ON INJECTIVE MODULES AND COGENERATORS

by

F. Kasch, H.-J. Schneider
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1. Introduction

In this paper all rings are assumed with identity and all modules are unitary. For a ring R denote by M_R respectively $_RN$ a right respectively left R-module.

A module M_R is a generator iff for every right R-module A_R

$$A = \sum_{\varphi \in \mathrm{Hom}_R(M,A)} \mathrm{ima}(\varphi) ,$$

where $\mathrm{ima}(\varphi)$ denotes the image of the homomorphism φ. Dually M_R is a cogenerator iff for every right R-module A_R

$$0 = \bigcap_{\varphi \in \mathrm{Hom}_R(A,M)} \ker(\varphi) ,$$

where $\ker(\varphi)$ denotes the kernel of the homomorphism φ.

For a ring R the modules R_R and $_RR$ are projective and generators. In general these modules are neither injective nor cogenerators.

The representation theory of finite groups is to a large extent based on the fact that the group ring of a finite group with coefficients in a field is on both sides injective and a cogenerator.

In this connection there exists the following well known theorem (see [9],[4]): If R is Noetherian or Artinian on one

*The results in this paper arose out of a seminar in the Spring of 1969 at Carnegie-Mellon University.

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side and if R is injective or a cogenerator on one side, then R is Artinian, injective and a cogenerator on both sides.

A ring with these properties is called a Quasi-Frobenius-ring (= QF ring).

Several authors ([1],[6],[8],[9]) have considered the hypotheses of the above theorem dropping the assumptions on chain conditions.

There exist examples which show that being either injective or a cogenerator on one side does not imply all these properties for the ring on both sides.

The question arises as to which combination of injective and cogenerator properties have to be assumed to ensure that R is an injective cogenerator on both sides.

A still open (well known) conjecture in this direction states: $_RR$ is injective and a cogenerator iff R_R is injective and a cogenerator.

The following results have been established.

<u>Theorem 1</u> (T. Onodera [8], T. Kato [6]): The following statements are equivalent:

(1) R_R and $_RR$ are cogenerators.

(2) R_R and $_RR$ are injective and R is a left and right S-ring.*

<u>Theorem 2</u> (F. Sandomierski, unpublished): The following statements are equivalent:

(1) R_R is injective and $_RR$ is a cogenerator.

(2) R_R is a cogenerator and $_RR$ is injective.

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*For the definition of left respectively right S-ring see 3.3.

In this paper we generalize some of the results in the literature from rings to modules. Some of our proofs when restricted to the case of rings are simpler than those in the literature. Our setting is the following: Let R be a ring and M_, a right R-module. Set r = End(M_R) with r operating on the left side of M_R in the usual manner. Thus M = Jl_R is a F-R bimodule. If $M_R = R_R$ we have the special case R = ${}_R R_R$ which is the one dealt with in the literature. We remark that there exists a ring homomorphism R»r »» (Mam •» mreM) e End(${}_\Gamma$M). If this mapping is an isomorphism then we say that R is naturally ringisomorphic to End(-M).

In what follows we use a theorem of P. Pahl. Since this result is unpublished we include a proof of

Theorem 3 (P. Pahl [10]): Let M_, be a generator. Then the following statements are equivalent:

(1) M_R is selfinjective.

(2) M_ is injective.

(3) ry, is injective.

Generalizing theorem 1 and 2 we obtain

Theorem 4: The following statements are equivalent:

(1) M_R and ${}_\Gamma M$ are projective cogenerators and R is naturally ringisomorphic to End(_pi) .

(2) M_R and Jl are injective generators, R is a right S-ring and T is a left S-ring.

and

Theorem 5: The following statements are equivalent:

(1) M_R is an injective generator and $_T$# is a cogenerator.

(2) M_R is a finitely generated, projective cogenerator, Jl is injective and R is naturally ringisomorphic to End(_pi).

(3) M_R, jji, R_R, $_RR$, r_r and ^T are injective, projective, finitely generated, generators, cogenerators and semiperfect.

We intend to make this paper as self-contained as possible. Thus we Include several proofs and facts which are already in the literature. Since these are spread over several papers and some are not even available we think that this may prove useful to the reader.

2. Changing sides.

The setting described in section 1 holds throughout this paper.

