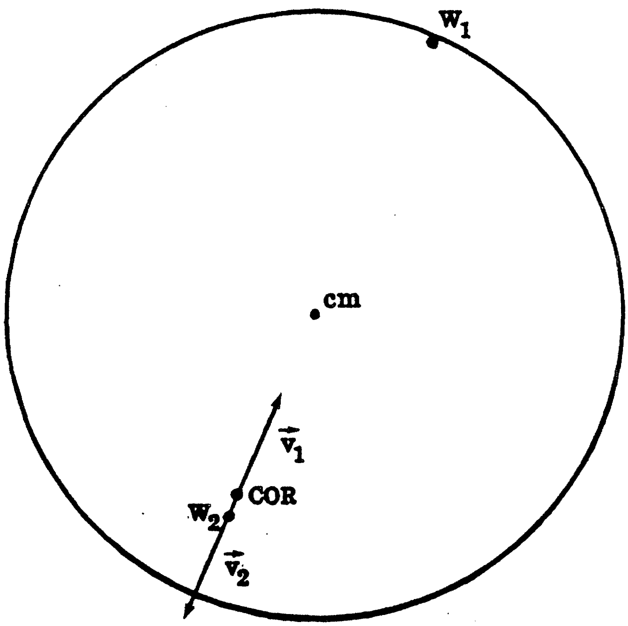


Figure 4-2: Dipod responsible for a negative value of $\vec{v}_r$, for $r < a$

ow consider d_r (equation 9). Clearly if the support distribution is concentrated at the CM, eighted distance from the COR to the support distribution is just r. In fact r is the smallest va hich d_r can attain. In the configuration shown in figure 4-2,

$$d_r = S_1 \cdot (a + r) + S_2 \cdot 0 = r . \qquad ($$

$_r$ takes on its maximum value when the support distribution consists of two points as in figure 4 hat value is

$$d_r = (r^2 + a^2)^{1/2} . \qquad ($$

ince $\vec{q}_r$ is the quotient of $\vec{v}_r$ and d_r, extreme values of $|\vec{q}_r|$ occur when $\vec{v}_r$ is maximal and inimal, and when $\vec{v}_r$ is minimal and d_r maximal.

.2. Numerical Solution for the Quotient Locus

aving found the extreme values of $\vec{q}_r$ above, we can find the locus of all possible quotie umerically. It is much easier to find the $\{\vec{q}_r\}$ locus (for a given value of $\vec{r}$) than it is to find the C cus. No iteration is required; for a given tripod, $\vec{v}_r$ and d_r can be calculated immediately. Figu 3 and 4-4 show typical $\{\vec{q}_r\}$ loci for $r < a$ and $r > a$, respectively. The dots are values of $\vec{q}_r$ fou umerically, while the solid curve is the empirical boundary of the locus as described below.

he dots in figures 4-3 and 4-4 represent over 3,000,000 and 500,000 randomly chosen tripo espectively. The solid curves which appear to bound the dots are generated by two classes pods, discussed below. Since dipods are a special case of tripods, the solid curve is in $\{\vec{q}_r\}$. e basis of numerical studies such as shown in these figures, we believe that no value of enerated by a tripod falls outside the dipod curve. Therefore the dipod curve is the exact bound $\{\vec{q}_r\}$. We have not been able to prove analytically that no tripod value of $\vec{q}_r$ falls outside the dip urve, so the boundaries should be considered empirically justified only.

3. Boundary for |COR| < a

e observe that for $r < a$ the boundary of the locus is a circle. This empirical boundary can enerated by two-point support distributions (dipods) of the type shown in figure 4-5, where the an can vary. These dipods are a generalization of the one shown in figure 4-2. (The distance from $\vec{r}$

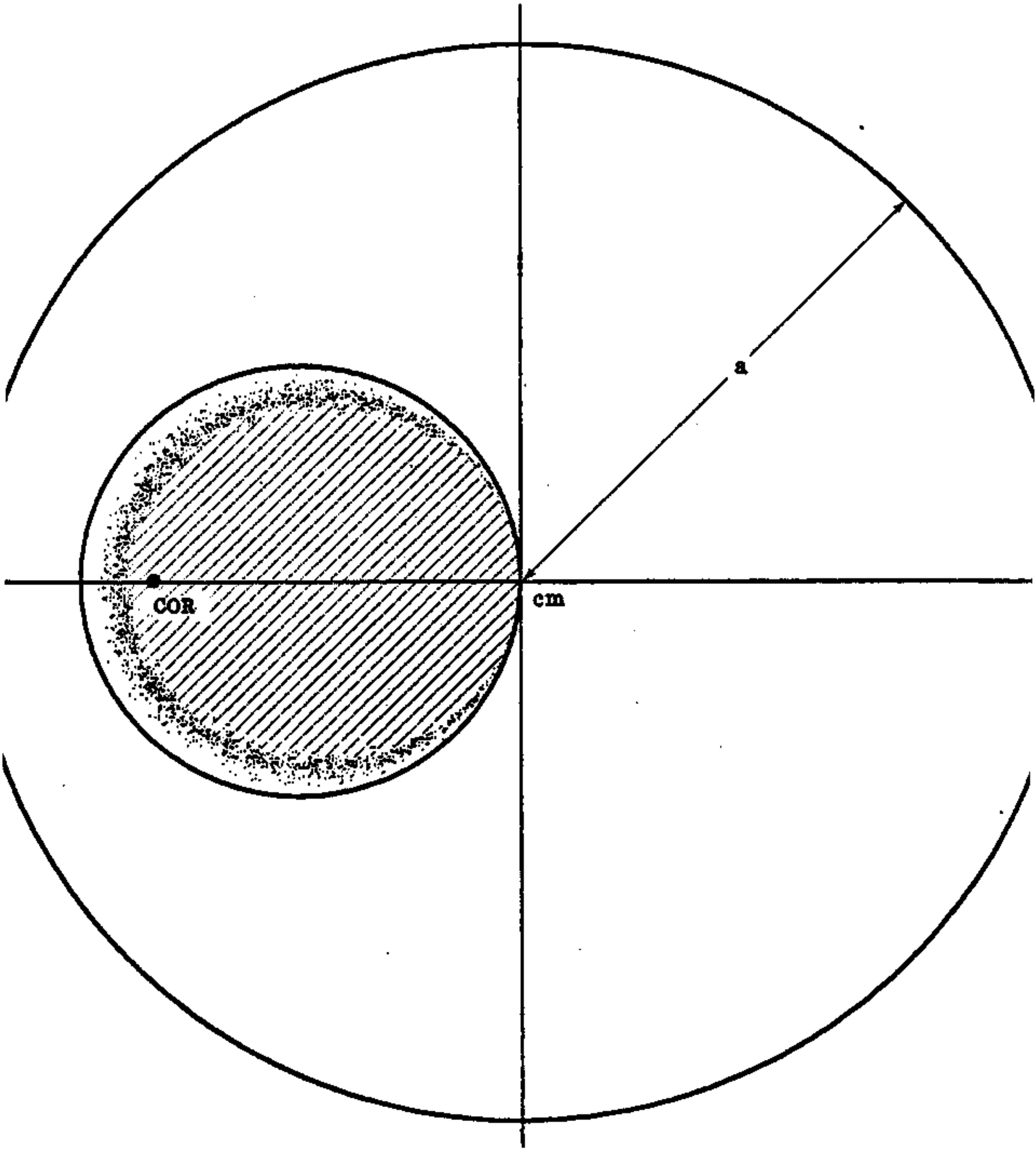


Figure 4-3: Quotient locus $\{t_r\}$ (dots), and empirical boundary (solid), **for** $r<a$

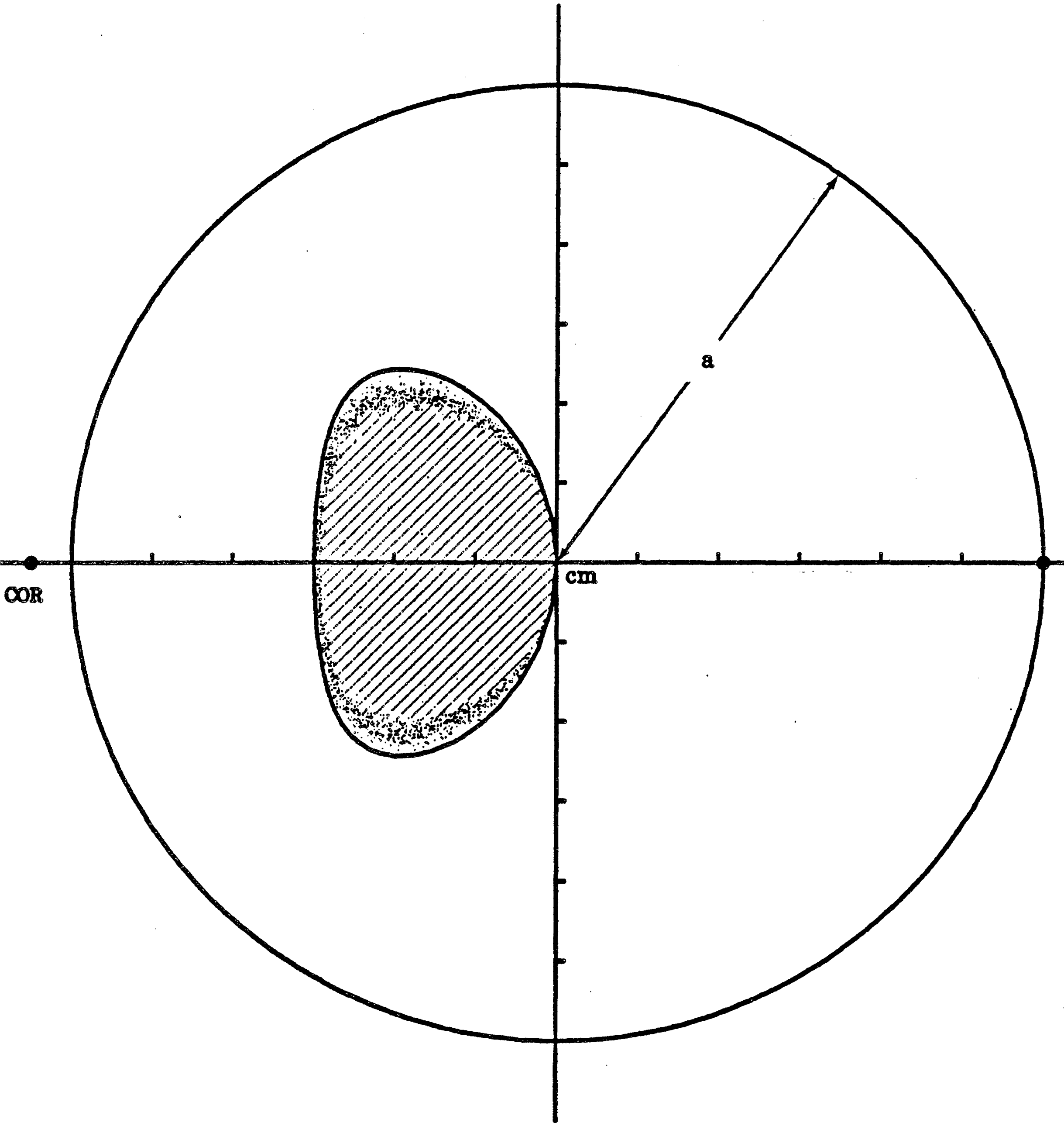


Figure 4-4: Quotient locus $\{\vec{q}_r\}$ (dots), and empirical boundary (solid), for $r > a$

