UNIVALENCE CONDITIONS AND STURM-LIOUVILLE EIGENVALUES

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Let f(z) be an analytic function in the unit disk, and denote by {f,z} its Schwarzian derivative

(l.D (f,z) =<fr-)- - ^(jї-) •

It is known that f(z) will be univalent in |z| < l--i.e., f(z1) ^ f(z~) if z1, z? are distinct points in the unit disk-- if {fjZ} is subjected to conditions of the type |{f,z}| £ S(|z|), where S is a suitably chosen positive function on the interval [0,1]([3,4]). In particular, the univalence of f(z) is guaranteed by the conditions

(1.2) I {f,z} | ^——^T~σ> 1 z 1 < r>
(1-r)z

([3]) and

4
1-r

([6]). Both these inequalities are best possible in the sense that the constants on the right-hand side cannot be replaced by smaller numbers (for the case of (1.2), cf. [2]). Condition (1.2) occupies a special position since it becomes a necessary condition for the univalence of f if the constant 2 on the right-hand side of (1.2) is replaced by 6 ([3]). Also, condition (1.2) has been found to be of importance in the theory of quasi-conformal mapping ([1]).

The principal aim of the present paper is to establish the following strengthened versions of these univalence criteria.

Theorem 1.1. *Let* $R(r)$ *be a nonincreasing nonnegative function on* $[0,1)$ *for which*

$$\int_0^1 R(r)\,dr \le 1. \tag{1.4}$$

If

$$|\{f,z\}| \le \frac{2R(r)}{(1-r^2)^2}, \qquad |z| < r, \tag{1.5}$$

then $f(z)$ *is univalent in* $|z| < 1$.

Theorem 1.2. *Let* $R(r)(1-r^2)^{-1}$ *be nonincreasing in* $[0,1)$ *and let*

$$\int_0^1 \frac{R(r)}{1+r}\,dr \le 1. \tag{1.6}$$

If (1.5) *holds, then* $f(z)$ *is univalent in* $|z| < 1$.

It will be observed that condition (1.2) follows from Theorem 1.1 for $R(r) \equiv 1$, while condition (1.3) corresponds to the case $R(r) = 2(1-r^2)$ of Theorem 1.2.

2. It was shown in [4] that the condition $|\{f,z\}| \le 2S(r)$ will guarantee univalence if $S(r)$ has the following two properties: (a) the function $(1-r^2)^2 S(r)$ is nonincreasing on $[0,1)$; (b) the differential equation

$$y'' + S(r)\,y = 0$$

has a solution y which is positive and nonincreasing on $[0,1)$. Equivalently, property (b) can be expressed by the condition

$$A \geq 1, \tag{2.1}$$

where A is the lowest eigenvalue of the problem

$$V'' + AS(r)v = 0, \quad V(0) = v(1) = 0. \tag{2.2}$$

If $S(r)$ is not defined for $r = 1$, A will be understood to mean $\lim_{r \to 1} A_r$, where A_r is the lowest eigenvalue of equation (2.2) for the boundary conditions $V(0) = v(r) = 0$, $0 < r < 1$.

In the case of Theorems 1.1 and 1.2, the function $R(r) = (1 - r^2)^2 S(r)$ is nonincreasing, i.e., $S(r)$ has the property (a). To prove these Theorems, it thus is only necessary to show that, under our assumptions, the lowest eigenvalue of (2.2) is subject to the condition (2.1). The remainder of this paper has therefore nothing to do with complex function theory; all we are concerned with is to obtain lower bounds for the first eigenvalue of the problem (2.2), where the coefficient $S(r)$ is subject to certain restrictions. The required bounds will be provided by the following result, which is also of independent interest as a comparison theorem of a rather unusual type for second-order linear differential equations.