<u>2.1 Lemma:</u> Let x e M, xR simple and xR contained in an injective submodule of M_,. Then *Tx* is simple.

Proof: Let y_ox ^ 0 for some $y_o$el\ Then y_oxR / 0 and since xR is simple it follows that

$$xR3\ xr\ y\text{-*}\ y_0\ xr\ €\ y_0 xR$$

is an isomorphism. Since xR is contained in an injective submodule of M_R, there exists y_1 e *T* such that $y_1 y_o$x = x. Hence Ty_ox = *Tx* and *Tx* is simple.

<u>2.2 Corollary</u>: If M_R is injective then the socle of M_R is contained in the socle of $_\Gamma M$.

<u>2.3 Lemma</u>: Let x,y e M, yR ≅ xR and xR contained in an injective submodule of M_R. Then Tx is isomorphic to a submodule of TV.

Proof: By assumption there exists *7\ e T* such that

$$yR\ 3\ yr\ \text{M}\ Ayr\ e\ xR$$

is the given isomorphism yR ≅ xR. Hence there exist r_1^r_o e R such that Ay = xr^, A$\dot{y}r_2$ = x. Consider the T-homomorphism

$$\Gamma x \ni \gamma x \mapsto \gamma x r_1 = yAy\ e\ TV.$$

Since yxr_1 = 0 implies $\gamma x r_1 r_2$ = *yx = o}* this is a monomorphism.

2.4 Corollary; Let M_R be injective, x,y e M, yR $\cong$ xR and xR simple. Then Fx is simple and isomorphic to Iy»

Proof: By 2.1 and 2.3.

2.5 Corollary; Let x,y e M, yR $\cong$ xR and yR and xR contained in injective submodules of M_R. Then the injective hulls of IV and Fx are isomorphic.

Proof: By 2.3 Fx is isomorphic to a submodule of Fy and TV is isomorphic to a submodule of Fx. Hence an injective hull of Fx is isomorphic to a submodule of an injective hull of Fy and vice versa. By [3] two injective modules which are isomorphic to submodules of each other are isomorphic. This implies 2.5.

3. Annihilators.

Let U be a submodule of M_R. The left annihilator of U in F is given by

$$l_p(u) = \{yerlyU = 0\}.$$

Similarly for a right ideal Jl of *T* we have

$$l_\Gamma(\Omega) = \{\gamma \in \Gamma | \gamma\Omega = 0\}.$$

Then $1_{I'}(U)$ and $l_r(fl)$ are left ideals in I\ For a left ideal A of F we have two right annihilators

$$r_r(A) = [ye\ rl\ Ay = 0\},$$

$$r_M(A) = \{meM)An = 0\}.$$

Clearly $r_I(A)$ is a right ideal of *T* and $r_M(A)$ is a submodule of M_,. Obviously these definitions do not depend on the assumption that r is the endomorphism ring of M_R.

3.1 Lemma; Let *jJP* be a cogenerator and A a left ideal of T\ Then

$$1_\Gamma r_\Gamma(\Lambda) = 1_\Gamma r\Lambda\ \Lambda) = A.$$

Proof (see for example [8], [12]):

i) $l_I r_I(A) = A$: This is well known. We give a proof for complete-ness. By definition Ac_l_r (A). Assume £ e $l_I r_\Gamma(A)$ and 4 *4* A. Since $_I T$ is a cogenerator, there exists a T-homomorphism

$$f: r/A -* r$$

such that f(J) ^ *0,* where If is the coset of £ in I/A. Let

v* T - T/A

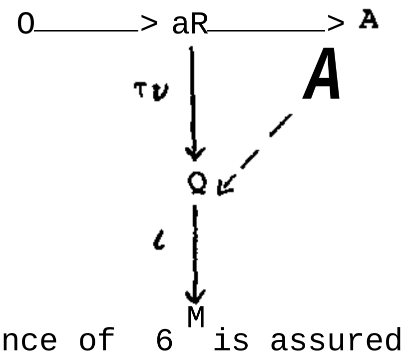
be the natural epimorphism. Then fv(£) / 0 and *fv* e Hom_(_T, _T).
Hence *fv* is given by multiplication on the right with an element
y_o = *fv(l)* e T. Since fv»(A) = 0 it follows that Ay_Q = 0, that
is y_o € r_r(A). Since fi/(£) ^ 0 we have $£y_Q$ ^ 0 which contra-
dicts £ e $l_r r_r$(A).

ii) $l\text{-}_r{}^\wedge{}_M$(A) = A: From r_r(A)M c r_M(A) it follows that lpF_M(A) c
$l_t(r_t$(A)M) = $l_t r_t$(A) = A. Since Ac ¹pʳM^^ the proof is complete.