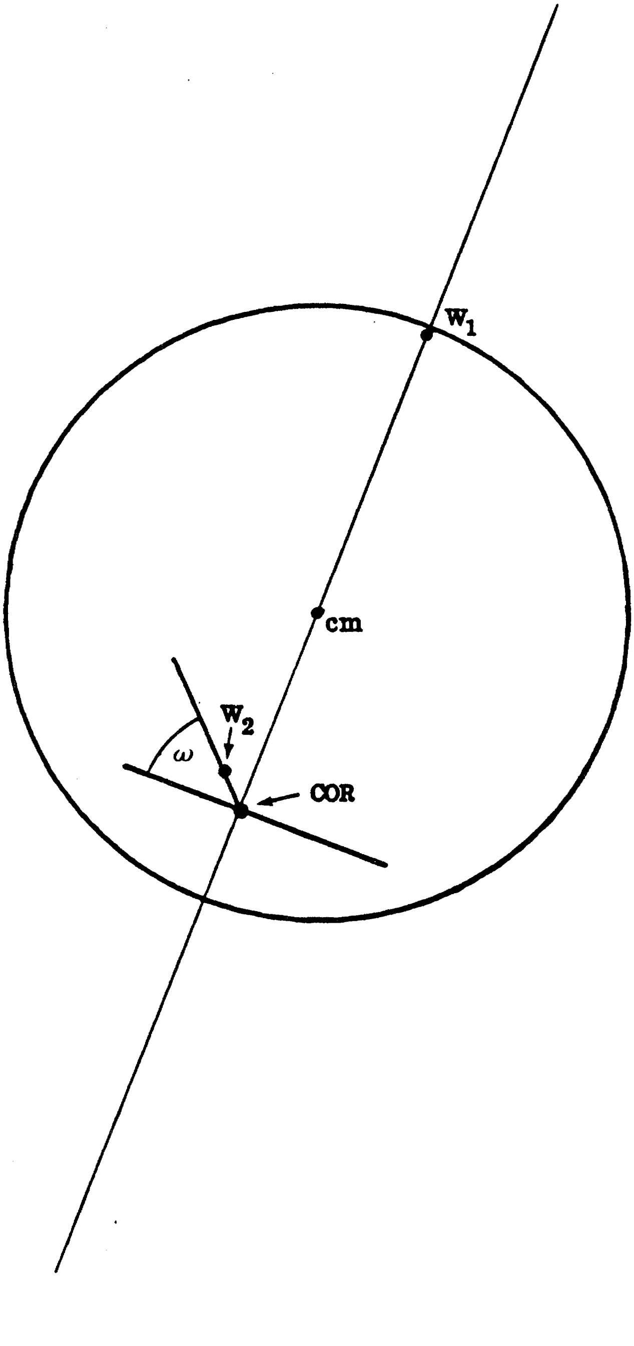


Figure 4-5: Dipods contributing to the boundary of $\{\vec{q}_r\}$, for $r < a$

$$q_r = w \frac{;—\backslash UOJ - r)}{r + a}$$

where <o = (cos w, sin <J)

his generates a circle cf radius

$$r + a'$$

.4. Boundary **for** |COR| > a

or $r > a$, the empirical boundary of the locus $\{\vec{q}_r\}$ is generated by dipods of the type shown in figi 6, where o> is allowed to vary. These dipods are a generalization of the dipod shown in figure 4 gain, the boundary can be calculated parametrically from w (via intermediate terms </+,rf~, y+, a ")as

$$d^{\pm} = (r^2 + a^2 \pm 2 a r \cos \omega)^{1/2}$$

$$\sin \gamma^{\pm} = \frac{a \sin \omega}{d^{\pm}}$$

$$\cos \gamma^{\pm} = (1 - \sin^2 \gamma^{\pm})^{1/2}$$

$$\vec{v}_r = \left(\frac{\cos \gamma^+ + \cos \gamma^-}{2}, \frac{\sin \gamma^+ - \sin \gamma^-}{2} \right)$$

$$d_r = \frac{d^+ + d^-}{2}$$

$$\vec{q}_r = r^2 \frac{\vec{v}_r}{d_r}.$$

is the boundaries of $\{\vec{q}_r\}$ that will be used (in section 5) to determine the boundaries of the CC cus. Therefore the boundaries of the COR locus, toos can be found by considering only dipoc lis is a stronger statement than Mason's theorem 5, which requires tripods. Additionally, we ha und the two points constiiuting the dipods. Howevc, it should be noted that the sufficiency pods ho!ds for any object, whereas dipods are sufficient only for a disk.

gures 4-3 and 4-4 demonstrate that the two classes of dipods considered above, and illustrated lures 4-5 and 4-6, generate extremal quotient moments. In other words, the locus {^} of values for all distributions of support $S(\vec{w})$ satisfying the conditions of section 2.2 fall inside the empirk

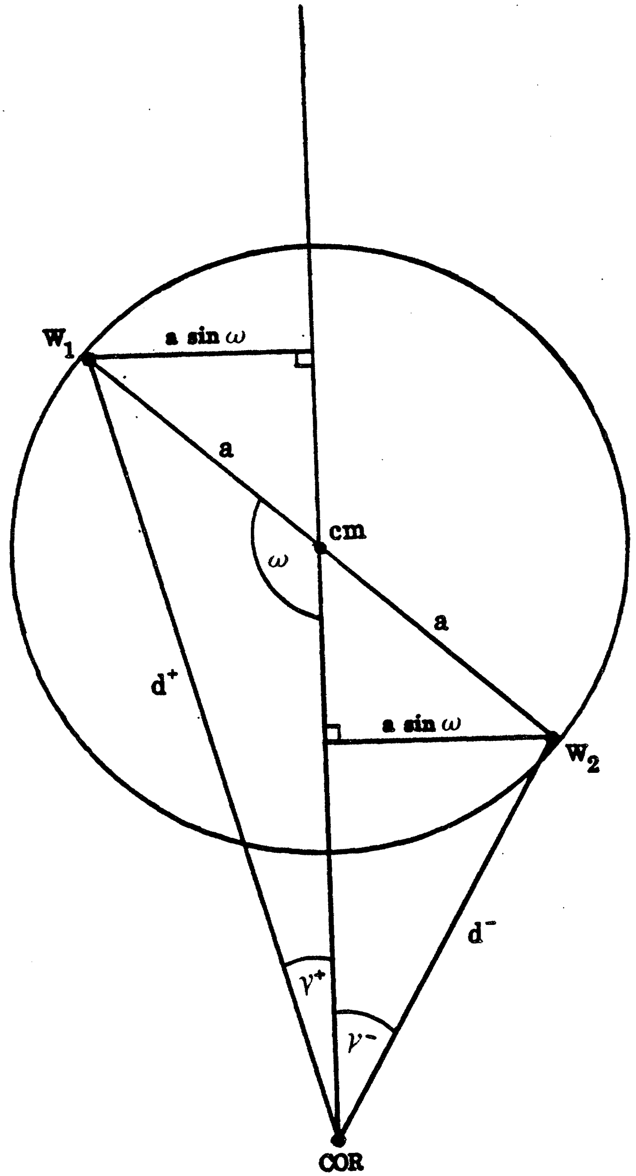


Figure 4-6: Dipods contributing to the boundary of $\{\vec{q}_r\}$, for $r > a$

\. The COR Locus

aving found a parametric representation of the {^.} locus, we can find the COR locus. Recall t jquirement for minimizing the energy lost to friction (equation 12):

$$r^2 a \quad = \quad q_f \quad [a(c- \quad r)]. \qquad (.$$

ie COR locus is the set of all *t* for which there exists a $\$_r$ c $\{\vec{q}_r\}$ satisfying equation 22.

quation 22 is a vector equation. The left side obtains its direction from $\vec{a}$. The right side obtains rection from 2^, since <?•(?- $\vec{r}$) is a scalar. To satisfy the vector equation ^ must have directi , We can rewrite equation 22 in scalar form, retaining the direction constraint on $\$_r$ separately:

'2 = Iff, I It-it-t)] C

where t_r € $it_r\}$

and $\vec{q}_r \parallel \vec{\alpha}$

e wish to find the locus of $\vec{r}$ for all distributions $S(\vec{w})$. It is best to imagine $\vec{r}$ to be an independe triable. Each value of *t* yields a locus $\{\vec{q}_r\}$, with one element t_r € {^} corresponding to ea stribution $S(\vec{w})$. For some values of $\vec{r}$ the value of $\vec{q}_r$ required to satisfy equation 23 is in $\{\vec{q}_r\}$; I her values it is not. The former values constitute the COR locus.

is confusing, but unavoidable, that the locus $\{\vec{q}_f\}$ shifts as we consider different locations of t inter of rotation 1\ In figure 5-1 we have plotted several $\{\vec{q}_r\}$ loci for different values of $\vec{r}$. Note t\ .rying the *magnitude* of ^continuously changes the shape or size of the $\{\vec{q}_r\}$ loch But changing t *rection* of $\vec{r}$^only causes a corresponding rotation of the {^} locus.

ie variables of equation 23 are shown geometrically in figures 5-2, 5-3, and 5-4. In each figure i tve plotted a value of $\vec{r}$'and the locus $\{\vec{q}_r\}$ for that $\vec{r}$. We then calculate and plot the value of quired to satisfy equation 23. In figure 5-2, the value of $\vec{q}_r$ required does not fall in {^}, so t lue of $\vec{r}$ shown is not in the COR locus. In figure 5-3, the value of $\vec{q}_r$ required does fall in {^}, 3 value of $\vec{r}$ shown is in the COR locus.