Theorem 2.2. *Let* $p(x), R(x)$ *be nonnegative, and let* $p(x)$ *be nondecreasing and* $R(x)$ *be nonincreasing on* $[0,1)$. *Denote by* A *and* Λ, *respectively, the lowest eigenvalues of the problems*

$$(2.3) \qquad u'' + \lambda R(x)p(x)u = 0, \quad u'(0) = u(1) = 0,$$

$$(2.4) \qquad v'' + \Lambda p(x)v = 0, \quad v'(0) = v(1) = 0.$$

If $\beta = \beta(x)$ is defined as the unique root of the equation

$$(2.5) \qquad -\frac{\beta v'(\beta)}{v(\beta)} = \frac{x}{1-x}$$

in $[0,1)$ (where $v(x)$ is the solution of (2.4)), and if we set

$$(2.6) \qquad \theta(x) = [\ \cdots\ ]^2,$$

then $\theta(x)$ is nondecreasing in $[0,1]$ and

$$(2.7) \qquad \lambda \int_0^1 R(x)\,d\theta(x) \geq \Lambda .$$

For $R(x) = 1$, (2.7) becomes an equality.

We first show that the estimates needed to complete the proofs of Theorems 1.1 and 1.2 indeed are special cases of inequality (2.7). In the case of Theorem 1.1, we set $p(x) = (1 - x^2)^{-2}$, and we note that the lowest eigenvalue and the corresponding eigensolution of (2.4) are, respectively, $\Lambda = 1$ and $v = (1 - x^2)^{1/2}$ (since $p(x)$ is not defined for $x = 1$, the eigenvalue has to be defined by the limiting procedure indicated above). Because of

$$\frac{\beta v'(\beta)}{v(\beta)} = -\frac{\beta^2}{1-\beta^2}$$

the function β defined in (2.5) is $\beta = x^{1/2}$, and (2.6) shows that $\theta(x) \equiv x$. Inquality (2.7) therefore reduces to

$$A\int_0^1 R(x)\,dx \ge 1,$$

and we conclude that $A \ge 1$ for functions $R(x)$ satisfying condition (1.4). Inequality (2.1) thus holds for the lowest eigenvalue of

$$u'' + \frac{AR(x)}{(1-x^2)^2}u = 0, \quad u'(0) = u(1) = 0,$$

if $R(x)$ is nonnegative and nonincreasing, and is subject to condition (1.4). By the result quoted at the beginning of this section, this establishes Theorem 1.1.

To obtain Theorem 1.2, we set $p(x) = (1 - x^2)^{-1}$ and, to avoid a clash with the notation of Theorem 1.2, we write $R_1(x)$ instead of $R(x)$ in (2.3). If $R(x)$ is the function appearing in Theorem 1.2, we then have

$$R(x) = (1 - x^2)R_1(x), \tag{2.8}$$

and we have to show that (2.1) holds for the lowest eigenvalue of

$$u'' + \frac{AR_1(x)}{1 - x^2}u = 0, \quad u'(0) = u(1) = 0, \tag{2.9}$$

if $R_1(x)$ is nonincreasing and--in view of (1.6) and (2.8)--satisfies the condition

$$\int_0^1 (1 - x)R_1(x)\,dx \le 1. \tag{2.10}$$

For $p(x) = (1 - x^2)^{-1}$, the lowest eigenvalue and the corresponding eigenfunction of (2.4) are, respectively, $\lambda = 2$ and $v = 1 - x^2$. Since

$$\frac{\beta v'(\beta)}{v(\beta)} = \frac{2\beta^2}{1-\beta^2}$$

equation (2.5) shows that, in this case,

$$fi^2 \;\text{"}\; \frac{x}{2 - x} \; *$$

Hence, by (2.6), $0(x) = x(2 - x)$, and inequality (2.7) takes the form

$$A \int_0^{L} (1 - x)R_L(x)dx \geq 1.$$

Because of (2.10), this implies (2.1) and thus completes the proof of Theorem (1.2) .

As these two examples show, any problem (2.4) which can be solved explicitly, and for which equation (2.5) can be solved for *ft*> , leads to a criterion of univalence (which is necessarily the best of its kind). Unfortunately, it is not too easy to find examples for which these two operations can be carried out explicitly and which, at the same time, are of interest from the function-theoretic point of view, it may be noted that, even in the case $p(x) \equiv 1$, (2.5) leads to the transcendental equation

f *(o) f - r^-K

for *ft* .