3.2 Lemma ([9]): M_. is a cogenerator iff for every simple right
R-module there exists a monomorphism into an injective submodule
Of M_R.

Proof: ^: Let U_R be a simple module and let T: U_R -• Q_R be an
injective hull of U_D. Since M_R is a cogenerator there exists
<*pi* Q -» M such that *(pr £* 0. Assume ker(qo) ^ 0. Since
T(U) is large in Q_., T(U) 0 ker(ca) ^ 0 and by simplicity of U,
T(U)_C ker((p) which implies #T = 0. Therefore <p is a mono-
morphism and *<pT* is a monomorphism of U into the injective sub-
module ima(<p) = <p(Q) of M.

$_{fe}$: Let A_R ^ 0 be an arbitrary module and let a e *A,*
a 7^ 0. Let B be a maximal submodule of the cyclic submodule
aR. By assumption there exists a monomorphism T: aR/B -. Q
where Q is an injective submodule of M_R. If *i>:* aR _ aR/B
is the natural epimorphism, then Tt/(a).^ 0. Consider the
diagram



The existence of 6 is assured since Q is injective. But then *15* € $Hom_R(A_{Rj}M_R)$ and *6(a) = Tv(a) ^ 0.

<u>Remark</u>: By Lemma 3.2 M_R is a cogenerator iff for every simple right R-module U there exists a monomorphism U -. M which can be factorized through an injective module. Dually M_R is a generator iff for every simple right R-module U there exists an epimorphism M -• U which can be factorized through a projective module.

3.3 A ring R is called a <u>left S-ring</u> iff for every proper left ideal A of R, $^r o(^A)$ ^ °* <u>Right S-rings</u> are defined similarly.

<u>Lemma</u> ([7]): R is a left S-ring iff for every simple left R-module there exists an isomorphic left ideal in R.

Proof: Let A be a left ideal of R. One verifies easily that

$$r_R(A) j x H. (R/A?\bar{r} i- rxeR) e Hom_R(R/A,R)$$

is a right R-isomorphism.

^: Every simple left R-module is isomorphic to R/A for a certain maximal left ideal A. Since $^r{}_R(^A)$ ^ *0* this implies $Hom_R(R/A,R)$ ^ 0 and using the simplicity of R/A we get the desired isomorphism.

«.: If R contains a copy of R/A, where A is maximal, then Hom_(R/A,R) ^ 0, hence r_o(A) ^ 0. Since every proper left ideal is contained in a maximal one R is a left S-ring.

3.4 Corollary: If _,R is a cogenerator, then R is a left
1 R
S-ring. If _,R is injective and R is a left S-ring then ᵣR
R *^
is a cogenerator.

Remark; This corollary gives a proof of the easy part of theorem 1. Note that the assumptions are only needed on one side in this case.

3.5 Proposition; Let tL be a cogenerator and *T* a left S-ring.
K
Then M_ is injective.

R.

Proof (Special case in [8], page 406):

Let T: M -» Q be a monomorphism of M_o into an injective module Q_R. Then A: = Hom_R(Q,M)T is a left ideal in I\ Assume A f T - Then there exists <p e *T, (p £* 0 such that *Aip* = 0. Furthermore T a monomorphism implies *r<p £* 0. But since *Vi*- is a cogenerator, there exists *rj* e Hom_R(Q,M) such that *r)r<p* ^ 0. This is a contradiction. Hence *A* = *T* and there exists T ' e Horn (Q,M) such that L= T'T e Jl.

Thus T(M) is a direct summand of Q, hence T(M) and as a consequence M are also injective.

3.6 Corollary: Let !L be a cogenerator and R a left S-ring. Then R_R is injective.

Remarks:

1) Corollary 3.6 finishes the proof of theorem 1.