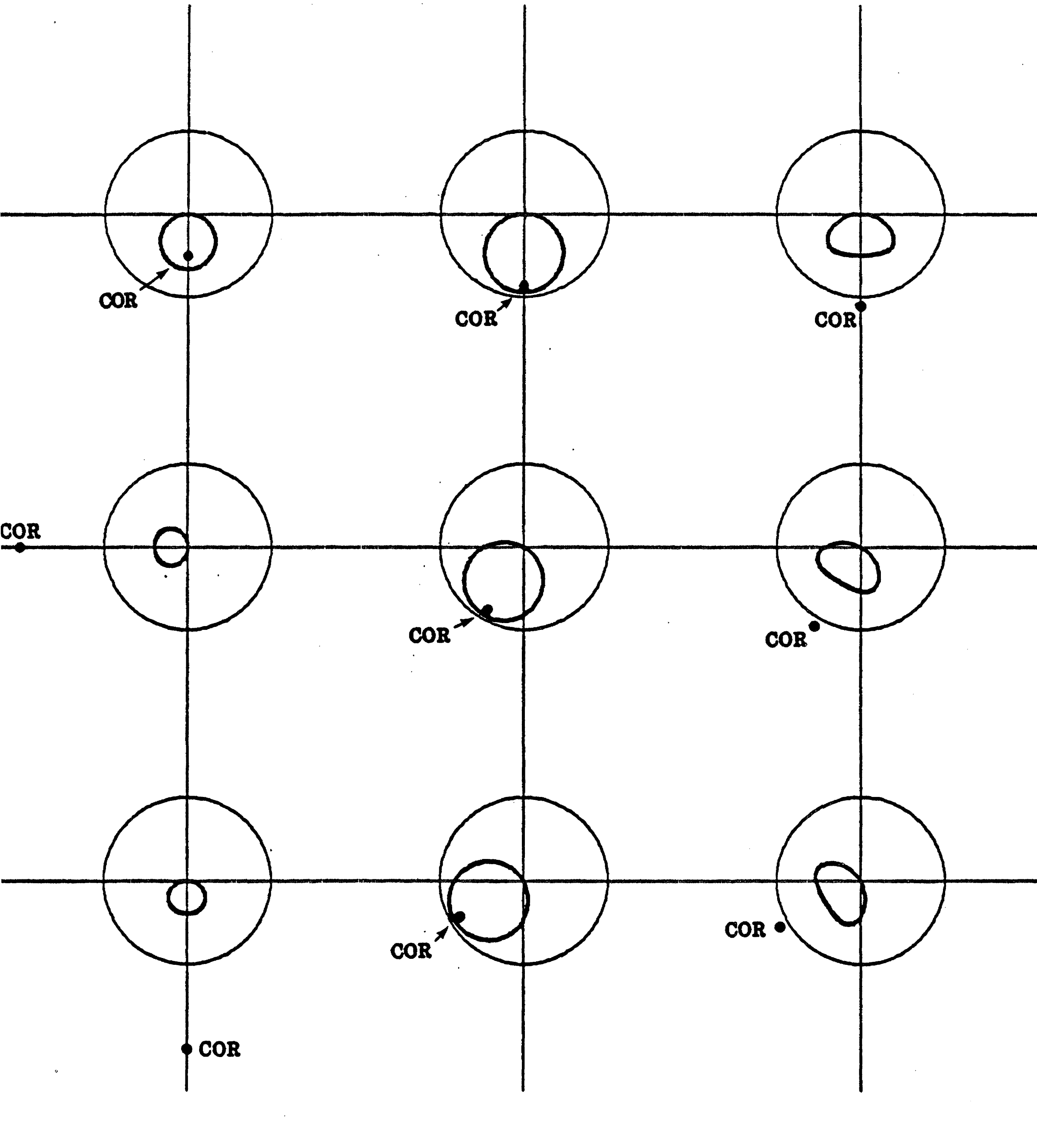


Figure 5-1: Boundaries of quotient loci $\{\vec{q}_r\}$ for various $\vec{r}$

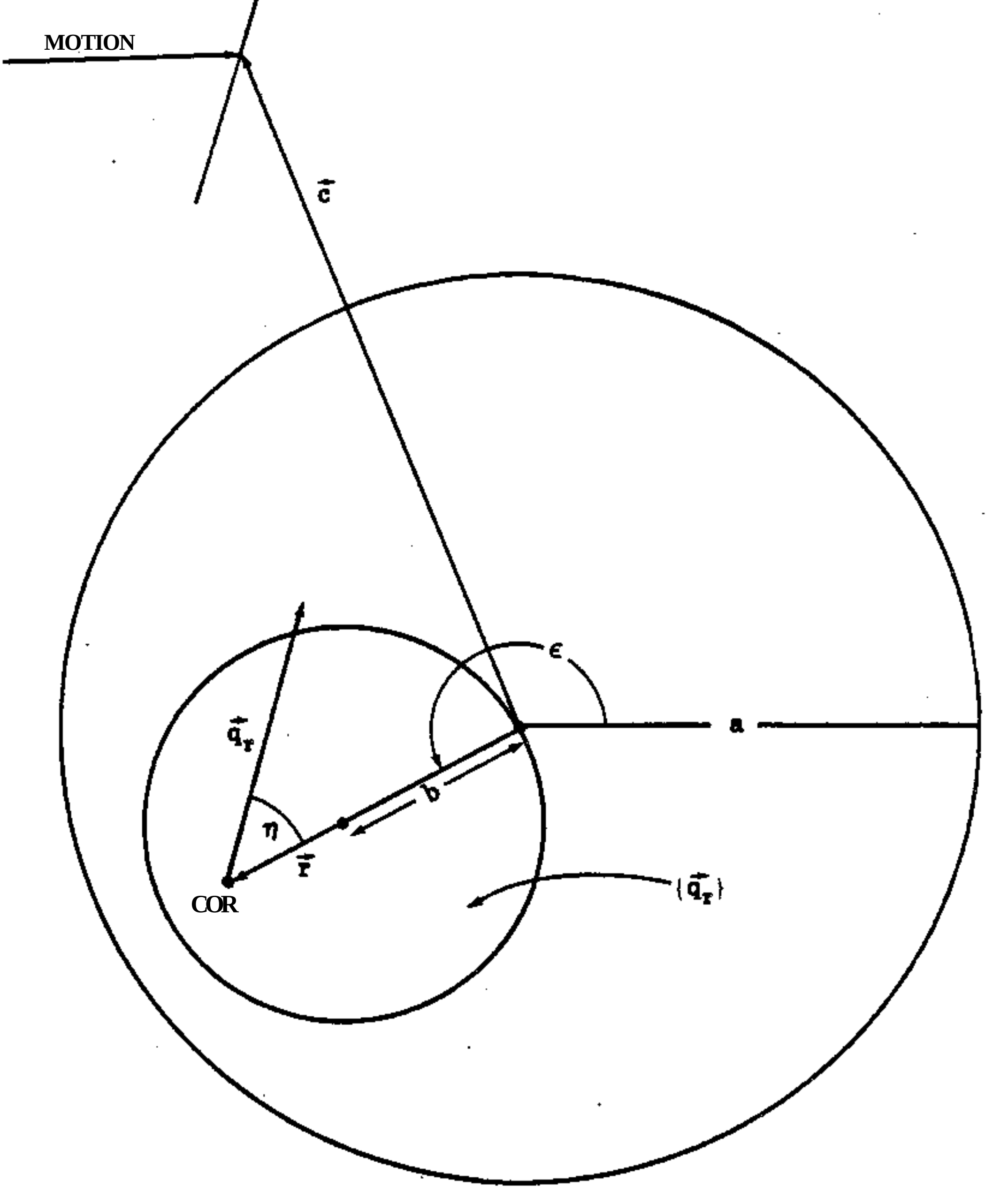


Figure 5-2: Variables of equation 23, for a value of ? not in the COR locus

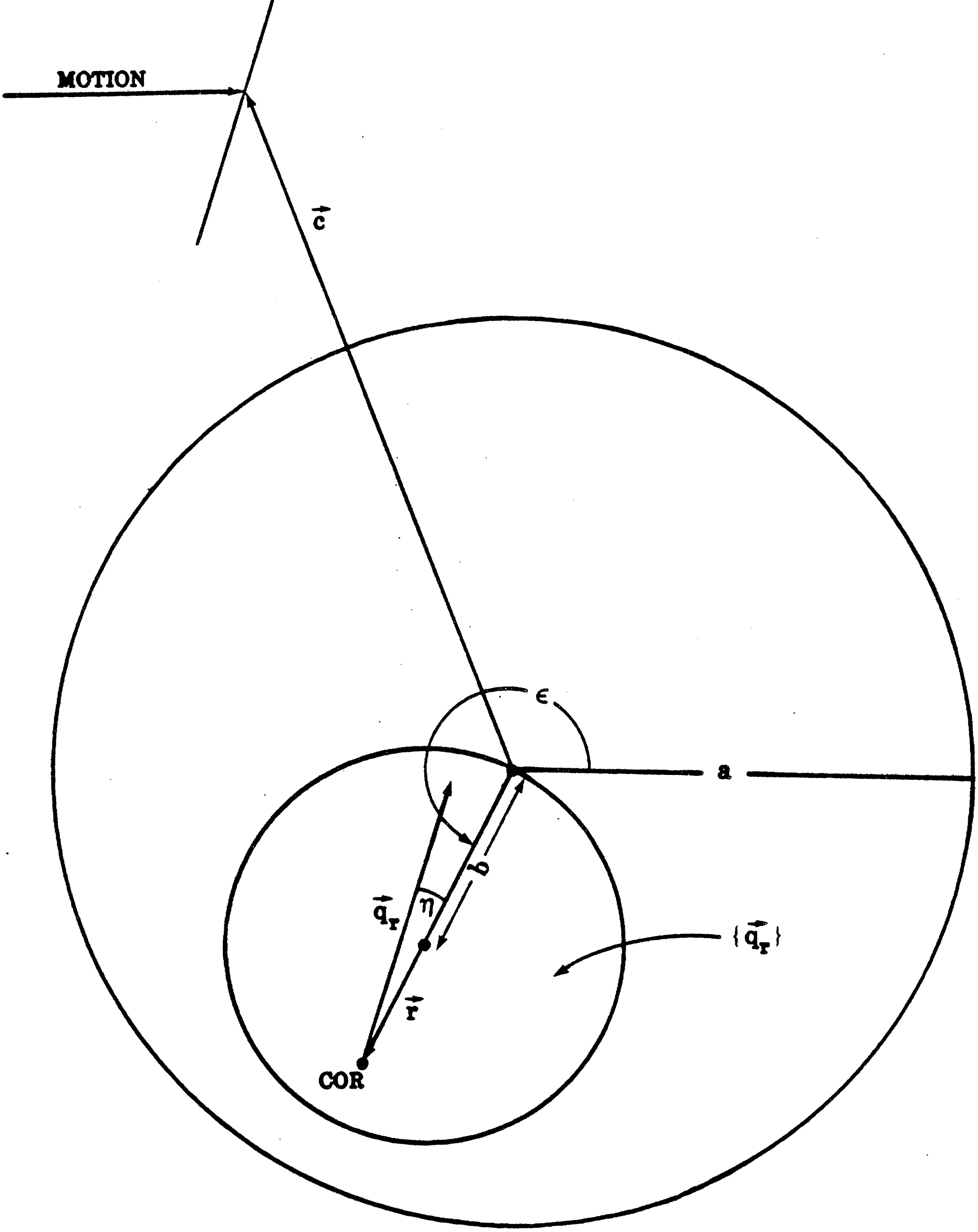


Figure 5-3: Variables of equation 23, for a value of $\vec{r}$ in the COR locus

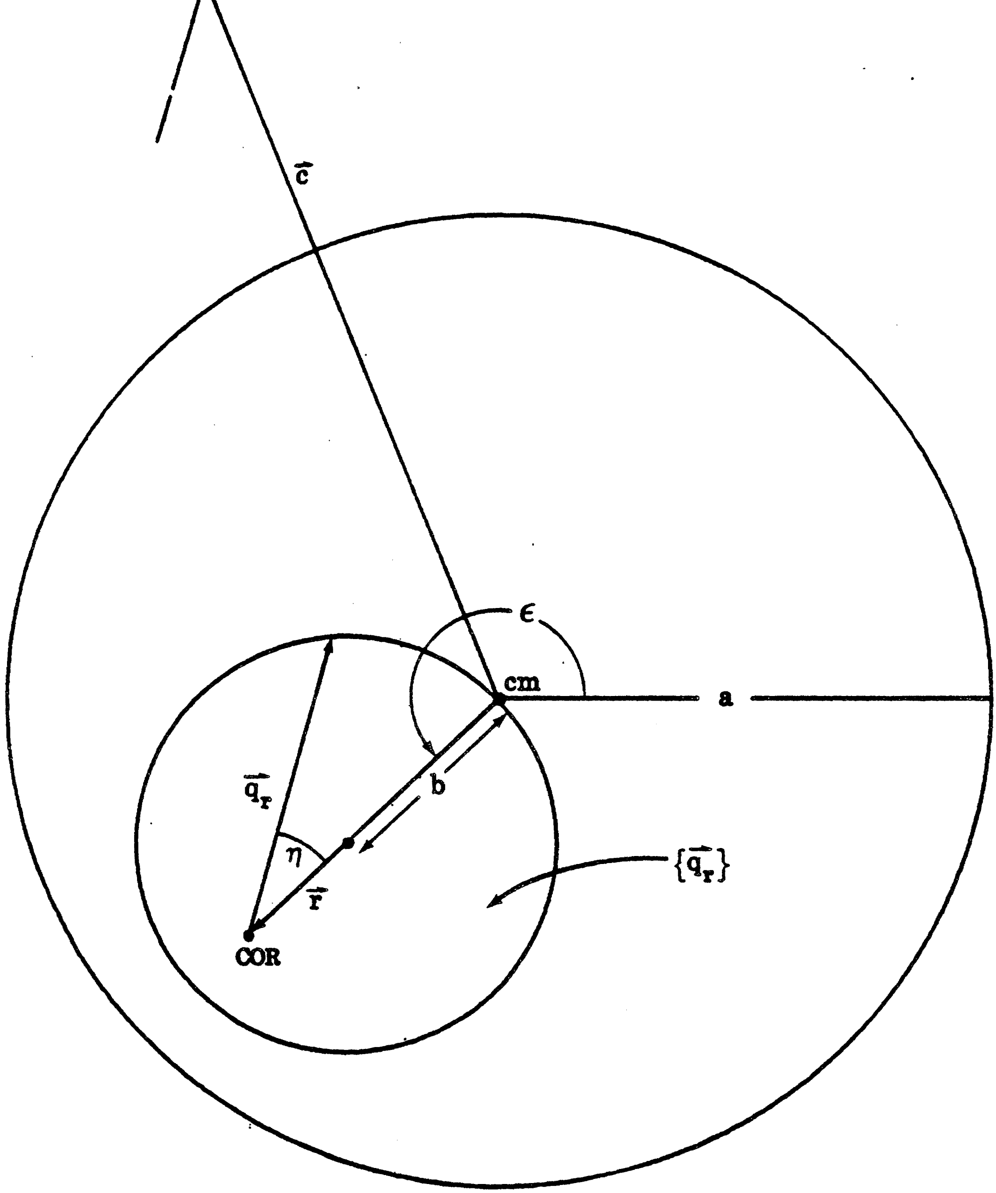


Figure 5-4: Variables of equation 23, for a value of $\vec{r}$ on the boundary of the COR locu

$\{\vec{q}_r\}$ locus. The boundary of the COR locus is generated by such cases. Interior points of the CO[R]
[lo]cus are generated when the $\vec{q}_r$ required is interior to the $\{\vec{q}_r\}$ locus, as in figure 5-3. Since we a[re]
[in]terested only in the boundary of the COR locus, we will consider only values of $\vec{q}_r$ which are on t[he]
[b]oundary of the $\{\vec{q}_r\}$ locus, as shown.

.1. |COR| < a

[It] will be convenient to represent the COR by its polar coordinates (r, ε), and to define the relat[ive]
[a]ngle η. Both angles are shown in figure 5-4. We have

$$\varepsilon = \pi + \alpha - \eta \, .$$ (

[For] $r < a$, the boundary of $\{\vec{q}_r\}$ is a circle. The condition that $\vec{q}_r$ lie on the circle can be expressed

$$\left| \, |\vec{q}_r| \; \vec{\alpha} + (r - b)\vec{\varepsilon} \right| = b \, .$$ (

[W]here b is the radius of the circle, from equation 20. Equation 25 can be expressed in terms of t[he]
[a]ngle η as

$$(|\vec{q}_r| - (r-b)\cos\eta)^2 + ((r-b)\sin\eta)^2 = b^2 \, .$$ (

[So]lving this quadratic equation for $|\vec{q}_r|$ we find

$$|\vec{q}_r| = (r-b)\cos\eta \pm (b^2 - ((r-b)\sin\eta)^2)^{1/2} \, .$$ (

[In]serting this value of $|\vec{q}_r|$ into equation 23 and eliminating the square root we obtain

$$\left(\frac{r^2}{\vec{\alpha}\cdot(\vec{c}-\vec{r})} - (r-b)\cos\eta\right)^2 = b^2 - ((r-b)\sin\eta)^2 \, .$$ (

[S]ubstituting b from equation 20 and simplifying we find