We also note that the assumption that $R(x)$ be nonincreasing is essential. If this assumption is omitted, the conclusion (2.7) of Theorem 2.2 does not necessarily follow. in fact, as the following example shows, the left-hand side of (2.7) may be made arbitrarily small if $R(x)$ is not required to be nonincreasing. We set $p(x) = (1 - x^2)^{-2}$ and

$$R(x) = (2n - 1)\, x^{2n-2} \left(\frac{1 - x^2}{1 - x^{2n}}\right)^2, \quad n > 1, \tag{2.11}$$

Since the equation

$$u'' + \frac{(2n - 1)\, x^{2n-2}}{(1 - x^{2n})^2}\, u = 0$$

has the solution $u = (1 - x^{2n})^{1/2n}$, the lowest eigenvalue λ of the corresponding problem (2.3) has the value 1. As shown above, the function $\theta(x)$ associated with the coefficient $p = (1 - x^2)^{-2}$ is $\theta(x) = x$, and $\mathcal{N} = 1$. If Theorem 2.1 were applicable to this case, we could conclude from (2.7) that $c_n \geq 1$, where, by (2.11),

$$c_n = (2n - 1) \int_0^1 x^{2n-2} \left(\frac{1 - x^2}{1 - x^{2n}}\right)^2 dx .$$

However, the easily confirmed inquality

$$\frac{1 - x^{2n}}{1 - x^2} \geq n x^{n-1} \qquad (0 < x < 1,\ n \geq 1)$$

shows that $c_n \leq 2/n$. Hence, not only does (2.7) not hold in general for functions $R(x)$ which are not nonincreasing: for such functions, the left-hand side of (2.7) does not possess a positive lower bound.

3. Turning now to the proof of Theorem 2.1, we first show that it is sufficient to prove (2.7) for the set of non-increasing functions $R(x)$ consisting of the characteristic function $\chi_t(x)$ of the intervals $[0,t]$, where $t \in (0,1)$. While this follows from a general result concerning certain functionals defined on convex sets [5], it is easy to give a direct proof which applies to the present case. If we denote the lowest eigenvalue of (2.3) by $\lambda(Rp)$, in order to indicate its dependence on the coefficient of the equation, it follows from classical results that

$$[\lambda(Rp)]^{-1} = \sup_y \int_0^1 Rpy^2 dx,$$

if y ranges over all functions of $D^1[0,1]$ which satisfy the boundary conditions $y(0) = y(1) = 0$ and the normalization condition

$$\int_0^1 y'^2 dx = 1.$$

If R is a convex combination $\alpha_1 R_1 + \ldots + \alpha_n R_n$ ($\alpha_k \geq 0$, $\alpha_1 + \ldots + \alpha_n = 1$) of functions $R_1, \ldots, R_n$, and $\phi(x)$ is a nondecreasing function of x in $[0,1]$, we have therefore

$$\left[\lambda(Rp) \int_0^1 R\, d\phi\right]^{-1} = \sup_y \frac{\sum_{k=1}^n \alpha_k \int_0^1 R_k p y^2 dx}{\sum_{k=1}^n \alpha_k \int_0^1 R_k\, d\phi}$$

$$\leq \sup_k \sup_y \frac{\int_0^1 R_k p y^2 dx}{\int_0^1 R_k\, d\phi}$$

$$= \sup_k [\lambda(R_k p) \int_0^1 R_k \, d\emptyset]^{-1}.$$

Accordingly, if the functions R belong to a convex set C, and it is desired to find the greatest lower bound of

(3.1) $$MRp) \int_0^1 R \, d\emptyset$$

for R€C, it is sufficient to consider the extremal points of C. If C is the set of nonincreasing nonnegative functions R on [0,1) which are normalized by the condition R(0) = 1 (this normalization is possible because the functional (3.1) remains unchanged if R is multiplied by a constant), C is convex, and it is easy to see that the extremal points of C correspond to the characteristic functions JL^{ν}_t of the intervals [0,t], where te(0,1) .

Accordingly, it is sufficient to prove Theorem 2.1 for the functions R = $J._t$. Since, because of (2.5), (2.6) and the fact that V (0) = 0, we have 0(0) = 0, inequality (2.7) reduces in this case to

(3.2) $$A_t \theta(t) \geq A.$$

Here, $\backslash_t$ denotes the eigenvalue of (2.3) in the case in which R = X_t- (2.3) may now also be formulated as an eigenvalue problem for the interval [0,t]. Since, in (t^l], u is necessarily proportional to 1 - x, it follows from the continuity of u and u' at x = t that

$$\frac{u'(t)}{u(t)} = - \frac{1}{1 - t}$$

In [0,t], (2.3) is therefore equivalent to the problem

(3.3) $\qquad u'' + \Lambda_t p(x)u = 0, \quad u'(0) = 0, \quad \frac{u'(t)}{u(t)} = - \frac{1}{1 - t}$.