2) Note that in the proof of proposition 3.5 we only need that M_R is a Q cogenerator and Q can be taken to be an injective hull of M_R.

3) By dualizing the proof we get: Let M_R be a generator and r a right S-ring. Then M_R is projective.

3.7 Lemma ([10]): Let M_R be injective and U,V submodules of I^*L_R. Then

$$i_r(unv) = i_I,(u) + i_r(v).$$

Proof: It is easily verified that $1_I,(U) + l_r(V) \subseteq l_I(UDV)$. Let a 6 $l_\Gamma(unv)$ and define

$$\varphi_1: U*u \mapsto ueM,$$

$$\varphi_2: V*v \mapsto (1+a)veM.$$

Then for w e U Pi V: $\varphi_1(w) = w = (l+a)w = \varphi_2(w)$. Thus we have a well-defined R homomorphism

$$\varphi: U + V \ni u + v \mapsto u + (1+a)v' \in M.$$

By assumption φ can be extended to an R homomorphism of M_R into M_R, that is there exists p e T such that pj (U+V) = φ. Hence pu = u for all u € U and pv = (l+a)v for all v e V. Thus a = $(P-1_M) + (l_M+a-p)$ 6 lp(U) + 1_I,(V) and the proof is complete.

3.8 Proposition: Let M_R be injective and $_\Gamma T$ a cogenerator. Then T is a semiperfect ring.

Proof: By [5] we need only show that JT is complemented with respect to addition. Let ^ ⊆ jj[1] and let $B_R \subseteq M_R$ be maximal with respect to $r_M(A) \cap B = 0$. It follows from 3.1 and 3.7 that

$$\Gamma = l_r(0) = l_r(r_M(A) \cap B) = l \wedge U) + l_r(B) =$$

$$= A + l_r(B).$$

Assume pjl ⊆ lp(B) and $T = A + Q$. Then $r_M(A) \cap r_M(Q) =$ r_M(A+fi) = r_M(D = 0. Since B ⊆ $r_M l_r(B) \subseteq r_M$(fi), the maximality of B implies B = r 1 (B) = r_M(ft). Hence by 3.1 $SI = l_r r_M(0) =$ 1_r(B) and l.p(B) is minimal with respect to F = A + 1_r(B).

3.9 Corollary; Assume R_ is injective and _R is a cogenerator. Then R is semiperfect and R_R is a cogenerator.

Proof: That R is semiperfect is immediate from 3.8. Thus there exists only a finite number, say n, of isomorphism classes of simple left R-modules and the same number for the isomorphism classes of simple right R-modules. Let $U_1, \ldots, U_n$ be a set of representatives for the isomorphism classes of simple left R-modules. By 3.2 there exist monomorphisms

$$T_i : U_i \to Q_i, \quad i = 1, \ldots, n$$

with $Q_i \subseteq _R R$ and Q_i injective. Let $U_i' = T_i(U_i)$. Then $U_i' \cong U_i$ and $U_i' \subseteq Q_i \subseteq R$. We may then consider $U_1, \ldots, U_n$ as a set of representatives of the n isomorphism classes and write $U_i = Rx_i$. It follows then from 2.1 that x_iR is also simple. Since R_R is injective x_iR ? xJR for i ^ j by 2.3 and then by 3.2 R_R is a cogenerator.

Remarks;

1) Theorem 2 follows from corollary 3.6 and corollary 3.9.

2) Note that in the above proof we need not use the fact that R is semiperfect but only that the number of isomorphism classes of simple R-modules is finite. This follows from the fact that $R/rad(R)$ is semisimple and this in turn is an easy consequence of the fact that $_RR$ is complemented with respect to addition. More generally we prove: If M_R is complemented with respect to addition, then $\bar{M} = M/rad(M)$ is a semisimple R-module. To see this let $\nu: M \to \bar{M} = M/rad(M)$ be the natural epimorphism and $\bar{A}$ a submodule of $\bar{M}$. Set $A = \nu^{-1}(\bar{A})$. Let A' be a complement of A such that $M = A + A'$ and A' minimal in this equation. Then it is easy to see that $A \cap A'$ is a small submodule of M and hence $A \cap A' \subseteq rad(M)$. It follows then that $\bar{M} = \bar{A} \oplus \bar{A'}$ and $\bar{M}$ is semisimple.