$$r^2(a+r) + (r-a)\,[\vec{\alpha}\cdot(\vec{c}-\vec{r})]^2 - 2\,r^2\,[\vec{\alpha}\cdot(\vec{c}-\vec{r})]\cos\eta = 0$$ (

where $[\vec{\alpha}\cdot(\vec{c}-\vec{r})] = \vec{\alpha}\cdot\vec{c} + r\cos\eta \, .$

[Eq]uation 29 is a cubic in r and quadratic in $\cos\eta$. The solution for $\cos\eta$ is

$$\cos \eta = \frac{r\{(r+a)^2 + \vec{}(a\text{-}?)^1)^{1/2} - a(\vec{a}\text{-}?)}{r(r+a)}.$$ (.

ie other quadratic root is invalid. Since TJ is related by equation 24 to the polar angle e, equation ascribes the boundary of the COR locus in the polar coordinates r, *e,* for *Ka.* A typical COR loc aerated using equation 30 is shown in figure 5-5.

.1.1. Extremal Radius of **the COR** Locus Boundary **for** |C0R|<a

ie minimum radius of the COR locus boundary occurs at e = a, which corresponds to TJ = TT. Frc juation 29 we find

$$r_{min} = \frac{a(\vec{\alpha}\cdot\vec{c})}{2a + (\vec{\alpha}\cdot\vec{c})}.$$ (:

)te that r_{mm} . is not the minimum distance from the CM to an *element* of the COR locus; that distan zero, r_{tnin} . is the minimum distance from the CM to the *boundary* of the COR locus, r_{tntn} . is indicat figure 5-5.

will also be useful to have the angles at which the COR locus boundary intersects the di •undary. From equation 30 we obtain

$$\cos \eta_{r=a} = \frac{((\vec{\alpha}\cdot\vec{c})^2 + 4a^2)^{1/2} - (\vec{\alpha}\cdot\vec{c})}{2a}$$ (:

1.2. Curvature of the COR Locus Boundary at r_{mjn}

om equation 29 we can find the curvature of the COR locus boundary at r_{min}:

$$\frac{d^2r}{dt^2} = \frac{2a^2(\vec{\alpha}\cdot\vec{c})((\vec{\alpha}\cdot\vec{c})^2 + 2a(\vec{\alpha}\cdot\vec{c}) + 2a^2)}{((\vec{\alpha}\cdot ?) + 2a)^4}$$ (:

lich is equivalent **to** a radius **of** curvature **of**

$$s = \frac{a(\vec{a}\cdot ?)((\vec{a}\cdot ?) + 2a)^2}{(\vec{\alpha}\cdot\vec{c})^3 + 4a(\vec{\alpha}\cdot ?)^2 + 8a^2(\vec{\alpha}\cdot\vec{c}) + 4a^3}.$$ (:

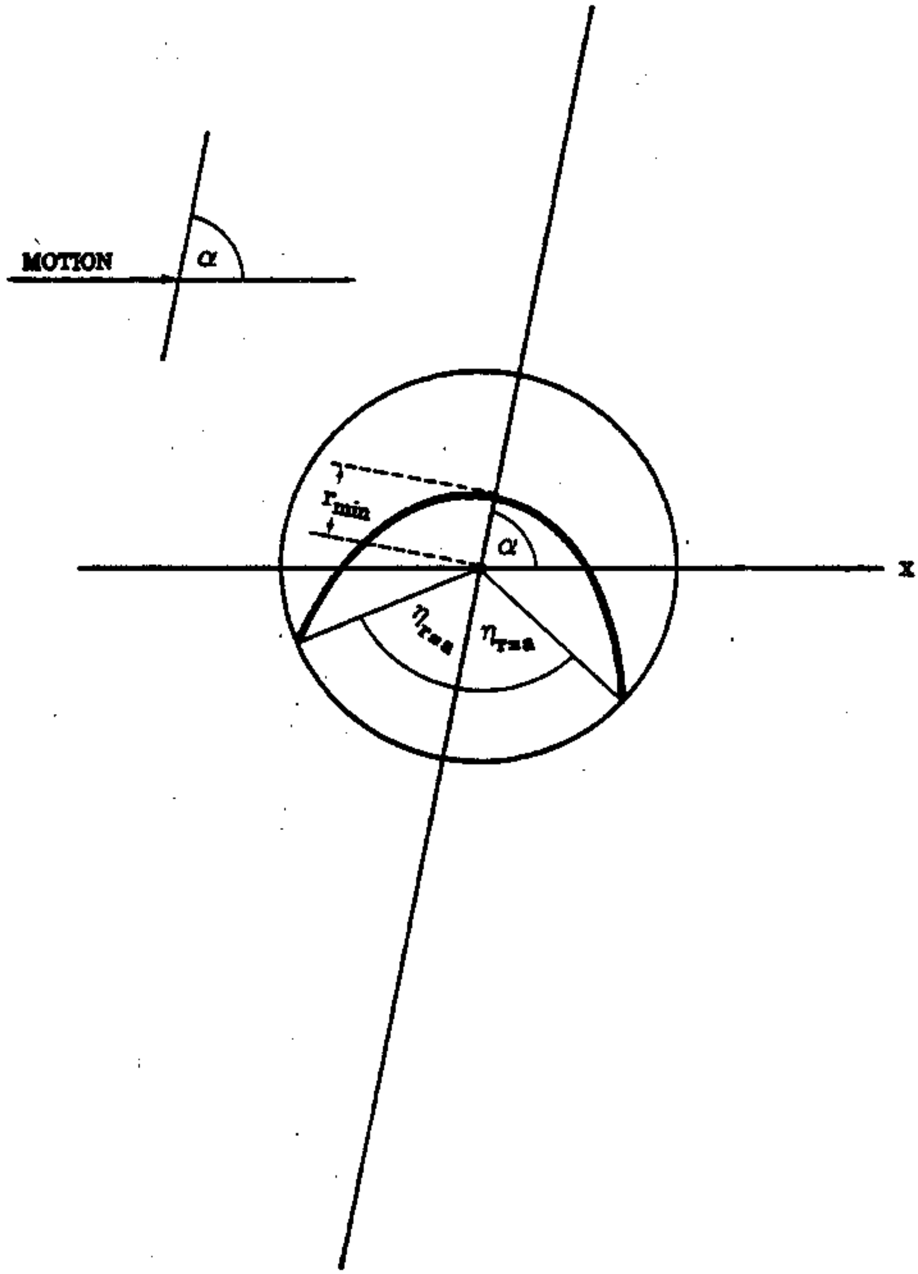


Figure 5-5: COR locus boundary for *Ka*

.2. |COR| > a

$r > a$, we cannot find a simple equation analogous to equation 25 constraining $\vec{q}_r$ to the boundary $\{\vec{q}_r\}$. An effective approach is to parametrize the boundary of the $\{\vec{q}_r\}$ locus by the angle ω quation 21, and solve for both ε and r by binary search.

or each ω the following procedure is used: We guess a value of r, in the range $a < r < r_{tip}$, where an upper bound to be found in section 5.2.1. Equation 21 is then used to calculate a value of ngle η is related to the terms of equation 21 by

$$\eta = \arctan \frac{-\vec{v}_x}{\vec{v}_y} \qquad ($$

nd so can be computed from ω. Equation 23 can be written in terms of the angle η as

$$r^2 = |\vec{q}_r| \, (\vec{\alpha}\cdot\vec{c} + r\cos\eta) \qquad ($$

hich is easily tested. If it is satisfied, we have found angle η and magnitude r describing a point e boundary of the COR locus. ε is then obtained from η using equation 24.

the left-hand side of equation 36 is greater (resp. less) than the right-hand side, we increase (re ecrease) the value of r guessed above. In this way we perform a binary search, quickly convergi a solution for r and ε.

gure 5-6 shows the boundary of the COR locus for various $\vec{c}$ and α. The part of the boundary insi e disk was computed using equation 30, while the part outside the disk was found by binary sear outlined here. Calculation of each locus required about 2 CPU seconds on a VAX-780.

2.1. Tip Line

e can calculate the extremum of the COR locus analytically. For many purposes this may be all t required. Additionally, it gives us a range within which to conduct the binary search discussed ction 5.2. By symmetry, r takes on an extremal value when $\eta = 0$. In figure 4-6 this corresponds $= 0$, which in turn occurs only when $\omega = 0$ or $\omega = \frac{\pi}{2}$.

$\omega = 0$ we find from equation 21

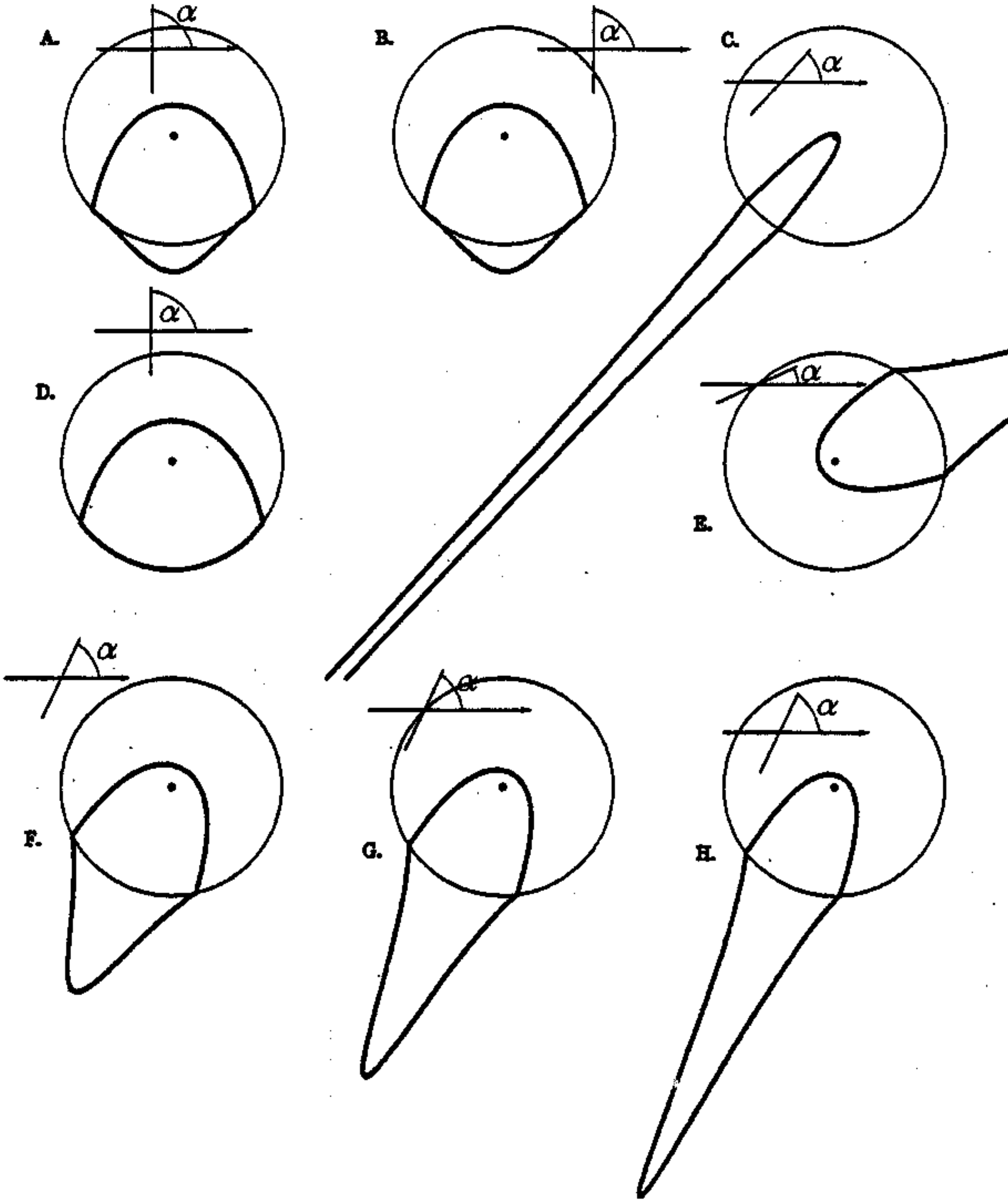


Figure 5-6: Boundaries of COR loci for various $\bar{c}^*$ and a

ubstituting this into equation 23 we obtain

$$r = \frac{\vec{\alpha} \cdot \vec{c}}{2}. \qquad ($$

his extremum has no apparent meaning.

t $\omega = \frac{\pi}{2}$ we find from equation 21

$$\vec{q}_r = \vec{\alpha}\frac{r^3}{r^2 + a^2}. \qquad ($$

t this value equation 23 yields

$$r_{tip} = \frac{a^2}{\vec{\alpha} \cdot \vec{c}}. \qquad ($$

his is the greatest distance $\vec{r}$ may be from the CM, and it occurs at polar angle $\varepsilon = \pi + \alpha$. In figu 7 we plot r_{tip} vs. contact angle α, for a given value of $\vec{c}$. As α is varied, the tip of the COR locus stance r_{tip} from the CM traces out straight line, the *tip line*.

he use of this graphical construction is illustrated in figure 5-7. For a given value of α, as shown, at the intersection of the tip line described above with a ray from the CM at angle $\pi + \alpha$.

n interesting case occurs when $\vec{\alpha}$ becomes perpendicular to $\vec{c}$. (Note that this does not requ $= \frac{\pi}{2}$.) As $\vec{\alpha} \cdot \vec{c} \to 0$, we have $r_{tip} \to \infty$. The COR at infinity corresponds to pure translati erpendicular to $\vec{a}$.

ason (Mason, 1985) showed that if $\vec{\alpha} \cdot \vec{c} = 0$, pure translation is assured. In that case the *entire* C cus (not just its farthest point) must go to infinity. In all other cases $\vec{\alpha} \cdot \vec{c} \neq 0$, and the CM (for or mains in the COR locus. Figure 5-6c shows a case in which $\vec{\alpha}$ is almost perpendicular to ustrating the transition from a COR locus entirely at infinity to one which includes the CM.

e now have the ability to quickly compute the COR locus for any $\vec{c}$ and α.