Theorem 2.1 will be proved if we can show that the eigenvalue Λ_t of (3.3) satisfies the inequality (3.2), where $-\Lambda$, is the eigenvalue of (2.4) and $\theta(t)$ is determined by (2.6) and (2.5).

To do so, we require the two following auxiliary results:

(a) the function $\phi(x)$ defined in (2.5) satisfies the inequality

(3.4) $\qquad \phi(x) \geq x \qquad x \in [0,1)$

and it increases monotonically from 0 to 1 as x varies from 0 to 1; (b) the function $\theta(x)$ is non-decreasing.

To prove (a), we note that, with

(3.5) $\qquad V = -\frac{v'}{v}$,

(2.4) is equivalent to the Riccati equation

(3.6) $\qquad V' = V^2 + \Lambda p$,

and V will be positive in (0,1) if we assume (as we may) that v is positive in this interval. By (3.6),

$$V^2 \sim ,$$

and thus,

$$\text{vdb- - w i r } \wedge^x i -^x \quad (0 < x < x_1 < 1).$$

Since $V(x_1) \to \infty$ for $x_1 \to 1$, this implies

$$V(x) \leq \frac{1}{1-x} .$$

Hence, by (2.5) and (3.5),

$$\frac{x}{1-x} \leq \frac{\beta}{1-\beta} ,$$

which is equivalent to (3.4). To show the monotonicity of β, we note that, because of (3.5), (2.5) may be written

$$\beta V(\beta) = \frac{x}{1-x} . \tag{3.7}$$

Differentiating this with respect to x and simplifying the result with the help of (3.6) and (3.7), we obtain

$$\beta' \quad (1-x)^2 \quad \geq \quad \frac{1}{(1-x)^2} \tag{3.8}$$

and this shows that β is increasing. The fact that $\beta(0) = 0$ and $\beta(1) = 1$ is obvious from (3.7). Assertion (b) also follows from (3.8). Since $\phi = x^2\beta^{-2}$, we have

$$\frac{x}{2}\,\frac{\phi'}{\phi} = 1 - \frac{x\beta'}{\beta}$$

and thus, by (3.8)

$$\frac{x}{2(1-x)^2}\,\frac{\phi'}{\phi} = \Lambda \beta \beta' p .$$

Since $\Lambda \geq 0$, ϕ is thus found to be nondecreasing.

If we set

$$U = \frac{u'}{u} , \quad tU(t\xi) = \sigma(\xi) ,$$

the system (3.3) takes the form

(3.9) tf' = <^{2} + A_tp(t$), d'(0) = 0, 6(1) = $\frac{t}{1-t}$,

where the prime now denotes differentiation with respect to ξ.
Similarly, setting

$$\beta V(\beta\xi) = \tau(\xi) \qquad (\beta = \beta(t))$$

and noting that V(0) = 0, we can replace (3.6) by

(3.10) <£< = T^2 + (b^2AP0K 'C(0) =0, Z(l) = $\beta V(\beta)$.

Since, in accordance with (3.7), /i is determined from the equation

$$\beta V(\beta) = \frac{t}{1-t},$$

the boundary conditions in (3.9) and (3.10) are identical. By assumption, p is nondecreasing, and it follows therefore from (3.4) that p(t"5) $\leq$ p(ft^) . If, in addition, it were true that $t^2 A_t \leq$ f>^{2}.A, equations (3.9) and (3.10) would imply that
2 2 t

$$\frac{d}{d\xi}\left[(\sigma - \tau)e^{-\int(\sigma+\tau)\,ds}\right] \leq 0.$$

This, however, is absurd since, because of the boundary conditions satisfied by <T and t, <T - H vanishes both at ξ = 0 and ξ= 1. Hence, we must have t ~h_t $\geq$ ^> A- Because of the definition (2.6) of 0, this establishes inequality (3.2) and thus completes the proof of Theorem 2.1.

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