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4. Generators.

In this section we consider certain results on generators which appear in an unpublished paper of P. Pahl [10]. For the convenience of the reader we include here the proofs which are to some extent related to the Morita theorem (see [2], Chapter II or [11], 4.11).

4.1 Consider the R-bihomomorphism

$: Hom_(M_,R_,) 8 M J J ?) . <g>m.i-» £ <p.m. e R
K K . i t _< p 1 . 1 1 1 . 1 1

and the r-bihomomorphism

>!': M g Honu(M_,R_) 3 £ m. ® <o . i» (M 3 x ^ £ m <p. (x)) € T-
R Tl R R ± 1 ' i i

The following relations are easily verified:

Mlp mj&p^m = p ^(m^^igw^^Jm = £ m^. Φ(φ i⊗m) ,

φ 1 Ψ(m⊗φ) = Φ(φ 1⊗m)φ for

φ i, φ 1, <P e Hom R(M R * R R^ and m i ' m e M-

We will use these relations througout this section.

4.2 Lemma: (1) if M_R is a generator then pi is finitely generated and projective.

(2) If Mj^ is finitely generated and projective then J4 is a generator.

(3) If Mg is a generator then R is naturally ringisomorphic to End(pi).

Proof: (1) If M_R is a generator then it follows that Φ is an epimorphism. Hence there exist

$$\eta_1,\ldots,\eta_n \in \mathrm{Hom}_R(M_R,R_R) \quad \text{and} \quad e_1,\ldots,e_n \in M$$

such that

$$\Phi\left(\sum_j \eta_j \otimes e_j\right) = \sum_j \eta_j e_j = 1 \in R.$$

Then for arbitrary $m \in M$

$$m = m \cdot 1 = m \sum_j \Phi(\eta_j \otimes e_j) = \sum_j \Psi(m \otimes \eta_j) e_j .$$

Since $\Psi(-\otimes \eta_j) \in \mathrm{Hom}_T({}_TM, {}_TT)$ we get by the dual basis lemma that ${}_TM$ is finitely generated and projective.

(2) If ${}_TM$ is finitely generated and projective then there exist by the dual basis lemma

$$\varphi_1,\ldots,\varphi_k \in \mathrm{Hom}_R(M_R,R_R) \quad \text{and} \quad m_1,\ldots,m_k \in M$$

such that for all $m \in M$

$$m = \sum_j m_j \varphi_j(m) = \sum_j m_j \Phi(\varphi_j \otimes m) =$$

$$= \sum_j \Psi(m_j \otimes \varphi_j) m .$$

Hence $1_T = \sum_j \Psi(m_j \otimes \varphi_j)$ and since $\Psi(-\otimes \varphi_j) \in \mathrm{Hom}_T({}_TM, {}_TT)$ we get that ${}_TM$ is a generator.

(3) Let M_R be a generator. Consider the ringhomomorphism

$$\rho: R \ni r \mapsto (M \ni x \mapsto xr \in M) \in \mathrm{Hom}_T({}_TM, {}_TM) .$$

Let $f \in \mathrm{Hom}_T({}_TM, {}_TM)$ and $m \in M$. Since M_R is a generator there exist $\eta_1,\ldots,\eta_n \in \mathrm{Hom}_R(M_R,R_R)$ and $e_1,\ldots,e_n \in M$ such that

mf = (£ ^(m®rj.)e.)f = £ tf(m®i7.,)e .f =
j 3 3 3 3 3

= m £ $(rj.®e.f) = m«r_f where r_f 6 R.
j D D

Hence p is an epimorphism.

Let r € R and Mr = 0. Then $e_j r = 0$ and

r = lr = £ $(T7.j®e.)r = £ ${r}.®e-r) = 0.
3 J J 3 J J

Therefore p is a monomorphism.