2.2. Curvature of the COR Locus Bound at the Tip

he radius of curvature of the COR locus boundary at the tip can be found analytically, and will pro

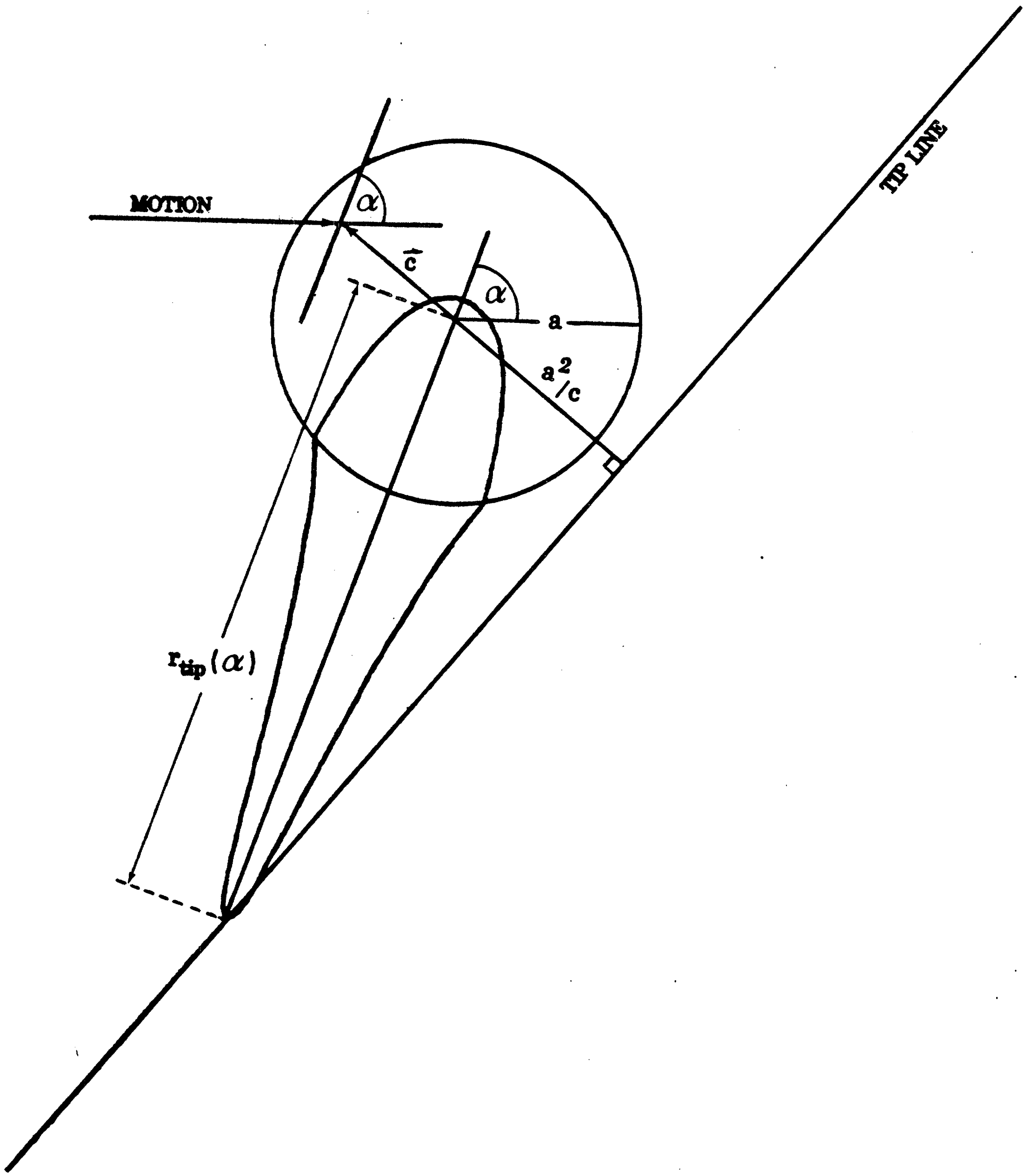


Figure 5-7: $r_{tip}(\alpha)$ vs. α, and construction of the tip line

rom equations 21 and 35 we compute taylor expansions for η and q_r to lowest order ω^2, at ω = he expansions are substituted into equation 36, and the result expanded about $\omega = \frac{\pi}{2}$ to low rder ω^2. Finally we expand about $r = r_{tip}$ to order r^1.

/e find

$$\frac{d^2r}{d\varepsilon^2} = r_{tip}\left(\frac{1}{2} - \frac{r_{tip}^4}{a^4}\right) \quad (\text{}$$

hich corresponds to a radius of curvature

$$s = \frac{r_{tip}}{\frac{1}{2} + \frac{r_{tip}^4}{a^4}}. \quad (\text{}$$

.3. Summary

this section we have found the boundary of the COR locus for any choice of $\vec{c}$ and α. Within t sk the boundary is given by a simple formula relating r and ε, the polar coordinates of the bound quation 30). Outside of the disk, the polar coordinates of the boundary are found by binary sear outlined in section 5.2. We have found the minimum and maximum distances from the CM to t oundary (r_{min} and r_{tip}), and the curvature of the boundary at those points. And we have found t olar angles at which the boundary of the COR locus intersects the boundary of the disk (equati ?).

. Applications

useful application of the results found above is to the problem of aligning an object by pushing it. jure 1-2 a misoriented rectangle is being pushed by a fence. The fence is moving in a directi rpendicular to its front edge. Evidently the rectangle will rotate CW as the fence advances (Maso 85), and will cease to rotate when the edge of the rectangle comes into contact with the front ed the fence (Brost, 1985). The problem is to find how far the fence must advance to assure that t N motion is complete.

e geometry of this problem differs from the geometry used in previous sections. Previously a po sher made contact with a straight object edge. Here the straight edge of the pusher makes conta ıject is normal to the edge, regardless of whether the edge is that of the pusher or that of the objec

nce the motion of the object can depend only on the force applied to it, the angle of the fence tak e place of the angle of the object edge (α), and all the results derived above remain unchanged.

this section we will generalize the problem slightly, relative to the problem illustrated in figure 1-2:

- The object pushed is arbitrary, not a rectangle.
- The motion of the fence is not necessarily perpendicular to its face.

rst we circumscribe a disk of radius a about the object. The disk is centered at the CM of the obje gure 6-1). Note that the contact point need not be on the perimeter of circumscribed disk.

e know (Mason, 1985) that the object will rotate CW, and will cease to rotate when the fir ınfiguration shown in figure 6-2 is reached.

e now ask the rate of rotation of the object about the COR, with unit advance of the pusher. Let t ıgle of the CM from the direction of motion of the pusher be β. This is also the angle between the e and the perpendicular to the line of motion. (Both angles are indicated in figure 6-1). We have

$$dx = y\, d\beta \qquad (4$$

ıere y is the distance from the line of motion to the COR. To find the minimum rate of rotation ust find the maximum value of y for any COR in the COR locus.

gures 6-1 and 6-2 also show the COR loci for the pushed disk. The COR furthest from the line otion is at point A in figure 6-2 . The tip of the COR locus is at point B. The maximum value of y ıy COR in the locus is the distance from the line of motion to point A. It is well approximated by t rtical distance from the line of motion to point B. We will first solve the fence-push problem usi is approximation, and then bound the difference between the answer thus obtained and the exa ıswer.