4.3 Lemmat M_D is a generator iff $ is an isomorphism.

Proof: ^: We mentioned already in the proof of lemma 4.2 that

* is an epimorphism. Assume that $(£ <p.<8*n.) = 0. Then
i i i

E p. ® m = £ S *(T?.®e.)<p.(8im. =
i x i i j 3 D i i

= Z p T7 ^(e_j®p_i) ® nu =
i j j i

= Σ Σ $\eta_j \otimes \Psi(e_j \otimes \varphi_i) m_i$ =
i j

= £ E T?^ ® e . *(<p.®m.) = ^ T? • ® e. (£ $(<p.«jn.)) = 0,
i j] J 1 : L j D J i X 1

where T^J and e_j are chosen as in the proof of lemma 4.2. Thus
j

$ is a monomorphism.

e.: M_R is a generator since $ is an epimorphism and

R_R is a generator.

4.4 Lemma: Let M_R be a generator and A a right ideal of R.

Denote by t : A_R -. R_R the inclusion map. Then

Hom(l_M,C) ® I..

(1) 0 -. $Hom_R(M_R,A_R)$ ®_Γ M ______1________^ $Hom_R(M_R,R_R)$ ®_Γ M is exact

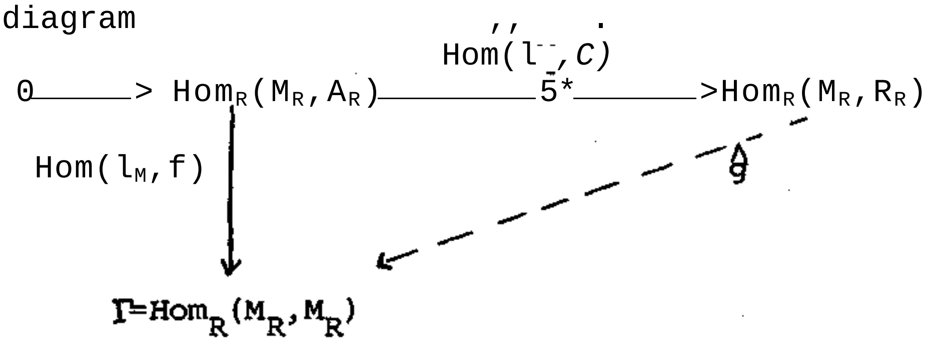
and

(2) $\Phi^{-1}(A) = \mathrm{Hom}_R(M_R, A_R) \otimes_\Gamma M$ (where we are identifying $\mathrm{Hom}_R(M_R, A_R) \otimes_\Gamma M$ with a submodule of $\mathrm{Hom}_R(M_R, R_R) \otimes_\Gamma M$ via (1)).

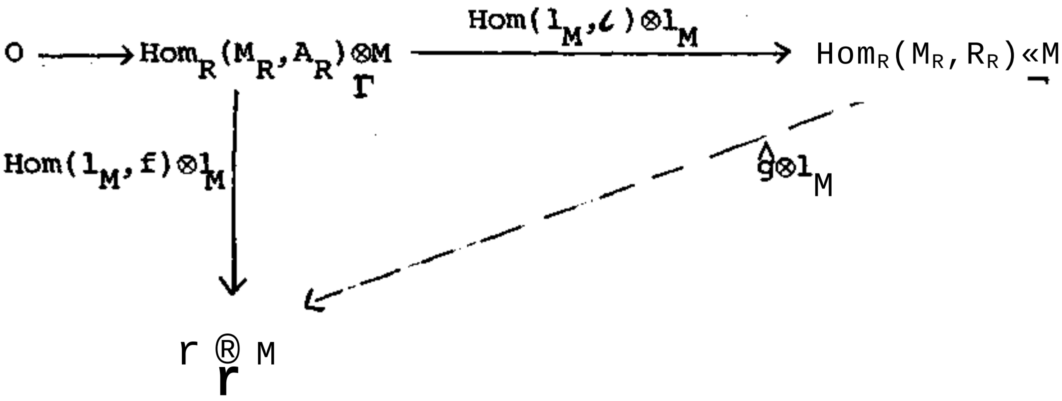
Proof: Since Hom is left exact and by lemma 4.2 (1) $_\Gamma M$ is projective hence flat, we obtain (1). Since M_R is a generator $\Phi(\mathrm{Hom}_R(M_R, A_R) \otimes_\Gamma M) = A$ and since Φ is an isomorphism by lemma 4.3, (2) follows.

<u>4.5 Lemma</u>: If M_R is a generator and Γ_Γ is injective, then M_R is injective.

Proof: Let A be a right ideal of R and $f \in \mathrm{Hom}_R(A_R, M_R)$. We must find $g \in \mathrm{Hom}_R(R_R, M_R)$ such that $g|A = f$. Consider the following diagram



where $\iota : A_R \to R_R$ is the inclusion map. In this diagram the row is exact and all mappings are right Γ-homomorphisms. Since Γ_Γ is injective $\hat{g}$ exists making the diagram commutative. From this we get a commutative diagram with exact row (see lemma 4.4):



Using the isomorphism

6: r O M3» z 7i ® m M H y m e M
T i i

and the fact that $ is an isomorphism by lemma 4.3 we set

g = efS®]^)*""[1]. If a e A then #"[1](a) = £ ^KSHIK with (/?ieHom$_R$(M$_R$,A$_R$)
i

(by lemma 4.4). Hence g(a) = 6(gVL$_M$) (T, (p^Om^) = 6(£ fpJSim^ =
i i

= T f(<p.m.) = f(~£<p m.) = f(a) and g is the desired homomorphism.
i i

A module M$_D$ is called selfinjective iff for every exact
R

sequence O -. A$_n$ H M$_n$ and every R-homomorphism f: A$_o$ -* M . there
R R K R

exists g: M$_D$ - M$_D$ such that f = get.

4.6 Proposition (P. Pahl): Let M$_R$ be selfinjective and -fl flat.
i

Then r$_y$ is injective.

Proof: Let A be a right ideal of T, <-t Ap -» T-p the inclusion map and f: A -» T an arbitrary homomorphism. We have to show that there exists y_o e V such that f(A) = y A for all A e A.
o

Since M is flat 0 -• A ® M —C<S>1M→* r ® M is exact. Thus A ® M
1 r r r

will be considered as a submodule of r ® M. Let 6: r ® M -. M
r r

be the isomorphism described above. Its restriction

6$_o$: A ® M 3 S A . ® m. *-> j A m. e AM
r ± i i i i i

is an isomorphism. Set ĝ = 6(f®l$_M$)6$_o^{-1}$. Then ĝ is a right R-homomorphism of AM into M. Since M$_R$ is selfinjective there exists y_Q e T such that y$_Q$|AM = §. Then ĝ(Am) = f(A)m = y Am where m e M, A e A and thus f(A) = y A for all A e A.

4.7 Proof of theorem 3 (as stated in the introduction):

(2) =» (1): by definition.

(1) =» (3): by proposition 4.6.

(3) =» (2): by lemma 4.5.

Remark: (1) =» (2) is a special case of the fact that if a module is injective with respect to a generator, then it is injective„

5. Proofs of theorems 4 and 5.

We again refer the reader to the introduction for the statements of theorems 4 and 5.

5.1 Lemma: Let M_R be a projective cogenerator. Then R_R is a cogenerator.

Proof: Since M_D is projective there exists a monomorphism r?: M_R, -• © $_i R_i$ where $R_i = {}_{"}R_{—}$ for all i e I. Let A_R be an arbitrary module and 0 ^ a e A. We have to find g: A_R -* R_R such that g(a) ^ 0. Since $M_{.K}$ is a cogenerator there exists f: A -. M such that f(a) ^ 0. Since *rt* is a monomorphism we get t7f(a) ^ 0. Hence there exists a projection w_3: $@_{iel}$ R_x -» R_1 such that $Tr_J f]f(a)$ ^ 0. Thus we can set g = ir_J^f.

5.2 Lemma; Let R_R be a cogenerator and R a left S-ring. If M is faithful then M, is a generator.

Proof: ([8], page 407): Consider the left ideal A of R defined by A = E ima(p)$_{<peHom_R(M_R,R_R)}$. Assume A c_F R. Since R is a left S-ring there exists r_o e R, r_o ^ 0 such that Ar_o = 0. It follows then that <p(Mr_Q) = 0 for all *<p e* $Hom_R(M_R,R^R)$. Since R_R is a cogenerator this implies Mr^o = 0 and since M^K is faithful we get r_Q = 0. Hence A = R and ^ is a generator.