om figure 6-3 we find

$$y_{approx} = c \sin\beta + r_{tip} \sin\alpha \qquad (4$$

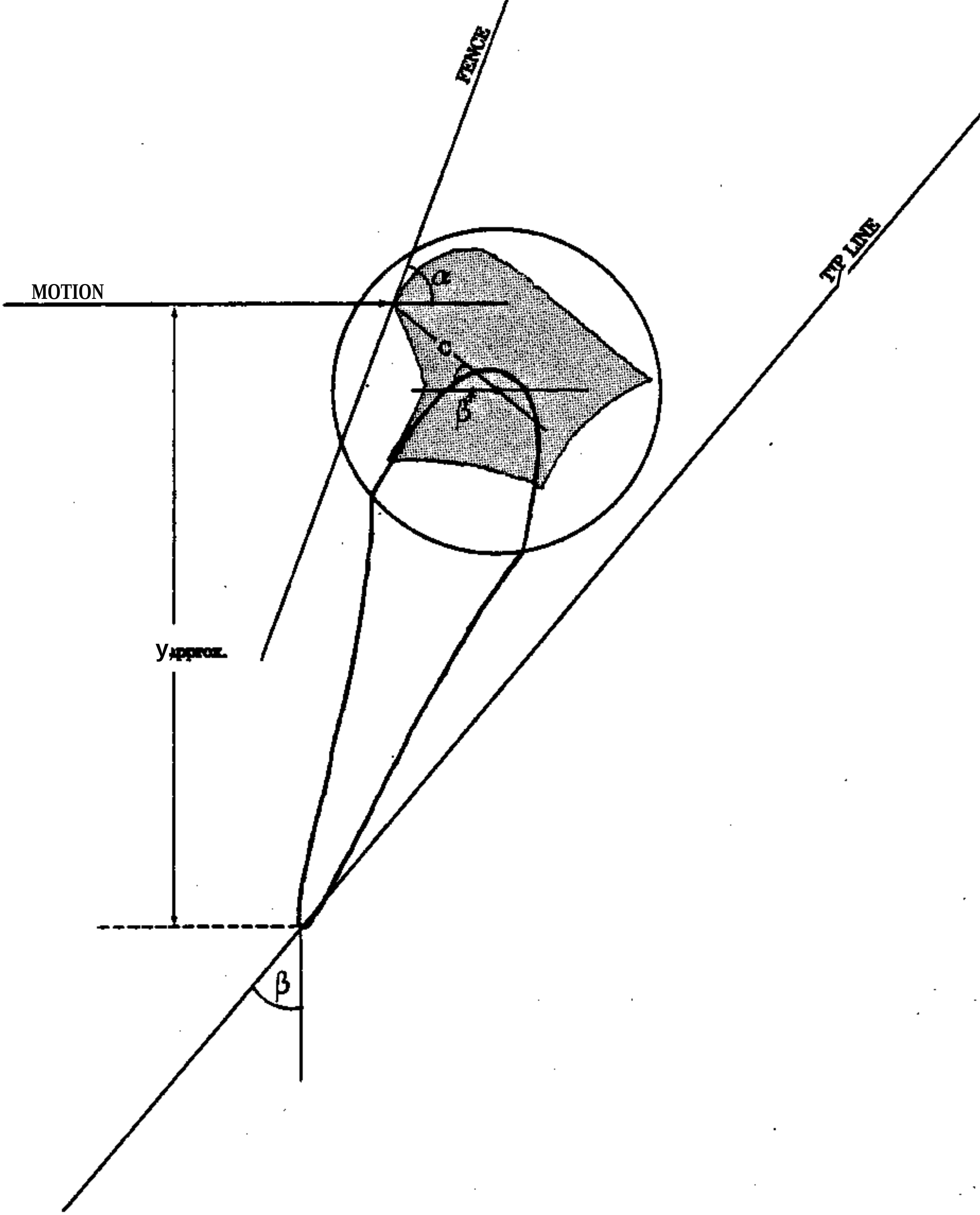


Figure 6-1: Initial configuration of object and fence, and resulting COR locus

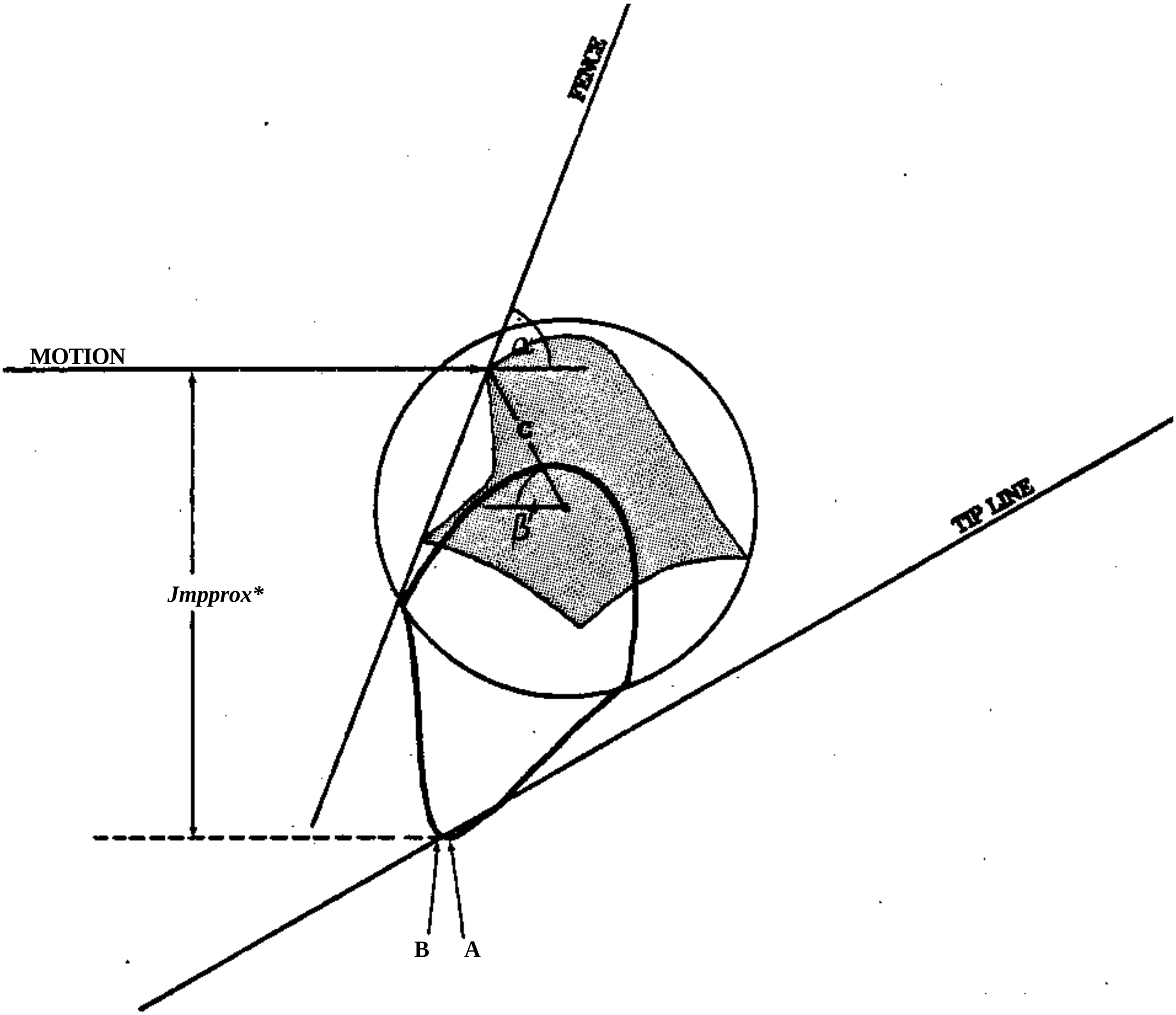


Figure 6-2: Final configuration of object and fence, and resulting COR locus

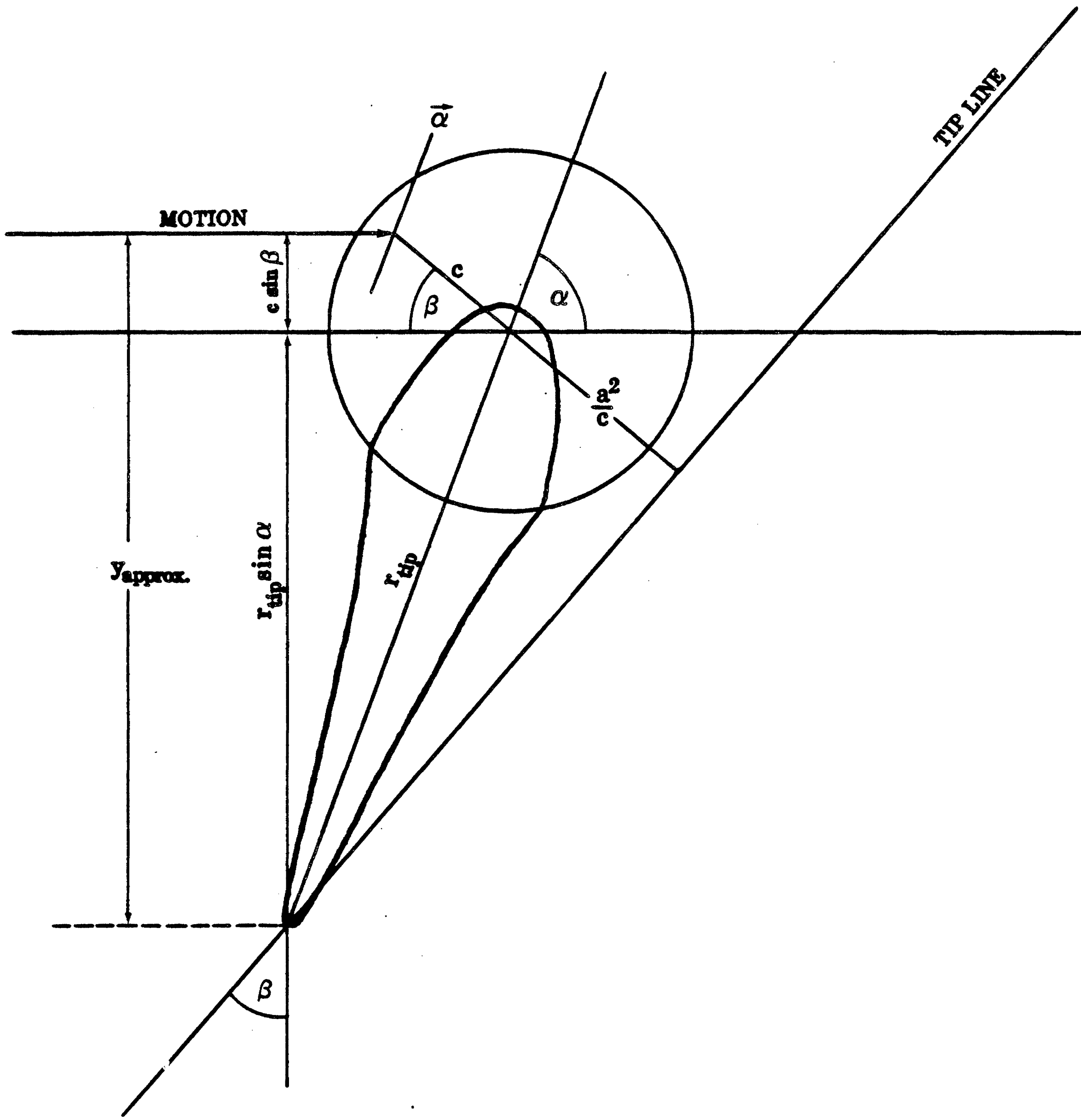


Figure 6-3: y_{approx} is the distance from the line of motion to the tip of the COR locus

e can now integrate $dx = y\ d\beta$ to obtain the indefinite integral

$$x = -c\cos\beta - \frac{a^2 \sin\alpha}{2c} \log\left|\frac{1+\sin(\alpha+\beta)}{1-\sin(\alpha+\beta)}\right| . \qquad (4$$

) find the maximum pushing distance, $\triangle x$, required to cause the object to rotate from its init
ınfiguration shown in figure 6-1 to its final configuration shown in figure 6-2, we simply substitu
e initial and final values of β into equation 45, and take the difference $x_{final} - x_{initial}$.

ıe value of y used (equation 44) is slightly less than the true maximum distance from the line
otion to any point of the COR locus. Using the construction of figure 6-4 we can approximate t
ue distance y from the line of motion to the lowest point of the COR locus (point A):

$$y_{true} = c\sin\beta + (r_{tip} - s)\sin\alpha + s \qquad (4$$

serting the radius of curvature s from equation 42 in equation 46, we find that $dx = y\ d\beta$ cannot
tegrated in closed form. However if we make the approximation

$$s = \frac{r_{tip}}{\frac{1}{2} + \frac{r_{tip}^4}{a^4}} \approx \frac{a^4}{r_{tip}^3} \qquad (4$$

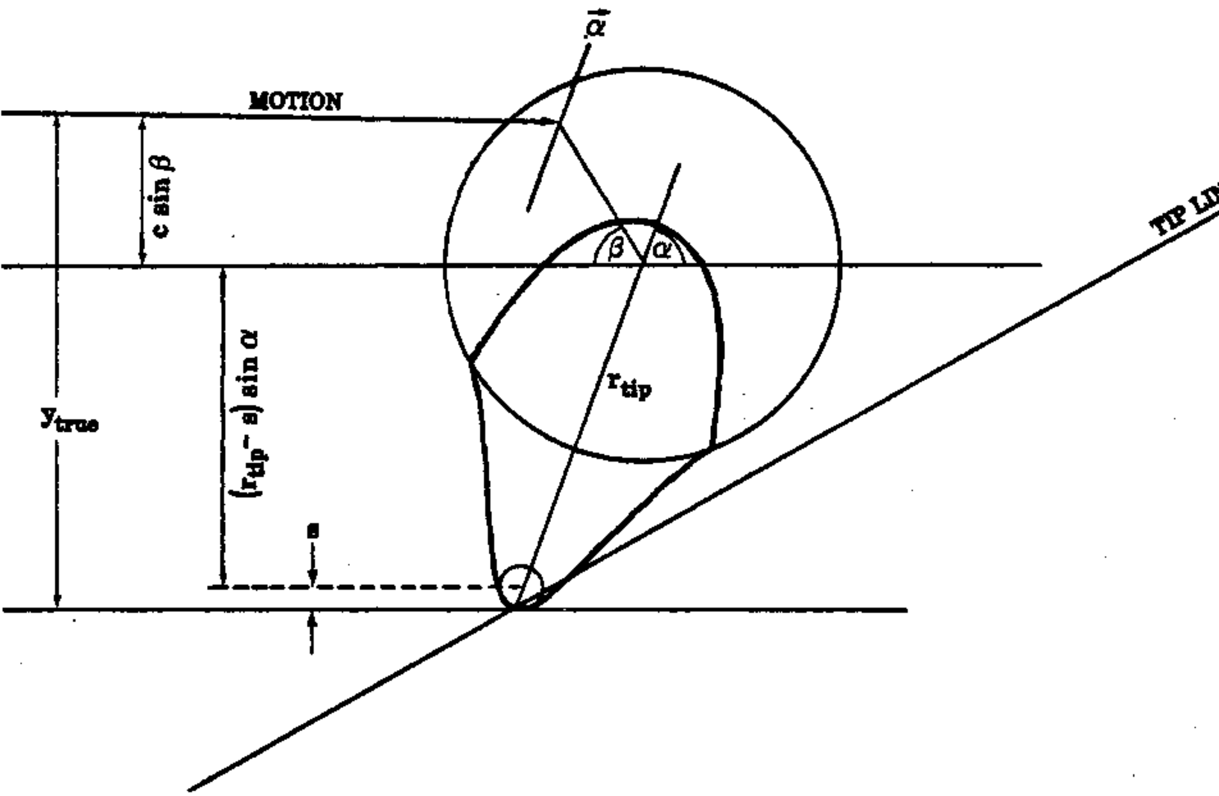
en $dx = y\ d\beta$ can be integrated to yield an additional required pushing distance of

$$x_{adnl} = \frac{-c^3}{a^2}(1-\sin\alpha)\left(\sin(\alpha+\beta) - \frac{\sin^3(\alpha+\beta)}{3}\right). \qquad (4$$

ıe approximation made in equation 42 slightly *overestimates* the radius of curvature s, leading to
ight overestimate of y_{true}, and therefore to a upper bound for the required additional pushi
stance x_{adnl}.

ıe sum of equations 45 and 48 therefore provides an upper bound for the required pushi
stance. For most purposes equation 45 alone will suffice.

the case of figures 6-1 and 6-2, if the disk diameter is a, we have α = 70 degrees, $c = .8125\,a$, β_{in}
40 degrees, and β_{final} = 60 degrees. equation 45 yields a required pushing distance of 1.054



Figu re 6-4: $y_{(m}$ **is a bound for the distance from the line of motion to the true lowest point**

Conclusion

'e have found the locus of all possible centers of rotation for a pushed object sliding on a le
ırface. A major assumption is the neglect of friction at the point of contact between the pusher a
e object. A forthcoming paper will show how the effect of such friction can be calculated. It will
e necessary to modify the derivations above to incorporate friction.

nother major assumption is that the support force distribution is confined to a disk. The loci fou
bove are exact bounds on the center of rotation of a disk with an arbitrary distribution of supp
rces. For objects other than disks, the loci found above are outer bounds on the center of rotatio
ıt tighter bounds are possible.

ı immediate application of the results derived here is to the problem of the distance a sliding obj
ust be pushed by a fence before it rotates into alignment with the fence. This problem requires us
ıd the *slowest* possible rotation of the sliding object.

ne application of the *fastest* possible rotation would be to find the maximum misalignment of
oject when it is pushed from a known configuration.

nother possible application is the calculation of the maximum torque a grasp can withstand befo
e grasped object slips. Just as the support force distribution is unknown in the sliding proble
eated above, the distribution of pressure exerted by a gripper on a grasped object is unknown.
ay be possible to use similar methods to find the strength of the grasp.

. Acknowledgements

e wish to acknowledge useful discussions with Randy Brost and with Matt Mason. We especia
ank Randy Brost for several readings of this paper, and for many suggestions. Figure 1-1 w
opied from (Mason, 1985). Several equations were obtained or simplified using MACSYMA.

. Appendix - Validity of Energy Minimization

ome question has arisen about the validity of the assumption that the system will rotate about a C
hich minimizes the frictional energy loss. (The assumption is known to be valid in conservat
stems, but the presence of friction makes the sliding system dissipative.) Here we show that t

ince the sliding system is assumed to be quasi-static, we have zero acceleration, and Newton's la mply require total force and total torque to be zero. In other words the applied force and torque d the pusher must be opposed by an equal and opposite force or torque created by frictional forc etween the object and the surface it is sliding on.

nce the coefficient of friction at the pusher/object contact is zero, the applied force is direct erpendicular to $\vec{\alpha}$. Torque may be measured with respect to any origin. We choose to measure out the COR. Denoting the applied force $\vec{F}_A$, the torque is

$$\tau_r = |\vec{F}_A \times (\vec{c} - \vec{r})| = |\vec{F}_A| \ \vec{\alpha} \cdot (\vec{c} - \vec{r}) . \qquad (4$$

e opposing torque developed by frictional forces is

$$\tau_r = \mu_s \int S(\vec{w}) \, |\vec{w} - \vec{r}| \, d\vec{w}. \qquad (5$$

etting the opposing torques equal we can solve for the applied force

$$|\vec{F}_A| = \frac{\mu_s}{\vec{\alpha} \cdot (\vec{c} - \vec{r})} \int S(\vec{w}) \, |\vec{w} - \vec{r}| \, d\vec{w} . \qquad (5$$

e also require translational forces to cancel:

$$\vec{F}_A = - \int F(\vec{w}) d\vec{w} \qquad (5$$

ere we know that $\vec{F}(\vec{w})$, the element of frictional force at $\vec{w}$, must be directed opposite to t rection of motion at $\vec{w}$. The direction of motion at $\vec{w}$ is perpendicular to the vector $(\vec{w} - \vec{r})$, where the COR. The magnitude of the element of frictional force at $\vec{w}$ is $\mu_s \ S(\vec{w})$. So we have

$$\vec{F}(\vec{w}) = \mu_s S(\vec{w}) \left[\frac{-(\vec{w} - \vec{r})_y \ \vec{x}}{|\vec{w} - \vec{r}|} + \frac{(\vec{w} - \vec{r})_x \ \vec{y}}{|\vec{w} - \vec{r}|} \right] . \qquad (5$$

tegrating over all $d\vec{w}$

$$\vec{F}_A = - \mu_s \int S(\vec{w}) \left[\frac{-(\vec{w} - \vec{r})_y \vec{x}}{|\vec{w} - \vec{r}|} + \frac{(\vec{w} - \vec{r})_x \vec{y}}{|\vec{w} - \vec{r}|} \right] d\vec{w} . \qquad (5$$

$$\vec{F}_A = |\vec{F}_A|(-\alpha_x\vec{y} + \alpha_y\vec{x}) \, . \qquad ($$

ormally exchanging $\vec{x}$ with $\vec{y}$, and equating the right-hand-sides of equations 54 and 55 we have

$$|\vec{F}_A|\,\vec{\alpha} = \mu_s \int S(\vec{w})\,\frac{\vec{w}-\vec{r}}{|\vec{w}-\vec{r}|}\,d\vec{w} \, . \qquad ($$

ombining the result from consideration of total torque (equation 51) with the result fr onsideration of total translational force (equation 56) yields

$$\frac{\vec{\alpha}}{\vec{\alpha}\cdot(\vec{c}-\vec{r})}\int S(\vec{w})\,|\vec{w}-\vec{r}|\,d\vec{w} = \int S(\vec{w})\,\frac{\vec{w}-\vec{r}}{|\vec{w}-\vec{r}|}\,d\vec{w} \qquad ($$

hich can be cast in a form equivalent to equation 8:

$$d_r\,\vec{\alpha} = \vec{v}_r\,\vec{\alpha}\cdot(\vec{c}-\vec{r}) \qquad ($$

where

$$d_r = \int S(\vec{w})\,|\vec{w}-\vec{r}|\,d\vec{w}$$

and

$$\vec{v}_r = \int S(\vec{w})\,\frac{\vec{w}-\vec{r}}{|\vec{w}-\vec{r}|}\,d\vec{w} \, .$$

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