5.3 Proof of theorem 4:

(1) => (2) : Since $_I$M and *t^* are projective cogenerators it follows by lemma 5.1 that R_R and $J"_I$ are cogenerators. This implies by corollary 3.4 that R is a right S-ring and T is a

left S-ring. By proposition 3.5 we get that M_R is injective and since $_T fl$ is flat it follows by proposition 4.6 that *Tj,* is injective. Using the assumption that R is naturally ring-isomorphic to End($_T$Jl) we get similarly that $_T$# and $_R R$ are injective. Then it follows by corollary 3.9 that $_R R$ and $r U_T$ are cogenerators. Hence ^ and M_R are generators by lemma 5.2.

(2) => (1) : Let U_0 be simple. Then there exists $x \in R$ such that $U_D \cong xR$ since R is a right S-ring. Since M_, is a generator M_, is faithful and there exists $m \in M$ such that mx ^ 0. Then $U_R \cong xR \cong mxR \subseteq M$. Using the assumption that M_R is injective we get by lemma 3.2 that ML is a cogenerator. By lemma 4.2 R is naturally ring isomorphic to Endf^-M) and M_R is projective. Similarly it follows that $_T$JI is a projective cogenerator.

5.4 Proof of theorem 5:

(1) => (3) ; By lemma 4.2 $_T fl$ is projective. Hence by lemma 5.1 it follows that $_T T$ is a cogenerator. Since M_R is an injective generator we get by theorem 3 that r_T is injective. Hence by corollary 3.9 *T* is semiperfect, r_T is a cogenerator and by corollary 3.6 $_T F$ is injective. Using the fact that $_T$J4 is finitely generated and projective it follows that $_T N$ is injective (since $_T J''$ is injective) and $_T M$ is semiperfect (since *T* is semiperfect) . Obviously $_T$J^l is faithful. Hence $_T$J4 is a generator by lemma 5.2. Then it follows by lemma 4.2 that M is a finitely generated projective generator.

By the Morita theorem it follows that the functor -ig> M^ is an equivalence between module categories. Since $\Gamma \to M_R = M_R$ and $I L_\Gamma$ is a cogenerator we then get that M_K is a cogenerator.

We now apply the above results under the assumptions $_I M$ is an injective generator and M_R is a cogenerator, where R is viewed as the endomorphism ring of $_I M$ via lemma 4.2. This allows us to obtain the rest of the statements in (3).

(3) => (1): trivial.

(2) => (3): Since M_R is finitely generated and projective we get by lemma 4.2 that $_I M$ is a generator. By assumption R can be identified with the endomorphism ring of $_I M$. Switching sides we then copy the proof of (1) => (3) to obtain (3).

(3) => (2): Since $_R L$ is a generator it follows by lemma 4.2 that R is naturally ringisomorphic to End($_I L$). The other statements in (2) follow trivially.

Carnegie-Mellon University
Pittsburgh, Pennsylvania
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References

[I] Asumaya, G.: Completely faithful modules and self-injective rings, Nagoya Math. J. 27, 697-708 (1966).

[2] Bass, H.: Algebraic K-theory, W. A. Benjamin, New York, 1968.

[3] Bumby, R.: Modules which are isomorphic to submodules of each other, Arch. Math. 16, 184-185 (1965).

[4] Faith, C: Rings with ascending condition on annihilators, Nagoya Math. J. 27, 179-191 (1966).

[5] Kasch, F. and E. A. Mares, Eine Kennzeichnung semi-perfekter Modules, Nagoya Math. J. 27, 525-529 (1966).

[6] Kato, T.: Torsionless modules, Tôhuku Math. J. 20, 234-243 (1968).

[7] Morita, K.: On S-rings in the sense of F. Kasch, Nagoya Math. J. 27, 687-695 (1966).

[8] Onodera, T.: Über Kogeneratoren, Arch. Math. 19, 402-410 (1968).

[9] Osofsky, B. L.: A generalization of quasi-Frobenius rings, J. Algebra 4, 373-387 (1966).

[10] Pahl, P.: Quasi-Frobenius-Generatoren und verallgemeinerte Dualitätsfragen, Dissertation München 1964.

[II] Pareigis, B.: Kategorien und Funktoren, B. G. Teubner, Stuttgart, 1969.

[12] Sugano, K.: A note on Azumaya's theorem, Osaka J. Math. 4, 157-160 (1967).